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Algebraic Cycles on Hypersurfaces in P^N

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Dedicated to Professor Masayoshi Nagata on his sixtieth birthday

§ 0. Introduction

The main theme of this paper is to exploit an idea that the Abel-Jacobi map can be studied in close analogy with the cycle map as far as the behavior of the image is concerned.

Given a nonsingular complex projective variety X, let $CH^r(X)$ be the Chow group of codimension r algebraic cycles on X modulo rational equivalence ($0 \leq r \leq \dim X$). The cycle map is a homomorphism

$$\gamma^r\colon\ CH^r(X) \longrightarrow H^{2r}(X, \boldsymbol{Z}) \simeq H_{2(n-r)}(X, \boldsymbol{Z})$$

and the Abel-Jacobi map is a homomorphism

$$\psi^r\colon\ CH^r(X)_{\text{hom}} \overset{\text{def}}{=} \operatorname{Ker}(\gamma^r) \longrightarrow J^r(X)$$

where $J^r(X)$ is the r-th intermediate Jacobian of X. Set

$$\mathscr{C}^r(X) = \operatorname{Im}(\gamma^r), \qquad J^r_h(X) = \operatorname{Im}(\psi^r).$$

Further, letting $CH^r(X)_{\text{alg}}$ denote the subgroup of $CH^r(X)$ of algebraic cycles algebraically equivalent to zero, we set

$$J^r_a(X) = \psi^r(CH^r(X)_{\text{alg}}).$$

It is known that $J^r_a(X)$ is a complex subtorus of $J^r(X)$ having structure of abelian variety and that the quotient group $J^r_h(X)/J^r_a(X)$ is at most countable.

Now we propose to study the behavior of $J^r_a(X)$ for X hypersurfaces of dimension $2r-1$ in a projective space in comparison with that of $\mathscr{C}^r(X')$ for X' hypersurfaces of dimension $2r$. Such an idea must have been known for some time, but personally it has occurred to us in trying to understand Griffiths' theorem on the image of the Abel-Jacobi map for generic hypersurfaces of odd dimension ([G3]). We see that there is a strong analogy between Griffiths' result and the classical theorem of Max Noether. Inspired by this, we have recently found some explicit example of hypersurfaces defined over $\boldsymbol{Q}$ (the field of rational numbers) such that $J^r_a(X) = 0$

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([S5]). The pursuit of the said analogy turns out to be quite useful, and the aim of this paper is to make it clear.

The contents of the paper are as follows. After recalling some preliminary facts in Section 1, we give a unified proof for the theorems of Griffiths and of M. Noether in Section 2. Here emphasis is put on showing how close the proofs of these theorems are. In Section 3, we review the variants for hypersurfaces defined over $\boldsymbol{Q}$; namely we are concerned with the existence of hypersurfaces defined over $\boldsymbol{Q}$ which satisfy the same property, called (G) or (N) in text, as generic ones. A result of S. Bloch along this line is contained in the appendix. In Section 4, we turn to the other side of the problem, i.e., we find hypersurfaces for which the Abel-Jacobi map has nontrivial image of positive dimension. Our approach here again follows similar results on the cycle map, dealing with those hypersurfaces having "inductive structure".

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§ 1. Preliminaries

First let us briefly recall the definition of the Abel-Jacobi map

$$\psi^r\colon\ CH^r(X)_{\mathrm{hom}} \longrightarrow J^r(X).$$

The intermediate Jacobian $J^r(X)$ is a complex torus which is isomorphic to $H^{2r-1}(X, \boldsymbol{R})/H^{2r-1}(X, \boldsymbol{Z})$ as real torus and whose complex structure is given by

$$J^r(X) = H^{2r-1}(X, \boldsymbol{C})/\{F^r H^{2r-1}(X, \boldsymbol{C}) + H^{2r-1}(X, \boldsymbol{Z})\},$$

where $F^{\boldsymbol{\cdot}}$ denotes the Hodge filtration: if $H^i(X, \boldsymbol{C}) = \sum_{p+q=i} H^{p,q}$ is the Hodge decomposition, then $F^p H^i = \sum_{a \geq p} H^{a,i-a}$. Now take an element η of $CH^r(X)_{\mathrm{hom}}$ and let Y be an algebraic cycle on X representing η. Then there is a topological $2(n-r)+1$ chain, say Γ, such that $\partial\Gamma = Y$. The linear functional

$$\omega \longrightarrow \int_\Gamma \omega \qquad (\omega \in F^{n-r+1} H^{2n-2r+1})$$

determines an element of $H^{2r-1}(X, \boldsymbol{C})/F^r H^{2r-1}$ by the Poincaré duality, and its image in $J^r(X)$ is by definition the image $\psi^r(\eta)$ of η under the Abel-Jacobi map.

