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Introduction to the Minimal Model Problem

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Dedicated to Professor Nagayoshi Iwahori on his sixtieth birthday

Contents

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Introduction

Our aim is to present a program for the construction of minimal models in any dimension. This theory has been recently developed by Ando, Benveniste, Kawamata, Kollár, Miyaoka, Mori, Nakayama, Reid, Shokurov, Tsunoda and others. We put special emphasis on its application to the classification theory of higher dimensional algebraic varieties.

In this paper the ground field k is assumed to be algebraically closed and of characteristic zero unless otherwise stated.

We shall quickly review the theory of minimal models for surfaces, namely for complete algebraic varieties of dimension 2, using the flow chart in Figure 1.

We start with a nonsingular projective surface S having the Picard number $\rho=\rho(S)$. Our first question is whether the canonical divisor K_S of S is *nef*, i.e., whether the intersection number of K_S with any reduced irreducible curve on S is nonnegative, or not. If the answer is YES, we call S the *minimal model* (in our sense). In this case the Kodaira dimension $\kappa(S)$ of S is nonnegative and K_S is semi-ample, i.e., the linear system $|mK_S|$ is base point free some $m \in N$. This implies that the canonical ring $R(S):=\bigoplus_{m\geq 0} H^0(S, \mathcal{O}_S(mK_S))$ of S is a finitely generated algebra over k, and we have the natural morphism

$$\psi:=\Phi_{|mK_S|}: S \longrightarrow \operatorname{Proj} R(S) \qquad \text{for some } m\gg 0.$$

When $\kappa(S)=\dim S=2$, i.e., when S is of general type, ψ is a birational morphism which is the contraction of all the (-2)-curves to rational double points (cf. [A1], [Mf], [Kod3], [Bo]). When $\kappa(S)=1$, ψ gives an elliptic fibration onto a nonsingular projective curve whose singular fibers are thoroughly studied by Kodaira (cf. [Kod2]). When $\kappa(S)=0$, we have $mK_S\sim 0$ for some $m \in N$ (N denotes the set of positive integers in this paper); more precisely, S is either an abelian surface, a $K3$ surface, a hyperelliptic surface or an Enriques surface.

If the answer to the first question is NO, our second question is whether there exists a (-1)-curve E on S. If the answer is No, then $\kappa(S)=-\infty$; furthermore, S is isomorphic either to $\boldsymbol{P}^2$ or to a minimal ruled surface over a nonsingular projective curve C. If the answer to the

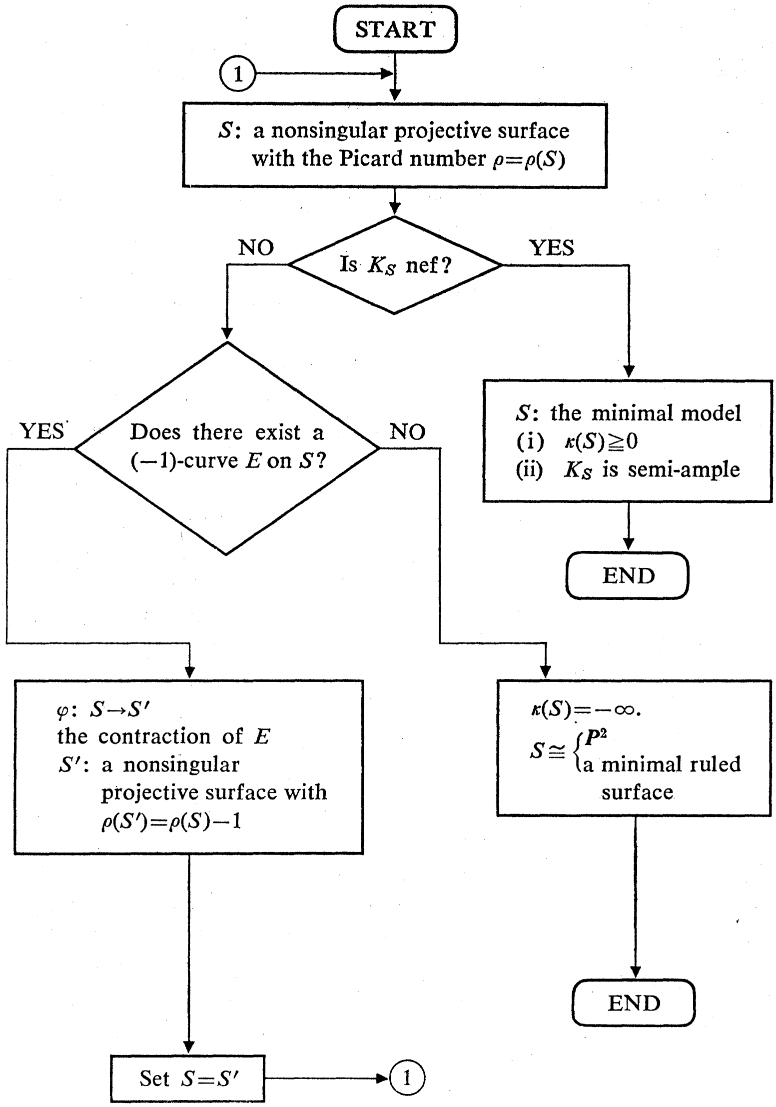


Figure 1

second question is YES, we have Enriques-Castelnuovo's contraction morphism of S onto a nonsingular projective surface S' with the Picard number $\rho(S')=\rho(S)-1$. In this case, we go back to the starting point with the Picard number decreased by one. This procedure must come to an end after a finite number of repetitions. Thus starting with an arbitrary nonsingular projective surface, we end up either with a minimal model, $\boldsymbol{P}^2$ or with a minimal ruled surface which is birational to S. We note that the minimal model of S is uniquely determined by its birational class when $\kappa(S)\geqq 0$.

Our main purpose is to draw an analogous flow chart and prove the existence of the minimal models for higher dimensional varieties which are not uniruled. Our main results to carry out this program are the *Base Point Free Theorem* (or the *Contraction Theorem*) and the *Cone Theorem*. Now we look at the flow chart for higher dimensional varieties in Figure 2.

Even though we are concerned with a nonsingular projective variety X of dimension d, we allow X at the starting point of the flow chart to have at most $\boldsymbol{Q}$-factorial terminal singularities in order for the inductive procedure to work. (For the precise definition of $\boldsymbol{Q}$-factorial terminal singularities, see Section 0–2.)

Our first question is whether the canonical divisor K_X of X is *nef* or not. If the answer is YES, we call X a *minimal model.* In this case, we conjecture that $\kappa(X)\geqq 0$ and that K_X is semi-ample (the *Abundance Conjecture*). If the Abundance Conjecture is true, the canonical ring $R(X):=\bigoplus_{m\geqq 0}H^0(X, \mathcal{O}_X(mK_X))$ is a finitely generated algebra over k, and we obtain the natural morphism $\psi:=\Phi_{|mK_X|}: X\to \mathrm{Proj}\, R(X)$. When $\kappa(X)=\dim X$, i.e., when X is of *general type*, ψ is the birational morphism onto its unique *canonical model* $X_{\mathrm{can}}:=\mathrm{Proj}\, R(X)$, which has only canonical singularities. When $0<\kappa(X)<\dim X$, ψ has a structure of an algebraic fiber space onto a normal projective variety with only rational singularities, whose generic fiber X_η is itself a minimal model with $\kappa(X_\eta)=0$. When $\kappa(X)=0$, we have $mK_X\sim 0$ for some $m\in \boldsymbol{N}$.

If the answer to the first question is NO, the Cone Theorem guarantees the existence of an extremal ray R with a supporting function H, and the Base Point Free Theorem implies that $\varphi:=\Phi_{|mH|}: X\to X'$ for $m\gg 0$ gives a morphism, the *contraction* of R, onto a normal projective variety X'. Then our second question is whether $\dim X'=d$ or not. If the answer is NO, then φ is said to be of *fiber type*. In this case, we have $\kappa(X)=-\infty$; more precisely, X is uniruled, i.e., there esists a generically finite and generically surjective rational map $\Psi: Y\times \boldsymbol{P}^1 \cdots\to X$ for some algebraic variety Y, by a result of Miyaoka and Mori (cf. [MM]). If the answer to the second question is YES, we go on to the third question which asks whether the dimension of the exceptional locus E of φ is equal to $d-1$ or not, the answer to which is always YES in the case of surfaces. If the answer to this question is YES, $\varphi: X\to X'$ is the contraction of some unique prime divisor and X' is a normal projective variety having

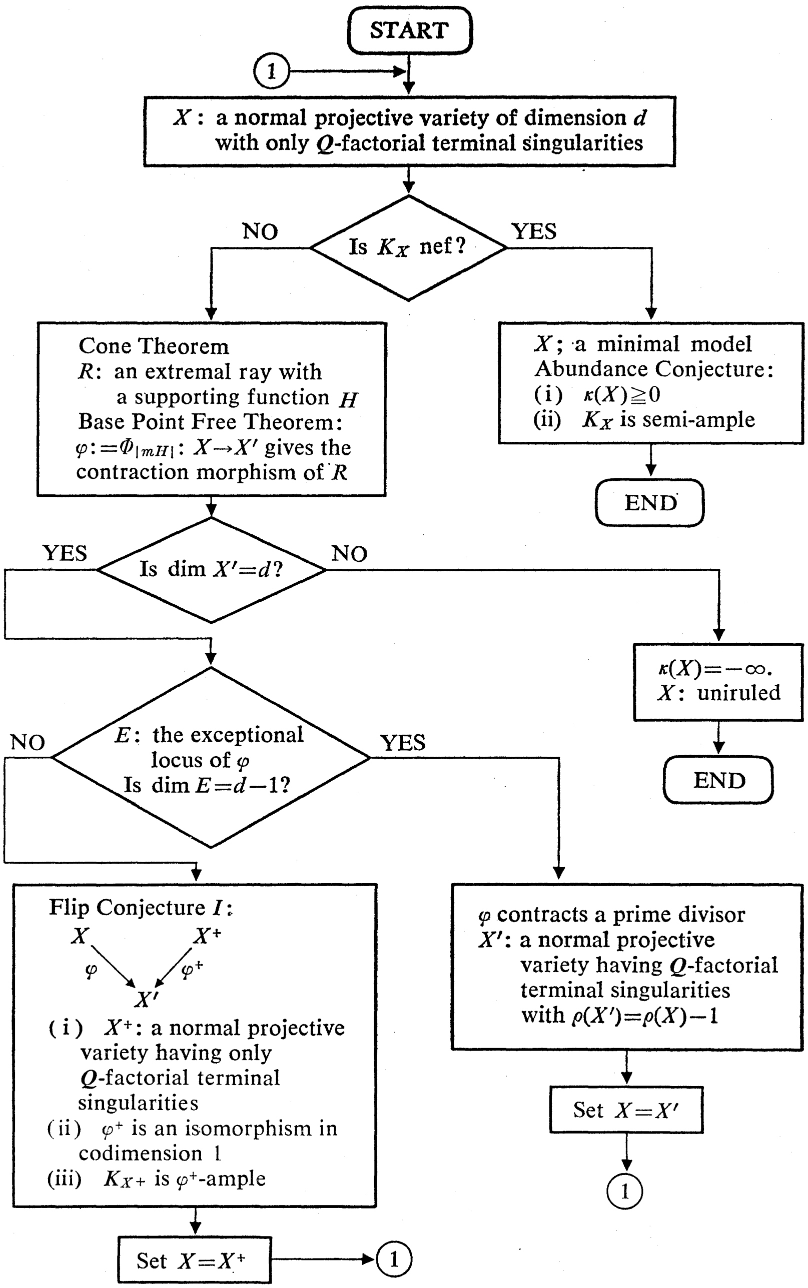

START
1
X : a normal projective variety of dimension d with only Q-factorial terminal singularities
NO
Is K_X nef?
YES
Cone Theorem
R: an extremal ray with a supporting function H
Base Point Free Theorem:
φ:=Φ_|mH|: X→X′ gives the contraction morphism of R
X; a minimal model
Abundance Conjecture:
(i) κ(X)≧0
(ii) K_X is semi-ample
END
YES
Is dim X′=d?
NO
κ(X)=−∞.
X: uniruled
END
NO
E: the exceptional locus of φ
Is dim E=d−1?
YES
Flip Conjecture I:
X
X+
φ
φ+
X′
(i) X+: a normal projective variety having only Q-factorial terminal singularities
(ii) φ+ is an isomorphism in codimension 1
(iii) K_X+ is φ+-ample
φ contracts a prime divisor
X′: a normal projective variety having Q-factorial terminal singularities with ρ(X′)=ρ(X)−1
Set X=X′
1
Set X=X+
1

Figure 2

only $\boldsymbol{Q}$-factorial terminal singularities with the Picard number $\rho(X')=\rho(X)-1$. Then φ is called a *divisorial contraction.* In this situation, we go back to the starting point of the flow chart with the Picard number decreased by one.

