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Constructible Sheaves Associated to Whittaker Functions

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Introduction

Let X_0 be a proper smooth geometrically connected curve over the field $\boldsymbol{F}_q$ with q elements. Let K be the function field of X_0 over $\boldsymbol{F}_q$, $\boldsymbol{A}$ the adele ring of K, and ℓ a prime number prime to the characteristic of $\boldsymbol{F}_q$. Let $\pi_1(X_0)$ be the fundamental group of X_0. (For the fundamental group, see [8, p. 39].) We always assume that a continuous representation

$$\rho\colon \pi_1(X_0) \longrightarrow \mathrm{GL}(n, \bar{\boldsymbol{Q}}_\ell) \qquad (\bar{\boldsymbol{Q}}_\ell\colon \text{an algebraic closure of } \boldsymbol{Q}_\ell)$$

of $\pi_1(X_0)$ factors through

$$\rho\colon \pi_1(X_0) \longrightarrow \mathrm{GL}(n, E),$$

where E is a finite extension of $\boldsymbol{Q}_\ell$.

Such a ρ gives rise to an L-function

$$L(\rho, s) = \prod_{v \in |X_0|} \det(1 - \mathrm{Nm}(v)^{-s} \rho(\mathrm{Fr}_v))^{-1} \in \bar{\boldsymbol{Q}}_\ell[[q^{-s}]],$$

where $|X_0|$ is the set of closed points of X_0, and Fr_v is the geometric Frobenius substitution at v.

Langlands ([6, p. 211]) asked whether it is an automorphic L-function. (For the definition of automorphic L-function, see [2, p. 49]). Drinfeld (cf. [3]) has solved this problem for $n=2$. First he expressed the Whittaker function associated to ρ by the trace of the Frobenius substitution on some constructible sheaf. Next, he proved geometrically that the Shalika transform (cf. [9]) of the Whittaker function turns out to be an automorphic form.

For a representation ρ as above, we can associate a function f on $\mathrm{GL}(n, \boldsymbol{A})$ called the Whittaker function for ρ. By the functional equation satisfied by the Whittaker function, it can be regarded as a function on $U_K \backslash \mathrm{GL}(n, \boldsymbol{A}) / \mathrm{GL}(n, \hat{O})$, where U_K is the subgroup of upper triangular

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matrices in $\mathrm{GL}(n, K)$. On the other hand, we can define some moduli scheme $(J\times_P \mathrm{Flag}_{\mathscr{L}}^{d,0})_0$ over $\boldsymbol{F}_q$, whose $\boldsymbol{F}_q$-rational points can be identified with some elements of $U_K\backslash \mathrm{GL}(n, A)/\mathrm{GL}(n, \hat{O})$. The purpose of this paper is to construct a constructible sheaf $\mathrm{Wh}_{\mathscr{L}}^{d}(\rho)$ on $(J\times_P \mathrm{Flag}_{\mathscr{L}}^{d,0})_0$ with the following property: The value of the Whittaker function f at g corresponding to the element w of $(J\times_P \mathrm{Flag}_{\mathscr{L}}^{d,0})_0$, can be expressed in terms of the trace of the Frobenius substitution at w on the geometric fiber $\mathrm{Wh}_{\mathscr{L}}^{d}(\rho)_{\bar{w}}$ of $\mathrm{Wh}_{\mathscr{L}}^{d}(\rho)$ at $\bar{w}$.

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§ 1. Motivation and group theoretic background

1.1. *L*-functions of class 1 principal series

We here recall necessary results of Godement-Jacquet [5] and Zevelinsky-Bernstein [1].

Let K_v be a nonarchimedean local field with a finite residue field $\boldsymbol{F}_q$ with q elements. Let O_v be the ring of integers of K_v, t_v a uniformizing parameter, and $\| \quad \|$ its nonarchimedean absolute value.

Assume that we are given unramified characters

$$\pi_i\colon\ K_v^*/O_v^* \longrightarrow \boldsymbol{C}^* \qquad \text{for } i=1, \cdots, n,$$

satisfying the following condition:

$$(*) \qquad \pi_i(t_v)\neq q\pi_j(t_v) \qquad \text{for all } i\neq j.$$

We then define a representation $\pi(\pi_1, \cdots, \pi_n)$ induced by $\pi_1, \cdots, \pi_n$ as follows. Let $\pi(\pi_1, \cdots \pi_n)$ be the vector space of $\boldsymbol{C}$-valued functions on $\mathrm{GL}(n, K_v)$ satisfying the following conditions (1) and (2):

$$(1) \qquad f\left(\begin{bmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{bmatrix} g\right)=\prod_{\Delta_+\ni\alpha}\|\alpha(a_1, \cdots, a_n)\|\prod_{i=1}^{n}\pi_i(a_i)f(g)$$

for all $g \in \mathrm{GL}(n, K_v)$. Here, Δ_+ is the set of positive roots of $\mathrm{GL}(n, K_v)$ with respect to the Borel subgroup of upper triangular matrices in $\mathrm{GL}(n, K_v)$,

(2) $\{h \in \mathrm{GL}(n, K_v) | f(gh)=f(g) \text{ for all } g \in \mathrm{GL}(n, K_v)\}$ is an open subgroup of $\mathrm{GL}(n, K_v)$.

$GL(n, K_v)$ acts on this space by right translation, and this space gives an irreducible representation which belongs to the class 1 principal series (cf. [1, p. 454]).

The L-function of this representation is defined by Godement-Jacquet ([5, p. 163]) as follows.

Definition (the spherical function of a class 1 principal series). Let (π, V) be an irreducible representation of $GL(n, K_v)$ in the class 1 principal series and (π', V') its dual. We can choose $v_0 \in V$, $v'_0 \in V'$ such that

$$\pi(g)v_0 = v_0,\ \pi(g)v'_0 = v'_0 \text{ for all } g \in GL(n, O_v) \text{ and } \langle v_0, v'_0 \rangle = 1.$$

We define the spherical function f_0: $GL(n, K_v) \to C$ of π by

$$f(g) = \langle \pi(g)v_0, v'_0 \rangle.$$

Note that f_0 is uniquely determined by (π, V), because v_0 and v'_0 satisfying the above conditions are unique up to constant multiple for an irreducible representation in class 1 principal series.

Definition (L-function). Let Φ be the characteristic function of $M(n, O_v) \cap GL(n, K_v)$, dx^* the Haar measure of $GL(n, K_v)$ normalized by $dx^*(GL(n, O_v)) = 1$. We define the L-function of an irreducible representation in the class 1 principal series (π, V) by using the spherical function f_0 of π in the following way:

$$L(\pi, s) = \int_{GL(n,K_v)} \Phi(x) f_0(x) \| \det(x) \|^{s+(n-1)/2} dx^*.$$

We can also describe the L-function in terms of the Hecke algebra

$H_0 = \{$bi-$GL(n, O_v)$-invariant C-valued functions with compact support on $GL(n, K_v)\}$.

H_0 becomes an algebra under the convolution product. An element φ of the Hecke algebra H_0 acts on V by

$$T(\varphi)v = \int_{GL(n,K_v)} \varphi(x)(\pi(x)v) dx^* \qquad \text{for } v \in V.$$

Let $v_0 \in V$ be an eigenvector with respect to the Hecke algebra H_0. If Φ_m is the characteristic function of $\{x \in M(n, O_v) \mid \|\det(x)\| = q^{-m}\}$, and if $T(\Phi_m)v_0 = \lambda(\Phi_m)v_0$, then

$$L(\pi, s) = \sum_{n=0}^{\infty} q^{-m(s+(n-1)/2)} \lambda(\Phi_m)$$

holds. Let δ_i be the characteristic function of

$$\mathrm{GL}(n, O_v)\begin{pmatrix} \overbrace{t_v \quad \ddots \quad t_v}^{i} & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}\mathrm{GL}(n, O_v),$$

where t_v is a uniformizing parameter of K_v and let $\lambda(\delta_i)$ be the eigenvalue of $T(\delta_i)$:

$$T(\delta_i)v_0=\lambda(\delta_i)v_0.$$

If $\mu_1, \cdots, \mu_n$ are the roots of the equation

$$\sum_{i=0}^{n}(-1)^i q^{i(i-1)/2}\lambda(\delta_i)x^i=0, \tag{1.1}$$

in x, then we have (cf. [5, p. 77]).

$$L(\pi, s)=\prod_{j=1}^{n}(1-\mu_j q^{-(n-1)/2-s})^{-1},$$

which is known to be a rational function of q^{-s} (cf. [5]).