We refer to [G2], [G3], [Li] or [M] for the above as well as for the following:

Proposition 1.1. *With the same notation as before, let*

$$J_a^r(X)=\psi^r(CH^r(X)_{\mathrm{alg}}), \qquad J_h^r(X)=\mathrm{Im}(\psi^r).$$

Then

(i) *the quotient $J_h^r(X)/J_a^r(X)$ is at most countable.*

(ii) *$J_a^r(X)$ is a complex subtorus of $J^r(X)$ having the structure of an abelian variety.*

(iii) *When one identifies the tangent space to $J^r(X)$ (at the origin) with the subspace $H^{r-1,r}+\cdots+H^{0,2r-1}$ of $H^{2r-1}(X, \boldsymbol{C})$, the tangent space T to $J_a^r(X)$ is contained in $H^{r-1,r}$. In particular,* $\dim J_a^r(X)\leq h^{r-1,r}$.

(iv) *The direct sum $T+\bar{T}$ defines a rational sub-Hodge structure of $H^{2r-1}(X, \boldsymbol{Q})$.*

Remark 1.2. (a) According to Grothendieck's modified version of the general Hodge conjecture ([Gr], [S3]), the sub-Hodge structure mentioned in (iv) above will be the maximal sub-Hodge structure of $H^{2r-1}(X, \boldsymbol{Q})$ of type $(r, r-1)+(r-1, r)$ (see [M]).

(b) The definition of the intermediate Jacobian given above is due to Griffiths, and it is different from the definition due to Weil [W1]. However the statements about $J_a^r(X)$ hold in either sense, and in particular the notation $J_a^r(X)$ has the same meaning as that of Lieberman [Li]. According to [W2], the above fact (iii) was already known to Hodge in early 1950's.

(c) The quotient in (i) is not necessarily finitely generated, as was shown by Clemens [C].

Proposition 1.3. *Suppose that Z is an algebraic correspondence between nonsingular projective varieties Y and X. The maps induced by Z on the Chow groups and on the cohomology groups:*

$$\chi^d\colon CH^d(Y)\longrightarrow CH^{d+e}(X)$$

$$\eta^i\colon H^i(Y)\longrightarrow H^{i+2e}(X) \qquad (e=\dim X-\dim Y)$$

are compatible with the cycle map if $i=2d$, and with the Abel-Jacobi map if $i=2d-1$.

The last statement for $i=2d-1$ means that the restriction of χ^d to the part "algebraically equivalent to zero" fits in the commutative diagram

$$\begin{array}{ccc} CH^d(Y)_{\mathrm{alg}} & \xrightarrow{\chi^d} & CH^{d+e}(X)_{\mathrm{alg}} \\ \psi^d \downarrow & & \downarrow \psi^{d+e} \\ J^d(Y) & \longrightarrow & J^{d+e}(X) \end{array}$$

where the vertical maps are the Abel-Jacobi maps and where the lower horizontal map has the associated tangent map

$$\sum_{\substack{p+q=2d-1 \\ p<q}} H^{p,q}(Y) \longrightarrow \sum_{\substack{a+b=2(d+e)-1 \\ a<b}} H^{a,b}(X)$$

which is compatible with η^i for $i=2d-1$.

For the proof, we refer to Lieberman [Li], Section 6; cf. [M].

Proposition 1.4. *Let X be a nonsingular hypersurface of degree m in $\boldsymbol{P}^{n+1}$. Then, in case $n=2r$, $H^n(X, \boldsymbol{C})$ has pure Hodge type (r, r) if and only if $m<2+2/r$. In case $n=2r-1$, $H^n(X, \boldsymbol{C})$ has Hodge type $(r, r-1)+(r-1, r)$ if and only if either $r=1$ or $r>1$ and $m<2+3/(r-1)$.*

This can be checked without difficulty since the Hodge numbers $h^{p,q}$ of such X are known. For the convenience of the reader, we recall that $h^{p,q}$ is equal to the number of $(n+2)$-tuples of positive integers less than m whose sum equals $m(q+1)$, increased by 1 if $p=q$ (cf. [S1]).

§ 2. Griffiths' theorem versus Noether's theorem

From now on we consider the case of nonsingular hypersurfaces X in a projective space. Let

$$(*) \qquad n=\dim X, \qquad m=\deg X.$$

The space of nonsingular hypersurfaces for fixed n, m forms a Zariski-open set of $\boldsymbol{P}^N$ ($N={}_{n+m+1}C_m-1$). Following the standard usage, we say that a property holds for a *generic* hypersurface of given dimension and degree if it holds for all X belonging to the complement of a countable union of proper analytic subsets of the said space.

Theorem 2.1 (Theorem of Max Noether). ([Le], p. 108; [D]). *Assume that* (i) *n is even*: *$n=2r$ and* (ii) *$m\geq 2+2/r$. Then for a generic hypersurface with $(*)$, the space $\mathscr{C}^r(X)$ of algebraic cohomology classes of the middle dimension on X is generated by a rational multiple of the class of linear space sections. Hence its middle Picard number $\rho^r(X)=\mathrm{rank}\ \mathscr{C}^r(X)$ satisfies*

$$\text{(N)} \qquad \rho^r(X)=1.$$

Theorem 2.2 (Theorem of Griffiths) ([G3], § 13; cf. [K]). *Assume that* (i) *n is odd*: $n=2r-1$ *and* (ii) $r>1$ *and* $m\geq 2+3/(r-1)$. *Then for a generic hypersurface X with* $(*)$, *one has*

