# The Rationality of the Moduli Spaces of Vector Bundles of Rank 2 on $P^2$
### (with an appendix by Isao Naruki)

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## Introduction

Let $k$ be an algebraically closed field and $M(c_1, c_2)$ the moduli space of vector bundles $E$ of rank 2 on $P_k^2$ with $c_1(E) = c_1$ and $c_2(E) = c_2$. Through tensoring a suitable line bundle, $M(c_1, c_2)$ is isomorphic to one of $M(0, a)$ or $M(1, b)$. It is known that

1)  $M(0, a) = \phi$ unless $a \geq 2$ while $M(1, b) = \phi$ unless $b \geq 1$,

2)  $M(0, a)$ and $M(1, b)$ are non-singular, irreducible, quasi-projective varieties for all $a \geq 2$ and all $b \geq 1$,

3)  all the $M(0, a)$ and $M(1, b)$ are unirational.

W. Barth [1] stated that $M(0, a)$ is rational for every $a \geq 2$ while the rationality of $M(1, b)$ was proved by K. Hulek [4]. Recently we found a serious gap in the proof of Barth. We come upon a similar gap in Hulek's proof though it can be fortunately corrected in an obvious way (see the footnote on p. 266 of [4]). On the other hand, G. Ellingsrud and S.A. Strømme [3] showed the following results by a method completely different from Barth's and Hulek's.

**Theorem 0.1.** (1)  $M(1, b)$ is rational for every $b \geq 1$.
(2)  $M(0, a)$ is rational if $a \geq 2$ and $a$ is odd.
(3)  If $a$ is even and $a \geq 2$, then there is a $P^1$-bundle in the étale topology (see Remark 3.8) over a dense open set of $M(0, a)$ which is rational.

If we try to fix the proof of Barth, then we encounter the problem of rationality of the quotient of the affine cone over a Grassmann variety by an action of a finite group which is a semi-direct product of $S_n$ by $(Z/2Z)^{\oplus n}$. The situation is going to be explained in Section 1. Combining the above theorem, Theorem 7.17 of [7], Theorem 2 of [5] and Theorem 3.17 of [8], we see that $M(c_1, c_2)$ is rational if it is fine. We shall give a

proof of Theorem 0.1 in Section 3 which shows, comparing it with the account in Section 2 about the existence of universal families, that the fact is not accidental at all. Our proof of Theorem 0.1 is essentially based on the same idea as that in the proof of the unirationality of $M(c_1, c_2)$ in Section 7 of [7].

Throughout this work $P$ denotes the projective plane over an algebraically closed field $k$. For a $k$-scheme $X$, we shall use the notation $X(k)$ for the set of $k$-rational points of $X$. If $f: X \to Y$ is a morphism of schemes and $F$ a coherent sheaf on $X$, then $X(y)$ denotes the (geometric) fibre of $f$ over a (geometric) point $y$ of $Y$ and $F(y)$ does $F \otimes_{\mathscr{O}_X} \mathscr{O}_{X(y)}$. For a coherent sheaf $E$ on a $k$-scheme $Z$, $h^i(Z, E)$ or $h^i(E)$, for short, means, as usual, $\dim_k H^i(Z, E)$. $E^*$ denotes the dual $\mathscr{H}om_{\mathscr{O}_X}(E, \mathscr{O}_X)$ of a coherent sheaf $E$ on a scheme $X$.

## § 1. Barth's proof

Let $H$ be an $n$-dimensional vector space over $C$ and $H^*$ its dual space. For the affine cone $C_n$ over the Grassmann variety $\mathrm{Gr}(1, n-1)$, $C_n^* = C_n - \{0\}$ can be identified with $H^* \wedge H^* - \{0\}$ and, by using the Plücker coordinates, we have a natural embedding $\sigma$ of $C_n^*$ into the space of bilinear forms $M_n(C)$. Let $\varDelta$ be the closed set in $(C^*)^n$ of $n$-tuples $(\lambda_1, \cdots, \lambda_n)$ with not all $\lambda_i$ different. We have a morphism $f$ of $X_n = ((C^*)^n - \varDelta) \times (C^*)^n \times C_n^*$ to $Y_n = M_n(C) \times M_n(C) \times M_n(C)$ as follows;

For $\lambda = (\lambda_1, \cdots, \lambda_n)$ in $(C^*)^n - \varDelta$, $\mu = (\mu_1, \cdots, \mu_n)$ in $(C^*)^n$ and $x$ in $C_n$, we set

$$f(\lambda, \mu, x) = \left( \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \sigma(x) + \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix} \right).$$

Then it is easy to see that $f$ is an immersion.

Let $\mathrm{GL}(H)$ act on $Y_n$ in the following way:

$(\omega, (A, B, C)) \longmapsto ({}^t\omega A\omega, {}^t\omega B\omega, {}^t\omega C\omega)$   for $\omega$ in $\mathrm{GL}(H)$ and

for $(A, B, C)$ in $Y_n$.

An $\omega$ in $\mathrm{GL}(H)$ sends $f(X_n)$ to itself if and only if

(1.1.1) $\qquad {}^t\omega \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \omega \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$   and

$$(1.1.2) \qquad {}^t\omega \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \omega = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix} \qquad \text{for all } (\lambda_1, \cdots, \lambda_n) \text{ and}$$

$(\mu_1, \cdots, \mu_n)$ in $(C^*)^n - \Delta$.

Thus the group $G_n = \{\omega \in \mathrm{GL}(H) \mid \omega$ satisfies (1.1.1) and (1.1.2)$\}$ acts on $X_n$. By using a beautiful relationship between $M(0, n)$ and the rank-2 nets of quadrics, Barth showed that $M(0, n)$ is birational to the quotient of $X_n$ by $G_n$. However, he inferred incorrectly that $G_n$ was the image of $S_n$ by the regular representation $\rho$. Indeed the correct group is described as follows. The group $S_n$ acts on $(Z/2Z)^{\oplus n}$ as the permutation of the direct factors. This action defines a semi-direct product $G'_n$.

$$1 \longrightarrow (Z/2Z)^{\oplus n} \longrightarrow G'_n \longrightarrow S_n \longrightarrow 1.$$

Sending $(\varepsilon_1, \cdots, \varepsilon_n)$ in $(Z/2Z)^{\oplus n}$ to

$$\begin{pmatrix} (-1)^{\varepsilon_1} & & 0 \\ & \ddots & \\ 0 & & (-1)^{\varepsilon_n} \end{pmatrix}$$

in $\mathrm{GL}(H)$, we obtain a representation $\tau$ of $(Z/2Z)^{\oplus n}$ in $\mathrm{GL}(H)$. The map which sends $(g, h)$ of $G'_n$ to $\rho(h)\tau(g)$ in $\mathrm{GL}(H)$ induces an isomorphism of $G'_n$ to $G_n$.