The major difficulty to carry out the program in the higher dimensional case arises when the answer to the third question is NO, i.e., when φ is an isomorphism in codimension one. There actually exist such cases. We call this φ a *flipping contraction.* Then we have the so-called *Flip Conjecture* which consists of two parts. The first part, the *Flip Conjecture* I, claims that there exists a commutative diagram

$$\begin{array}{ccc} X & \overset{\mathrm{tr}_\varphi}{\dashrightarrow} & X^+ \\ {\scriptstyle\varphi}\searrow & & \swarrow{\scriptstyle\varphi^+} \\ & X' & \end{array}$$

which satisfies the following properties:

(i) X^+ is a normal projective variety with only $\boldsymbol{Q}$-factorial terminal singularities, which is isomorphic to X in codimension one via tr_φ, and

(ii) the canonical divisor K_{X^+} is φ^+-ample.

Note that the Flip Conjecture I is equivalent to saying that

$$\bigoplus_{m\geqq 0} \varphi_* \mathcal{O}_X(mK_X)$$

is finitely generated as an $\mathcal{O}_{X'}$-algebra. This procedure to obtain $\varphi^+: X^+ \to X'$ from $\varphi: X \to X'$ is simply called a *flip* (if it exists). Once we get $\varphi^+: X^+ \to X'$, we go back to the starting point of the flow chart with X^+ in place of X.

The second part, *Flip Conjecture* II, claims that a sequence of flips has to terminate after finitely many steps. Those two parts of the conjecture combined together imply that there exists a finite chain of flips

$$\begin{array}{ccccccccc} X & \dashrightarrow & X^+ & \dashrightarrow & X^{++} & \dashrightarrow \cdots & \cdots \dashrightarrow & X^{(+n)}=Z \\ & \searrow \ \swarrow & & \searrow \ \swarrow & & & \searrow \ \swarrow & \\ & X' & & X'' & & & X^{('n)} & \end{array}$$

such that either

1) K_Z is nef,
2) Z has a contraction of fiber type, or
3) Z has a contraction of divisorial type.

We give the affirmative answer to the Flip Conjecture II in case $\dim X=3$ or 4. If the Flip Conjecture holds, we can go on to the other loops of the flow chart, and finally we obtain a minimal model of X unless X is uniruled.

In the following chapters we discuss the program above in full details with rigorous proofs. Moreover, we extend our objectives in the following two directions:

(i) Not only working with varieties over k, but we also work with varieties over some fixed variety S. Then for example, we can apply our theory to the deformation of varieties over the parameter space S.

(ii) We consider a pair (X, Δ) consisting of a variety X and a divisor Δ on it satisfying certain conditions. (See § 0–2.) As a consequence of this generalization, open varieties with their boundaries can be treated in our theory according to philosophy of Iitaka [I4].

The first and second cases are called the *relative* and *logarithmic* cases (the latter being called the *log* case for short), respectively. Note that even when we work in the category of varieties over a fixed field k, it is often more natural (and necessary) to get into the relative and/or logarithmic categories. Our flow chart is applicable in both extended cases.

The specific contents of this paper are as follows.

In Chapter 0, we introduce some concepts necessary to state our results in this paper, while mentioning such basic facts as Kleiman's criterion for ampleness. Section 0–3 gives a characterization of canonical varieties due to Reid [R1], followed by Section 0–4 which explains what our main goal, the Minimal Model Conjecture, is.

Chapter 1 proves several vanishing theorems. The Covering Lemma in Section 1–1 leads to the Vanishing theorem of Kawamata and Viehweg which plays a tricky but essential role in our whole paper. The Rationality of weak log-terminal singularities is an easy corollary to the Vanishing Theorem of Elkik and Fujita in Section 1–3.

Chapter 2 is devoted to the Non-Vanishing Theorem due to Shokurov [S1].

In Chapter 3 we prove the Base Point Free Theorem, which is presented in another form as the Contraction Theorem in Section 3–2. The canonical ring of a general type variety is finitely generated as a k-algebra once it has a minimal model, as proved in Section 3–3 as a corollary to the Base Point Free Theorem.

In Chapter 4, we present the Rationality Theorem, from which the Cone Theorem (with the discreteness of extremal rays) follows immediately.

We discuss the Flip Conjecture in Chapter 5, classifying the types of contractions of extremal rays and studying their properties. We give a proof to the termination of flips in case dimension ≤ 4 using the notion of difficulty due to Shokurov [S1]. Section 5–2 gives a good evidence for the Flip Conjecture, working with toric morphisms and giving some other examples.

Chapter 6 is somewhat independent of the other chapters and discusses what the main results should be if nef and big divisors are replaced by nef and abundant divisors. Then with these results in hand, we formulate the Abundance Conjecture.

Chapter 7 is devoted to some applications of our theory, related to the problem of the classification of higher dimensional varieties.

This paper is a survey as a whole but contains some generalizations of the existing theorems; the Vanishing, Base Point Free and Cone Theorems are stated in a relative category with weak log-terminal singularities, while the Vanishing Theorem of Elkik and Fujita is formulated in a local form. The termination of flips in dimension 4 is a new result. Many other improvements are made to simplify the proofs.

This paper is an expanded version of a course of lectures given by the first author at the University of Tokyo from September 1984 to March 1985. Section 1–3 and Chapter 7 were written up by the second author and all the rest by the third.

Chapter 0. Notation and Preliminaries

§ 0-1. Kleiman's criterion for ampleness

Let X be a normal variety over k of dimension d, where a variety means an integral seperated scheme which is of finite type over k, and let $\pi\colon X\to S$ be a proper morphism onto a variety S. We use the following notation:

$Z_{d-1}(X)$:=the group of Weil divisors, i.e., the free abelian group generated by prime divisors on X.

$\mathrm{Div}(X)$:=the group of Cartier divisors on X, which is naturally isomorphic to $H^0(X, \mathrm{Rat}(X)^\times/\mathcal{O}_X^\times)$, where $\mathrm{Rat}(X)^\times$ is the sheaf of nonzero rational functions on X.

$\mathrm{Pic}(X)$:=the group of line bundles on X.

Take a complete curve C on X which is mapped to a point by π. For $D \in \mathrm{Pic}(X)$, we define the intersection number $(D.C):=\deg_{\bar{C}} f^*D$ where $f\colon \bar{C}\to C$ is the normalization of C. Via this intersection pairing, we introduce a bilinear form

$$(\ .\ )\colon \mathrm{Pic}(X)\times Z_1(X/S)\longrightarrow \mathbf{Z},$$

where

$Z_1(X/S)$: = the free abelian group generated by reduced irreducible curves which are mapped to points on S by π.

Now we have the notion of numerical equivalence both in $Z_1(X/S)$ and in $\mathrm{Pic}(X)$, which is denoted by $\approx$, and we obtain a perfect pairing

$$N^1(X/S) \times N_1(X/S) \longrightarrow \boldsymbol{R}$$

where

$$N^1(X/S) := \{\mathrm{Pic}(X)/\approx\} \otimes \boldsymbol{R} \quad \text{and} \quad N_1(X/S) := \{Z_1(X/S)/\approx\} \otimes \boldsymbol{R},$$

namely $N^1(X/S)$ and $N_1(X/S)$ are dual to each other through this intersection pairing. It is well-known that $\dim_{\boldsymbol{R}} N^1(X/S) = \dim_{\boldsymbol{R}} N_1(X/S) < \infty$. We define

$\overline{NE}(X/S)$: = the closed convex cone in $N_1(X/S)$ generated by reduced irreducible curves on X which are mapped to points on S by π,

$\overline{NE}_D(X/S) := \{z \in \overline{NE}(X/S); (D, z) \geqq 0\}$ for $D \in N^1(X/S)$.

When $S = \mathrm{Spec}\, k$, we drop /Spec k from the notation, e.g., we simply write $N_1(X)$ in stead of $N_1(X/\mathrm{Spec}\, k)$. The notion of numerical equivalence depends on $\pi: X \to S$; $N_1(X)$ and $N_1(X/S)$ are different even in case X itself is complete.

Definition 0-1-1. An element $D \in N^1(X/S)$ is called *π-nef* (or *relatively nef for π*), if $D \geqq 0$ on $\overline{NE}(X/S)$. When $S = \mathrm{Spec}\, k$, we simply say that D is *nef* (or *numerically effective*, or *numerically semi-positive*).

Theorem 0-1-2 (Kleiman's criterion for ampleness, cf. [K1, Chapter IV. § 4. Theorem 1]). *Let $\pi: X \to S$ be a projective morphism between algebraic schemes. Then $H \in \mathrm{Pic}(X)$ is π-ample if and only if the numerical class of H in $N^1(X/S)$ gives a positive function on $\overline{NE}(X/S) - \{0\}$.*

In this paper, we deal not only with the usual divisors but also with the divisors with rational coefficients, which turn out to be fruitful and natural (cf. [Ka2]).

Definition 0-1-3. An element of $Z_{d-1}(X) \otimes \boldsymbol{Q}$ (resp. $\mathrm{Div}(X) \otimes \boldsymbol{Q}$) is called a *$\boldsymbol{Q}$-divisor* (resp. a *$\boldsymbol{Q}$-Cartier divisor*). Two elements $D, D' \in Z_{d-1}(X) \otimes \boldsymbol{Q}$ are said to be *$\boldsymbol{Q}$-linearly equivalent*, denoted by $D \sim_{\boldsymbol{Q}} D'$, if there exists $r \in \boldsymbol{N}$ (the set of positive integers) such that $rD, rD' \in Z_{d-1}(X)$ and that rD and rD' are linearly equivalent in the ordinary sense, i.e.,

$rD - rD' = \mathrm{div}(\varphi)$ in $Z_{d-1}(X)$ for some $\varphi \in \mathrm{Rat}(X)^\times$. (We define $\mathrm{div}(\varphi) := \sum v_\Gamma(\varphi)\Gamma$, where v_Γ denotes the valuation of φ at the prime divisor Γ, and the Γ run through all the prime divisors on X.) $D \in \mathrm{Div}(X) \otimes \boldsymbol{Q}$ is said to be *π-ample* if there exists $r \in \boldsymbol{N}$ such that $rD \in \mathrm{Div}(X)$ and rD is *π-ample* in the ordinary sense.