$$\text{(G)} \qquad J^r_a(X)=0.$$

Furthermore, if W is a nonsingular hypersurface of degree m in $\boldsymbol{P}^{2r+1}$ such that $\rho^r(W)>1$ and if $\{X_t\}$ is a Lefschetz pencil of hyperplane sections of W, then the generic member X_t satisfies not only (G) *but also*

$$\text{(G}'\text{)} \qquad \operatorname{rank} J^r_h(X_t)\geq \rho^r(W)-1>0.$$

The first part of Theorem 2.2 has strong analogy with Theorem 2.1. To emphasize this fact, we shall sketch a unified proof of (N) and (G) which is based on 1) the Hodge theory and 2) the monodromy theory of Lefschetz pencils (cf. [D], [K]). Before doing so, we note that the second part of Theorem 2.2 implies the celebrated result that the Griffiths group

$$\mathrm{Griff}^r(X)=CH^r(X)_{\mathrm{hom}}/CH^r(X)_{\mathrm{alg}}$$

can be nontrivial, that is, algebraic and homological equivalence can differ for algebraic cycles of codimension $r>1$. As an example of W, one can take the Fermat variety of dimension $2r$, since $\rho^r(W)-1$ is then not smaller than the cardinality of the set $\mathscr{D}^n_m$ consisting of $(n+2)$-tuples of integers which are permutations of $(a_0, m-a_0, \cdots, a_{r+1}, m-a_{r+1})$ for some positive integers $a_0, \cdots, a_{r+1}$ less than m (cf. [S1]).

Proof. Taking a Lefschetz pencil $\{X_t | t \in \boldsymbol{P}^1\}$ of hypersurfaces of dimension n and degree m, we shall prove (N) and (G) for a generic member $X=X_t$. Let $S=\{t_1, \cdots, t_k\}$ be the subset of $\boldsymbol{P}^1$ corresponding to the singular members, and fix a reference point $o \in \boldsymbol{P}^1-S$. The fundamental group $\pi_1=\pi_1(\boldsymbol{P}^1-S, o)$ has standard generators σ_i $(1\leq i\leq k)$ represented by a positively oriented loop around each t_i. By the Lefschetz theory (cf. [Le], [G3], [La]), the monodromy representation of π_1 on the homology group $H=H_n(X_o, \boldsymbol{Z})$ has the following properties:

(L1) There exist "vanishing cycles" $\delta_1, \cdots, \delta_k$ in H such that σ_i induces the Picard-Lefschetz transformation

$$\sigma_i^*(\gamma)=\gamma\pm(\gamma, \delta_i)\delta_i \qquad (\gamma \in H).$$

(L2) δ_i's are conjugate under π_1.

(L3) δ_i's generate the primitive part of H, i. e. H itself if n is odd and H modulo the linear space sections if n is even.

In short, the monodromy representation of π_1 on the space generated over

$\boldsymbol{Q}$ by vanishing cycles (which coincides with the space of primitive cycles) is irreducible.

On the other hand, let M_t denote the maximal sub-Hodge structure of $H^n(X_t, \boldsymbol{Z})$ of type $(r, r-1)+(r, r-1)$ if $n=2r-1$ and of type (r, r) if $n=2r$. Then we have

(H1) $M_t \otimes \boldsymbol{Q}$ gives a Hodge-theoretic upper bound for $\mathscr{C}^r(X_t)$ or $J_a^r(X_t)$ according to the parity of n. (This follows from the well-known fact that algebraic cycles are Hodge cycles and from Proposition 1.1).

(H2) The condition (ii) of Theorem 2.1 (or Theorem 2.2) fails exactly when $H^n(X, \boldsymbol{C})$ is of type (r, r) (or of type $(r, r-1)+(r-1, r)$), and hence $M_t \neq H^n(X_t, \boldsymbol{Z})$ under the condition (ii). (See Proposition 1.4.)

(H3) The Hodge filtration $F^{\cdot}H^n(X_t, \boldsymbol{C})$ depends holomorphically on t (cf. [G1], [G3]).

Now identify $H^n(X_t, \boldsymbol{Z})$ with $H_n(X_t, \boldsymbol{Z})$ by the Poincaré duality, and denote it by H_t. Take a primitive element γ of H_o. Suppose that t is in a small neighborhood U of o in $\boldsymbol{P}^1 - S$ and regard γ as an element $\gamma_t \in H_t$. Then γ_t belongs to M_t if and only if it is orthogonal to $F^{r+1}H^n(X_t, \boldsymbol{C})$, i.e.

$$(\sharp) \qquad \int_{\gamma_t} \omega = 0 \qquad (\omega \in F^{r+1}H^n(X_t, \boldsymbol{C})).$$

By (H3), this is an analytic condition on $t \in U$; hence the set of $t \in U$ such that $\gamma_t \in M_t$ is either a countable subset of U or U itself. In the latter case, ($\sharp$) holds for any $t \in \boldsymbol{P}^1 - S$ and for any γ_t obtained by continuous deformation of γ. In particular, (L1) implies that $(\gamma, \delta_i)\delta_i$ belongs to M_o. By (L3), $(\gamma, \delta_i) \neq 0$ for some i and hence $\delta_i \in M_o \otimes \boldsymbol{Q}$. Then (L2) implies that all $\delta_j \in M_o \otimes \boldsymbol{Q}$, and (L3) shows that $M_o = H_o$. But this is excluded under the condition (ii) by (H2).