Since $\rho(S_n)$ acts on the first direct factor $(C^*)^n - \Delta$ of $X_n$ as the permutation of the coordinates, the quotient $\{(C^*)^n - \Delta\}/\rho(S_n)$ is rational and the action is free. Then, by descent theory, it is not difficult to prove that $X_n/\rho(S_n)$ is rational. To our regret, the correct group is not $\rho(S_n)$ and moreover $\tau((Z/2Z)^{\oplus n})$ acts trivially on the first and the second direct factors of $X_n$. Then, the problem of rationality of $X_n/G_n$ is, as far as the author knows, quite difficult.

**Observation 1.2.** $\tau((Z/2Z)^{\oplus n})$ acts trivially on $(C^*)^n$ and $(C^*)^n - \Delta$ while the action of $G_n/\tau((Z/2Z)^{\oplus n}) \simeq \rho(S_n)$ on both spaces is permutation of the coordinates. Thus $(C^*)^n$ and $(C^*)^n - \Delta$ are embedded into $C^n$ equivariantly. Moreover, the extended action of $\rho(S_n)$ is linear. Therefore, if (a) the action of $\bar{G}_n = G_n/\mathrm{center}$ on $C_n$ is generically free and (b) $C_n/\bar{G}_n$ is rational, then $M(0, n)$ is rational by virtue of the descent theory of vector bundles.

Let $\{Y_1, \cdots, Y_n\}$ and $\{Z_1, \cdots, Z_n\}$ be two sets of indeterminates over $C$. If we set $T_{ij} = Y_i Z_j - Y_j Z_i$, then the subring $R_n = C[T_{ij} \mid 1 \leq i < j \leq n]$ of $C[Y_1, \cdots, Y_n, Z_1, \cdots, Z_n]$ is the affine ring of $C_n$. When $n = 2$,

we have $\bar{G}_2 \simeq Z/2Z \oplus Z/2Z$ and the action of $\bar{G}_2$ on $C_2$ is not effective. But the quotient of $(C^*)^2 \times C_2$ by $\bar{G}_2$ is rational; the function field is generated by $W_1 + W_2$, $W_1 W_2$, $T_{12}^2$, where $W_1$, $W_2$ are the coordinates of $(C^*)^2$. Thus $M(0, 2)$ is rational. In fact, it is the space of non-singular conics in $P^2$. Assume that $n \geq 3$. Then the ring of invariants of $R_n$ by $\tau((Z/2Z)^{\oplus n})$ contains $\{T_{ij}^2 \mid 1 \leq i < j \leq n\}$ and hence we see easily (a) in these cases. When $n = 3$, the function field of $C_3/\tau((Z/2Z)^{\oplus 3})$ is generated by $T_{12}^2$, $T_{13}^2$ and $T_{12}T_{23}T_{13}$. We set

$$U_1 = T_{12}^2 + T_{23}^2 + T_{13}^2, \quad U_2 = T_{12}^2 T_{23}^2 + T_{23}^2 T_{13}^2 + T_{12}^2 T_{13}^2,$$

$$U_3 = T_{12}^2 T_{23}^2 T_{13}^2, \quad \delta = (T_{12}^2 - T_{23}^2)(T_{23}^2 - T_{13}^2)(T_{12}^2 - T_{13}^2) \quad \text{and}$$

$$V = \delta T_{12} T_{23} T_{13}.$$

Then, the function field $L$ of $C_3/\bar{G}_3$ is generated by $U_1$, $U_2$, $U_3$ and $V$ over $C$. Since the transcendence degree of $L$ over $C$ is 3, we must have one relation among the generators. The discriminant $\delta^2$ is a polynomial $D(U_1, U_2, U_3)$ of $U_1$, $U_2$, $U_3$ and the relation is $V^2 = U_3 D(U_1, U_2, U_3)$. Since $M(0, 3)$ is birational to the product $C^6 \times (C_3/\bar{G}_3)$, the function field $L$ is stably rational. Therefore, we come to the following question which seems to be quite interesting in view of the work [2].

**Question 1.3.** Let $Z$ be the affine variety defined by the equation $V^2 = U_3 D(U_1, U_2, U_3)$ in $C^4$ with coordinates $V$, $U_1$, $U_2$, $U_3$. Is $Z$ rational?[*]

**Remark 1.4.** In the proof of the rationality of $M(c_1, c_2)$ with $c_1$ odd, Hulek made the same mistake as Barth. In fact, $\Sigma_{n-1}$ in the proof of (2.1) of [4] should be replaced by $G_{n-1}$ in the above. But $G_{n-1}$ acts faithfully on the last direct factor $C^{n-1}$ of $X$ and the function field of $C^{n-1}/G_{n-1}$ is generated by the elementary symmetric polynomials in $b_1^2$, $b_2^2$, $\cdots$, $b_{n-1}^2$. Thus the conditions similar to those we required in (a), (b) of Observation 1.2 are satisfied, which implies the rationality of $M(c_1, c_2)$ when $c_1$ is odd.

## § 2. Universal family

Let $H(x)$ be a numerical polynomial in one variable $x$ of degree $n$. Then, for every integer $m$, we can write $H(m)$ in the form

$$H(m) = \sum_{i=0}^{n} a_i \binom{m+i}{i} \quad \text{with } a_0, \cdots, a_n \text{ integers.}$$

We set

$$\delta(H) = \text{G.C.D.} \{a_0, \cdots, a_n\}.$$

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[*] $Z$ is rational (see Appendix).

Pick a positive integer $r$ which is a divisor of $a_n$ and set

$$\delta'(H, r) = \text{G.C.D.} \{a_0, \cdots, a_n, r\}.$$

Obviously $\delta'(H, r)$ is a divisor of $\delta(H)$, a fortiori, $\delta'(H, r) \leqq \delta(H)$.

Let $f: X \to S$ be a smooth, projective, geometrically integral morphism of schemes of finite type over a universally Japanese ring. Fix an $f$-ample invertible sheaf $\mathcal{O}_X(1)$ and set $M_{X/S}(H)$ to be the moduli space of stable sheaves $E$ on $(X, \mathcal{O}_X(1))$ with $\chi(E(m)) = H(m)$ (see [6] and [7]). In [7] we proved the following (Theorem 6.11).

**Theorem 2.1.** *Every quasi-compact open set $U$ of $M_{X/S}(H)$ has a universal family if $\delta(H) = 1$.*

Let $M_{X/S}(H)_0$ be the open set of $M_{X/S}(H)$ whose points correspond to locally free sheaves. If $f: X \to S$ has a section, for example, $S = \text{Spec}(k)$ with $k$ an algebraically closed field, then we have the following result which is stronger than Theorem 2.1 for $M_{X/S}(H)_0$.