Definition 0-1-4. $D \in \mathrm{Div}(X)$ is said to be *π-generated* if the natural homomorphism $\pi^*\pi_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$ is surjective. $D \in \mathrm{Div}(X)\otimes\boldsymbol{Q}$ is said to be *π-semi-ample* if there exists $r \in \boldsymbol{N}$ such that $rD \in \mathrm{Div}(X)$ and that rD is π-generated. When $S = \mathrm{Spec}\, k$, D is said to be *semi-ample*. For $D \in \mathrm{Div}(X)$ with $\pi_*\mathcal{O}_X(D) \neq 0$, we define the *$\pi$-fixed locus* of D to be the unique effective divisor $F \in Z_{d-1}(X)$ such that

$$\mathrm{Im}\,(\pi^*\pi_*\mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D))^\wedge = \mathcal{O}_X(D-F),$$

where $\wedge$ denotes the double dual.

Definition 0-1-5. Let $f\colon Y \to X$ be a morphism from a variety Y and $D \in \mathrm{Div}(X)\otimes\boldsymbol{Q}$ such that $f(Y) \not\subset \mathrm{supp}\, D$. Then we define the *pull-back* of D by f to be $f^*D := (1/r)f^*(rD) \in \mathrm{Div}(X)\otimes\boldsymbol{Q}$, where r is some positive integer which makes $rD \in \mathrm{Div}(X)$ and the pull-back on the right hand side is the one defined for usual Cartier divisors. We define the strict transform of a prime divisor D_0 on X by a birational map $\alpha\colon X \dashrightarrow Y$ as follows. Let X_0 be the maximal open subset of X on which α is regular. Then the *strict transform* $\alpha_*(D_0)$ is defined to be the closure of $\alpha(D_0 \cap X_0)$. By linearity, we can also define the *strict transform* $\alpha_*(D)$ of a $\boldsymbol{Q}$-divisor D on X.

Remark 0-1-6. (1) For any morphism $f\colon Y \to X$, we can always define the pull-back homomorphism $f^*\colon \mathrm{Pic}(X)\otimes\boldsymbol{Q} \to \mathrm{Pic}(Y)\otimes\boldsymbol{Q}$.

(2) We have the natural homomorphisms

$$\mathrm{Div}(X) \longrightarrow Z_{d-1}(X) \quad \text{and} \quad \mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow N^1(X/S),$$

and the ones tensored with $\boldsymbol{Q}$. The first homomorphism is injective since X is normal. We say that $D \in Z_{d-1}(X)\otimes\boldsymbol{Q}$ is a *$\boldsymbol{Q}$-Cartier divisor* if D is in the image of the first homomorphism tensored with $\boldsymbol{Q}$.

Definition 0-1-7. If the natural homomorphism $\mathrm{Div}(X)\otimes\boldsymbol{Q} \to Z_{d-1}(X)\otimes\boldsymbol{Q}$ is surjective, a normal variety X is called *$\boldsymbol{Q}$-factorial.*

Definition 0-1-8. Let $D = \sum a_i D_i \in Z_{d-1}(X)\otimes\boldsymbol{Q}$, where the a_i are rational numbers and the D_i are mutually distinct prime divisors on X. We define

$$[D]:=\sum [a_i]D_i, \quad \text{the integral part of } D,$$
$$\ulcorner D\urcorner:=\sum \ulcorner a_i\urcorner D_i=-[-D], \quad \text{the round up of } D,$$
$$\langle D\rangle:=\sum \langle a_i\rangle D_i=D-[D], \text{ the fractional part of } D,$$

where for $r \in \boldsymbol{R}$, we define $[r]:=\max\{t \in \boldsymbol{Z};\ t\leqq r\}$.

We conclude this section with pointing out the one to one correspondence between the linear equivalence classes of Weil divisors on X and the isomorphism classes of reflexive sheaves of rank one.

Definition 0-1-9. For a sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules, $\mathscr{F}^*$ denotes the dual $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$ and $\mathscr{F}^\wedge:=\mathscr{F}^{**}$. $\mathscr{F}$ is said to be *reflexive* if $\mathscr{F}=\mathscr{F}^\wedge$, i.e., if the natural homomorphism $\mathscr{F}\to\mathscr{F}^\wedge$ is an isomorphism.

Lemma 0-1-10 (cf. [R1, Proposition 2]). *For a coherent sheaf $\mathscr{F}$ on X, the following conditions are equivalent*:

(1) *$\mathscr{F}$ is reflexive of rank one,*

(2) *if $X^0\subset X$ is a nonsingular open subvariety such that $X-X^0$ has codimension $\geqq 2$, then $\mathscr{F}|_{X^0}$ is invertible and $\mathscr{F}=i_*(\mathscr{F}|_{X^0})$, where i denotes the inclusion i: $X^0\to X$.*

Proof. See [Ha2, Proposition 1.6].

Proposition 0-1-11 (cf. [R1, Theorem 3]). *The correspondence*

$$Z_{d-1}(X)/\sim \longrightarrow \{\textit{reflexive sheaves of rank } 1\}/\cong$$

given by $D\to\mathcal{O}_X(D)$ is a bijection, where for any $D=\sum n_\Gamma \Gamma \in Z_{d-1}(X)$ with $n_\Gamma \in \boldsymbol{Z}$ and the Γ being mutually distinct prime divisors, the sheaf $\mathcal{O}_X(D)$ is defined by

$$\Gamma(U, \mathcal{O}_X(D))=\{f \in \mathrm{Rat}(X);\ v_\Gamma(f)+n_\Gamma\geqq 0 \textit{ for all codimension } 1 \textit{ points } \Gamma \in U, \textit{ where } v_\Gamma(f) \textit{ is the valuation of } f \textit{ at } \Gamma\}.$$

Proof. Immediate from Lemma 0-1-10.

§ 0-2. Definitions of terminal, canonical, and (weak) log-terminal singularities

Let X be a normal variety of dimension d.

Definition 0-2-1. The *canonical divisor* K_X on X is an element of $Z_{d-1}(X)$ such that $\mathcal{O}_{X_{\mathrm{reg}}}(K_X)=\Omega^d_{X_{\mathrm{reg}}}$, where X_{reg} is the nonsingular locus of X.

Remark 0-2-2. (1) Since $\operatorname{codim}(X-X_{\mathrm{reg}})\geqq 2$, the canonical divisor K_X is well-defined up to linear equivalence. We use the following notation:

$$\omega_X := \omega_X^{[1]} := \mathcal{O}_X(K_X), \quad \omega_X^{[r]} := \mathcal{O}_X(rK_X) \quad \text{for } r \in \boldsymbol{Z}.$$

Then by Proposition 0-1-11, $\omega_X^{[r]}$ is a reflexive sheaf of rank 1 for any $r \in \boldsymbol{Z}$, and hence $\omega_X^{[r]} = i_*((\Omega^d_{X_{\mathrm{reg}}})^{\otimes r})$ where i is the natural inclusion $i\colon X_{\mathrm{reg}} \to X$.

(2) For the dualizing complex $\omega_X^{\boldsymbol{\cdot}}$ of X, we have $\omega_X = H^{-d}(\omega_X^{\boldsymbol{\cdot}})$.

Definition 0-2-3. X is called a ***Q**-Gorenstein* variety if the canonical divisor K_X is a $\boldsymbol{Q}$-Cartier divisor, i.e., if $\mathcal{O}_X(rK_X)$ becomes invertible for some $r \in \boldsymbol{N}$.

Remark 0-2-4. (1) The condition for X to be a Cohen-Macaulay variety is equivalent to the condition that $H^i(\omega_X^{\boldsymbol{\cdot}})=0$ for $i \neq -d$.

(2) When X is a $\boldsymbol{Q}$-Gorenstein variety and $\min\{r \in \boldsymbol{N}\colon rK_X \in \operatorname{Div}(X)\} = e$, we call X an *e-Gorenstein* variety and e the *index* of X. It is obvious that if X is a Gorenstein variety (i.e., ω_X is invertible and X is Cohen-Macaulay), then X is a 1-Gorenstein variety, while the converse is not necessarily true. (cf. [Is])

Definition 0-2-5 (cf. [R1]). Let X be a $\boldsymbol{Q}$-Gorenstein normal affine variety of index r with a nowhere vanishing section ω of $\mathcal{O}_X(rK_X)$. Then ω defines a structure of an $\mathcal{O}_X$-algebra on $R := \bigoplus_{i=0}^{r-1} \mathcal{O}_X(-iK_X)$. Namely R is a quotient ring of the algebra $\tilde{R} := \bigoplus_{m\geqq 0} \mathcal{O}_X(-mK_X)$ divided by the ideal $(\omega - \mathrm{id})\tilde{R}$, where we denote the operation of multiplication by ω with the same symbol ω. We put $V :=$ Spec R. The natural finite Galois cover $\tau\colon V \to X$ of degree r is called the *canonical cover* of X. Since the $\mathcal{O}_X(-iK_X)$ are reflexive sheaves and since τ is étale in codimension 1, V is normal. By the choice of r, V is irreducible. If $\tau'\colon V' \to X$ is the canonical cover obtained from another choice of ω, then $V' \times_X X' \cong V \times_X X'$ for some étale cover X' of X. In particular, the canonical cover is locally unique up to complex analytic isomorphisms, if $k=\boldsymbol{C}$. Since τ is étale on the nonsingular locus of X, we have $K_V = \tau^* K_X$. On the other hand by [Ha1], $\tau_* \omega_V \cong \bigoplus_{i=1}^{r} \mathcal{O}_X(iK_X) \cong R \otimes \mathcal{O}_X(rK_X)$. Hence ω_V is invertible, i.e., K_V is a Cartier divisor.

Definition 0-2-6 (cf. [R1]). A normal variety X is said to have only *canonical* (resp. *terminal*) singularities if the following two conditions are satisfied:

(i) X is a $\boldsymbol{Q}$-Gorenstein variety,

(ii) there exists a resolution of singularities $f\colon Y \to X$ such that $K_Y = f^* K_X + \sum a_i E_i$ for $a_i \in \boldsymbol{Q}$ with $a_i \geqq 0$ (resp, $a_i > 0$) for all i, where the E_i

vary among all the prime divisors which are exceptional with respect to f. Note that the equality above holds up to $\boldsymbol{Q}$-linear equivalence, while the divisor $\sum a_i E_i$ is uniquely determined in $Z_{d-1}(X)\otimes \boldsymbol{Q}$ ($d=\dim X$).

Remark 0-2-7. (1) In the formula $K_Y=f^*K_X+\sum a_i E_i$ in the definition above, a_i is called the *discrepancy* at E_i. Note that the discrepancy does not depend on the choice of resolution but is determined only by the divisor E_i, i.e, by the D.V.R. in $\mathrm{Rat}(X)$ associated to E_i.

(2) If a normal variety X has only canonical (resp. terminal) singularities, then the condition (ii) is satisfied for an arbitrary resolution.

(3) When $\dim X=2$, X has only canonical (resp. terminal) singularities if and only if X has only rational double or nonsingular points (resp. nonsingular points). In particular, all the canonical singularities in dimension 2 have the index 1.

(4) Reid [R2] showed that the canonical covers of terminal singularities of 3-folds have at most isolated compound du Val singularities (i.e., they are the 1-parameter deformation spaces of rational double points), or they are nonsingular. The former case is studied in more detail by Mori [Mo3], while the latter by Morrison and Stevens [MS] and by Danilov [D2].