1.2. Shintani's formula and a formulation of Langlands' problem

Let K_v be a nonarchimedean local field and t_v, O_v its uniformizing parameter and the ring of integers, respectively. Let ψ be a nontrivial $\boldsymbol{C}$-valued additive character of K_v, and U_{K_v} the subgroup of $\mathrm{GL}(n, K_v)$ of unipotent upper triangular matrices. We define a character $\bar{\psi}$ of U_{K_v} by

$$\bar{\psi}\left(\begin{bmatrix} 1 & u_1 & & * \\ & \ddots & \ddots & \\ & & \ddots & u_{n-1} \\ 0 & & & 1 \end{bmatrix}\right)=\psi(u_1+\cdots+u_{n-1}).$$

We define the space ω of Whittaker functions by

$$\omega=\{f \mid f \text{ is a locally constant function on } \mathrm{GL}(n, K_v) \text{ such that } f(ug)=\bar{\psi}(u)f(g) \text{ for all } g\in \mathrm{GL}(n, K_v),\ u\in U_{K_v}\}.$$

This space is a representation of $\mathrm{GL}(n, K_v)$ under the right translation of $\mathrm{GL}(n, K_v)$. Any irreducible representation π_v of $\mathrm{GL}(n, K_v)$ in the class 1 principal series can be realized as a unique subrepresentation (π_v, ω_v) of ω (cf. [4, p. 315]). ω_v is called the Whittaker model of π_v.

Theorem 1.1 (Shintani [11]). *Suppose that ψ is trivial on O_v and non-trivial on $t_v^{-1}O_v$. In other words, the conductor of ψ is O_v. Let π_v be an irreducible representation in the class* 1 *principal series, (π_v, ω_v) the space defined as above, and μ_i the complex numbers defined in* (1.1). *Let f be an element of ω_v fixed under the action of π_v $(\mathrm{GL}(n, O_v))$ such that $f(e)=1$.*

Then the value $f(\mathrm{diag}(t_v^{f_1}, \cdots, t_v^{f_n}))$ of f at the diagonal matrix $\mathrm{diag}(t_v^{f_1}, \cdots, t_v^{f_n})$ *is equal to*

$$(1.2) \qquad q^e \chi_Y(\mu) \quad \text{if} \quad Y=(f_1, \cdots, f_n), \quad f_1 \geqq \cdots \geqq f_n, \quad f_i \in \boldsymbol{Z},$$

where $e=\sum_{i=1}^n (i-n) f_i$, while $f(\mathrm{diag}(t_v^{f_1}, \cdots, t_v^{f_n}))=0$ otherwise. Here χ_Y is the irreducible character of $\mathrm{GL}(n, \boldsymbol{C})$ *associated to the Young diagram Y, and μ is the conjugacy class represented by the diagonal matrix* $\mathrm{diag}(\mu_1, \cdots, \mu_n)$.

Notice that f is uniquely determined because π_v belongs to class 1 principal series.

Remark 1 (cf. [11, p. 180]). Using the Cartan decomposition

$$\mathrm{GL}(n, K_v) = U_{K_v} \cdot T_{K_v} \cdot \mathrm{GL}(n, O_v)$$

with

$$T_{K_v} = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \in \mathrm{GL}(n, K_v) \right\},$$

the values of a Whittaker function f on $\mathrm{GL}(n, K_v)$ are determined by the above formula (1.2). Conversely for given non-zero complex numbers $\mu_1, \cdots, \mu_n$, the Whittaker function determined by (1.2) generates an irreducible representation in the class 1 principal series contained in ω provided that

$$(*') \qquad \mu_i \neq q\mu_j \quad \text{for} \quad i \neq j.$$

Remark 2. Let f be a Whittaker function with respect to ψ, and a an element of K_v^*. The function $\gamma_a(f)$ on $\mathrm{GL}(n, K_v)$ given by

$$(\gamma_a(f))(g) = f(\mathrm{diag}(1, a, \cdots, a^{n-1})g)$$

is a Whittaker function with respect to $\psi \circ a^{-1}$. This transformation γ_a gives an equivalence of representations between the Whittaker models with respect to ψ and those with respect to $\psi \circ a^{-1}$.

Now we formulate the problem of Langlands. Let K be a global field of characteristic $p>0$ and A its adele ring. Let χ be an unramified C-valued character of A^*/K^* with absolute value 1. We define the space $L_0^2(\mathrm{GL}(n, K)\backslash \mathrm{GL}(n, A), \chi)$ of cusp forms with a central character χ as the space of locally constant functions f on $\mathrm{GL}(n, A)$ satisfying the following four conditions:

i) $f(\gamma x)=f(x)$ for all $x \in \mathrm{GL}(n, A)$, $\gamma \in \mathrm{GL}(n, K)$.

ii) $f(zx)=\chi(z)f(x)$ for all $z \in A^*$, $x \in \mathrm{GL}(n, A)$.

iii) $\int_{A^*\mathrm{GL}(n,K)\backslash \mathrm{GL}(n,A)} |f(\dot{x})|_C^2 d\dot{x}<\infty$,

where $d\dot{x}$ is the measure induced by a Haar measure of $\mathrm{GL}(n, A)$ and $|\ \ |_C$ is the complex absolute value.

iv) For the unipotent radical U of any proper parabolic subgroup P of $\mathrm{GL}(n, K)$, we have

$$\int_{U_K\backslash U_A} f(ux)du=0 \text{ for almost all } x \in \mathrm{GL}(n, A),$$

where du is the measure induced by a Haar measure of U_A.

Let ℓ be a prime number different from p. From now on, we fix an identification of C and $\bar{Q}_\ell$. Consider a continuous representation $\rho\colon \pi_1(X_0) \to \mathrm{GL}(n, \bar{Q}_\ell)$, and assume that the following conditions hold:

(1.3) $|\det(\rho(Fr_v))|_C=1$ for all $v \in |X_0|$ under the fixed identification of C and $\bar{Q}_\ell$.

(1.4) For $v \in |X_0|$ let $\mu_1 q^{-(n-1)/2}, \cdots, \mu_n q^{-(n-1)/2}$ be the inverse of the eigenvalues of $\rho(Fr_v)$. Then the condition $(*')$ holds for $\{\mu_i\}_{i=1,\cdots,n}$.

Let us formulate Langlands' problem. Let ψ'_v be an additive character of K_v with the conductor O_v. The eigenvalues $\mu_1, \cdots, \mu_n$ of $\rho(\mathrm{Fr}_v)$ define a Whittaker function f' by Remark 1 to Theorem 1.1. Any additive character ψ_v of K_v can be written as $\psi'_v \circ a^{-1}$, where the conductor of ψ'_v is O_v and a an element of K_v. Thus by Remark 2 to Theorem 1.1, $\gamma_a(f')$ generates an irreducible subrepresentation (π_v, ω_v) of $\mathrm{GL}(n, K_v)$ in the space ω of Whittaker functions with respect to ψ_v.

Langlands' Problem. *Let $\psi=\prod_v \psi_v$ be a quasi-character of A/K. Consider the Whittaker model (π_v, ω_v) with respect to ψ_v as above. Then is $\pi=\otimes_v \pi_v$ equivalent to some constituent of $L_0^2(\mathrm{GL}(n, K)\backslash \mathrm{GL}(n, A)$, $\det \rho)$ as a representation of $\mathrm{GL}(n, A)$?*

1.3. The Global Whittaker function and the Shalika transform

Let K be a global field of characteristic $p>0$, X_0 the corresponding curve over F_q, and ρ a continuous representation of $\pi_1(X_0)$ of degree n over $\bar{Q}_\ell$. Assume that ρ satisfies the conditions (1.3), (1.4) of the previous paragraph. We also assume that the genus of X_0 is positive. Let K_v be the completion of K at v, and O_v the ring of integers of K_v. Put $\hat{O}=\prod_v O_v$. We fix a nontrivial additive character $\psi=\prod_v \psi_v$ of $A/(K+\hat{O})$. Then the conductor of ψ_v is O_v for almost all v. For all v the additive character ψ_v of K_v can be written as $\psi'_v \circ u_v^{-1}$, where the conductor of ψ'_v is O_v and u_v an element of K_v. The eigenvalues $\mu_1, \cdots, \mu_n$ of $\rho(\mathrm{Fr}_v)$ determine a Whittaker function f_v in view of Remark 1 to Theorem 1.1. Let $\tilde{f}_v:=\gamma_{u_v}(f_v)$ and define the global Whittaker function f on $\mathrm{GL}(n, A)$ associated to ρ by

$$f(g)=\prod_v \tilde{f}_v(g_v) \qquad \text{for} \quad g=(g_v)\in \mathrm{GL}(n, A).$$

We can define the global Whittaker model associated to ρ in the following way: Define a character $\bar{\psi}$ of U_A by

$$\bar{\psi}\left(\begin{bmatrix} 1 & u_1 & & * \\ & \ddots & \ddots & \\ & & \ddots & u_{n-1} \\ 0 & & & 1 \end{bmatrix}\right):=\psi(u_1+\cdots+u_{n-1}).$$

Let U_A be the subgroup of $\mathrm{GL}(n, A)$ of unipotent upper triangular matrices and ω_K the space consisting of locally constant functions f on $\mathrm{GL}(n, A)$ such that $f(ug)=\bar{\psi}(u)f(g)$ for all $u\in U_A$ and $g\in \mathrm{GL}(n, A)$. $\mathrm{GL}(n, A)$ acts on the space ω_K by the right translation. We can easily show that the global Whittaker function f belongs to ω_K. The subrepresentation of ω_K generated by this f is called the Whittaker model associated to ρ. It is irreducible, because ρ satisfies the condition (1.4) in the previous paragraph.