**Theorem 2.2.** *Assume that $f: X \to S$ has a section $g: S \to X$ and the degree $d$ of $X$ with respect to $\mathcal{O}_X(1)$ is constant. Then $r = a_n/d$ is a positive integer. A quasi-compact open set $U$ of $M_{X/S}(H)$ has a universal family if (1) $\delta'(H, r) = 1$ and if (2-a) $U$ is contained in $M_{X/S}(H)_0$ or (2-b) $U$ is locally factorial and $U \cap M_{X/S}(H)_0$ is dense in $U$.*

*Proof.* There are an open subscheme $Q$ of a Quot-scheme $\text{Quot}(\mathcal{O}_X^{\oplus N}/X/S)$ and a surjective morphism $\phi: Q \to U$ such that (a) $\text{PGL}(N, S)$ acts on $Q$, (b) $\phi$ is a principal fibre bundle with group $\text{PGL}(N, S)$, (c) the universal quotient sheaf $\tilde{E}$ on $X \times_S Q$ carries a $\text{GL}(N, S)$-linearization when $X \times_S Q$ is regarded as a $\text{GL}(N, S)$-scheme through the action of $\text{PGL}(N, S)$ on $Q$ and (d) the action of the center of $\text{GL}(N, S)$ on $\tilde{E}$ is the multiplication by constants. By (c) and (d) we see that if there is an invertible sheaf $L$ on $Q$ with $\text{GL}(N, S)$-linearization such that the action of the center of $\text{GL}(N, S)$ on $L$ is the multiplication by constants, then the action of the center on $\tilde{E}' = \tilde{E} \otimes_{\mathcal{O}_Q} L^*$ is trivial and hence $\tilde{E}'$ carries a $\text{PGL}(N, S)$-linearization. Then, thanks to descent theory, (b) implies that $\tilde{E}'$ descends to a sheaf on $X \times_S U$ which is a universal family up to the tensor of a line bundle on $X$. In the proof of Theorem 2.1 ([7, Theorem 6.11]) we showed that there is a line bundle $L_1$ on $Q$ with a $\text{GL}(N, S)$-linearization such that the action of the center of $\text{GL}(N, S)$ is the multiplication by the $\delta(H)$-th power of constants. On the other hand, we have the section $g' = g \times_S 1_Q$: $Q \to X \times_S Q$ which is a $\text{GL}(N, S)$-morphism. $E' = g'^*(\tilde{E})$ carries a $\text{GL}(N, S)$-linearization. Let $E_0$ be the restriction of $E'$ to $U_0 = M_{X/S}(H)_0 \cap U$. Then $E_0$ is a vector bundle of rank $r$. Thus $L_2 = \bigwedge^r E_0$ is a

GL($N$, $S$)-sheaf on which the center acts as the multiplication by the $r$-th power of constants. Assuming (2-b), the double dual $L_2' = (\bigwedge^r E')^{**}$ is an invertible GL($N$, $S$)-sheaf with the same action of the center as $L_2$ because $U$ and hence $Q$ are locally factorial. Since $\delta'(H, r) = 1$ in any case, there are integers $\alpha$, $\beta$ such that $\alpha\delta(H) + \beta r = 1$. Therefore, $L = L_1^{\otimes\alpha} \otimes L_2^{\otimes\beta}$ in the former case and $L' = L_1^{\otimes\alpha} \otimes L_2'^{\otimes\beta}$ in the latter case carry GL($N$, $S$)-linearizations such that the actions of the center are the multiplication by constants.                    Q.E.D.

The following is a corollary to the proof of Theorem 2.1 and Theorem 2.2.

**Corollary 2.2.1.** *Let $U$ be a quasi-compact open subscheme of $M_{X/S}(H)$ and $\phi: Q \to U$ the principal PGL($N$, $S$)-bundle which appeared in the proof of Theorem 2.2. $U$ has a universal family if and only if there is a line bundle $L$ on $Q$ with GL($N$, $S$)-linearization such that the action of the center of GL($N$, $S$) is the multiplication by constants.*

*Proof.* Since we have the universal quotient sheaf $E$ on $X \times_S Q$, there is a morphism $f_E$ of $Q$ to $U$ such that for every geometric point $x$ of $Q$, $f_E(x)$ corresponds to $E(x)$. By the construction of $f_E$ in [6, § 5], $f_E$ is nothing but the given $\phi$. Assume that we have a universal family $F$ on $X \times_S U$. Then, there is a line bundle $L$ on $Q$ such that $(1 \times f_E)^*(F) \otimes_{\mathcal{O}_Q} L \simeq E$ (see [8, the proof of Theorem 3.17]). Since $f_E = \phi$, $G = (1 \times f_E)^*(F)$ carries a PGL($N$, $S$)-linearization, that is, a GL($N$, $S$)-linearization for which the action of the center of GL($N$, $S$) is trivial. For a sufficiently large integer $m$, $H_1 = p_{2*}(p_1^*(\mathcal{O}_X(m)) \otimes G)$ and $H_2 = p_{2*}(p_1^*(\mathcal{O}_X(m)) \otimes E)$ are locally free and $H_1 \otimes L \simeq H_2$, where $p_i$ is the i-th projection of $X \times_S Q$. On $P(H_1) = P(H_2)$ we have two tautological line bundles $\mathcal{O}_{P(H_1)}(1)$ and $\mathcal{O}_{P(H_2)}(1)$. Since $\mathcal{O}_{P(H_i)}(1)$ are GL($N$, $S$)-linearized, so is $\pi^*(L) \simeq \mathcal{O}_{P(H_2)}(1) \otimes \mathcal{O}_{P(H_1)}(-1)$, where $\pi$ is the projection of $P(H_1)$ to $Q$. Moreover, the action of the center of GL($N$, $S$) on $\pi^*(L)$ is the multiplication by constants. Now obviously $L \simeq \pi_*\pi^*(L)$ meets our requirement. The converse is contained in the proof of Theorem 2.1.                    Q.E.D.

The proof of Theorem 2.2 suggests another result which can be applied to the case of $X = P_k^2$.