(5) Canonical singularities of index 1 are the same as rational Gorenstein singularities (cf. [E]).

Proposition 0-2-8. *Let X be a normal variety. Then X has only canonical singularities if and only if*

(i′) *there exists an integer $r>0$ such that $\omega_X^{[r]}$ is invertible, and*

(ii′) *$f\colon Y\to X$ being a desingularization of X, there exists a natural homomorphism $\rho_s\colon f^*\omega_X^{[s]}\to\omega_Y^{\otimes s}$ for any $s\in \boldsymbol{N}$ such that ρ_s restricted to $Y-E$ is the identity, where E is the exceptional locus of f.*

Proof. Straightforward and left to the reader (cf. [R1]).

As we shall see later, canonical (resp. terminal) singularities are the ones which appear on the canonical model of a variety of f.g. general type (resp. a minimal model of a variety). Thus it is quite natural for us to define the logarithmic version of canonical (resp. terminal) singularities as follows, when we want to consider the log-canonical (resp. log-minimal) model of an open variety with a boundary as explained in Introduction.

Definition 0-2-9. Let X be a nonsingular variety of $\dim X=d$. A reduced effective divisor $D\in \mathrm{Div}(X)$ is said to have only *simple normal crossings* (resp. *normal crossings*) if for each closed point p of X, a local defining equation f of D at p can be written as $f=z_1\cdots z_{j_p}$ in $\mathcal{O}_{X,p}$ (resp. $\hat{\mathcal{O}}_{X,p}$), where $\{z_1,\cdots,z_{j_p}\}$ is a part of a regular system of parameters.

Definition 0-2-10. (1) Let X be a normal variety of dimension d, and $\varDelta \in Z_{d-1}(X)\otimes \boldsymbol{Q}$ an effective $\boldsymbol{Q}$-divisor on X. Then the pair $(X, \varDelta)$ is said to have *log-terminal* (resp. *log-canonical*) singularities if the following conditions are satisfied:

(i) $K_X+\varDelta \in \mathrm{Div}(X)\otimes \boldsymbol{Q}$.

(ii) $[\varDelta]=0$ (resp. (ii′) $\ulcorner \varDelta \urcorner$ is a reduced divisor, i.e., a sum of mutually distinct prime divisors with all the coefficients being one).

(iii) (resp. (iii′)) There exists a resolution of singularities $f\colon Y\to X$ such that the union of the exceptional locus and f^{-1} (supp $\varDelta$) is a divisor with only normal crossings, and that

$$K_Y=f^*(K_X+\varDelta)+\sum a_jF_j \quad \text{for } a_j \in \boldsymbol{Q} \quad (*)$$

with the condition that $a_j>-1$ (resp. $a_j\geqq -1$) whenever F_j is exceptional for f.

(2) Let X be a normal variety of dimension d, and let $\varDelta \in Z_{d-1}(X)\otimes\boldsymbol{Q}$ be an effective $\boldsymbol{Q}$-divisor on X. Then the pair $(X, \varDelta)$ is said to have *weak log-terminal* singularities if the conditions (i), (ii′), (iii) with the following condition (iv) are satisfied:

(iv) There is an f-ample divisor $A \in \mathrm{Div}(Y)$ whose support coincides with that of the exceptional locus of f.

Remark 0-2-11. (1) In the condition (iv) of the definition above, it follows that $-A$ is effective.

(2) We remark that from the view point of the classification of open varieties with boundaries the weak log-terminal singularities should have been called log-terminal. But the word "log-terminal" is commonly used in the current literature in the sense above and we simply wanted to avoid any confusion (cf. [Ka7]).

(3) In the formula $(*)$, a_j is called the *log-discrepancy* at F_j.

(4) The condition (iv) is necessary to guarantee the rationality of weak log-terminal singularities (cf. § 1.3). For example, let C be a non-singular projective elliptic curve, let L be a negative line bundle on C, and let Y be the total space of the vector bundle $L\oplus L$. We embed C in Y as a zero section, and let F_1 and F_2 be two hypersurfaces of Y which are images of injections $L\to L\oplus L$ given by $x\mapsto(x, 0)$ and $x\mapsto(0, x)$, respectively. Let X be a normal 3-fold obtained by contracting C from Y, and let $\varDelta$ be the strict transform of F_1+F_2 on X. Then the pair $(X, \varDelta)$ and the resolution $Y\to X$ satisfies conditions (i), (ii′) and (iii), but not (iv). In fact, X is not a Cohen-Macauley variety.

Lemma 0-2-12. *If the pair* $(X, \varDelta)$ *has only log-terminal or log-canonical singularities, then the formula* $(*)$ *in the condition* (iii) *or* (iii′)

above is satisfied with $a_j > -1$ *or* $a_j \geqq -1$, *respectively, for an arbitrary desingularitzation.*

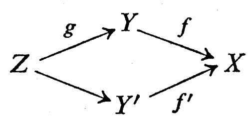
Proof. We need the following lemma.

Lemma 0-2-13 (Logarithmic Ramification Formula [I3, Theorem 11.5]). *Let V and W be nonsingular varieties of dimension d, and let D and B be divisors on V and W, respectively, with B having only normal crossings. Let* $f: V \to W$ *be a generically surjective morphism such that* $f^{-1}(B) \subseteqq D$. *Then there is a natural injective homomorphism* $f^*\Omega_W^p(\log B) \to \Omega_V^p(\log D)$ *for all* $p \geqq 0$. *In particular,*

$$K_V + D = f^*(K_W + B) + E$$

for some effective divisor E.

We go back to the proof of Lemma 0-2-12. Now suppose that $f': Y' \to X$ is another desingularization. Since we can take a third desingularization Z which dominates both Y and Y' as in the diagram (cf. [Hi2]),



it is sufficient to see that the same condition holds for $f \circ g: Z \to X$ in order to see that the formula (∗) in the condition (iii) with $a_j > -1$ or $a_j \geqq -1$ holds for $f': Y' \to X$. By Lemma 0-2-13, there exists some effective divisor E on Z such that

$$K_Z + (g^*(\sum_{a_j<0} F_j))_{\text{red}} = g^*(K_Y + \sum_{a_j<0} F_j) + E,$$

which implies

$$K_Z = (f \circ g)^*(K_X + \Delta) + g^*(\sum_{a_j \geqq 0} a_j F_j) + g^*(\sum_{a_j<0} (1 + a_j) F_j)$$
$$+ E - (g^*(\sum_{a_j<0} F_j))_{\text{red}}.$$

We calculate the log-discrepancies by using the right hand side of the formula above. For any prime component G of $g^*(\sum_{a_j<0} F_j)$, the coefficient of it in $(g^*(\sum_{a_j<0} F_j))_{\text{red}}$ is 1 by definition and $\geqq 0$ in $g^*(\sum_{a_j<0}(1+a_j)F_j)$ since $a_j \geqq -1$. This completes the proof in the case of log-canonical singularities. To see the lemma in the case of log-terminal singularities, take an exceptional divisor H for $f \circ g$. If $g(H) \not\subset \operatorname{supp}(\sum_{a_j<0} F_j)$, the log-discrepancy at H is clearly nonnegative. If $g(H) \subset \operatorname{supp}(\sum_{a_j<0} F_j)$, since the assumption $[\varDelta]=0$ implies that $a_j>-1$ whenever $a_j<0$, the log-discrepancy at H is greater than -1. q.e.d.

Remark 0-2-14. The formula $(*)$ in the condition (iii) *does* depend on the choice of resolution in the case of weak log-terminal singularities. For example, take a nodal curve C on a nonsingular surface Y. Then $X=Y$, $\varDelta=C$ and $f=$identity actually satisfy the conditions (i), (ii′), (iii) and (iv) of Definition 0-2-10, while if we take $f=$the blow up of Y at the node of C, then the exceptional divisor has the log-discrepancy -1.

Lemma 0-2-15. *When $(X, \varDelta)$ has log-terminal singularities, for an arbitrary effective $\boldsymbol{Q}$-Cartier divisor $\varDelta' \in \mathrm{Div}(X)\otimes\boldsymbol{Q}$, the pair $(X, \varDelta+\varepsilon\varDelta')$ has also log-terminal singularities for any sufficiently small positive rational number ε.*

Proof. Trivial.

Proposition 0-2-16. *Let X be a $\boldsymbol{Q}$-Gorenstein normal variety with a canonical cover $\tau\colon V\to X$. Then $(X, 0)$ has only log-terminal singularities, if and only if V has rational Gorenstein singularities.*

Proof. See [R1, Proposition 1.7] and [Ka7, Proposition 1.7].

Remark 0-2-17. (1) One of the direct consequences that follow from the proposition above is the fact that all quotient singularities are log-terminal singularities. Furthermore, we can construct all the log-terminal singularities X with $\varDelta=0$ by taking the quotients of rational Gorenstein singularities (cf. [Ka7]).

(2) As a corollary to Proposition 0-2-16, we obtain the result that all the log-terminal singularities with $\varDelta=0$ are rational singularities. In Section 1-3, we shall show that weak log-terminal singularities are always rational singularities, while this is not necessarily the case with log-canonical singularities (cf. [Ka1], [Ts1], [Is]).

(3) When $k=\boldsymbol{C}$, the followng conditions for a germ (X, p) of a normal complex analytic surface singularity are equivalent:

(i) (X, p) is a quotient singularity,

(ii) (X, p) is a log-terminal singularity with $\varDelta=0$ (cf. [Ts1], [Ka7]).

§ 0-3. Canonical varieties

Let X be a normal complete variety of dimension d. The complete linear system associated with $D \in \mathrm{Div}(X)$ is defined by

$$|D|:=\{D+\mathrm{div}(\varphi);\ \varphi \in \mathrm{Rat}(X)^{\times} \text{ with } D+\mathrm{div}(\varphi)\geqq 0\}.$$

When $|D|\neq\phi$, we define the base locus of $|D|$ to be

$$\mathrm{Bs}|D|:=\bigcap_{D'\in|D|}\mathrm{supp}(D').$$

We have the rational map $\Phi_{|D|}:=X\cdots\to P^N$ which is a morphism on $X-\mathrm{Bs}|D|$ that takes a closed point $x\in X-\mathrm{Bs}|D|$ to the point $(\varphi_0(x):\varphi_1(x):\cdots:\varphi_N(x))$ on P^N, where $\varphi_0,\varphi_1,\cdots,\varphi_N$ form a basis of $H^0(X,\mathcal{O}_X(D))$. The image $\Phi_{|D|}(X)$ is defined to be the closure of $\Phi_{|D|}(X-\mathrm{Bs}|D|)$ in P^N.

Definition 0-3-1 (cf. [I1]). The *Iitaka dimension* $\kappa(X, D)$ of the pair (X, D) is defined as follows:

$$\kappa(X,D):=\begin{cases}\max_{m\in N}\{\dim\Phi_{|mD|}(X)\} & \text{if } |mD|\neq\phi \text{ for some } m\in N\\ -\infty & \text{otherwise.}\end{cases}$$

We define the graded algebra associated to D to be

$$R(X,D):=\bigoplus_{m\geqq 0}H^0(X,\mathcal{O}_X(mD)).$$

We remark that $\kappa(X, D)$ is characterized by the property that there exist $\alpha,\beta>0$ and $m_0\in N$ such that the inequalities

$$\alpha m^\kappa\leqq h^0(X,\mathcal{O}_X(mm_0D))\leqq\beta m^\kappa$$

hold for any sufficiently large $m\in N$ (see [I1], [I3], [U]). We have also

$$\kappa(X,D)=\begin{cases}\text{trans. deg}_k R(X,D)-1 & \text{if } R(X,D)\neq k,\\ -\infty & \text{otherwise.}\end{cases}$$

Definition 0-3-2. $D\in\mathrm{Div}(X)$ is called *big* if $\kappa(X,D)=\dim X$. Let $\pi: X\to S$ be a proper morphism onto a variety S. Then $D\in\mathrm{Div}(X)$ is called *π-big* if D_η is big on X_η, η being the generic point of S.