We omit the proof for the following, since it is standard.

Proposition 1.2 (Shalika transform). *Let f be the global Whittaker function associated to ρ. The summation*

$$\varphi(g)=\sum_{U_{n-1,K}\backslash \mathrm{GL}(n-1,K)\ni\gamma} f\left(\begin{bmatrix}\gamma & 0\\ 0 & 1\end{bmatrix}\right) \qquad \text{for} \quad g\in \mathrm{GL}(n, A)$$

is essentially finite and defines a function on $\mathrm{GL}(n, A)$, *where* $U_{n-1,K}$ *is the subgroup of* $\mathrm{GL}(n-1, K)$ *of unipotent upper triangular matrices. Moreover, the following equality holds for some constant* $c\neq 0$:

$$f(g)=c\int_{U_K\backslash U_A} \bar{\psi}(u^{*-1})\varphi(u^*g)du^*,$$

where du^* *is the measure induced by a Haar measure of* U_A.

Theorem 1.3 (Shalika [9]). *Let* $f \in L_0^2(\mathrm{GL}(n, K)\backslash \mathrm{GL}(n, A), \chi)$ *and put*

$$W_f(g) := \int_{U_K \backslash U_A} \bar{\psi}(u^{*-1}) f(u^* g) du^*.$$

Then we have

$$f(g) = \sum_{U_{n-1,K}\backslash \mathrm{GL}(n-1,K) \ni \gamma} W_f\left(\begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix} g\right).$$

Question. *Is* φ *defined in Proposition* 1.2 *invariant under the left translation for* $\mathrm{GL}(n, K)$?

The following sections are devoted to the geometric interpretation for the global Whittaker function.

§ 2. The construction of the Whittaker sheaves

2.1. Representability

Let X be a proper smooth absolutely irreducible curve over a field k. For an integer $n \geqq 2$, let $\mathscr{L}$ be a locally free sheaf of rank n over X. We write $\boldsymbol{d} := (d_1, \cdots, d_{n-1})$ for integers $d_1, \cdots, d_{n-1}$. Consider the following functor

$$\boldsymbol{Flag}_{\mathscr{L}}^{\boldsymbol{d}} : (\mathrm{Sch}/k)^{\circ} \longrightarrow (\mathrm{Sets})$$

which sends T to the set of sequences $\mathrm{pr}_1^* \mathscr{L} = \mathscr{L}_0 \supset \mathscr{L}_1 \supset \cdots \supset \mathscr{L}_{n-1}$ of subsheaves of $\mathrm{pr}_1^* \mathscr{L}$ over $X \times_k T$ such that

(i) $\mathscr{L}_i$ is a locally free sheaf of rank $n-i$ over $X \times_k T$,

(ii) $\mathscr{L}_0 / \mathscr{L}_i$ is flat over T, and

(iii) $\deg(\mathscr{L}_i|_{X \times \{t\}}) = d_i$ for all $t \in T$.

Theorem 2.1. *The functor* $\boldsymbol{Flag}_{\mathscr{L}}^{\boldsymbol{d}}$ *is represented by a proper scheme over* k.

For the proof of the above theorem, we show the following:

Proposition 2.2. *Let* n, m *and* d *be integers such that* $n \geqq m \geqq 1$. *Let* $\mathscr{L}$ *be a locally free sheaf of rank* n. *Then the functor*

$$\boldsymbol{Flag}_{\mathscr{L},m}^{d} : (\mathrm{Sch}/k)^{\circ} \longrightarrow (\mathrm{Sets})$$

which sends T *to the set of locally free subsheaves* $\mathscr{L}_1$ *of the locally free sheaf* $\mathrm{pr}_1^* \mathscr{L}$ *over* $X \times_k T$ *such that*

(i) rank $\mathscr{L}_1 = m$,
(ii) $\mathrm{pr}_1^* \mathscr{L}/\mathscr{L}_1$ *is flat over* T, *and*
(iii) $\deg(\mathscr{L}_1|_{X\times\{t\}}) = d$ *for all* $t \in T$.
is represented by a proper scheme over k.

Lemma 2.3. *Assume that* X *has a* k*-rational point* x_0 *which determines an invertible sheaf* $O(x_0)$ *of degree* 1. *Then there exists a natural number* k_0 *depending only on* X *and* $\mathscr{L}$, d, m *such that any locally free subsheaf* $\mathscr{L}_1$ *of* $\mathscr{L}$ *of degree* d *and rank* m *have the properties that* $\mathscr{L}(kx_0)$ *and* $\mathscr{L}_1(kx_0)$ *are generated by global sections and that* $h^1(\mathscr{L}_1(kx_0)) = 0$ *for any* $k \geqq k_0$.

Proof. Step 1. Let g be the genus of X, and $\mathscr{L}_1$ a locally free subsheaf of $\mathscr{L}$ of rank m and degree d. Then any invertible quotient sheaf of $\mathscr{L}_1$ has degree greater than or equal to $d - h^0(\mathscr{L}) - (m-1)(g-1)$. Indeed, let $\mathscr{L}'$ be a invertible quotient sheaf of $\mathscr{L}_1$, and let $\mathscr{L}'' := \ker(\mathscr{L}_1 \to \mathscr{L}')$. Then by the Riemann-Roch theorem we have

$$\deg \mathscr{L}'' = h^0(\mathscr{L}'') - h^1(\mathscr{L}'') + (m-1)(g-1).$$

Thus,

$$\deg \mathscr{L}'' \leqq h^0(\mathscr{L}) + (m-1)(g-1).$$

Hence

$$\deg \mathscr{L}' = d - \deg \mathscr{L}'' \geqq d - h^0(\mathscr{L}) - (m-1)(g-1).$$

Step 2. In the notation of Step 1, there exists a natural number k_0 such that $H^1(\mathscr{L}_1(kx_0))$ and $H^1(\mathscr{L}_1(kx_0 - x))$ vanish, while $\mathscr{L}_1(kx_0)$ is generated by global sections for all $x \in X$ and $k \geqq k_0$. Indeed, by the Serre duality, we have

$$\begin{aligned} H^1(X, \mathscr{L}_1(kx_0 - x))^\vee &\simeq H^0(X, \mathscr{L}_1(kx_0 - x)^\vee \otimes \Omega_X^1) \\ &\simeq \mathrm{Hom}(\mathscr{L}_1, \Omega_X^1(-kx_0 + x)), \\ H^1(X, \mathscr{L}_1(kx_0))^\vee &\simeq \mathrm{Hom}(\mathscr{L}_1, \Omega_X^1(-kx_0)). \end{aligned}$$

Fix a natural number k_0 such that

$$2g - 2 - k_0 + 1 < d - h^0(\mathscr{L}) - (m-1)(g-1) - m.$$

Then by Step 1 we have $\mathrm{Hom}(\mathscr{L}_1, \Omega_X^1(-k_0x_0 + x)) = \mathrm{Hom}(\mathscr{L}_1, \Omega_X^1(-k_0x_0)) = 0$, for all $k > k_0$. In this situation, the homomorphism

$$H^0(X, \mathscr{L}_1(kx_0)) \longrightarrow H^0(X, \mathscr{L}_1(kx_0) \otimes k(x)) \simeq \mathscr{L}_1(kx_0) \otimes k(x)$$

is surjective for all $k \geqq k_0$ and $x \in X$. Therefore $\mathscr{L}_1(kx_0)$ is generated by global sections.

To show the rest of the lemma, it is enough to choose k_0 large enough that $\mathscr{L}(kx_0)$ is generated by global sections for all $k \geqq k_0$. q.e.d.

Proof of Proposition 2.2. For the proof of the representability, we may assume that X has a rational point x_0, for otherwise choose a separable finite extension of k over which X has a rational point and use descent theory. Let us fix a natural number k greater than k_0 as in Lemma 2.3. We have an isomorphism $\boldsymbol{Flag}^{d}_{\mathscr{L},m} \simeq \boldsymbol{Flag}^{d+km}_{\mathscr{L}(kx_0),m}$ of functors. By Lemma 2.3, we may assume that $\mathscr{L}$ as well as any locally free subsheaf $\mathscr{L}_1$ of $\mathscr{L}$ of degree d and rank m are generated by global sections, and that $h^1(\mathscr{L}) = h^1(\mathscr{L}_1) = 0$.

$$\boldsymbol{Flag}^{d}_{\mathscr{L},m}(T) \ni (\mathscr{L}_1/X \times T) \longrightarrow (\mathrm{pr}_{2*}\mathscr{L}_1/T) \in \boldsymbol{Grass}\,(T)$$

gives an injective morphism of functors, where $\boldsymbol{Grass}$ is the Grassmannian functor with $\boldsymbol{Grass}\,(T)$ consisting of subvectorbundles of rank e in $H^0(\mathscr{L}) \otimes O_T$, where $e = d + m(1-g)$.