Therefore, for a given $\gamma \in H_o$, the subset of $t \in U$ with $\gamma_t \in M_t$ is at most countable; call this set A_γ. The union A of all A_γ for γ primitive is again a countable subset of U. So we conclude that, for any $t \in U - A$, $M_t = \{0\}$ if n is odd and $M_t \otimes \boldsymbol{Q}$ is the one-dimensional space spanned by the class of linear space sections if n is even. This proves (N) or (G) for a generic member of the Lefschetz pencil $\{X_t\}$ we started with. q.e.d.

In this paper, we consider only the case of hypersurfaces, but the same argument as above works also for the case of complete intersections in a projective space. For the proof using etale cohomology, we refer to [D], [K] and [Su].

§ 3. Variants for hypersurfaces over $\boldsymbol{Q}$

The theorems of Noether or of Griffiths are certainly the most fundamental results for algebraic cycles on hypersurfaces in a projective space, but one is immediately led to the following type of questions:

Question I. *Can one find some nongeneric X, especially some X defined over $\boldsymbol{Q}$, satisfying the property such as* (N) *or* (G)?

Question II. *Can one give some X for which $\mathscr{C}^r(X)$ or $J_a^r(X)$ is nontrivial, or can one find a nontrivial lower bound for $\rho^r(X)$ or* $\dim J_a^r(X)$ *for a given X?*

We discuss the former in this section and the latter in the next section.

As for the Question I, there are two approaches, i.e. one is to construct explicit examples and the other is to argue existence abstractly. Let us review the recent results on this subject. The results concerning (G) below have all been motivated by the corresponding results concerning (N) under the "close analogy".

First, to find explicit examples, one possibility is to use the action of some automorphisms of a hypersurface on its cohomology, which should play the role of monodromy in the proof of Theorem 2.1 or 2.2 given in the pervious section. Namely, if one finds a hypersurface X admitting a group G of projective transformations such that the primitive part of $H^n(X, \boldsymbol{Q})$ has no G-invariant sub-Hodge structure of type $(r, r-1)+(r-1, r)$ if $n=2r-1$ or of type (r, r) if $n=2r$ other than $\{0\}$, then the same argument as in Section 2 will show that X satisfies the condition (G) or (N).

The following results are based on such an idea:

Theorem 3.1 ([S4]). *Assume that* (i) $n=2r$, (ii) g.c.d. $(m, 6)=1$ *and* (iii) *there is a prime divisor of $n+1$ not dividing m. Then the hypersurface*

$$Y_m^n\colon x_0x_1^{m-1}x_1x_2^{m-1}+\cdots+x_nx_0^{m-1}+x_{n+1}^m=0$$

satisfies the condition (N), *i.e.* $\rho^r(Y_m^n)=1$.

Theorem 3.2 ([S5]). *Assume that* (i) $n=2r-1$, (ii) $r>1$ *and* $m\geq 2+3/(r-1)$, *and* (iii) *the integer*

$$d_0=\{(m-1)^{n+2}+1\}/m$$

is a prime number. Then the hypersurface

$$Z_m^n\colon\ x_0x_1^{m-1}+x_1x_2^{m-1}+\cdots+x_nx_{n+1}^{m-1}+x_{n+1}x_0^{m-1}=0$$

satisfies the condition (G), *i.e.* $J_a^r(Z_m^n)=0$.

The additional assumptions on m and n in the above theorems are imposed to assure a kind of irreducibility of the action of the following automorphism of Y_m^{2r} or Z_m^{2r-1}:

$$g:\begin{cases} x_i'=\exp(2\pi\sqrt{-1}(1-m)^{2r-1}/d)\cdot x_i & (0\leq i\leq 2r) \\ x_{2r+1}'=x_{2r+1} \end{cases}$$

where $d=(m-1)^{2r+1}+1$, which replaces the irreducibility of the monodromy representation.

Remark 3.3. (a) The automorphism g above can be viewed as the monodromy along the circle $|t|=1$ of a pencil $\{Y_t\}$ or $\{Z_t\}$ defined by

$$Y_t\colon\ x_0x_1^{m-1}+\cdots+x_nx_0^{m-1}+t\cdot x_{n+1}^m=0$$

and similarly for Z_t. Of course, these are *not* Lefschetz pencils.

(b) We do not know whether or not the full image of the Abel-Jacobi map $J_h^r(X)$ is trivial for $X=Z_m^n$ in Theorem 3.2. There are $n+2$ linear subspaces of codimension r on X

$$L_i\colon\ x_i=x_{i+2}=\cdots=x_{i+2(r-1)}=0 \qquad (0\leq i\leq n+1)$$

(suffix being taken modulo $n+2$), which are mutually homologically equivalent. It is easy to show that the cycles L_i-L_j give torsion elements of $CH^r(X)$ of order dividing d_0. We expect that L_i, L_j are *not rationally equivalent.* (At any rate, this is true in the 1-dimensional case $n=r=1$, where L_i-L_j gives $\boldsymbol{Q}$-rational points of order d_0 of J, the Jacobian of the curve Z_m^1. Indeed, $cl(L_1-L_0)$ generates the torsion part $J(\boldsymbol{Q})_{\text{tor}}$ of the Mordell-Weil group of J over $\boldsymbol{Q}$ provided $d_0=m^2-3m+3$ is prime and $m>4$, as can be shown by a result of Gross-Rohrlich [G-R].)