The following lemma gives a characterization of big divisors.

Lemma 0-3-3 (Kodaira's Lemma). *Let X be a normal complete variety of dimension d, and $D\in\mathrm{Div}(X)$ with $\kappa(X,D)=d$. Then for an arbitrary divisor $M\in\mathrm{Div}(X)$, we have $|nD-M|\neq\phi$ for $n\gg 0$.*

Proof. By Hironaka [Hi2], there is a birational morphism $\mu: X'\to X$ from a nonsingular projective variety X'. Considering the pull-backs μ^*D and μ^*M, we may assume that X is projective. Then it is sufficient to show that for a very ample divisor $A\in\mathrm{Div}(X)$ there exists $l\in N$ such that $|lD-A|\neq\phi$. Since we have the exact sequence

$$0\longrightarrow\mathcal{O}_X(mm_0D-A)\longrightarrow\mathcal{O}_X(mm_0D)\longrightarrow\mathcal{O}_Y(mm_0D)\longrightarrow 0,$$

where Y is a general member of $|A|$, and since there exist positive real numbers α, β and a positive integer m_0 (which we can actually take to be equal to 1 as a result of this lemma) such that $h^0(X, \mathcal{O}_X(mm_0D)) \geqq \alpha m^d$ and that $h^0(Y, \mathcal{O}_Y(mm_0D)) \leqq \beta m^{d-1}$ for sufficiently large $m \in N$, we obtain the result $h^0(X, \mathcal{O}_X(lD-A)) \neq \phi$ for $l = mm_0$ where m is sufficiently large. q.e.d.

Corollary 0-3-4. *Let X be a normal variety with a proper morphism $\pi: X \to S$ onto a variety S. If $D \in \mathrm{Div}(X)$ is π-big, then for an arbitrary divisor $M \in \mathrm{Div}(X)$, we have $\pi_* \mathcal{O}_X(nD-M) \neq 0$ for $n \gg 0$.*

From the lemma above, Kleiman's criterion for ampleness, and from Hironaka's resolution theorem, we can easily deduce the facts below.

Corollary 0-3-5 (cf. [Ka4]). *Let X be a nonsingular projective variety and let $D \in \mathrm{Div}(X) \otimes \boldsymbol{Q}$ be nef and big. Then there exists an effective $\boldsymbol{Q}$-divisor D_0 such that $D - \delta D_0$ is ample for all $\delta \in \boldsymbol{Q}$ with $0 < \delta < 1$.*

Corollary 0-3-6. *Let $\pi: X \to S$ be a proper surjective morphism of normal varieties, and let D be a π-nef and π-big $\boldsymbol{Q}$-Cartier divisor on X. Then there exists a proper birational morphism $\mu: Y \to X$ from a nonsingular variety Y projective over S and a family of divisors $\{F_j\}$ on Y such that the union of the support of $\mu^* D$ and $\bigcup F_j$ is a divisor with only simple normal crossings and such that $\mu^* D - \sum_j \delta_j F_j$ is $\pi \circ \mu$-ample for some $\delta_j \in \boldsymbol{Q}$ with $0 < \delta_j \ll 1$.*

Definition 0-3-7 (cf. [I1]). Let X be a variety. Then we define the *Kodaira dimension* $\kappa(X)$ of X to be $\kappa(X) := \kappa(X', K_{X'})$ where X' is a nonsingular complete variety birational to X, whose existence is guaranteed by the resolution theorem of Hironaka [Hi2]. The *canonical ring* of X is defined to be $R(X) := R(X') := \bigoplus_{m \geqq 0} H^0(X', \mathcal{O}_{X'}(mK_{X'}))$. Then

$$\kappa(X) = \begin{cases} \text{trans. deg}_k R(X) - 1 & \text{if } R(X) \neq k \\ -\infty & \text{otherwise.} \end{cases}$$

Remark 0-3-8. For any two nonsingular complete varieties X' and X'' which are birational to each other, the canonical rings $R(X')$ and $R(X'')$ are isomorphic, and therefore the Kodaira dimension of an arbitrary variety is well-defined and is a birational invariant.

Definition 0-3-9. A variety X is said to be of *general type* if $\kappa(X) = \dim X$, i.e., if the canonical divisor $K_{X'}$ is big for some nonsingular complete birational model X' of X.

In the rest of this section, we shall give a characterization of a canonical variety, which justifies our naming of canonical singularities, by extending an argument in [R1].

Definition 0-3-10 (cf. [R1]). A variety X (resp. a pair (X, Δ)) with a projective morphism $\pi: X \to S$ is called a *canonical variety* (resp. a *log-canonical variety*) over S if X has only canonical singularities (resp. if (X, Δ) has only long-canonical singularities) and if K_X (resp. $K_X+\Delta$) is π-ample.

Definition 0-3-11 (cf. [R1]). Let X be a nonsingular variety with a proper morphism $\pi: X \to S$ onto a variety S, and let Δ be an effective $\boldsymbol{Q}$-divisor on X whose support has only simple normal crossings and such that $\ulcorner \Delta \urcorner$ is a reduced divisor. X (resp. (X, Δ)) is said to be of *f.g. general type* (resp. of *f.g. log-general type*) over S if K_X (resp. $K_X+\Delta$) is π-big, and if

$$R(X/S) := \bigoplus_{m \geqq 0} \pi_* \mathcal{O}_X(mK_X)$$
$$(\text{resp. } R(X/S, K_X+\Delta) := \bigoplus_{m \geqq 0} \pi_* \mathcal{O}_X([m(K_X+\Delta)]))$$

is finitely generated as an $\mathcal{O}_S$-algebra. In this case, we can define the *canonical model* X_{can} of X (resp. the *log-canonical model* $(X, \Delta)_{\text{can}}$ *of* (X, Δ)) to be Proj $R(X/S)$ (resp. (Proj $R(X/S, K_X+\Delta), \Delta'$) where Δ' is the strict transform of Δ).

Theorem 0-3-12 (cf. [R1, Proposition 1.2]). *X (resp. (X, Δ)) is a canonical variety (resp. a log-canonical variety) over S if and only if there exists a nonsingular complete variety X' of f.g. general type (resp. a pair (X', Δ') of f.g. log-general type) over S such that $X \cong X'_{\text{can}}$ (resp. $(X, \Delta) \cong (X', \Delta')_{\text{can}}$).*

Proof. We shall give a proof of the theorem in the logarithmic case. First we prove the "if" part of the theorem. Let (X', Δ') with $\pi': X' \to S$ be a pair of f.g. log-general type with the relative log-canonical ring $R(X'/S, K_{X'}+\Delta')) = \bigoplus_{m \geqq 0} R_m$, which is a finitely generated $\mathcal{O}_S$-algebra by definition. Then there exists a positive integer r such that $r(K_{X'}+\Delta') \in \text{Div}(X')$ and that $R^{(r)} := \bigoplus_{m \geqq 0} R_{mr}$ is generated by $R_1^{(r)} = R_r$ as an $\mathcal{O}_S$-algebra, and thus $X := \text{Proj } R(X'/S, K_{X'}+\Delta') = \text{Proj } R^{(r)}$ is projectively normal. Resolving the singularities and eliminating the indeterminacy of the rational map associated to the natural homomorphism

$$\pi'^* \pi'_* \mathcal{O}_{X'}(r(K_{X'}+\Delta')) \longrightarrow \mathcal{O}_{X'}(r(K_{X'}+\Delta')),$$

we obtain a proper birational morphism $f: Y \to X'$ such that

$$r(K_Y + \sum_{a_j<0} (-a_j)F_j) = r(f^*(K_{X'}+\Delta') + \sum_{a_j \geqq 0} a_j F_j)) = D + G,$$

where $K_Y = f^*(K_{X'}+\Delta') + \sum a_j F_j$, G is the $\pi' \circ f$-fixed part of $r(f^*(K_{X'}+$

$\varDelta')+\sum_{a_j\geqq 0} a_jF_j)$ and D is $\pi'\circ f$-generated. Note here that supp $(\sum_{a_j\geqq 0} a_jF_j)$ is exceptional for f. Then the morphsim

$$\psi\colon Y \longrightarrow \psi(Y)=X\subset P((\pi'\circ f)_*\mathcal{O}_Y(D))$$

associated to the surjection $(\pi'\circ f)^*(\pi'\circ f)_*\mathcal{O}_Y(D)\twoheadrightarrow \mathcal{O}_Y(D)$ coincides with the natural morphism $Y\to X$ and $\mathcal{O}_Y(D)=\psi^*\mathcal{O}_X(1)$. Since $R^{(r)}$ is generated by $R_1^{(r)}=\pi'_*\mathcal{O}_{X'}(r(K_{X'}+\varDelta'))$ and since $\mathcal{O}_Y(D)$ is the pull back of a π-very-ample sheaf $\mathcal{O}_X(1)$, $\pi\colon X\to S$ being the structure morphism, it follows that codim supp $\psi(G)\geqq 2$. Since ψ is a proper birational morphism between normal varieties, there exists a closed subscheme X_1 of X with codim $X_1\geqq 2$ such that $\psi\colon Y-\psi^{-1}(X_1)\cong X-X_1$. Setting $X_0:=\psi(G)\cup X_1$, we obtain

$$\begin{aligned}\psi^*\mathcal{O}_X(1)|_{Y-\psi^{-1}(X_0)}&=\mathcal{O}_Y(D)|_{Y-\psi^{-1}(X_0)}\\&=\mathcal{O}_Y(r(K_Y+\sum_{a_j<0}(-a_j)F_j))|_{Y-\psi^{-1}(X_0)}.\end{aligned}$$

Hence $\mathcal{O}_X(1)=(\mathcal{O}_X(r(K_X+\varDelta)))^\wedge$, where $\varDelta=\psi_*(\sum_{a_j<0}(-a_j)F_j)$, since codim $X_0\geqq 2$. Therefore $(X, \varDelta)$ has only log-canonical singularities and $K_X+\varDelta$ is π-ample, i.e., $(X, \varDelta)$ is a log-canonical variety over S.

To see the "only if" part of the theorem, let $f\colon Y\to X$ be a desingularization of X as in the definition of log-canonical singularities. Then $R(Y/S,\ K_Y+\sum_{a_j<0}(-a_j)F_j)\cong R(X/S,\ K_X+\varDelta)$, the latter being finitely generated as an $\mathcal{O}_S$-algebra since $K_X+\varDelta$ is π-ample. It is clear that

$$X\cong \operatorname{Proj} R(X/S,\ K_X+\varDelta)\cong \operatorname{Proj} R(Y/S,\ K_Y+\sum_{a_j<0}(-a_j)F_j)$$

and that $f_*(\sum_{a_j<0}(-a_j)F_j)=\varDelta$. q.e.d.