**Proposition 2.3.** *Let $U$ be a quasi-compact, irreducible, locally factorial open subscheme in $M_{X/S}(H)$. For a line bundle $L$ on $X$, set $h(L) = \min\{h^0(X(s), E(s) \otimes L) \mid E(s)$ is the stable sheaf corresponding to $s\}$, where $s$ ranges over all geometric points of $U$. We define $\bar{\delta}(U)$ and $\bar{\delta}'(U)$ to be G.C.D.$\{h(L) \mid L \in \mathrm{Pic}(X)\}$ and G.C.D.$\{\bar{\delta}(U), r\}$, respectively, where $r$ is the*

*rank of a sheaf corresponding to a point of U.*

(1)  *If $\bar{\delta}(U)=1$, then U has a universal family.*

(2)  *If $\bar{\delta}'(U)=1$ and if $f: X \rightarrow S$ has a section, then U has a universal family.*

*Proof.*  Let us consider again $\phi: Q \rightarrow U$ and $\widetilde{E}$ in the proof of Theorem 2.2.  There are members $L_1, \cdots, L_t$ of $\mathrm{Pic}(X)$ and integers $\alpha_1, \cdots, \alpha_t$ such that $\sum_{i=1}^{t} \alpha_i h(L_i) = \bar{\delta}(U)$.  Set $E_i = L_i \otimes_{\mathcal{O}_X} \widetilde{E}$.  By virtue of the upper semi-continuity of cohomologies, we have a non-empty open set $V$ of $Q$ such that $h^0(E_i(z))$ is constant and equal to $h(L_i)$ for all $z$ in $V$.  Since $Q$ is reduced, for the projection $p: X \times_S Q \rightarrow Q$, $F_i = p_*(E_i)$ is locally free and of rank $\beta_i = h(L_i)$ on $V$.  Then $M_i = (\bigwedge^{\beta_i} F_i)^{**}$ is an invertible $GL(N, S)$-sheaf on which the center acts as the multiplication by the $\beta_i$-th power of constants because $Q$ is locally factorial.  $M_1^{\otimes \alpha_1} \otimes \cdots \otimes M_t^{\otimes \alpha_t}$ carries a $GL(N, S)$-linearization such that the action of the center of $GL(N, S)$ is the multiplication by the $\bar{\delta}(U)$-th power of constant.  Now we can apply the same argument as in the proof of Theorem 2.2 to both cases (1) and (2).                                              Q.E.D.

**Example 2.4.**  We can apply Proposition 2.3, (2) to the case of $X = P_k^2$.  In this case, if $r = 2$ and if one of $h(\mathcal{O}_X(m))$ is odd, then the moduli space has a universal family.

### § 3.  Proof of Theorem 0.1

We shall identify the set of $k$-rational points of $M(i, a)$ $(i = 0, 1)$ with the set of stable vector bundles $E$ with $c_1(E) = i$, $c_2(E) = a$ and $r(E) = 2$.  First of all let us recall the key lemma in [7, § 7].

**Lemma 3.1.**  *Let E be a stable vector bundle of rank 2 on $P = P_k^2$ with $c_1(E) = i = 0$ or $1$ and $c_2(E) = a$.*

(1)  *There is an integer $l$ such that $(\sqrt{4a+1-i}-1-i)/2 \geqq l \geqq 1-i$, $H^0(P, E(l)) \neq 0$ and $H^0(P, E(l-1)) = 0$.*

(2)  *For a general E in $M(i, a)(k)$ and for $l_0 = [(\sqrt{4a+1-i}-1-i)/2]$, we have $H^0(P, E(l_0-1)) = 0$, where for a real number $\alpha$, $[\alpha]$ means the largest integer which is not greater than $\alpha$.*

For the proof of the above, see Lemma 7.3 and Corollary 7.5.1 of [7].

Let $J$ be the ideal of a 0-dimensional scheme in $P = P_k^2$ such that $h^0(\mathcal{O}_P/J) = \alpha_i(l) = l^2 + il + a$, where $i = 0$ or $1$ and $a \geqq 2$ or $1$ according as $i = 0$ or $1$.  Since $h^0(\mathcal{O}_P(2l-3+i)) = 2l^2 + (2i-3)l - i + 1$, we have $H^0(P, J(2l-3+i)) = 0$ if $J$ is general enough and if $A(l) = l^2 + (i-3)l - i +$

$1-a=h^0(\mathcal{O}_P(2l-3+i))-\alpha_i(l)\leqq0$. And then $h^1(J(2l-3+i))=-A(l)$
$\geqq0$. If $h^0(\mathcal{O}_P(2l+i))-h^0(\mathcal{O}_P/J)=l^2+(3+i)l+2i+1-a\geqq0$ and if $J$ is
sufficiently general, then we have

$$h^0(J(2l+i))=l^2+(3+i)l+2i+1-a=B(l).$$

**Lemma 3.2.** *Let $l_0$ be the integer given in Lemma 3.1, (2) for the $i$
and $a$.*
  (1)  *$A(l_0)<0$ and $A(l_0+1)<0$.*
  (2)  *If $i=0$, then $B(l_0)$ is even or odd according as $a$ is odd or even.*
  (3)  *If $i=1$, then $B(l_0)$ or $B(l_0+1)$ is even.*

*Proof.* The positive root of the quadratic equation $A(l)=0$ is $r_1=$
$\{(3-i)+\sqrt{4a+5-i}\}/2$. Hence

$$r_1-l_0\geqq\{(3-i)+\sqrt{4a+5-i}\}/2-(\sqrt{4a+1-i}-1-i)/2>(3+1)/2=2.$$

We see therefore that $A(l_0)<0$ and $A(l_0+1)<0$. Since $l^2+3l+1$ is always
odd if $l$ is an integer, we obtain the assertion (2). Assume that $i=1$.
Then $l^2+(3+i)l+(2i+1)=l^2+4l+3$ is even or odd according as $l$ is odd
or even. This shows (3).                                       Q.E.D.

We know that $M(1,1)$ consists of one point $T_P(-1)$. Thus we fix
the couple $(i,a)$ so that $i=0,1$ and $a\geqq2$. When $i=0$, we set $r=l_0$. If
$i=1$, then we put $r=l_0$ or $l_0+1$ so that $B(r)$ is even (see (3) of the above
lemma). For a general member $E$ of $M(i,a)$, we have an exact sequence
by virtue of Lemma 3.1

$$0\longrightarrow\mathcal{O}_P\longrightarrow E(l_0)\longrightarrow I(2l_0+i)\longrightarrow0,$$

where $I$ is an ideal of $\mathcal{O}_P$ such that $\mathcal{O}_P/I$ is supported by a finite set
of points and $h^0(\mathcal{O}_P/I)=\alpha_i(l_0)$. Since $h^0(I(2l_0+i))\geqq B(l_0)$, we have
$H^0(P,I(2l_0+i))\neq0$ if $r\neq l_0$. This implies that $E(r)/s\mathcal{O}_P$ is torsion free for
a general section $s$ in $H^0(P,E(r))$. Thus we obtain an exact sequence

(3.3)        $$0\longrightarrow\mathcal{O}_P(-3)\longrightarrow E(r-3)\longrightarrow J_0(2r-3+i)\longrightarrow0$$

with $J_0$ the ideal of a finite subscheme in $P$ such that $h^0(\mathcal{O}_P/J_0)=\alpha_i(r)$.
This exact sequence and Lemma 3.1, (2) show that $H^0(P,J_0(2r-3+i))=0$
and hence $h^1(J_0(2r-3+i))=-A(r)$.