Let $\mathscr{M}$ be the universal locally free subsheaf of $O_{\boldsymbol{Grass}} \otimes_k H^0(\mathscr{L})$ on $\boldsymbol{Grass}$ and $p: X \to \operatorname{Spec} k$ the structure morphism. Consider the following natural homomorphisms of sheaves on $X \times_k \boldsymbol{Grass}$.

$$\mathrm{pr}_2^*\mathscr{M} \longrightarrow H^0(\mathscr{L}) \otimes_k O_{X \times \boldsymbol{Grass}} \simeq \mathrm{pr}_1^* p^* p_* \mathscr{L} \longrightarrow \mathrm{pr}_1^* \mathscr{L}.$$

Let T be the stratum corresponding to the Hilbert polynomial $P(t) = \deg \mathscr{L} + n(1-g) - (d + m(1-g)) + t(n-m)$ of the flattening stratification of $\operatorname{Coker}(\mathrm{pr}_2^*\mathscr{M} \to \mathrm{pr}_1^*\mathscr{L})$ on $\boldsymbol{Grass}$. For $\mathscr{L}_1 := \operatorname{Im}(\mathrm{pr}_2^*\mathscr{M}|_T \to \mathrm{pr}_1^*\mathscr{L}|_T)$, we can regard $\mathscr{L}_1 \otimes_{O_T} k(t)$ as a subsheaf of $\mathscr{L} \otimes_k k(t)$ for all $t \in T$. The Hilbert polynomial of $\mathscr{L}_1 \otimes k(t)$ is $d + m(1-g) + mt$ and this $\mathscr{L}_1$ and T represent $\boldsymbol{Flag}^{d}_{\mathscr{L},m}$.

We now prove the properness of $\boldsymbol{Flag}^{d}_{\mathscr{L},m}$ by the valuative criterion. Let R be a discrete valuation ring over k, and K the field of fractions. Let $\mathscr{M}$ be a locally free subsheaf of degree d and rank m of $\mathscr{L} \otimes_k K$ over $X \times_k K$. Put $V := \Gamma(\mathscr{M}) \cap \Gamma(\mathscr{L} \otimes_k R)$ and consider the subsheaf $\mathscr{F}'$ of $\mathscr{L} \otimes_k R$ generated by V. Let $\mathscr{C}$ be $\operatorname{Coker}(\mathscr{F}' \to \mathscr{L} \otimes R)$ modulo its R-torsion and let $\mathscr{F} := \operatorname{Ker}(\mathscr{L} \otimes R \to \mathscr{C})$. The Hilbert polynomial of $\mathscr{F}_t$ ($t \in \operatorname{Spec} R$) is independent of t, because $\mathscr{F}$ is R-flat and $O_X \otimes R$ is coherent. $\mathscr{F}_t$ is a subsheaf of $\mathscr{L} \otimes_k k(t)$ for $t \in \operatorname{Spec} R$, because $\mathscr{C}$ is R-flat. Therefore $\mathscr{F}_t$ is a locally free sheaf over $X \times_k k(t)$. q.e.d.

Proof of Theorem 2.1. Put $\boldsymbol{d} = (d_1, \cdots, d_{n-1})$ and $Y = X \times_k \boldsymbol{Flag}^{d_1}_{\mathscr{L},n-1} \times \cdots \times \boldsymbol{Flag}^{d_{n-1}}_{\mathscr{L},1}$. For the universal sheaf $\mathscr{L}_i$ on $X \times_k \boldsymbol{Flag}^{d_i}_{\mathscr{L},n-i}$, its pull-back $\mathscr{M}_i = pr^*_{1,i+1}\mathscr{L}_i$ ($i = 2, \cdots, n$) is a locally free sheaf on Y. For each i, let T_i be the stratum corresponding to the Hilbert polynomial

$p(t)=0$ of the flattening stratification of $\boldsymbol{Flag}^{d_1}_{\mathscr{L},n-1}\times\cdots\times\boldsymbol{Flag}^{d_{n-1}}_{\mathscr{L},1}$ for $\mathscr{M}_i+\mathscr{M}_{i+1}/\mathscr{M}_i$.

$(\mathscr{M}_i+\mathscr{M}_{i+1}/\mathscr{M}_i)\otimes k(t)=0$ if and only if $t\in T_i$. Therefore $T=\bigcap_i T_i$ represents the functor $\boldsymbol{Flag}^{d}_{\mathscr{L}}$. Let us prove the closedness of each T_i hence of T by using the valuative criterion. Let R be a discrete valuation ring over k and K be the field of fractions. If the locally free sheaves $\mathscr{L}_0$, $\mathscr{L}_i$, $\mathscr{L}_{i+1}$ over $X\times\operatorname{Spec} R$ satisfy the conditions

a) $\mathscr{L}_0\supset\mathscr{L}_i$. $\mathscr{L}_0\supset\mathscr{L}_{i+1}$,

b) $\mathscr{L}_0/\mathscr{L}_i$, $\mathscr{L}_0/\mathscr{L}_{i+1}$ are R-flat,

c) $\mathscr{L}_i\otimes K\supset\mathscr{L}_{i+1}\otimes K$,

then $\mathscr{L}_i\supset\mathscr{L}_{i+1}$ holds. This proves the closedness of T_i. q.e.d.

Corollary 2.4. *The functor* $\boldsymbol{Flag}^{d,0}_{\mathscr{L}}$: $(\mathrm{Sch}/k)^\circ\to(\mathrm{Sets})$ *which sends T to the set*

$$\{(\mathscr{L}_0\supset\cdots\supset\mathscr{L}_{n-1})\in\boldsymbol{Flag}^{d}_{\mathscr{L}}(T)\,|\,\mathscr{L}_i/\mathscr{L}_{i+1}\text{ is invertible on }X\times_k T\text{ for any }i\}$$

is represented by an open subscheme of $\boldsymbol{Flag}^{d}_{\mathscr{L}}$.

2.2. A double coset decomposition and the Lang sheaf

We use the same notation as in Sections 1.3 and 2.1.

Let U_K be the subgroup of $\mathrm{GL}(n, K)$ consisting of unipotent upper triangular matrices. We now show that

(2.1) $U_K\backslash\mathrm{GL}(n, A)/\mathrm{GL}(n, \hat{O})$ is in one-to-one correspondence with the set consisting of $(\mathscr{L}_0\supset\cdots\supset\mathscr{L}_{n-1};\, \gamma_1, \cdots, \gamma_n)$ where $\mathscr{L}_i$ runs through locally free sheaves of rank $n-i$ over X such that $\mathscr{L}_{i-1}/\mathscr{L}_i$ is invertible for all i and γ_i rational sections of $\mathscr{L}_{i-1}/\mathscr{L}_i$.

This correspondence is given as follows. For a given element $g=(g_v)_{v\in|X_0|}$ of $\mathrm{GL}(n, A)$, and $v\in|X_0|$, the stalk at v of the corresponding flag $\mathscr{L}_0, \cdots, \mathscr{L}_{n-1}$ is given by

$$\{w\in K^n\,|\,wg\in O_v^n\}\supset\{w\in 0\oplus K^{n-1}\,|\,wg\in O_v^n\}\supset\cdots$$
$$\supset\{w\in 0\oplus\cdots 0\oplus K\,|\,wg\in O_v^n\}.$$

γ_i is the rational section corresponding to $(0, \cdots, \overset{i}{\check{1}}, \cdots, 0)$. This correspondence is well defined and one to one. The following proposition is easy to prove.

Proposition 2.5. *Under the above correspondence* (2. 1), *let*

$$g=\begin{bmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$$

correspond to $(\mathscr{L}_0 \supset \cdots \supset \mathscr{L}_{n-1}; \gamma_1, \cdots, \gamma_n)$. *Then*

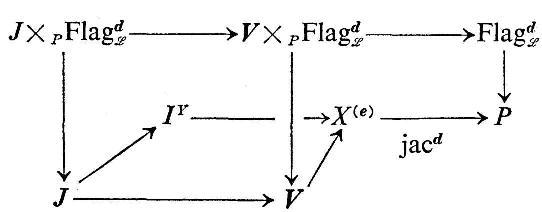
(1) γ_i *is a global section of* $\mathscr{L}_{i-1}/\mathscr{L}_i$ *if and only if* $\mathrm{ord}_v a_i \geqq 0$ *for all* v.

(2) $\mathrm{ord}_v\, a_i \geqq \mathrm{ord}_v\, a_{i+1}$ *if and only if* $\mathrm{ord}_v \gamma_i \geqq \mathrm{ord}_v\, \gamma_{i+1}$.