§ 0-4. Minimal Model Conjecture

Definition 0-4-1. Let X be a normal variety with only canonical singularities and let $\pi\colon X\to S$ be a proper morphism onto a variety S. Then X is called a *minimal variety* over S if K_X is π-nef. Moreover, if X has only terminal (resp. $\boldsymbol{Q}$-factorial terminal) singularities, then X is called a *terminal minimal* (*strictly minimal*) *variety*. A minimal variety, birational to a given variety $\pi'\colon X'\to S$ is called a *minimal model* of π. A *terminal minimal model* and a *strictly minimal model* are defined similarly.

Remark 0-4-2. If X is a strictly minimal variety over S, then X is *maximal* in the category of minimal varieties over S in the following sense: if another minimal variety X' over S has a proper birational morphism $f\colon X'\to X$ over S, then f must be an isomorphism.

Definition 0-4-3. Let X be a normal variety of dimension d with $\Delta \in Z_{d-1}(X) \otimes \mathbf{Q}$ such that the pair (X, Δ) has only (weak) log-terminal singularities, and let $\pi: X \to S$ be a proper morphism onto a variety S. Then (X, Δ) is said to be a (weak) log-minimal variety over S if $K_X + \Delta$ is π-nef. A (weak) log-minimal variety (X, Δ) birational to a given pair (X', Δ') (in the sense that X is birational to X' over S and that Δ is the strict transform of Δ'), is called a (*weak*) *log-minimal model* of (X', Δ') if no divisors on X corresponds to lower dimensional subvarieties of X'. *Strictly* (*weak*) *log-minimal varieties* and *models* are defined as in Definition 0-4-1.

Our main problem is to prove the following conjectures.

Conjecture 0-4-4 (Minimal Model Conjecture). *Let $\pi: X \to S$ be a proper surjective morphism of algebraic varieties. Assume that the irreducible components of the geometric generic fibre $X_{\bar{\eta}}$ are not uniruled.* (*For the definition of uniruled varieties, see Chapter* 5). *Then there is a minimal model $\pi': X' \to S$ of π.*

Conjecture 0-4-5. (Log-Minimal Model Conjecture). *Let $\pi: X \to S$ be a proper surjective morphism from a nonsingular variety X of dimension d onto a variety S. Let $\Delta \in Z_{d-1}(X) \otimes \mathbf{Q}$ be an effective $\mathbf{Q}$-divisor such that $\ulcorner \Delta \urcorner$ is a reduced divisor with only normal crossings. Assume that $\kappa(X_\eta, K_{X_\eta} + \Delta_\eta) \geqq 0$ for the generic fiber X_η. Then there exists a weak log-minimal model (X', Δ') of the pair (X, Δ) over S.*

Chapter 1. Vanishing Theorems

§ 1-1. Covering Lemma

Theorem 1-1-1 (Covering Lemma, cf. [Ka3, Theorem 17]). *Let X be a nonsingular projective variety of dimension d and let $D \in Z_{d-1}(X) \otimes \mathbf{Q}$ be a $\mathbf{Q}$-divisor such that the fractional part $\langle D \rangle$ has support with only simple normal crossings. Then there exists a finite Galois morphism $\pi: Y \to X$ from a nonsingular variety Y with Galois group* $G = \mathrm{Gal}(\mathrm{Rat}(Y)/\mathrm{Rat}(X))$ *which satisfies the following conditions*:

(i) *$\tau^* D \in Z_{d-1}(Y)$, i.e., $\tau^* D$ becomes integral,*

(ii) $\mathcal{O}_X([D]) \cong (\tau_* \mathcal{O}_Y(\tau^* D))^G$, $\mathcal{O}_X(K_X + \ulcorner D \urcorner) \cong (\tau_* \mathcal{O}_Y(K_Y + \tau^* D))^G$,

where G acts naturally on $\tau_* \mathcal{O}_Y(\tau^* D)$ and on $\tau_* \mathcal{O}_Y(K_Y + \tau^* D)$. Via these isomorphisms $\mathcal{O}_X([D])$ (resp. $\mathcal{O}_X(K_X + \ulcorner D \urcorner)$) turns out to be a direct summand of $\tau_* \mathcal{O}_Y(\tau^* D)$ (resp. $\tau_* \mathcal{O}_Y(K_Y + \tau^* D)$).

Proof. Take a positive integer m such that $m\langle D \rangle \in Z_{d-1}(X)$ and let $\langle D \rangle = \sum_{i \in I} a_i \Gamma_i$ be the decomposition of $\langle D \rangle$ into mutually distinct prime components. Now take a very ample divisor M on X such that

$mM-\Gamma_i$ becomes also very ample for all $i\in I$. Then general members $H_k^{(i)}\in|mM-\Gamma_i|$ with $i\in I$ and $1\leqq k\leqq d$ make $\operatorname{supp}\langle D\rangle\cup\operatorname{supp}(\sum_{i,k}H_k^{(i)})$ a divisor with only simple normal crossings. Let $X=\bigcup_{\alpha\in A}U_\alpha$ be an affine open cover of X with the transition functions $\{a_{\alpha\beta};\ a_{\alpha\beta}\in H^0(U_\alpha\cap U_\beta,\mathcal{O}_X^*)\}$ of M and local sections $\{\varphi_{k\alpha}^{(i)};\ \varphi_{k\alpha}^{(i)}\in H^0(U_\alpha,\mathcal{O}_X)\}$ such that $(H_k^{(i)}+\Gamma_i)|_{U_\alpha}=\operatorname{div}(\varphi_{k\alpha}^{(i)})$ on U_α and that $\varphi_{k\alpha}^{(i)}=a_{\alpha\beta}^m\cdot\varphi_{k\beta}^{(i)}$. Then we have only to take the normalization of X in $\operatorname{Rat}(X)[(\varphi_{k\alpha}^{(i)})^{1/m}]_{i,k}$ for some $\alpha\in A$ as Y. (Note that $\operatorname{Rat}(X)[(\varphi_{k\alpha}^{(i)})^{1/m}]_{i,k}=\operatorname{Rat}(X)[(\varphi_{k\beta}^{(i)})^{1/m}]_{i,k}$ for any $\alpha,\beta\in A$.) Since $\operatorname{Rat}(Y)/\operatorname{Rat}(X)$ is a Kummer extension, $\tau\colon Y\to X$ is a finite Galois morphism.

We shall show that Y is nonsingular. Take any closed point $x\in U_\alpha$. Set $I_x:=\{i\in I;\ x\in\Gamma_i\}$. Then for each $i\in I_x$ there exists k_i with $1\leqq k_i\leqq d$ such that $x\notin H_{k_i}^{(i)}$. Now the set

$$R_x:=\{\varphi_{k_i\alpha}^{(i)};\ i\in I_x\}\cup\{\varphi_{k\alpha}^{(i)};\ i\notin I_x,\ x\in H_k^{(i)}\}\cup\{\varphi_{k\alpha}^{(i)}/\varphi_{k_i\alpha}^{(i)};\ i\in I_x,\ x\in H_k^{(i)}\}$$

forms a part of a regular system of parameters of $\mathcal{O}_{X,x}$. Set

$$T_x:=\{\varphi_{k\alpha}^{(i)}/\varphi_{k_i\alpha}^{(i)};\ i\in I_x,\ x\notin H_k^{(i)}\}\cup\{\varphi_{k\alpha}^{(i)};\ i\notin I_x,\ x\notin H_{k\alpha}^{(i)}\}.$$

Any element $\psi\in T_x$ is a unit in $\mathcal{O}_{X,x}$. Therefore it is sufficient to show the following lemma in order to see that Y is nonsingular, i.e., $\mathcal{O}_{Y,y}$ is a regular local ring for any y with $\tau(y)=x$, and to see that $\tau^*\Gamma_i=m((\tau^*\Gamma_i)_{\mathrm{red}})$ for all $i\in I$.

Lemma 1-1-2. *Let R be a regular local k-algebra of dimension d with the maximal ideal M such that $R/M=k$. Let $\{z_1,z_2,\cdots,z_d\}$ be a regular system of parameters and $u_1,u_2,\cdots,u_s$ be units of R. Let m be a positive integer. Fix $e\in N$ with $1\leqq e\leqq d$. Then for any maximal ideal M_1 of $R_1:=R[z_1^{1/m},z_2^{1/m},\cdots,z_e^{1/m},u_1^{1/m},\cdots,u_s^{1/m}]$, the localization of R_1 by M_1, denoted by R_{1,M_1}, is a regular local ring with a regular system of parameters $\{z_1^{1/m},\cdots,z_e^{1/m},z_{e+1},\cdots,z_d\}$.*

Proof. Since M_1 is generated by $z_1^{1/m},z_2^{1/m},\cdots,z_e^{1/m},z_{e+1},\cdots,z_d$, $u_1^{1/m}-\alpha_1,\cdots,u_s^{1/m}-\alpha_s$, where α_i is an element of k such that $\alpha_i=u_i^{1/m}$ mod M_1, it is sufficient to show $u_i^{1/m}-\alpha_i\in(z_1^{1/m},\cdots,z_e^{1/m},z_{e+1},\cdots,z_d)R_{1,M_1}$. Noting that char $k=0$, we have $u_i-\alpha_i^m=(u_i^{1/m}-\alpha_i)\cdot v_i$ for a unit v_i of R_{1,M_1}, which implies the required statement. q.e.d.

Now (i) is obvious from the argument above. We go on to prove (ii). Noting that $\tau^*\Gamma_i=m((\tau^*\Gamma_i)_{\mathrm{red}})$, for any nonempty Zariski open subset U of X we have

$$\begin{aligned}\Gamma(U,(\tau_*\mathcal{O}_Y(\tau^*D))^G)&=\{\varphi\in\operatorname{Rat}(Y);\ (\operatorname{div}(\varphi)+\tau^*D)|_{\tau^{-1}(U)}\geqq0\}^G\\&=\{\psi\in\operatorname{Rat}(X);\ (\operatorname{div}(\psi)+[D])|_U\geqq0\}\\&=\Gamma(U,\mathcal{O}_X([D])).\end{aligned}$$

Thus $\mathcal{O}_X([D]) \cong (\tau_* \mathcal{O}_Y(\tau^* D))^G$. Similarly, since

$$K_Y = \tau^* K_X + (m-1)(\sum (\tau^* \Gamma_i)_{\text{red}} + \sum (\tau^* H_k^{(i)})_{\text{red}}),$$

we have

$$\begin{aligned} \Gamma(U, (\tau^* \mathcal{O}_Y(K_Y + \tau^* D))^G) &= \{\varphi \in \text{Rat}(Y);\ (K_Y + \tau^* D + \text{div}(\varphi))|_{\tau^{-1}(U)} \geqq 0\}^G \\ &= \{\psi \in \text{Rat}(X);\ (K_X + \ulcorner D \urcorner + \text{div}(\varphi))|_U \geqq 0\} \\ &= \Gamma(U, \mathcal{O}_X(K_X + \ulcorner D \urcorner)). \end{aligned}$$

Thus $\mathcal{O}_X(K_X + \ulcorner D \urcorner) \cong (\tau_* \mathcal{O}_Y(K_Y + \tau^* D))^G$. Since char $k=0$, we obtain the last statement. q.e.d.

One of the easy but important applications of the Covering Lemma is a generalization of Kodaira's vanishing theorem which we shall discuss in the next section.

§ 1-2. Vanishing theorem of Kawamata and Viehweg

First we recall the famous vanishing theorem of Kodaira.