Now the assumption on m and n in Theorem 3.1 or 3.2 is more restrictive than that of Theorem 2.1 or 2.2, and it will be very nice if one could find some explicit examples for each possible value of m, n. But no other examples have been known so far. So we turn to non-explicit approach to the problem. The first result in this direction is due to Terasoma:

Theorem 3.4 ([T]). *For any choice of* (m, n) *satisfying the assumption of Theorem* 2.1, *there exists a nonsingular hypersurface defined over* $\boldsymbol{Q}$, *of dimension* n *and of degree* m, *which satisfies the condition* (N).

The outline of the proof is as follows. Take a Lefschetz pencil of hypersurfaces $\{X_t\}$, defined over $\boldsymbol{Q}$, and consider the associated monodromy representation of the algebraic fundamental group of $\boldsymbol{P}^1-S$ on the ℓ-adic cohomology group $H^n(X_{\bar{t}}, \boldsymbol{Q}_\ell)$ as in Deligne's proof of Noether's theorem [D]. Here $\bar{t}$ is chosen as a $\bar{\boldsymbol{Q}}$-valued point of $\boldsymbol{P}^1-S$ over a $\boldsymbol{Q}$-rational point t. The key of Terasoma's proof is to compare the image ℓ-adic group of the monodromy representation with that of the Galois representation of $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ on $H^n(X_{\bar{t}}, \boldsymbol{Q}_\ell)$ and to show that, for a suitable choice of a $\boldsymbol{Q}$-rational point t of $\boldsymbol{P}^1-S$, these ℓ-adic groups coincide. Then the proof reduces to the result of [D].

The corresponding fact for (G) has been recently shown by Suwa:

Theorem 3.5 ([Su]). *For any choice of* (m, n) *satisfying the assumption of Theorem* 2.2, *there exists a nonsingular hypersurface defined over* $\boldsymbol{Q}$, *of dimension* n *and degree* m, *which satisfies the condition* (G).

More recently, just after the Sendai Symposium, S. Bolch has remarked that the method of Terasoma can be used to prove:

Theorem 3.6. *Under the same assumption as before, there exists a nonsingular hypersurface over* $\boldsymbol{Q}$ *which satisfies not only* (G) *but also* (G$'$). *i.e. which has* $J_a^r(X)=0$ *but* $J_h^r(X)\neq 0$. *Thus such an* X *over* $\boldsymbol{Q}$ *has the Griffiths group of rank*>0.

Bloch's proof will be found in the appendix at the end of this paper.

§ 4. Construction of hypersurfaces with nontrivial $J_a^r(X)$

Now we turn to the Question II stated at the beginning of Section 3. The example of hypersurfaces X with nontrivial $\mathscr{C}^r(X)$ ($\dim X=2r$) which have been best studied so far is perhaps the Fermat case $X=X_m^n$: $\sum_{i=0}^{n+1} x_i^m=0$. The Fermat varieties of a fixed degree m and of various dimensions are related by what is called *the inductive structure*, which allows one to construct nontrivial algebraic cycles on a higher dimensional Fermat variety starting from those on lower dimensional ones (cf. [K-S], [R], [S1], [S2]). In this section, we show that the method of inductive structure is also useful in constructing hypersurfaces with nontrivial $J_a^r(X)$. Indeed we can give some lower bound of the dimension of $J_a^r(X)$ for X admitting the inductive structure. Namely we have

Theorem 4.1. *Suppose that a nonsingular hypersurface* X *of odd dimension* $n=2r-1$ *has a defining equation of the form*

$$f(x_0, x_1, \cdots, x_{2s})+h(x_{2s+1}, \cdots, x_{2r})=0$$

for some $0<s<r$. *Let* U *and* V *denote the hypersurfaces in* $\boldsymbol{P}^{2s}$ *and* $\boldsymbol{P}^{2r-2s-1}$ *defined by*

$$U\colon f(x_0, x_1, \cdots, x_{2s})=0 \qquad (\dim U=2s-1)$$

$$V\colon h(y_0, y_1, \cdots, y_{2r-2s-1})=0 \qquad (\dim V=2(r-s-1)).$$

Then

$$\dim J_a^r(X) \geq \dim J_a^s(U) \cdot \{\rho^{r-s-1}(V)-1\} \tag{4.1}$$

where $\rho^{r-s-1}(V)$ *is the middle Picard number of* V.