Next we define some moduli schemes. Let S_m be the symmetric group of degree m which acts on X_0^m as permutations of factors. We write the quotient X_0^m/S_m by $X_0^{(m)}$. Let $X := X_0 \otimes \bar{F}_q$ and let $\mathrm{Pic}^m = \mathrm{Pic}^m(X)$ be the Picard variety of X of degree m. Denote by v: $\mathrm{Flag}^{\boldsymbol{d}}_{\mathscr{L}} \to P := \mathrm{Pic}^{e_1} \times \cdots \times \mathrm{Pic}^{e_n}$ the map which sends $(\mathscr{L}_0, \cdots, \mathscr{L}_{n-1})$ to $(\det \mathscr{L}_0 \otimes \det \mathscr{L}_1^{-1}, \cdots, \det \mathscr{L}_{n-2} \otimes \det \mathscr{L}_{n-1}^{-1}, \mathscr{L}_{n-1}) \in P$, where $e_1 = d_0 - d_1, \cdots, e_{n-1} = d_{n-2} - d_{n-1}$, $e_n = d_{n-1}$. Let us denote $X^{(e)}$ by $X^{(e_1)} \times \cdots \times X^{(e_n)}$ where $\boldsymbol{e} = (e_1, \cdots, e_n)$. The variety $X^{(e)} = X_0^{(e)} \otimes \bar{F}_q$ represents the set of effective divisors of degree e on X. Denote by jac^e the Albanese map from $X^{(e)}$ to Pic^e and $\mathrm{jac}^{(e)}$ the map $\mathrm{jac}^{e_1} \times \cdots \times \mathrm{jac}^{e_n}$ from $X^{(e)}$ to P. If $Y = (e_1, \cdots, e_n)$ satisfies $e_1 \geqq \cdots \geqq e_n \geqq 0$, we can define the incidence variety I^Y as the closed subscheme of $X^{(e_1)} \times \cdots \times X^{(e_n)}$ defined by

$$I^Y = \{(x_1, \cdots, x_n) \in X^{(e_1)} \times \cdots \times X^{(e_n)} \mid x_1 \geqq x_2 \geqq \cdots \geqq x_n \text{ as divisors}\}.$$

The fiber of the morphism jac^e at $\mathscr{A} \in \mathrm{Pic}^e$ is identified with the set of effective divisors of degree e rationally equivalent to $\mathscr{A}$ and it is identified with the projective space $\boldsymbol{P}(H^0(X, \mathscr{A}))$ associated to $H^0(X, \mathscr{A})$. Therefore the fiber of $\mathrm{jac}^{\boldsymbol{d}}$ at $(\mathscr{A}_1, \cdots, \mathscr{A}_n)$ is identified with $\boldsymbol{P}(H^0(X, \mathscr{A}_1)) \times \cdots \times \boldsymbol{P}(H^0(X, \mathscr{A}_n))$. Let $\mathscr{M}_i$ be the universal line bundle over $X \times \mathrm{Pic}^{e_i}$, f_i: $X \times \mathrm{Pic}^{e_i} \to \mathrm{Pic}^{e_i}$ the natural projection and $\boldsymbol{V}_i$ the variety Spec $(\mathrm{Sym}(f_{i*}\mathscr{M}_i^\vee))$ over Pic^{e_i}. For $X^{(e_i)}$ is naturally isomorphic to Proj (Sym $(f_{i*}\mathscr{M}_i^\vee))$, there is a natural morphism from $\boldsymbol{V}_i$ to $X^{(e_i)}$. Let $\boldsymbol{V} := \boldsymbol{V}_1 \times \cdots \times \boldsymbol{V}_n$ and $\boldsymbol{J} := \boldsymbol{V} \times_{X^{(e)}} I^Y$. Consider the following diagram.



Proposition 2.6. *Let* $(\boldsymbol{J} \times_P \boldsymbol{Flag}^{\boldsymbol{d},0}_{\mathscr{L}})_0$ *denote* $\boldsymbol{J} \times_P \boldsymbol{Flag}^{\boldsymbol{d},0}_{\mathscr{L}}$ *over* $\boldsymbol{F}_q$. *In the same notation as above, let* $B^{\boldsymbol{d}}_{A,\mathscr{L}}$ *be the subset of* $\mathrm{GL}(n, A)$ *consisting of upper triangular matrices*

$$g=\begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \in \mathrm{GL}(n, A)$$

with

(1) $\deg a_i = e_i$ *for* $i=1, \cdots, n$,

(2) $\mathrm{ord}_v a_i \geqq 0$ *for all* $v \in |X_0|$ *for* $i=1, \cdots, n$, *and*

(3) $\mathrm{GL}(n, K)\, g\mathrm{GL}(n, \hat{O})$ *defines the isomorphism class of* $\mathscr{L}$.

Let $JB^d_{A,\mathscr{L}}$ *be the subset of* $B^d_{A,\mathscr{L}}$ *consisting of elements*

$$g=\begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \in \mathrm{GL}(n, A)$$

with $\mathrm{ord}_v a_i \geqq \mathrm{ord}_v a_{i+1}$ *for all* $v \in |X_0|$ *and* $i=1, \cdots, n$. *Then under the correspondence of* (2.1), *we have the following identifications*:

$$U_K \backslash U_K B^d_{A,\mathscr{L}}\, \mathrm{GL}(n, \hat{O})/\mathrm{GL}(n, \hat{O}) \simeq (\boldsymbol{V} \times_P \boldsymbol{Flag}^{d,0}_{\mathscr{L}})_0(\boldsymbol{F}_q),$$
$$U_K \backslash U_K JB^d_{A,\mathscr{L}}\, \mathrm{GL}(n, \hat{O})/\mathrm{GL}(n, \hat{O}) \simeq (J \times_P \boldsymbol{Flag}^{d,0}_{\mathscr{L}})_0(\boldsymbol{F}_q).$$

Proof. Let $(\mathscr{L}_1, \cdots, \mathscr{L}_{n-1}; \gamma_1, \cdots, \gamma_n)$ be the subbundles of $\mathscr{L}$ and the rational section γ_i of $\mathscr{L}_{i-1}/\mathscr{L}_i$ corresponding to an element $g=\begin{pmatrix} a_1 & * \\ \cdot & \cdot \\ 0 & a_n \end{pmatrix}$ of $B^d_{A,\mathscr{L}}$. Then the invertible sheaf $\mathscr{L}_{i-1}/\mathscr{L}_i$ with the rational section γ_i corresponds to the invertible sheaf $O(-\Sigma_v \mathrm{ord}\,(a_{i,v}))(v))$ with the rational section $1 \in O \otimes K \simeq O(-\Sigma_v \mathrm{ord}\,(a_{i,v})(v)) \otimes K$. Therefore γ_i corresponds to a global section of $\mathscr{L}_{i-1}/\mathscr{L}_i$ if and only if $\mathrm{ord}\,(a_{i,v}) \geqq 0$ for all $v \in |X_0|$. Therefore the set on the left is identified with the set of pairs $(\mathscr{L}_1, \cdots, \mathscr{L}_{n-1}; \gamma_1, \cdots, \gamma_n)$ such that $\mathscr{L}_i$ is a subbundle of $\mathscr{L}$ and γ_i is a global section of the invertible sheaf $\mathscr{L}_{i-1}/\mathscr{L}_i$. On the other hand, the set of $\boldsymbol{F}_q$-rational points of $\boldsymbol{V}$ corresponds to the set of invertible sheaves $\mathscr{A}_i$ with their global sections γ_i. Thus the set on the left is in one-to-one correspondence with the set of $\boldsymbol{F}_q$-rational points of $\boldsymbol{V} \times_P \boldsymbol{Flag}^{d,0}_{\mathscr{L}}$. q.e.d.

By the above proposition, the restriction of a Whittaker function to $U_K \backslash U_K JB^d_{A,\mathscr{L}}\, \mathrm{GL}(n, \hat{O})/\mathrm{GL}(n, \hat{O})$ can be regarded as a function on $(J_P \times \boldsymbol{Flag}^{d,0}_{\mathscr{L}})_0(\boldsymbol{F}_q)$.

In the rest of this paragraph, we define the Lang sheaf. Fix $a_1, \cdots, a_n \in A^*$. We can define the map α from

$$U_K \backslash U_K \left\{ g=\begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \in \mathrm{GL}\,(n, A) \right\} \mathrm{GL}\,(n, \hat{O})/\mathrm{GL}\,(n, \hat{O})$$

to

$$\oplus_{i=1}^{n-1} A/(K+a_i/a_{i+1}\hat{O})$$

sending the class of

$$g=\begin{pmatrix} 1 & u_1 & & * \\ & \cdot & \cdot & \\ & & \cdot & u_{n-1} \\ 0 & & & 1 \end{pmatrix}\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

to the class of $(u_1, \cdots, u_{n-1})$ in $\oplus_{i=1}^{n-1} A/(K+a_i/a_{i+1}\hat{O})$.