Theorem 1-2-1 (cf. [Kod1]). *Let X be a compact complex manifold with a positive line bundle $\mathcal{L}$. Then $H^i(X, \mathcal{L} \otimes \omega_X) = 0$ for $i > 0$, where ω_X is the canonical bundle of X.*

By the Lefschetz principle, Kodaira's vanishing theorem holds for any nonsingular projective variety and an ample line bundle on it over an arbitrary field of characteristic zero. Once we have Theorem 1-2-1 in hand, the following Corollary 1-2-2 easily follows from the Covering Lemma.

Corollary 1-2-2 (cf. [Ka2]). *Let X be a nonsingular projective variety and $D \in \text{Div}(X) \otimes \boldsymbol{Q}$. Assume the following conditions:*

(i) *D is ample,*

(ii) *$\langle D \rangle$ has support with only simple normal crossings.*

Then $H^i(X, \mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0$ for $i > 0$.

Proof. Take a finite Galois morphism $\tau: Y \to X$ as in the Covering Lemma. Since $\tau^* D$ is ample, the vanishing theorem of Kodaira gives $H^i(Y, \mathcal{O}_Y(K_Y + \tau^* D)) = 0$ for $i > 0$. Therefore by Theorem 1-1-1 (ii) we obtain

$$\begin{aligned} H^i(X, \mathcal{O}_X(K_X + \ulcorner D \urcorner)) &\cong H^i(X, (\tau_* \mathcal{O}_Y(K_Y + \tau^* D))^G) \\ &\cong H^i(X, \tau_* \mathcal{O}_Y(K_Y + \tau^* D))^G = 0 \quad \text{for } i > 0. \end{aligned}$$

q.e.d.

Now we state the main theorem of this section.

Theorem 1-2-3 (Vanishing theorem of Kawamata and Viehweg, cf. [Ka4], [V]). *Let X be a nonsingular variety and let $\pi: X \to S$ be a proper morphism onto a variety S. Assume that a $\boldsymbol{Q}$-divisor $D \in \mathrm{Div}(X) \otimes \boldsymbol{Q}$ satisfies the following conditions*:

(i) *D is π-nef and π-big.*

(ii) *$\langle D \rangle$ has support with only normal crossings (not necessarily simple).*

Then $R^i \pi_* \mathcal{O}_X(K_X + \ulcorner D \urcorner) = 0$ for $i > 0$.

Proof. First we prove the theorem under the conditions:

(i′) D is π-ample.

(ii′) $\langle D \rangle$ has support with only simple normal crossings.

Since the statement is local, we may assume that S is affine. For some positive integer m with $mD \in \mathrm{Div}(X)$, the morphism ψ associated to the surjection $\pi^* \pi_* \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ gives a closed immersion $\psi: X \to P(\pi_* \mathcal{O}_X(mD))$ such that $mD = \psi^* \mathcal{O}(1)$. Then we can take suitable completions $\pi': X' \to S'$ where X' and S' are both projective (taking some desingularization in order to make X' nonsingular) with $\pi'|_X = \pi$ and a π-ample $\boldsymbol{Q}$-divisor D' with $D'|_X = D$. Thus we may further assume that both X and S are projective and that D is ample.

Take an ample Cartier divisor H on S and a positive integer m. Consider the following spectral sequence

$$E_2^{j,i} = H^j(S, R^i \pi_* \mathcal{O}_X(K_X + \ulcorner D \urcorner + m\pi^* H)) \Longrightarrow H^{j+i}(X, \mathcal{O}_X(K_X + \ulcorner D \urcorner + m\pi^* H)).$$

Then by Serre's vanishing theorem $E_2^{j,i} = 0$ for $j > 0$ and for any sufficiently large integer $m \gg 0$, which implies $E_2^{0,i} = E_\infty^i$. Thus

$$H^0(S, R^i \pi_* \mathcal{O}_X(K_X + \ulcorner D \urcorner + m\pi^* H)) = H^i(X, \mathcal{O}_X(K_X + \ulcorner D \urcorner + m\pi^* H)) = 0 \quad \text{for } i > 0,$$

by Corollary 1-2-2. Since H is ample on S and $m \gg 0$,

$$R^i \pi_* \mathcal{O}_X(K_X + \ulcorner D \urcorner + m\pi^* H) \cong R^i \pi_* \mathcal{O}_X(K_X + \ulcorner D \urcorner) \otimes \mathcal{O}_S(mH)$$

is generated by global sections. Therefore we finally obtain the result $R^i \pi_* \mathcal{O}_X(K_X + \ulcorner D \urcorner) = 0$ for $i > 0$.

Now we prove the theorem under the conditions (i) and (ii). By Corollary 0-3-6, there exists a proper birational morphism $f: Y \to X$ from another nonsingular Y projective over S and some family of divisors $\{F_\alpha\}$ on Y such that the union of $\langle f^* D \rangle$ and $\bigcup F_\alpha$ is a divisor with only simple normal crossings and that $f^*D-\sum\delta_\alpha F_\alpha$ is $\pi\circ f$-ample for some $\delta_\alpha\in\boldsymbol{Q}$ with $0<\delta_\alpha\ll 1$. Then by the first part of the proof applied to f, we have

$$0=R^if_*\mathcal{O}_Y(K_Y+\ulcorner f^*D-\sum\delta_\alpha F_\alpha\urcorner)=R^if_*\mathcal{O}_Y(K_Y+\ulcorner f^*D\urcorner)\quad\text{for } i>0.$$

Since the Logarithmic Ramification Formula implies $f_*\mathcal{O}_Y(K_Y+\ulcorner f^*D\urcorner)=\mathcal{O}_X(K_X+\ulcorner D\urcorner)$, we have, by the first part of the proof again,

$$\begin{aligned}0&=R^i(\pi\circ f)_*\mathcal{O}_Y(K_Y+\ulcorner f^*D-\sum\delta_\alpha F_\alpha\urcorner)=R^i\pi_*(f_*\mathcal{O}_Y(K_Y+\ulcorner f^*D\urcorner))\\&=R^i\pi_*\mathcal{O}_X(K_X+\ulcorner D\urcorner)\quad\text{for } i>0.\end{aligned}$$

q.e.d.

We note that there are some works such as [Ra1], [Ra2], [Mi1], [Ka2], etc. preceding to the above theorem. We note also that the following theorem is its easy corollary.

Corollary 1-2-4 (cf. [GR]). *Let f: $Y\to X$ be a proper, generically finite and generically surjective morphism from a nonsingular variety Y to a variety X. Then $R^if_*\omega_Y=0$ for $i>0$, where ω_Y is the canonical bundle of Y.*

Theorem 1-2-5. *Let X be a normal variety of dimension d with $\Delta\in Z_{d-1}(X)\otimes\boldsymbol{Q}$ such that the pair (X,Δ) has only weak log-terminal singularities, let π: $X\to S$ be a proper morphism onto a variety S, and let $D\in(\mathrm{Div}(X)\otimes\boldsymbol{Q})\cap Z_{d-1}(X)$ be a $\boldsymbol{Q}$-Cartier integral Weil divisor. Assume that $D-(K_X+\Delta)$ is π-ample. Then*

$$R^i\pi_*\mathcal{O}_X(D)=0\qquad\textit{for } i>0.$$

Proof. We take a resolution of singularities f: $Y\to X$ satisfying the following properties:

(0) there exists a family of divisors $\{F_j\}$ such that the union of $\mathrm{supp}(\bigcup F_j)$ and $f^{-1}(\Delta)$ has support with only normal crossings,

(1) $f^*(D-(K_X+\Delta))+\delta f^{-1}(\Delta)-\sum\delta_jF_j$ is $\pi\circ f$-ample for some rational numbers $\delta,\delta_j\in\boldsymbol{Q}$ with $0<\delta\ll\min_{\delta_j\neq 0}\delta_j\ll 1$ and $\mathrm{supp}(\sum\delta_jF_j)$ is exceptional for f, and

(2) $K_Y+\delta f^{-1}(\Delta)=f^*(K_X+\Delta)+E$ where E is a $\boldsymbol{Q}$-divisor with $\ulcorner E\urcorner\geqq 0$ and $f_*\ulcorner E\urcorner=0$.

Then $f_*\mathcal{O}_Y(\ulcorner f^*D+E\urcorner)=\mathcal{O}_X(D)$, and we have

$$\begin{aligned}&R^if_*\mathcal{O}_Y(\ulcorner f^*D+E\urcorner)\\&\quad=R^if_*\mathcal{O}_Y(K_Y+\ulcorner f^*(D-(K_X+\Delta))+\delta f^{-1}(\Delta)-\sum\delta_jF_j\urcorner)=0\end{aligned}$$

for $i>0$ by Theorem 1-2-3. Therefore Theorem 1-2-3 again gives

$$\begin{aligned} 0 &= R^i(\pi \circ f)_* \mathcal{O}_Y(K_Y + \ulcorner f^*(D-(K_X+\Delta)) + \delta f^{-1}(\Delta) - \sum \delta_j F_j \urcorner) \\ &= R^i(\pi \circ f)_* \mathcal{O}_Y(\ulcorner f^*D + E \urcorner) \\ &= R^i \pi_* \mathcal{O}_X(D) \qquad \text{for } i>0. \end{aligned}$$

q.e.d.

Remark 1-2-6. The assertion of Theorem 1-2-5 holds in case $D-(K_X+\Delta)$ is π-nef and π-big if the pair (X, Δ) has only log-terminal singularities.

The following theorem is deduced from the vanishing theorem of Tankeev [Ta] and Kollár [Kol3] by the same argument as above, and it turns out to be quite useful, e.g., when we discuss the invariance of plurigenera under deformation.

Theorem 1-2-7 (cf. [Kol3]) and [Ta]). *Let X be a normal variety of dimension d with $\Delta \in Z_{d-1}(X) \otimes \boldsymbol{Q}$ such that the pair (X, Δ) has only log-terminal singularities. Let $\pi: X \to S$ be a proper morphism onto a variety S. If $D \in (\mathrm{Div}(X) \otimes \boldsymbol{Q}) \cap Z_{d-1}(X)$ is a $\boldsymbol{Q}$-Cartier integral Weil divisor such that $D-(K_X+\Delta)$ is π-semi-ample, then $R^i \pi_* \mathcal{O}_X(D)$ is a torsion free $\mathcal{O}_S$-module for any $i \geqq 0$.*

A simplified proof of the vanishing theorem in [Kol3] (in a generalized form) can be found in [Ar], [EV] or [Mw1]. See also [Sa].

§ 1-3. Vanishing theorem of Elkik and Fujita

In this section we shall prove a vanishing theorem due to Elkik [E] and Fujita [Ft4] in a slightly generalized form, using the Grothendieck duality theorem (cf. [Ha1]). The rationality of weak log-terminal singularities follows immediately from this theorem.

Theorem 1-3-1. *Let f: $Y \to X$ be a proper birational morphism from a nonsingular variety Y onto a variety X with $L, \tilde{L} \in \mathrm{Div}(Y)$. Assume that there exist $\boldsymbol{Q}$-divisors $D, \tilde{D} \in \mathrm{Div}(Y) \otimes \boldsymbol{Q}$ and an effective divisor $E \in \mathrm{Div}(Y)$ such that the following conditions are satisfied*:

(i) $\mathrm{supp}\,(D)$ *and* $\mathrm{supp}\,(\tilde{D})$ *are divisors with only simple normal crossings, and* $[D]=[\tilde{D}]=0$,

(ii) *both* $-L-D$ *and* $-\tilde{L}-\tilde{D}$ *are f-nef,*

(iii) $K_Y \sim L+\tilde{L}+E$, *and*

(iv) *E is exceptional for f, i.e.,* $\mathrm{codim}_Y f(\mathrm{supp}\, E) \geqq 2$.