As a special but interesting case, we restate the case $n=3$:

Corollary 4.2. *Let* C *be a nonsingular plane curve of degree* m *with the equation* $f(x_0, x_1, x_2)=0$, *and let* D *be the* m *points in* $\boldsymbol{P}^1$ *defined by* $h(y_0, y_1)=0$, h *being a binary form of degree* m *without multiple factors. Assume* $m \geq 3$. *Then the threefold* X *in* $\boldsymbol{P}^4$ *defined by the equation*

$$f(x_0, x_1, x_2)+h(x_3, x_4)=0$$

has nontrivial $J_a^2(X)$; *more precisely, one has*

$$\dim J_a^2(X) \geq \tfrac{1}{2}(m-1)^2(m-2). \tag{4.2}$$

This result can be expressed in more geometric terms. Embed C and D into X as disjoint subsets as follows:

$$C=\{(x_0, x_1, x_2, 0, 0)\}, \qquad D=\{(0, 0, 0, x_3, x_4)\}$$

and let $L(P, Q)$ denote the line of $\boldsymbol{P}^4$ joining $P \in C$ and $Q \in D$, which is obviously contained in X. For a fixed Q, $\{L(P, Q) | P \in C\}$ forms an algebraic family of 1-cycles on X parametrized by C. Fixing a point P_0 of C, consider the map of C to the intermediate Jacobian $J^2(X)$ of X:

$$P \longrightarrow \psi^2(L(P, Q)-L(P_0, Q)),$$

and let α_Q be the induced homomorphism of the Jacobian $J(C)$ into $J_a^2(X)$. With this notation, we can rewrite Corollary 4.2 as follows:

Corollary 4.3. *Each homomorphism* $\alpha_Q\colon J(C) \longrightarrow J_a^2(X)$ *is an isogeny from* $J(C)$ *to its image. In particular, for any* $P_1 \in C$, *there exist only finitely many points* $P \in C$ *such that the lines* $L(P, Q)$ *and* $L(P_1, Q)$ *are rationally equivalent. Furthermore these* α_Q*'s are related by a single relation:* $\sum_Q \alpha_Q=0$, *and any* $m-1$ *of them are independent. Hence the images of* α_Q $(Q \in D)$ *generate an abelian subvariety of* $J_a^2(X)$ *whose dimension is equal to*

$$(m-1) \cdot \dim J(C)=(m-1)^2(m-2)/2.$$

Proof of Theorem 4.1. Let us prove this in several steps.

Step 1. First we recall the general setup of hypersurfaces with inductive structure (cf. [K-S], [S1]). Suppose $n=a+b$ with $a, b>0$, and consider hypersurfaces of a fixed degree m X^n, X^{a-1}, X^{b-1}, X^a and X^b in projective spaces of respective dimensions given as follows:

$$\begin{aligned}
&X^n\colon f(z_0, \cdots, z_a)+h(z_{a+1}, \cdots, z_{n+1})=0\\
&X^{a-1}\colon f(x_0, \cdots, x_a)=0\\
&X^{b-1}\colon h(y_0, \cdots, y_b)=0\\
&X^a\colon f(x_0, \cdots, x_a)+x_{a+1}^m=0\\
&X^b\colon h(y_0, \cdots, y_b)+y_{b+1}^m=0.
\end{aligned}$$

Then there is a diagram

$$\text{(4.3)}\qquad \begin{tikzcd} Z_1 \arrow[d, "\beta_1"] & \subset & Z \arrow[d, "\beta"] \arrow[dr, "\psi"] & \\ X^{a-1}\times X^{b-1} & \subset & X^a\times X^b \arrow[r, "\varphi"] & X^n \end{tikzcd}$$

where (i) φ is a rational map of degree m defined by

$$z_i=x_i\cdot y_{b+1}\quad (0\leq i\leq a),\qquad z_{a+1+j}=\varepsilon x_{a+1}\cdot y_j\quad (0\leq j\leq b)$$

ε being a fixed m-th root of -1; (ii) the fundamental locus of φ is the subvariety $x_{a+1}=y_{b+1}=0$ of $X^a\times X^b$ which can be identified with $X^{a-1}\times X^{b-1}$; (iii) $\beta\colon Z\to X^a\times X^b$ is the blowing-up along the center $X^{a-1}\times X^{b-1}$; (iv) $\psi=\beta\circ\varphi$ is a morphism of Z to X^n; and (v) $\beta_1=$the restriction of β to the exceptional divisor Z_1 (thus Z_1 is a $\boldsymbol{P}^1$-bundle over $X^{a-1}\times X^{b-1}$).

Step 2. Now we apply Proposition 1.3 to the correspondence Z_1 between $Y=X^{a-1}\times X^{b-1}$ and X^n. We further specialize to the case $n=2r-1$, $a=2s$, $b=2(r-s-1)+1$, $d=r-1$, $e=1$. Then we obtain a commutative diagram

$$\text{(4.4)}\qquad \begin{tikzcd} CH^s(X^{a-1})_{\mathrm{alg}}\times CH^{r-s-1}(X^{b-1}) \arrow[r] \arrow[d, "\psi^s\times\gamma^{r-s-1}"] & CH^{r-1}(Y)_{\mathrm{alg}} \arrow[r, "\chi^{r-1}"] \arrow[d, "\psi^{r-1}"'] & CH^r(X^n)_{\mathrm{alg}} \arrow[d, "\psi^r"'] \\ J_a^s(X^{a-1})\times \mathscr{C}^{r-s-1}(X^{b-1}) \arrow[r] & J_a^{r-1}(Y) \arrow[r] & J_a^r(X^n) \end{tikzcd}$$

such that the tangent map of the lower horizontal map is compatible with the map

$$H^{a-1}(X^{a-1})\times H^{b-1}(X^{b-1})\longrightarrow H^{n-2}(Y)\longrightarrow H^n(X^n).$$