Proposition 2.7. *For an element a_i of A^*, define an invertible sheaf $\mathscr{A}_i$ on X by*

$$\mathscr{A}_i(U)=\{K \ni f \mid \operatorname{ord}_v f+\operatorname{ord}_v a_i \geqq 0\ (v \in U)\}.$$

Then we have the equality:

$$A/(K+a_i a_{i+1}^{-1}\hat{O}) \simeq \operatorname{Ext}^1(\mathscr{A}_i, \mathscr{A}_{i+1}).$$

Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} U_K \backslash U_K \left\{ g=\begin{pmatrix} a_1 & * \\ & \ddots \\ 0 & a_n \end{pmatrix} \in \mathrm{GL}(n, A) \right\} \mathrm{GL}(n, \hat{O})/\mathrm{GL}(n, \hat{O}) & \xrightarrow[\alpha]{} & \oplus_{i=1}^{n-1} A/(K+a_i/a_{i+1}O) \\ \downarrow & & \downarrow \\ \{(\mathscr{L}_0 \supset \cdots \supset \mathscr{L}_{n-1}; \gamma_1, \cdots, \gamma_n) \mid \mathscr{L}_{i-1}/\mathscr{L}_i \simeq \mathscr{A}_i\} & \xrightarrow[\tilde{\alpha}]{} & \oplus_{i=1}^{n-1} \operatorname{Ext}^1(\mathscr{A}_i, \mathscr{A}_{i+1}), \end{array}$$

where $\tilde{\alpha}$ sends $(\mathscr{L}_0 \supset \cdots \supset \mathscr{L}_{n-1}; \gamma_1, \cdots, \gamma_n)$ to

$$(0 \rightarrow \mathscr{L}_{i+1}/\mathscr{L}_{i+2} \rightarrow \mathscr{L}_i/\mathscr{L}_{i+2} \rightarrow \mathscr{L}_i/\mathscr{L}_{i+1} \rightarrow 0)_i.$$

Proof. The first equality is derived from the exact sequence

$$0 \longrightarrow \operatorname{Hom}(\mathscr{A}_i, \mathscr{A}_{i+1}) \longrightarrow K \longrightarrow K/\operatorname{Hom}(\mathscr{A}_i, \mathscr{A}_{i+1}) \longrightarrow 0,$$

and

$$H^1(X, K)=0,\ H^0(X, K/\operatorname{Hom}(\mathscr{A}_i, \mathscr{A}_{i+1})) \simeq A/a_i a_{i+1}^{-1}\hat{O}.$$

The last assertion can be shown by chasing the correspondence of (2.1).

q.e.d.

Let us consider the additive character ψ: $A/(K+\hat{O}) \rightarrow \bar{Q}_\ell^*$. From now on, let us assume that there exists an additive character:

$$\varphi: F_q \longrightarrow \bar{Q}_\ell^*,$$

and a differential $\omega \in H^0(X_0, \Omega^1_{K_0}) \simeq \mathrm{Hom}(H^1(X_0, O_{X_0}), \boldsymbol{F}_q)$ such that $\psi = \varphi \circ \omega$. Let $\tilde{\mathcal{M}}_i$ be the universal line bundle on $X \times \mathrm{Pic}^{e_i}$ and $\mathcal{M}_i$ the pulled back sheaf over $X \times P$. Let $\mathcal{E}xt^1_P(\mathcal{M}_i, \mathcal{M}_{i+1})$ denote the sheaf of extensions over P. We will write W for $\mathrm{Spec}(\mathrm{Sym}\bigoplus_{i=1}^{n-1}\mathcal{E}xt^1_P(\mathcal{M}_i, \mathcal{M}_{i+1})))$. We can define a morphism τ over P from $\boldsymbol{Flag}_{\mathcal{L}}$ to W by sending $(\mathcal{L}_0 \supset \cdots \supset \mathcal{L}_{n-1})$ to $(0 \to \mathcal{L}_{i+1}/\mathcal{L}_{i+2} \to \mathcal{L}_i/\mathcal{L}_{i+2} \to \mathcal{L}_i/\mathcal{L}_{i+1} \to 0)_i$. Summing these up, we can define the following maps:

$$J \times_P \boldsymbol{Flag}^{d,0}_{\mathcal{L}} \xrightarrow[\mathrm{id}\times\tau]{} J \times_P W \xrightarrow[\beta]{} P \times (H^1(X_0, O))^{n-1} \xrightarrow[\mathrm{pr}_2]{} H^1(X_0, O))^{n-1}$$
$$\xrightarrow[\Sigma]{} H^1(X_0, O) \xrightarrow[\omega]{} \boldsymbol{A}^1,$$

where the map β from $J \times_P W$ to $P \times (H^1(X_0, O))^{n-1}$ on P is given fiberwise by the Serre duality

$$\begin{aligned}&((\mathrm{Hom}(\mathcal{A}_2, \mathcal{A}_1) - \{0\}) \times \cdots \times (\mathrm{Hom}(\mathcal{A}_n, \mathcal{A}_{n-1}) - \{0\})\\ &\quad \times (\mathrm{Hom}(O, \mathcal{A}_n) - \{0\}))\\ &\quad \times (\mathrm{Ext}^1(\mathcal{A}_1, \mathcal{A}_2) \times \cdots \times \mathrm{Ext}^1(\mathcal{A}_{n-1}, A_n))\\ &\quad \longrightarrow H^1(X_0, O)^{n-1}.\end{aligned}$$

We denote this composite by f. The Artin-Schreier covering

$$\boldsymbol{A}^1 \ni x \longrightarrow x^q - x \in \boldsymbol{A}^1$$

defines an étale covering of $\boldsymbol{A}^1$, with the covering transformation group equal to $\boldsymbol{F}_q$. φ defines a smooth étale sheaf $\bar{\mathcal{L}}_\varphi$ of rank one over $\boldsymbol{A}^1$. The pulled-back sheaf $\mathcal{L}_\varphi = f^*\bar{\mathcal{L}}_\varphi$ over $J \times_P \boldsymbol{Flag}^{d,0}_{\mathcal{L}}$ will be called the Lang sheaf.

2.3. The construction of the Whittaker sheaves

For a given representation of ρ: $\pi_1(X_0) \to \mathrm{GL}(n, \bar{\boldsymbol{Q}}^*_\ell)$, we define a smooth étale sheaf $\mathcal{F}(\rho)$ on X_0 associated to ρ (cf. [8, p. 43]). The symmetric group S_m of degree m acts on X_0^m as permutations of factors. There is an obvious equivariant action of S_m on $\mathrm{pr}_1^*\mathcal{F}(\rho) \otimes \cdots \otimes \mathrm{pr}_m^*\mathcal{F}(\rho)$, hence on $\pi_{m*}(\mathrm{pr}_1^*\mathcal{F}(\rho) \otimes \cdots \otimes \mathrm{pr}_m^*\mathcal{F}(\rho))$, where π_m is the natural projection from X_0^m to $X_0^{(m)} = X_0^m/S_m$. We define $\mathcal{E}^{(m)}(\rho)$ as the fixed subsheaf of $\pi_{m*}(\mathrm{pr}_1^*\mathcal{F}(\rho) \otimes \cdots \otimes \mathrm{pr}_m^*\mathcal{F}(\rho))$ under S_m.

Now for a Young diagram $Y = (e_1, \cdots, e_n)$ with $e_1 \geqq \cdots \geqq e_n \geqq 0$ and a representation ρ of $\pi_1(X_0)$ as above, we define a sheaf on $X_0^{(e_1)} \times \cdots \times X_0^{(e_n)}$ by $\mathcal{E}^Y(\rho) = \mathrm{pr}_1^*\mathcal{E}^{(e_1)}(\rho) \otimes \cdots \otimes \mathrm{pr}_n^*\mathcal{E}^{(e_n)}(\rho)$. We denote by $\mathrm{Sym}^Y(\rho)$ the restriction of $\mathcal{E}^Y(\rho)$ to the incidence variety I^Y.

Let $X^{(m)0}$ be the open subscheme of $X^{(m)}=X_0^{(m)}\otimes\bar{F}_q$ which corresponds to

$$\{x=x_1+\cdots+x_m \in X^{(m)}|x_i\neq x_j \qquad (i\neq j)\}.$$

The natural projection $\pi_m\colon X^m\to X^{(m)}$ induces an étale Galois covering π_m^0: $X^{m,0}=\pi_m^{-1}(X^{(m)0})\to X^{(m)0}$, with the Galois group S_m. If we put $f=(e_1-e_2, \cdots, e_n)$, then the incidence variety I^Y can be identified with $X^{(f)}$ by the map sending the element $(x_1, \cdots, x_n)$ of $X^{(\mathscr{F})}$ to the element $(\Sigma_{i=1}^n x_i, \Sigma_{i=2}^n x_i, \cdots, x_n)$ of $X^{(e)}$. Under this identification, let us define an open subvariety $I^0=(I^Y)^0$ of $I=I^Y$ by

$$I^0=X^{(e_1-e_2)0}\times\cdots\times X^{(e_n)0},$$

and an open set U of $X^{e_1-e_2}\times\cdots\times X^{e_n}$ by

$$U=X^{e_1-e_2,0}\times\cdots\times X^{e_n,0}.$$

We define a marking t of a Young diagram $Y=(e_1, \cdots, e_n)$ to be the diagram

$$t=\begin{array}{|l|}\hline t_1^1, \cdots\cdots t_{,e_1}^1 \\ \hline \cdots\cdots \\ \hline t_1^n, \cdots, t_{e_n}^n \\ \hline\end{array},$$

where $\{t_1^i, \cdots, t_{e_i}^i\}=\{1, \cdots, e_i\}$. For a given marking t, we can define the map G_t which sends the element $(x_{e_1}, \cdots, x_1)$ of $X^{e_1-e_2}\times\cdots\times X^{e_n}$ to the element $((x_{t_1^1}, \cdots, x_{t_{e_1}^1}), \cdots, (x_{t_1^n}, \cdots, x_{t_{e_n}^n}))$ of $X^{e_1}\times\cdots\times X^{e_n}$. Under this map we obtain the identification

$$\begin{aligned} G&=\mathrm{Gal}(U/I^0)\\ &\simeq\{h \in S_{e_1}\times\cdots\times S_{e_n}\subset\mathrm{Aut}(X^{e_1}\times\cdots\times X^{e_n})|h(\mathrm{Im}\,G_t)=\mathrm{Im}(G_t)\}.\end{aligned}$$