Let $Z$ be the Hilbert scheme $\mathrm{Hilb}_{P/k}^{\alpha_i(r)}$ and $\tilde{J}$ be the universal family of
ideals on $P\times Z$. If $Z_0$ is the open subscheme of $Z$ whose points corre-
spond to ideals $J$ with $H^0(P,J(2r-3+i))=0$, the $J_0$ in (3.3) gives rise to
a $k$-rational point $x$ of $Z_0$. Shrinking $Z_0$ to an affine open subscheme

which contains $x$, we have an isomorphism

$$\mathrm{Ext}^1_{\mathscr{O}_{P \times Z_0}}(\tilde{J}_0(2r-3+i), p_1^*(\mathscr{O}_P(-3))) \simeq \mathrm{Hom}_{\mathscr{O}_{Z_0}}(R^1 p_*(\tilde{J}_0(2r-3+i)), \mathscr{O}_{Z_0}),$$

where $\tilde{J}_0 = \tilde{J}|_{P \times Z_0}$ and $p_i$ is the $i$-th projection of $P \times Z_0$. On the $Z_0$-scheme $V(R^1 p_*(\tilde{J}_0(2r-3+i))) = W$, there is a universal section of $\pi^*(R^1 p_*(\tilde{J}_0(2r-3+i)))$ which defines a universal family of extensions

$$\mathrm{Ex}: 0 \longrightarrow \mathscr{O}_{P \times W}(-3) \longrightarrow \tilde{E} \longrightarrow (1 \times \pi)^*(\tilde{J}_0(2r-3+i)) \longrightarrow 0,$$

where $\pi$ is the structure morphism of $W$ (see [7, p. 600]). By the construction of $\tilde{E}$ and the fact that $R^1 p_*(\tilde{J}_0(2r-3+i))$ is a vector bundle of rank $-A(r)$, we have a point $w$ of $W$ over the $x$ in $Z_0$ such that $\mathrm{Ex}(w)$ is isomorphic to (3.3). Therefore, we see the following:

(3.4)  There is a non-empty open set $Z_1$ of $\mathrm{Hilb}_{P/k}^{\alpha_i(r)}$ such that for the ideal $J$ corresponding to any $z$ in $Z_1(k)$, there is a member $E$ in $M(i, a)(k)$ which fits in an exact sequence

$$0 \longrightarrow \mathscr{O}_P \longrightarrow E \longrightarrow J(2r+i) \longrightarrow 0.$$

Using the fact that $(\sqrt{4a+1-i}-1-i)/2$ is the positive root of the quadratic equation $l^2 + (3+i)l + 2i + 2 - a = 0$, we see that $u = \alpha_i(r)$ is greater than $v = [B(r)/2]$. Then $h^0(\mathscr{O}_P(2r+i)) = 3v + (u-v)$ or $3v + (u-v) + 1$ according as $B(r)$ is even or odd.

**Lemma 3.5.** *Let* $x_1, \cdots, x_v, x_{v+1}, \cdots, x_u$ *be sufficiently general, mutually distinct points in* $P$. *Then, we have:*

(3.5.0)  *The reduced subscheme* $\bigcup_{i=1}^u \{x_i\}$ *in* $P$ *is contained in* $Z_1(k)$. *Moreover, for* $z_1 = x_{v+1}, \cdots, z_{u-v} = x_u$, *we have:*

(3.5.1)  $x_1, \cdots, x_v, z_1, \cdots, z_{u-v}$ *are mutually distinct and*

$$\{s \in H^0(P, \mathscr{O}_P(2r+i)) \,|\, \mathrm{ord}_{x_k}(s) \geq 2 \text{ for all } 1 \leq k \leq v \text{ and } s(z_j) = 0$$
$$\text{for all } 1 \leq j \leq u-v\}$$

*is a vector space of dimension* 0 *or* 1 *according as* $B(r)$ *is even or odd,*

(3.5.2)  *The ideal* $K$ *defining the reduced subscheme*

$$\left(\bigcup_{j=1}^v \{x_j\}\right) \cup \left(\bigcup_{k=1}^{u-v} \{z_k\}\right)$$

*in* $P$ *satisfies* $H^0(P, K(2r-3+i)) = 0$ *or equivalently* $h^1(K(2r-3+i)) = -A(r)$.

(3.5.3)  $h^0(K(2r+i)) = B(r)$.

*Proof.* (3.5.0) is (3.4). Since $x_1, \cdots, x_u$ are mutually distinct, (3.5.2) and (3.5.3) are easy. For mutually distinct points $y_1, \cdots, y_a$ in $P$, we set

$\varXi(n_1 y_1, \cdots, n_\alpha y_\alpha) = \{s \in H^0(P, \mathcal{O}_P(2r+i)) \mid \mathrm{ord}_{y_i}(s) \geq n_i$ for all $1 \leq i \leq \alpha\}$
and denote by $|\varXi(n_1 y_1, \cdots, n_\alpha y_\alpha)|$ the linear system defined by $\varXi(n_1 y_1, \cdots, n_\alpha y_\alpha)$. We shall compute the codimension of $\varXi(2x_1, \cdots, 2x_v, x_{v+1}, \cdots, x_\alpha)$ in $H^0(P, \mathcal{O}_p(2r+i))$. Assume that for an $\alpha \geq v$, we can choose $x_1, \cdots, x_\alpha$ so that codim $\varXi(2x_1, \cdots, 2x_v, x_{v+1}, \cdots, x_\alpha) = 3v + (\alpha - v)$. If we pick $x_{\alpha+1}$ outside the set of base points of $|\varXi(2x_1, \cdots, 2x_v, x_{v+1}, \cdots, x_\alpha)|$, then codim $\varXi(2x_1, \cdots, 2x_v, x_{v+1}, \cdots, x_{\alpha+1}) = 3v + (\alpha + 1 - v)$. Thus it is enough to prove that for general $x_1, \cdots, x_v$, codim $\varXi(2x_1, \cdots, 2x_\alpha) = 3\alpha$ if $\alpha \leq v$.

**Claim.** If there is a member $D$ in $|\mathcal{O}_P(2r+i-1)|$ such that $x_1, \cdots, x_\alpha$ are, at least, double points of $D$, then for general $x_{\alpha+1}$, dim $\varXi(2x_1, \cdots, 2x_{\alpha+1}) = \dim \varXi(2x_1, \cdots, 2x_\alpha) - 3$.