Step 3. Letting the suffix "prim" denote the primitive part, one of the key cohomological facts on inductive structure asserts the injectivity of the map

$$H^{a-1}_{\mathrm{prim}}(X^{a-1})\otimes H^{b-1}_{\mathrm{prim}}(X^{b-1})\longrightarrow H^{n}_{\mathrm{prim}}(X^{n}). \tag{4.5}$$

This is well-known for the Fermat case (see [K-S], [S1] or [R]) and hence is valid for the present situation too since the property in question is a topological one. Therefore the "tangent map" of

$$J^{s}_{a}(X^{2s-1})\times \mathscr{C}^{r-s-1}_{\mathrm{prim}}(X^{2(r-s-1)})\longrightarrow J^{r}_{a}(X^{n})$$

is injective, and thus

$$\dim J^{r}_{a}(X^{n})\geq \dim J^{s}_{a}(X^{2s-1})\cdot \operatorname{rank} \mathscr{C}^{r-s-1}_{\mathrm{prim}}(X^{2(r-s-1)}).$$

This proves Theorem 4.1. q.e.d.

Proof of Corollaries 4.2 *and* 4.3. Let $n=3$, $r=2$, $s=1$ in the above. It suffices to observe that $D=X^0$ consists of m points so that $\mathscr{C}^{0}_{\mathrm{prim}}(D)$ has rank $m-1$ and that for a curve $C=X^1$ we have $J^1_a(C)=\mathrm{Jac}(C)$, whose dimension is $(m-1)(m-2)/2$. Hence (4.2) follows from (4.1), proving Corollary 4.2.

As for Corollary 4.3, note that the upper horizontal map in (4.4) sends the 0-cycle $(P-P_0)\times Q$ on $C\times D$ to the 1-cycle $L(P, Q)-L(P_0, Q)$ on X^3. Then the rest of the assertion follows from the injectivity of tangent map of

$$J^1_a(C)\times \mathscr{C}^{0}_{\mathrm{prim}}(D)\longrightarrow J^2_a(X^3)$$

stated in Step 3. q.e.d.

Remark 4.4. (a) Of course, the dimension of $J^2_a(X)$ for certain X can be strictly larger than the lower bound given by (4.1) or (4.2). For example, it is well-known that $\dim J^2_a(X)=5$ for any cubic 3-folds. Also, for Fermat 3-folds of small degree, we can determine $\dim J^2_a(X)$ since the general Hodge Conjecture holds for them (cf. [S3], Section 3).

(b) Suwa [Su] studies analogous problem for the Abel-Jacobi map in the algebraic context (i.e. the one defined by S. Bloch). His result for supersingular Fermat varieties in positive characteristic also makes use of inductive structure.

Finally we give an application to the rank of the Chow group $CH^r(X/K)$ which, for X defined over a number field K, denotes the group of K-rational algebraic cycles modulo rational equivalence.

Proposition 4.5. *With the notation of Corollary* 4.2, *assume further that* $f(x_0, x_1, x_2)$ *and* $h(y_0, y_1)$ *have coefficients in a number field* K. *Let* N *be the number of* K*-irreducible factors of* h, *and let* R *be the Mordell-Weil rank of the Jacobian* $J(C)$ *of* C *over* K. *Then*

$$\text{rank } CH^2(X/K) \geq 1+(N-1)R. \tag{4.6}$$

Proof. We may assume $N>1$. By Corollary 4.2, each K-irreducible factor of h defines an isogeny of $J(C)$ to its image in $J_a^2(X)$. Take R elements of $CH^1(X/K)_{\mathrm{alg}} = J(C)(K)$ which form a basis of the Mordell-Weil group modulo torsion. Then their images in $CH^2(X/K)$ map to R independent elements in $J_a^2(X)$, and hence they are themselves independent. Since the only relation among these isogenies of $J(C)$ into $J_a^2(X)$ is the relation "sum of all these isogenies is zero", we see that $CH^2(X/K)_{\mathrm{alg}}$ contains at least $(N-1)R$ independent elements. q.e.d.