We obtain the following diagram:

$$\begin{array}{ccccc} U & \xrightarrow{j} & X^{e_1-e_2}\times\cdots\times X^{e_n} & \xrightarrow{G_t} & X^{e_1}\times\cdots\times X^{e_n} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\bar{\pi}} & & \downarrow{\scriptstyle p=\pi_{e_1}\times\cdots\times\pi_{e_n}} \\ I^0 & \longrightarrow & I & \longrightarrow & X^{(e_1)}\times\cdots\times X^{(e_n)} \end{array}$$

The sheaf $j^*\bar{\pi}^*$ $(\mathrm{Sym}^Y(\rho))$ is equal to

$$j^*G_t^*(\mathrm{pr}_1^*\mathscr{F}(\rho)\otimes\cdots\otimes\mathrm{pr}_{e_1+\cdots+e_n}^*\mathscr{F}(\rho))$$

because π is étale for G acts on U freely. The natural map

$$G_t^*(\mathrm{pr}_1^*\mathscr{F}(\rho)\otimes\cdots\otimes\mathrm{pr}_{e_1+\cdots+e_n}^*\mathscr{F}(\rho))$$
$$\longrightarrow j_*j^*G_t^*(\mathrm{pr}_1^*\mathscr{F}(\rho)\otimes\cdots\otimes\mathrm{pr}_{e_1+\cdots+e_n}^*\mathscr{F}(\rho))$$

is an isomorphism because $G_t^*(\mathrm{pr}_1^*\mathscr{F}(\rho)\otimes\cdots\otimes\mathrm{pr}_{e_1+\cdots+e_n}^*\mathscr{F}(\rho))$ is a smooth sheaf. Thus we obtain the following composite:

$$\bar{\pi}^*(\mathrm{Sym}^Y(\rho))\longrightarrow j_*j^*\bar{\pi}^*(\mathrm{Sym}^Y(\rho))$$
$$\simeq j_*j^*G_t^*(\mathrm{pr}_1^*\mathscr{F}(\rho)\otimes\cdots\otimes\mathrm{pr}_{e_1+\cdots+e_n}^*\mathscr{F}(\rho))$$
$$\xleftarrow{\simeq} G_t^*(\mathrm{pr}_1^*\mathscr{F}(\rho)\otimes\cdots\otimes\mathrm{pr}_{e_1+\cdots+e_n}^*\mathscr{F}(\rho)).$$

Let H_t be the subgroup of $\mathrm{Aut}(X^{e_1}\times\cdots\times X^{e_n})$ consisting of $h\in S_{e_1+\cdots+e_n}\subset\mathrm{Aut}(X^{e_1}\times\cdots\times X^{e_n})$ which is a permutation of coordinates and which preserve the number written on the marking t. Then there is an isomorphism

$$H_t\simeq\underbrace{S_n\times\cdots\times S_n}_{e_n}\times\underbrace{S_{n-1}\times\cdots\times S_{n-1}}_{e_{n-1}-e_n}\times\cdots\times\underbrace{S_1\times\cdots\times S_1}_{e_1-e_2}.$$

which gives rise to a character sign_H of H_t defined as the product of signatures of all symmetric factor groups. $G_t^*(\mathrm{pr}_1^*\mathscr{F}(\rho)\otimes\cdots\otimes\mathrm{pr}_{e_1+\cdots+e_n}^*\mathscr{F}(\rho))$ is equal to $\mathrm{pr}_{t_1^1}^*\mathscr{F}(\rho)\otimes\cdots\otimes\mathrm{pr}_{t_{e_n}^n}^*\mathscr{F}(\rho)$. Therefore H_t acts on $G_t^*(\otimes_{i=1}^{e_1+\cdots+e_n}\mathrm{pr}_i^*\mathscr{F}(\rho))$ as a sheaf on I^Y. By this action we can define an endomorphism $\Sigma_{H_t\ni g}\,\mathrm{sign}_H(g)g$. Let $\mathscr{I}_t$ be the image of the composite map

$$\bar{\pi}^*(\mathrm{Sym}^Y(\rho))\longrightarrow G_t^*(\otimes_{i=1}^{e_1+\cdots+e_n}\mathrm{pr}_i^*\mathscr{F}(\rho))\xrightarrow[\Sigma_{H_t\ni g}\mathrm{sign}_H(g)g]{} G_t^*(\otimes_{i=1}^{e_1+\cdots+e_n}\mathrm{pr}_i^*\mathscr{F}(\rho)).$$

We have a natural map $\gamma\colon\bar{\pi}^*(\mathrm{Sym}^Y(\rho))\rightarrow\mathscr{I}_t$.

Definition. Let $\mathscr{E}(\chi^Y(\rho))$ be the sheaf on I^Y, the image sheaf of

$$\mathrm{Sym}^Y(\rho)\longrightarrow\bar{\pi}_*\bar{\pi}^*\,\mathrm{Sym}^Y(\rho)\xrightarrow{\bar{\pi}_*\gamma}\bar{\pi}_*\mathscr{I}_t.$$

Let D be an effective divisor of degree d. If $Y-d\delta=(e_1-d(n-1), e_2-d(n-2),\cdots,e_n)$ is a Young diagram, then we can define the map

$$i_{Y,D}\colon I^{Y-d\delta}\ni(x_1,\cdots,x_n)\longrightarrow(x_1+(n-1)D, x_2+(n-2)D,\cdots,x_n)\in I^Y.$$

From now on we fix a differential ω on X and let D be $\mathrm{div}(\omega)$. Then let $\mathscr{F}(\chi_Y(\rho)):=i_{Y,D*}(\mathscr{E}(\chi_{Y-d\delta}(\rho)))$. We fix an isomorphism between $\boldsymbol{C}$ and $\bar{\boldsymbol{Q}}_\ell$ and the additive character φ of $\boldsymbol{F}_q$. Then we can define the Lang sheaf by ω.

Proposition 2.8. *Let $Y=(e_1, \cdots, e_n)$ be a Young diagram which satisfies* (*) *as above. Let $g \in JB^d_{A,\mathscr{L}}$ be a diagonal matrix* $\operatorname{diag}(a_1, \cdots, a_n)$ *corresponding to $w \in (J\times_P Flag^{d,0}_{\mathscr{L}})_0(F_q)$ under the correspondence in Proposition* 2.2. *Let v be the image of w under the natural map $J\times_P Flag^{d,0}_{\mathscr{L}}\to I$ and $\bar{v}$ a geometric point over v. Let f be the global Whittaker function defined in Section* 1.3, *and* Fr_v *the Frobenius substitution on $\mathscr{F}(\chi_Y(\rho))_{\bar{v}}$. Then we have*

$$f(g)=q^e \operatorname{tr} \mathrm{Fr}_v|\mathscr{F}(\chi_Y(\rho))_{\bar{v}},$$

where $e=\Sigma^n_{i=1}(2i-n+1)(e_i-(2g-2)(n-i))/2$.

Definition. Let $\delta\colon J\times_P \boldsymbol{Flag}^{d,0}_{\mathscr{L}}\to I$ be the natural homomorphism. The Whittaker sheaf $\mathrm{Wh}^d_{\mathscr{L}}(\rho)$ is defined by

$$\mathrm{Wh}^d_{\mathscr{L}}(\rho)=\delta^*(\mathscr{F}(\chi_Y(\rho)))\otimes\mathscr{L}_\varphi,$$

where $\mathscr{L}_\varphi$ is the Lang sheaf defined in Section 2.2.

Theorem 2.9. *Let g be an element of $JB^d_{A,\mathscr{L}}$, and w the corresponding element of $(J\times_P \boldsymbol{Flag}^{d,0}_{\mathscr{L}})_0(\boldsymbol{F})$. In the same notation as in Proposition* 2.8, *we have*

$$f(g)=q^e \operatorname{tr} \mathrm{Fr}_v|\mathrm{Wh}^d_{\mathscr{L}}(\rho)_{\bar{w}},$$

where $\bar{w}$ is a geometric point over w.