Indeed, take a point $x_{\alpha+1}$ outside $D$ and mutually distinct lines $L_1, L_2, L_2'$ such that $L_2$ and $L_2'$ contain $x_{\alpha+1}$ but $L_1$ does not. Then $D + L_1$ is not a member of $|\varXi(2x_1, \cdots, 2x_\alpha, x_{\alpha+1})|$ but so are $D + L_2$ and $D + L_2'$. Since $D + L_2$ and $D + L_2'$ are smooth at $x_{\alpha+1}$, this means that the last linear system separates infinitely near points around $x_{\alpha+1}$. These imply our claim.

By the above claim it suffices to show that $h^0(\mathcal{O}_P(2r+i-1)) - 3(v-1) \geq \{4r^2 + (4i+2)r + 2i - 3r^2 - (3i+9)r - 6i + 3 + 3a\}/2 = \{r^2 + (i-7)r - 4i + 3 + 3a\}/2 > 0$. Since the discriminant of the equation $r^2 + (i-7)r - 4i + 3 + 3a = 0$ is $37 + 3i - 12a$, the above inequality holds if $a \geq 4$. It is easy to show that the condition in our claim is satisfied in the remaining cases except when $i = 1$ and $a = 3$. If $i = 1$ and $a = 3$, then $r = 2$ and $v = 6$. Up to $\alpha = 4$ it is easy to see the condition of the claim. For the five given points $x_1, \cdots, x_5$, there is a conic which passes through $x_1, \cdots, x_5$. Then $D = 2C$ satisfies the condition in our claim in this case, too.          Q.E.D.

Let us fix points $x_1, \cdots, x_u$ which satisfy all the conditions in the above lemma. The set of mutually distinct points $z_1, \cdots, z_{u-v}$ in $P$ is parametrized by an open set $X_0$ of the symmetric product of $u - v$ copies of $P$. $\{(z_1, \cdots, z_{u-v}) \in X_0 \mid x_1, \cdots, x_v, z_1, \cdots, z_{u-v}$ satisfy the conditions (3.5.1), (3.5.2) and (3.5.3)$\}$ is the set of $k$-rational points of an open set $X$ of $X_0$. On $P \times X$ we have the universal family of ideals $\tilde{K}'$ such that for a $z$ in $X(k)$, $\tilde{K}'(z)$ is the ideal of the reduced subscheme of $P$ corresponding to $z$. For the first projection $p_1: P \times X \to P$ and the defining ideal $J$ of the reduced scheme $\bigcup_{j=1}^v \{x_j\}$, we put $\tilde{K} = p_1^*(J)\tilde{K}'$. $\tilde{K}$ is the universal flat family of ideals defining reduced schemes

$$\left(\bigcup_{j=1}^v \{x_j\}\right) \cup \left(\bigcup_{k=1}^{u-v} \{z_k\}\right)$$

of $P$ such that $x_1, \cdots, x_v, z_1, \cdots, z_{u-v}$ satisfy the conditions (3.5.1), (3.5.2) and (3.5.3).

After shrinking $X$ to a suitable affine open subscheme which contains $(x_{v+1}, \cdots, x_u)$, as in [7, p. 600] or in the above, we can construct a universal family of extensions on $P \times V_0$

$$0 \longrightarrow \mathcal{O}_{P \times V_0}(-3) \longrightarrow \widetilde{F}_1 \longrightarrow (1 \times q)^*(\widetilde{K}(2r-3+i)) \longrightarrow 0,$$

where $V_0 = V(B)$ with $B = R^1 p_{2*}(\widetilde{K}(2r-3+i))$ and $q$ is the projection of $V_0$ to $X$. By (3.4) there is a non-empty open set $V$ of $V_0$ such that $\widetilde{F}_1|_{P \times V}$ is locally free and $\widetilde{F}_1(z)$ is stable for all $z$ in $V(k)$ because $(x_{v+1}, \cdots, x_u)$ is in $X$. Set $\widetilde{F} = \widetilde{F}_1(3-r)|_{P \times V}$. Then, for every $z$ in $V(k)$, $\widetilde{F}(z)$ fits in an exact sequence

$$0 \longrightarrow \mathcal{O}_P(-r) \longrightarrow \widetilde{F}(z) \longrightarrow \widetilde{K}(z)(r+i) \longrightarrow 0.$$

Since $c_2(\widetilde{K}(z)(r+i)) = \alpha_i(r)$, we have that $c_1(\widetilde{F}(z)) = i$ and $c_2(\widetilde{F}(z)) = a$.

**Lemma 3.6.** *For every $z$ in $V(k)$, there is an exact sequence*

$$0 \longrightarrow \mathcal{O}_P \longrightarrow J\widetilde{F}(z)(r) \longrightarrow (J\widetilde{K}(z))(2r+i) \longrightarrow 0.$$

*From this we deduce that $h^0(J\widetilde{F}(z)(r)) = 1$ or $2$ according as $B(r)$ is even or odd.*

*Proof.* From the exact sequence

$$0 \longrightarrow \mathcal{O}_P \longrightarrow \widetilde{F}(z)(r) \longrightarrow \widetilde{K}(z)(2r+i) \longrightarrow 0$$

we obtain the following exact commutative diagram

$$
\begin{array}{ccccccc}
& 0 & & 0 & & & \\
& \downarrow & & \downarrow & & & \\
& J & \longrightarrow & J\widetilde{F}(z) & \longrightarrow & J \otimes \widetilde{K}(z)(2r+i) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow \zeta & \\
0 \longrightarrow & \mathcal{O}_P & \overset{\psi}{\longrightarrow} & \widetilde{F}(z)(r) & \longrightarrow & \widetilde{K}(z)(2r+i) & \longrightarrow 0.
\end{array}
$$

The image of $\zeta$ is $J\widetilde{K}(z)(2r+i)$. Since $\widetilde{K}(z)$ is contained in $J$, the global section $\Gamma(\psi)(1)$ of $\widetilde{F}(z)(r)$ has zeros on $x_1, \cdots, x_v$ and hence $\psi(\mathcal{O}_P)$ is a subsheaf of $J\widetilde{F}(z)(r)$. This means that $\psi(\mathcal{O}_P)$ is contained in $D = \mathrm{Ker}(J\widetilde{F}(z)(r) \to J\widetilde{K}(z)(2r+i))$. Moreover, $\psi(\mathcal{O}_P)$ coincides with $D$ outside the set $\{x_1, \cdots, x_v\}$. Since $J\widetilde{F}(z)(r)/\psi(\mathcal{O}_P) \subseteq \widetilde{F}(z)(r)/\psi(\mathcal{O}_P) \simeq \widetilde{K}(z)(2r+i)$ and $\widetilde{K}(z)(2r+i)$ is torsion free, we see that $\psi(\mathcal{O}_P) = D$. The second assertion is deduced from (3.5.1) and the fact that $H^0(P, J\widetilde{K}(z)(2r+i))$ is exactly the vector space in (3.5.1). Q.E.D.