Example 4.6. Let $K=\boldsymbol{Q}$ and $X=X_m^3$ be the Fermat 3-fold of prime degree m. Then $h(y_0, y_1)=y_0^m+y_1^m$ has two irreducible factors over $\boldsymbol{Q}$, and $C=X_m^1$ is the Fermat curve of degree m. Hence we have

$$\text{rank } CH^2(X_m^3/\boldsymbol{Q}) \geq 1+\text{rank } J(C)(\boldsymbol{Q}).$$

By a result of Gross and Rohrlich [G-R], the rank of the Mordell-Weil group $J(C)(\boldsymbol{Q})$ tends to infinity as $m\to\infty$, which shows:

$$\lim_{m\to\infty} \text{rank } CH^2(X_m^3/\boldsymbol{Q}) = \infty. \tag{4.7}$$

In closing this paper, we would like to raise a question which is related to the above Corollary 4.3 and Remark 3.3(b):

Question. *Suppose that a nonsingular threefold* X *of degree* ≥ 3 *in* $\boldsymbol{P}^4$ *contains two lines* L, L'. *Then are* L *and* L' *rationally inequivalent?*

Appendix by Spencer Bloch

A remark on Griffiths groups*

Here is another application of the ideas in Terasoma's paper [T]. Let X be smooth and projective of dimension $2r$ defined over a number field k, and assume $H^{2r}(X_{\bar{k}}, \boldsymbol{Q}_\ell)$ contains a nontrivial primitive algebraic cycle class [Z] with Z defined over k (e.g. $X=$quadric in $\boldsymbol{P}^5$). Let Y be X blown up along the base of a Lefschetz pencil defined over k, so we have $Y\to\boldsymbol{P}^1$. Let

*) Extracted from a letter to T. Terasoma, dated July 1, 1985

$S \subset \boldsymbol{P}^1$ be the set of singular fibres. For $\bar{t} \in (\boldsymbol{P}^1 - S)(\boldsymbol{C})$ we have the Griffiths group

$$\operatorname{Griff}^r(Y(\bar{t})) = \{\text{cycles homologous to zero on } Y(\bar{t})\} \text{ modulo algebraic equivalence}$$

and $[Z|_{Y(\bar{t})}] \in \operatorname{Griff}^r(Y(\bar{t}))$. Set $Z(t) = Z|_{Y(t)}$.

Proposition. *Assume the monodromy representation on the vanishing cycles is irreducible. Assume also the Hodge structure on the vanishing cycles is not of type $(r, r-1)+(r-1, r)$. Then there exists an infinite number of points $t \in (\boldsymbol{P}^1 - S)(k)$ such that no multiple of $[Z(t)]$ is zero in* $\operatorname{Griff}^r(Y(\bar{t}))$ *for any geometric point $\bar{t}$ over t.*

Proof. Let V be the ℓ-adic vanishing cycle representation. As in Griffiths' work there is a nontrivial class

$$[Z] \in H^1(\pi_1(\boldsymbol{P}^1 - S), V).$$

In fact this class is of infinite order. Note $\boldsymbol{P}^1 - S$ is a $K(\pi, 1)$ so the continuous Galois cohomology here is the same as the topological cohomology. Let ρ denote the monodromy representation. If α is a 1-cocycle on $\pi_1(\boldsymbol{P}^1 - S)$ with values in V, then $g \to (\alpha(g), \rho(g))$ is a homomorphism from $\pi_1(\boldsymbol{P}^1 - S)$ to the semidirect product $V \times \operatorname{Im}(\rho)$. Denote this homomorphism by σ. By the argument of [T] there are lots of t's such that the composites $\operatorname{Gal}(\bar{\boldsymbol{Q}}/k) \to \operatorname{Im}(\sigma)$ (defined up to conjugation) are surjective. Note that for such a t, the map

$$\Gamma \underset{\text{def}}{=} \operatorname{Ker}(\operatorname{Gal}(\bar{\boldsymbol{Q}}/k) \longrightarrow \operatorname{Im}(\rho)) \longrightarrow \operatorname{Im}(\sigma) \cap V = \operatorname{Im}(\sigma|_{\operatorname{Ker}(\rho)})$$

is also surjective.

From the Hochschild-Serre spectral sequence we get

$$\begin{array}{ccccccc} 0 \longrightarrow & H^1(\operatorname{Im}(\rho), V) & \longrightarrow & H^1(\pi_1(\boldsymbol{P}^1 - S), V) & \longrightarrow & \operatorname{Hom}_{\operatorname{Im}(\rho)}((\operatorname{Ker}(\rho))^{ab}, V) \\ & \| & & \downarrow & & \downarrow \\ 0 \longrightarrow & H^1(\operatorname{Im}(\rho), V) & \longrightarrow & H^1(\operatorname{Gal}(\bar{\boldsymbol{Q}}/k), V) & \longrightarrow & \operatorname{Hom}_{\operatorname{Im}(\rho)}(\Gamma^{ab}, V). \end{array}$$

The image of α in the homomorphism group on the top right is induced by $\alpha|_{\operatorname{Ker}(\rho)}$ so the above mentioned surjectivity implies that α maps to a non-zero element $\beta(t)$ in $H^1(\operatorname{Gal}(\bar{\boldsymbol{Q}}/k), V)$.

If $Z(t)$ is algebraically equivalent to zero, then arguing as in my Crelle paper [B], there would be an abelian subvariety $A(t)$ of the intermediate Jacobian of $Y(t)$ such that $A(t)$ is defined over k, the Tate module $T_\ell(A(t))$

$\subset V$ as a Gal $(\bar{Q}/k)$ submodule, and β comes via the Kummer sequence from a k-point of $A(t)$. As Suwa [Su] has observed, irreduciblity of the representation ρ forces $A(t)=0$. This completes the proof.

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