Proof of Proposition 2.8. Let I_0 be the incidence variety defined over $\boldsymbol{F}_q$. First we look at the geometric fiber of $\mathscr{E}(\chi_Y(\rho))$ at a geometric point $\bar{v}$ over an element v of $I_0(\boldsymbol{F}_q)$. The point $\bar{v}$ can be expressed as an element $(v_1, \cdots, v_n)$ of $X^{(e_1)}\times\cdots\times X^{(e_n)}$. Let $x_1, \cdots, x_l$ be distinct closed points of X which appear in $\bar{v}$. Let $m_{i,j}$ be the multiplicity of x_i in v_j. Then $Y_i=(m_{i,1}, \cdots, m_{i,n})$ becomes a Young diagram. Under the componentwise sum of Young diagrams, we have $Y=Y_1+\cdots+Y_l$, i.e., $Y=(\sum^l_{i=1}m_{i,1}, \cdots, \sum^l_{i=1}m_{i,n})$. We denote the element $\bar{v}$ as $\bar{v}=\sum^l_{i=1}Y_ix_i$. $\sigma\in \operatorname{Gal}(\bar{\boldsymbol{F}}_q/\boldsymbol{F}_q)$ acts on $I_0(\bar{\boldsymbol{F}}_q)$ by $\sigma:\bar{v}\to\bar{v}^\sigma=\sum^l_{i=1}Y_ix_i^\sigma$, and $I_0(\boldsymbol{F}_q)$ can be regarded as the set of fixed elements in $I_0(\bar{\boldsymbol{F}}_q)$ under the action of $\operatorname{Gal}(\bar{\boldsymbol{F}}_q/\boldsymbol{F}_q)$. If $\bar{v}=\sum^l_{i=1}Y_ix_i$, then

$$\mathscr{E}(\chi_Y(\rho))_{\bar{v}}\simeq V_{Y_1}(\mathscr{F}(\rho)_{\bar{x}_1})\otimes\cdots\otimes V_{Y_l}(\mathscr{F}(\rho)_{\bar{x}_l}),$$

where $V_{Y_i}(\mathscr{F}(\rho)_{\bar{x}_i})$ is the representation space of $\mathrm{GL}(\mathscr{F}(\rho)_{\bar{x}_i})$ which corresponds to the Young diagram Y_i ([5, p. 129]). Moreover, the above isomorphism has the following meaning. Let $y_1, \cdots, y_k$ be the orbits of $x_1, \cdots, x_l$ under the action of $\operatorname{Gal}(\bar{\boldsymbol{F}}_q/\boldsymbol{F}_q)$. Then the Frobenius substitution Fr_{y_j} at y_j acts on the vector space $\otimes_{x_i \in y_i} V_{Y_i}(\mathscr{F}(\rho)_{\bar{x}_i})$. The action of the Frobenius at v on the left and that of $\mathrm{Fr}_{y_1} \otimes \cdots \otimes \mathrm{Fr}_{y_k}$ on the right are equivariant under the isomorphism.

Now let us look more closely at the action of Fr_{y_j} on the vector space $\otimes_{x_i \in y_j} V_{Y_i}(\mathscr{F}(\rho)_{\bar{x}_i})$. For a given étale $\bar{\boldsymbol{Q}}_\ell$-sheaf $\mathscr{F}$ over Spec $\boldsymbol{F}_q$, a map f: Spec $\boldsymbol{F}_{q^n} \to$ Spec $\boldsymbol{F}_q$, and $\tau \in \mathrm{Gal}\,(\boldsymbol{F}_{q^n}/\boldsymbol{F}_q)$, we have descent data $\sigma(\tau)$: $\tau_* f^* \mathscr{F} \to f^* \mathscr{F}$ on $f^* \mathscr{F}$ (cf. [8, p. 53]).

For $i \in \boldsymbol{Z}/n\boldsymbol{Z}$, let τ_i be the i-th power of the Frobenius in $\mathrm{Gal}(\bar{\boldsymbol{F}}_q/\boldsymbol{F}_q)$. The proof of the following lemma is an easy exercise of linear algebra.

Lemma 2.10. *Fix a geometric point* $\bar{v}$: Spec $\bar{\boldsymbol{F}}_q \to$ Spec $\boldsymbol{F}_{q^n}$. *Let A be a* $\mathrm{Gal}\,(\bar{\boldsymbol{F}}_q/\boldsymbol{F}_{q^n})$*-module and A_i be copies of A for $i=1, \cdots, n$. The sheaf* $\mathscr{G} = A_1 \otimes \cdots \otimes A_n$ *on* Spec $\boldsymbol{F}_{q^n}$ *has descent data*

$$\Gamma(\bar{v}^* \tau_{i*} \mathscr{G}) \simeq A_{1+i} \otimes \cdots \otimes A_{n+i} \longrightarrow A_1 \otimes \cdots \otimes A_n \simeq \Gamma(\bar{v}^* \mathscr{G})$$

which sends $(x_1 \otimes \cdots \otimes x_n)$ to $(x_1 \otimes \cdots \otimes x_n)$, where $A_j := A_{j-n}$ if $j > n$. If F is the descended sheaf on Spec $\boldsymbol{F}_q$, *then*

$$\mathrm{tr}\,\mathrm{Fr}_{F_q} | F_{\bar{v}} = \mathrm{tr}\,\mathrm{Fr}_{F_{q^n}} | A.$$

Applying the above lemma to $\otimes_{x_i \in y_j} V_{Y_i}(\mathscr{F}(\rho)_{\bar{x}_i})$, we have the following identity:

$$\begin{aligned}\mathrm{tr}\,\mathrm{Fr}_{y_j} | \otimes_{x_i \in y_j} V_{Y_i}(\mathscr{F}(\rho)_{\bar{x}_i}) &= \mathrm{tr}\,\mathrm{Fr}_{\mathrm{Im}(y_j)} | V_{Y_i}(\mathscr{F}(\rho)_{\bar{x}_i}) \\ &= \chi_{Y_i}(\rho(\mathrm{Fr}_{\mathrm{Im}(y_j)})),\end{aligned}$$

where $\mathrm{Im}\,(y_j)$ is the corresponding closed point of X and χ_Y the character of the representation V_Y.

We define $w = v + D\delta$ as the image of v under $i_{Y,D}$. Then we have the equality

$$\mathrm{tr}\,\mathrm{Fr}_w | \mathscr{F}(\chi_Y(\rho))_{\bar{w}} = \mathrm{tr}\,\mathrm{Fr}_v | \mathscr{E}(\chi_Y(\rho))_{\bar{v}},$$

hence

$$\mathrm{tr}\,\mathrm{Fr}_w | \mathscr{F}(\chi_Y(\rho))_{\bar{w}} = \prod_{j=1}^{k} (\chi_{Y_i - D_i\delta}(\rho(\mathrm{Fr}_{\mathrm{Im}(y_j)}))), \tag{2.2}$$

where $Y_i - D_i\delta$ is the Young diagram obtained from the multiplicity of $\bar{v}$ at $x_i \in y_j$. Now we compute the value of f at g.

$$\begin{aligned} f(g) &= \prod_v \tilde{f}_v(g_v) \\ &= \prod_v \gamma_{t_{\bar{v}}^{-D_v}} \circ f_v(g_v), \end{aligned}$$

where D_v is the multiplicity of D at v. Recall that we defined f_v in Section 1.3. using the eigenvalues $\mu_1, \cdots, \mu_n$ of $\rho(\mathrm{Fr}_v)$ and the equality (1.2). Therefore we have

$$\prod_v \gamma_{t_v^{-D_v}} \circ f_v(g_v) = \prod_{y_j} (q^{\sum_{r=1}^n (r-n)(m_{i,r} - D_i(n-r)) \deg y_i})$$
$$\times (\chi_{Y_i - D_i\delta}(\rho(\mathrm{Fr}_{\mathrm{Im}(y_j)}) q^{(n-1)\deg y_j/2}))$$
$$= \prod_{y_j} (q^{\sum_{r=1}^n ((r-n)(m_{i,r} - D_i(n-r)) + (n-1)\deg(Y_i - D_i\delta)/2) \deg y_j})$$
$$\times (\chi_{Y_i - D_i\delta}(\rho(\mathrm{Fr}_{\mathrm{Im}(y_j)})))$$

By the equality (2.2), it is equal to

$$q^e \operatorname{tr} \mathrm{Fr}_w | \mathscr{F}(\chi_Y(\rho))_{\bar{w}},$$

where

$$e = \sum_{j=1}^n (j-n)(e_j - \deg D(n-j)) + (n-1) \deg (Y - D\delta)/2)$$
$$= \sum_{j=1}^n (2j - n + 1)(e_j - (2g+2)(n-j)). \qquad \text{q.e.d.}$$

Proof of the Theorem. We have

$$f(g) = \psi(u_1 + \cdots + u_{n-1}) f\left(\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}\right) \quad \text{for } g = \begin{pmatrix} 1 & u_1 & & * \\ & \ddots & \ddots & \\ & & \ddots & u_{n-1} \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}.$$

By the commutativity of the Proposition 2.7 and the definition of the Lang sheaf $\mathscr{L}_\varphi$, we have

$$\psi(u_1 + \cdots + u_{n-1}) f\left(\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}\right) = (\operatorname{tr} \mathrm{Fr} | \mathscr{L}_{\varphi, \bar{w}}) \times (\operatorname{tr} \mathrm{Fr} | \delta^* \mathscr{F}(\chi_Y(\rho))_{\bar{w}}). \quad \text{q.e.d.}$$

Remark. The natural surjective morphism $\mathrm{Sym}^Y(\rho) \to \mathscr{E}(\chi_Y(\rho))$ splits. This can be shown by the specialization argument and by the Richardson rule for the representations of general linear groups (cf. [7]).

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