The vector bundle $\tilde{F}$ on $P \times V$ defines a morphism $\Phi_0$ of $V$ to $M(i, a)$. It is obvious that $\tilde{F}(z)$ and $\tilde{F}(\lambda z)$ are isomorphic for $z$ in $V(k)$ and $\lambda$ in $k - \{0\}$. Thus $\Phi_0$ passes through $\Phi: T \to M(i, a)$, where $T$ is an open subscheme of $P(B)$.

$$
\begin{array}{ccc}
V_0 \supseteq V & & \\
\rho \downarrow & \downarrow \rho_0 & \overset{\Phi_0}{\searrow} \\
P(B) \supseteq T & \xrightarrow{\quad \Phi \quad} & M(i, a)
\end{array}
$$

Since all the $\tilde{K}(z)$ are contained in $J$, the global section of $\tilde{F}(z)(r)$ obtained from the exact sequence $0 \to \mathcal{O}_P \to \tilde{F}(z)(r) \to \tilde{K}(z)(2r+i) \to 0$ is contained in $H^0(P, J\tilde{F}(z)(r))$. Thus Lemma 3.6 implies:

(3.7)   If $i = 1$ or if $i = 0$ and $a$ is odd, then $\Phi$ is injective. If $i = 0$ and $a$ is even, then the dimension of each fibre of $\Phi$ is one.

On the other hand, we have

$$
\dim P(B) = \dim Z + r(B) - 1 = 2(u - v) - A(r) - 1
$$
$$
= \begin{cases} 4a - 3 - i & \text{if } i = 1 \text{ or } i = 0 \text{ and } a \text{ is odd} \\ 4a - 2 & \text{if } i = 0 \text{ and } a \text{ is even.} \end{cases}
$$

Since $\dim M(i, a) = 4a - 3 - i$, this and (3.6) imply that $\Phi$ is generically surjective. We have therefore an injection $\Phi^*: k(M(i, a)) \to k(P(B))$.

Let us construct the inverse of $\Phi$ in the case where $i = 0$ and $a$ is odd or where $i = 1$. In both cases, we have a universal bundle $\tilde{U}_0$ on $P \times M(i, a)$ (see [7, Theorem 7.17]). For the ideal $J$ which appeared in the above construction of $(V, \tilde{F})$, we have proved that there is a non-empty open set $R$ of $M(i, a)$ such that $h^0(J\tilde{U}_0(z)(r)) = 1$ for all $z$ in $R(k)$. Setting $\tilde{U} = \tilde{U}_0(r)|_{P \times R}$, $M = \pi_{2*}(\pi_1^*(J)\tilde{U})$ is an invertible sheaf on $R$, where $\pi_i$ is the $i$-th projection of $P \times R$. By shrinking $R$ if necessary, we may assume that $M$ is isomorphic to $\mathcal{O}_R$. The base change theorem tells us that the canonical map $\xi: \mathcal{O}_{P \times R} \simeq \pi_2^*(M) \to \tilde{U}$ is injective and

$$
\xi(z): k \simeq H^0(P, \pi_2(M)(z)) \simeq H^0(P, J\tilde{U}(z)) \longrightarrow H^0(P, \tilde{U}(z))
$$

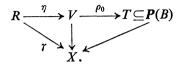is the natural injection for all $z$ in $R(k)$. Thus $\tilde{L} = (\tilde{U}/\xi(\mathcal{O}_{P \times R}))(-2r - i)$ is flat over $R$. Shrinking $R$ again, we may assume that $\tilde{L}$ is an ideal of $\mathcal{O}_{P \times R}$ such that for all $z$ in $R(k)$, $\tilde{L}(z)$ is contained in $J$ and defines a 0-dimensional subscheme

$$
\left( \bigcup_{j=1}^{v} \{x_j\} \right) \cup \left( \bigcup_{k=1}^{u-v} \{z_k\} \right)
$$

which satisfies (3.5.1), (3.5.2) and (3.5.3). Thus there is a morphism $\gamma$ of $R$ to $X$ such that $\gamma^*(\tilde{K}) \simeq \tilde{L}$. Furthermore, the extension

$$0 \longrightarrow \mathcal{O}_{P \times R} \longrightarrow \widetilde{U} \longrightarrow \widetilde{L}(2r+i) \longrightarrow 0$$

gives rise to a morphism $\eta$ of $R$ to $V$ such that $\eta^*(\widetilde{F}(r)) \simeq \widetilde{U}$. Moreover, the following diagram is commutative:

$$R \xrightarrow{\eta} V \xrightarrow{\rho_0} T \subseteq P(B)$$

with $r$ and $\rho_0$ mapping to $X$.

Set $\Psi = \rho_0 \eta$. Then, since $\Phi_0^*(\widetilde{U}(-r)) \simeq \widetilde{E}$, we have $(\Phi\Psi)^* \widetilde{U}(-r) \simeq \eta^* \rho^*(\Phi^* \widetilde{U}(-r)) \simeq \eta^*(\Phi_0^* \widetilde{U}(-r)) \simeq \widetilde{U}(-r)$. The universality of $(M(i, a), \widetilde{U}_0)$ and this imply that $\Phi\Psi = \mathrm{id}$. By Lemma 3.6, $\Psi$ is injective and hence generically surjective because $R$ and $P(B)$ have the same dimension. We obtain therefore an injection $\Psi^*: k(T) = k(P(B)) \to k(M(i, a))$ such that $\Psi^* \Phi^* = \mathrm{id}$. Thus $\Psi^*$ is birational. Since $P(B)$ is rational, this completes the proof of (1) and (2) of Theorem 0.1.

To outline our proof of Theorem 0.1, (3), let us consider the principal $PGL(N)$-bundle $\phi; Q \to M(0, a)$ which appeared in the proof of Theorem 2.2. We have a universal sheaf $\widetilde{U}$ on $P \times Q$ such that $c_1(\widetilde{U}(z)) = 0$ for a $z$ in $Q(k)$. Replacing $Q$ by a $PGL(N)$-invariant open set, we may assume that $\pi_{2*}(\pi_1^*(J)\widetilde{U}(r)) = G$ is a vector boundle of rank 2 by Lemma 3.6, where $\pi_i$ is the $i$-th projection of $P \times Q$. On $V(G^*)$ we have a universal section $\mathcal{O}_{V(G^*)} \to \nu^*(G^*)$, where $\nu: V(G^*) \to Q$ is the projection. Thus we get a natural homomorphism $\mathcal{O}_{P \times V(G^*)} \to q_2^* \nu^*(G) \to (1 \times \nu)^*(\widetilde{U}(r))$ with $q_2$ the second projection of $P \times V(G^*)$. Restricting this sequence to a suitable open subscheme $D$ of $V(G^*)$, we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_{P \times D} \longrightarrow (1 \times \nu)^*(\widetilde{U}(r))|_{P \times D} \longrightarrow \widetilde{S} \longrightarrow 0.$$

Shrinking $D$ if necessary, this provides us with a morphism $\Psi'$ of $D$ to $V$ such that $(1 \times \Psi')^*(\widetilde{F}) \simeq (1 \times \nu)^*(\widetilde{U})|_{P \times D}$.

$G$ carries a $GL(N)$-linearization and hence $P(G^*)$ descends to a $P^1$-bundle (in the étale topology) $H$ on an open subscheme of $M(0, a)$. Since the above $\Psi'$ is $PGL(N)$-invariant with respect to the trivial action of $PGL(N)$ on $V$, there is a morphism $\Psi$ of $H$ to $T$ which makes the following diagram commutative:

$$
\begin{array}{ccc}
V(G^*) & & \\
\cup & & \\
D & \xrightarrow{\Psi'} & V \\
{\scriptstyle \rho'}\downarrow & & \downarrow{\scriptstyle \rho_0} \\
H & \xrightarrow{\Psi} & T \subseteq P(B).
\end{array}
$$

On the other hand, the existence of $\widetilde{F}$ provides us with a morphism

$\Theta'$ of an open subscheme $V'$ of $V$ to $D$ such that $(1 \times \Theta')^*(\tilde{U}) \simeq \tilde{F}|_{P \times V'}$. Putting $D' = \Psi'^{-1}(V')$ and $\Psi'' = \Psi'|_{D'}$, we get an isomorphism $(1 \times \Psi'')^* (1 \times \Theta')^*(\tilde{U}) \simeq \tilde{U}|_{P \times D'}$. This supplies us with a $V'$-valued point $g$ of $GL(N)$ such that $\Theta'\Psi'' = g$. It is clear that there is a morphism $\Theta$ of an open subscheme of $P(B)$ to $H$ such that $\rho'\Theta' = \Theta\rho$. Hence $\Theta\Psi\rho'|_{D'} = \Theta\rho\Psi''$ $= \rho'g = \rho'|_{D'}$. Since $\rho'|_{D'}: D' \to \rho'(D')$ is faithfully flat, we deduce from this $\Theta\Psi = \mathrm{id}$. Lemma 3.6 shows that $\Psi$ and $\Theta$ are bijective. Therefore, $H$ and $P(B)$ are birational.

**Remark 3.8.** 1) By using the fact that $B(r)$ is even if $i = 1$ or if $i = 0$ and $a$ is odd, we can prove the existence of a universal family on $M(i, a)$ in the case of (1) and (2) of Theorem 0.1 (see Proposition 2.3 and Example 2.4).

2) The $P^1$-bundle in (3) of Theorem 0.1 is never a $P^1$-bundle in the Zariski topology. By an argument similar to the latter half of the proof of Corollary 2.2.1, we can show that if the bundle is a $P^1$-bundle in the Zariski topology, then there is a line bundle $L$ on a $GL(N)$-invariant open set which carries a $GL(N)$-linearization such that the center of $GL(N)$ acts as the multiplication by constants. Then we have a universal family on a non-empty open set of $M(0, a)$ and hence on $M(0, a)$. This contradicts the results of Le Potier in [5].

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## Appendix

### The variety in Question 1.3 is rational

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The purpose of this appendix is to prove the rationality of the variety defined in the affine space $(u_1, u_2, u_3, v)$ by the equation

(A) $$v^2 = u_3 D(u_1, u_2, u_3),$$

where $D = D(u_1, u_2, u_3)$ is the discriminant of the cubic equation $x^3 + 3u_1 x^2 + 3u_2 x + u_3 = 0$ (see Observation 1.2 and Question 1.3). It is classically known that $D$ is written (up to a constant factor) in the simple form

$$D = Q^2 - 4PR$$

if one puts

(A.1) $$\begin{cases} P = u_1^2 - u_2 \\ Q = u_1 u_2 - u_3 \\ R = u_2^2 - u_1 u_3. \end{cases}$$

Our first attempt is now to compare two fields $k(P, Q, R)$ and $k(u_1, u_2, u_3)$ ($k$: the ground field, $\mathrm{ch}(k) \neq 2, 3$). By the first two identities in (A.1) we obtain the expressions of $u_2, u_3$ in terms of $u_1, P, Q$ and by using these we eliminate $u_2, u_3$ from the last identity in (A.1):

(A.2) $$u_1^2 - \frac{Q}{P} u_1 + \frac{R}{P} - P = 0.$$

Now this quadratic equation in $u_1$ suggests us to introduce the following three algebraically independent elements in the function field:

(A.3) $$\begin{cases} s = Q/2P \\ t = 2R/Q \\ w = u_1 - (Q/2P) \quad (= u_1 - s). \end{cases}$$

One sees immediately from (A.2) that

(A.4) $$P = w^2 + st - s^2.$$

From (A.3) and (A.4) we further deduce

$$\begin{cases} Q=2sP=2s(w^2+st-s^2) \\ R=tQ/2=st(w^2+st-s^2) \\ u_1=s+w \\ u_2=u_1^2-P=(s+w)^2-(w^2+st-s^2)=s(2w+2s-t) \\ u_3=u_1u_2-Q=s\{(4s-t)w+4s^2-3st\}. \end{cases}$$

These imply in particular that we may use new variables $s$, $t$, $w$ instead of $u_1$, $u_2$, $u_3$. We have also

$$D=Q^2-4PR=4P^2s(s-t),$$

where $P$ is given by (A.4). Thus the equation (A) is equivalent to the following:

(A′)                    $v^2=4P^2s^2(s-t)\{(4s-t)w+4s^2-3st\}.$

Since $w$ is still contained in the expression of $P$, the right hand side of (A′) is cubic with respect to $w$. But this disadvantage can be removed when one introduces the new variable

$$\bar{v}=v/(2sP)$$

as a substitute for $v$. In fact the original variety is birational to the affine variety given by the equation

$$\bar{v}^2=(s-t)\{(4s-t)w+4s^2-3st\}$$

and this last identity shows that $w$ is rationally expressed in terms of $s$, $t$ and $\bar{v}$. This clearly implies that the variety defined by (A) is rational, which was to be proved.

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