Then $R^q f_* \mathcal{O}_Y(L)=0$ for $q>0$.

Proof. Since the assertion is local, we may assume that X is affine. Moreover, by taking generic hyperplane sections of X, we may also assume that $R^q := R^q f_* \mathcal{O}_Y(L)$ and $\tilde{R}^q := R^q f_* \mathcal{O}_Y(\tilde{L})$ are supported at a point $x \in X$, if they ever have a nonempty support at all. Noting that $(K_Y-\tilde{L})-(K_Y+\tilde{D})$ is f-nef (and also f-big because f is birational), we have by Theorem 1-2-3

$$\begin{aligned}(1)\qquad 0&=R^q f_* \mathcal{O}_Y(K_Y+\ulcorner(K_Y-\tilde{L})-(K_Y+\tilde{D})\urcorner)\\ &=R^q f_* \mathcal{O}_Y(K_Y-\tilde{L}) \quad \text{by the condition } [\tilde{D}]=0\\ &=R^q f_* \mathcal{O}_Y(L+E) \qquad \text{for } q>0.\end{aligned}$$

Lemma 1-3-2 (cf. [Ft4, Lemma 2.2]). *Let f: $Y\to X$ be a proper birational morphism from a nonsingular variety Y onto a variety X, let $L\in$* Div(Y), *let $D\in$* Div(Y)$\otimes\boldsymbol{Q}$, *and let $E\in$* Div(Y). *Assume that the support of D is a divisor with only simple normal crossings, $[D]=0$, $-L-D$ is f-nef, and that E is effective and exceptional for f. Then*

$$(2)\qquad f_*\mathcal{O}_E(L+E)=0.$$

Proof. For any reduced irreducible component E_j of E, we have the exact sequence

$$0\longrightarrow f_*\mathcal{O}_{E'}(L+E')\longrightarrow f_*\mathcal{O}_E(L+E)\longrightarrow f_*\mathcal{O}_{E_j}(L+E),$$

where $E'=E-E_j$. Thus, by induction on the number of irreducible components of E, we have only to prove that there exists a reduced irreducible component E_0 of E such that $f_*\mathcal{O}_{E_0}(L+E)=0$. We shall prove this by induction on $d=\dim X$.

First we deal with the case $d=2$. Write $E-D=A-B$, where A and B are effective $\boldsymbol{Q}$-divisors without common components. Since $[D]=0$, we have $A\neq 0$. Since $\operatorname{supp}(A)\subset\operatorname{supp}(E)$, A is exceptional for f. Hence by the Hodge index theorem, we have $(A.E_0)<0$ for some component E_0 of A. Then since $-L-D$ is f-nef, we have

$$((E+L).E_0)\leqq((E-D).E_0)\leqq(A.E_0)<0,$$

which implies $f_*\mathcal{O}_{E_0}(L+E)=0$.

Now suppose $d\geqq 3$. We shall derive a contradiction assuming that $f_*\mathcal{O}_{E_j}(L+E)\neq 0$ for any irreducible component E_j of E. Take a nonzero element s_j of $H^0(E_j, \mathcal{O}_{E_j}(L+E))$. In case $\dim f(E_j)=0$, we take a generic hyperplane section Y' of Y such that $E_j\cap Y'\not\subset\operatorname{div}(s_j)$. In case $\dim f(E)>0$, we take a generic hyperplane section X' of X, and set $Y'=f^{-1}(X')$. Then in either case, we have $f_*\mathcal{O}_{E_j\cap Y'}(L+E)\neq 0$, which is a contradiction by induction hypothesis. q.e.d.

Applying (1) and (2) to the long exact sequence derived from the following exact sequence

$$0 \longrightarrow \mathcal{O}_Y(L) \longrightarrow \mathcal{O}_Y(L+E) \longrightarrow \mathcal{O}_E(L+E) \longrightarrow 0,$$

we obtain the isomorphisms below:

(3) $$f_*\mathcal{O}_Y(L) \cong f_*\mathcal{O}_Y(L+E)$$

(4) $$R^q f_*\mathcal{O}_E(L+E) \cong R^{q+1} f_*\mathcal{O}_Y(L) \quad \text{for } q \geqq 0.$$

Lemma 1-3-3.

$$R^i \cong \begin{cases} \mathrm{Hom}(\tilde{R}^{d-i-1}, I) & \text{for } 0 < i \leqq d-2 \\ 0 & \text{for } i = d-1, d, \end{cases}$$

where I denotes an injective hull of k as an $\mathcal{O}_{X,x}$*-module.*

Proof. By the Grothendieck duality theorem ([Ha1, Chap. VII, Theorem 3.3]), $\mathcal{D}_X$ and $\mathcal{D}_Y$ being the dualizing functors, we have

$$\begin{aligned} \boldsymbol{R}f_*\mathcal{O}_Y(L) &\cong \boldsymbol{R}f_*\mathcal{D}_Y(\mathcal{O}_Y(\tilde{L}+E))[-d] \\ &\cong \mathcal{D}_X(\boldsymbol{R}f_*\mathcal{O}_Y(\tilde{L}+E))[-d] \\ &\cong \mathcal{D}_X(\tilde{\mathcal{F}})[-d] \quad \text{(by (1) with } L \text{ replaced by } \tilde{L}), \end{aligned}$$

where $\tilde{\mathcal{F}}$ denotes $f_*\mathcal{O}_Y(\tilde{L}+E)$. Thus,

(5) $$R^i = \mathrm{Ext}^{i-d}(\tilde{\mathcal{F}}, \omega_X^{\boldsymbol{\cdot}}).$$

On the other hand, since

$$\begin{aligned} \boldsymbol{R}f_*\mathcal{O}_Y(L+E) &\cong \boldsymbol{R}f_*\mathcal{D}_Y(\mathcal{O}_Y(\tilde{L}))[-d] \\ &\cong \mathcal{D}_X(\boldsymbol{R}f_*\mathcal{O}_Y(\tilde{L}))[-d], \end{aligned}$$

we have by (1)

$$H^i(\boldsymbol{R}\,\mathrm{Hom}(\boldsymbol{R}f_*\mathcal{O}_Y(\tilde{L}), \omega_X^{\boldsymbol{\cdot}})) = 0 \quad \text{for } i \neq -d.$$

Next we consider the following spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p(R^{-q}f_*\mathcal{O}_Y(\tilde{L}), \omega_X^{\boldsymbol{\cdot}}) \Longrightarrow H^{p+q}(\boldsymbol{R}\,\mathrm{Hom}(\boldsymbol{R}f_*\mathcal{O}_Y(\tilde{L}), \omega_X^{\boldsymbol{\cdot}})).$$

Since $R^{-q}f_*\mathcal{O}_Y(L)$ is a skyscraper sheaf for $q<0$, we have $E_2^{p,q}=0$ for $q<0$ and $p \neq 0$. It is clearly zero for $q>0$. Thus we obtain

(6) $$E_2^{-d+k,0} \cong E_2^{0,-d+k+1} \quad \text{for } k>0 \text{ and } E_2^{-1,0} = E_2^{0,0} = 0.$$

By (5), (6) and by the local duality ([Ha1, Chap. V, Theorem 6.2]), we have

$$R^i = \mathrm{Ext}^{-d+i}(\tilde{\mathscr{F}}, \omega_X^{\cdot}) = \mathrm{Ext}^0(\tilde{R}^{d-i-1}, \omega_X^{\cdot})$$
$$= \begin{cases} \mathrm{Hom}(\tilde{R}^{d-i-1}, I) & \text{for } 0 < i \leqq d-2 \\ 0 & \text{for } i = d-1, d. \end{cases}$$

This proves Lemma 1-3-3.

Lemma 1-3-4.

$$\tilde{R}^{d-i} = \mathrm{Hom}(R^{i+1}, I).$$

Proof. Since $\omega_E^{\cdot} = \mathcal{O}_E(K_E)[d-1] = \mathcal{O}_E((K_Y+E)|_E)[d-1]$, we have

$$\boldsymbol{R}\mathrm{Hom}(\boldsymbol{R}f_*\mathcal{O}_E(L+E), \omega_X^{\cdot}) = \mathscr{D}_X(\boldsymbol{R}f_*\mathcal{O}_E(L+E))$$
$$= \boldsymbol{R}f_*\mathscr{D}_E(\mathcal{O}_E(L+E)) = \boldsymbol{R}f_*\mathcal{O}_E(\tilde{L}+E)[d-1].$$

By taking the $(-i)$-th cohomology of both hand sides, we obtain

$$\mathrm{Hom}(R^{i+1}, I) = R^{-i+d-1}f_*\mathcal{O}_E(\tilde{L}+E)$$
$$= \tilde{R}^{d-i} \quad \text{(by (4) with } L \text{ replaced by } \tilde{L}\text{).} \qquad \text{q.e.d.}$$

Now suppose that $R^i = 0$. Then $0 = \mathrm{Hom}(R^i, I) = \tilde{R}^{d-i+1}$ by Lemma 1-3-4, and $0 = \mathrm{Hom}(\tilde{R}^{d-i+1}, I) = R^{i-2}$ by Lemma 1-3-3. Therefore we know that $R^i = 0$ implies $R^{i-2} = 0$. Since $R^d = R^{d-1} = 0$, this leads us to the result required in Theorem 1-3-1. q.e.d.

Remark 1-3-5. It follows easily that Theorem 1-3-1 holds under the following conditions (i′) and (ii′) (instead of (i) and (ii)):

(i′) $\mathrm{supp}(D)$ and $\mathrm{supp}(\tilde{D})$ are divisors with only normal crossings, $[D] = 0$, $\tilde{D}$ is effective and $\ulcorner \tilde{D} \urcorner$ is a reduced divisor,

(ii′) $-L-D$ is f-nef and $-\tilde{L}-\tilde{D}$ is f-ample.

Theorem 1-3-6. *All weak log-terminal singularities are rational.*

Proof. Let X be a normal variety of dimension d with $\Delta \in Z_{d-1}(X) \otimes \boldsymbol{Q}$ such that the pair (X, Δ) has only weak log-terminal singularities. Then we can take a resolution of singularities $f: Y \to X$ satisfying the following conditions:

(i) there exists a divisor $\sum F_j$ with only normal crossings whose support is the union of the exceptional locus for f and $f^{-1}(\mathrm{supp}\,\Delta)$,

(ii) $K_Y = f^*(K_X + \Delta) + \sum a_j F_j$ for $a_j \in \boldsymbol{Q}$, with the condition that $a_j > -1$ whenever F_j is exceptional for f, and

(iii) there exists an f-ample divisor $A = \sum b_j F_j$ for $b_j \in \boldsymbol{Q}$ where $b_j = 0$ if F_j is not exceptional for f.

Let

$$J' := \{j;\ F_j \text{ is exceptional for } f\} \quad \text{and}$$
$$J'' := \{j;\ F_j \text{ is not exceptional for } f\}.$$

Then

$$E' := \sum_{j \in J'} a_j F_j, \qquad E := \ulcorner E' \urcorner,$$
$$\tilde{D} := \sum_{j \in J''} (-a_j F_j) + E - E' - \delta A$$

for some sufficiently small positive rational number δ,

$$\tilde{L} := K_Y - E,$$
$$L := 0 \quad \text{and} \quad D := 0$$

satisfy the conditions (i′) and (ii′) of Remark 1-3-5 and (iii) and (iv) of Theorem 1-3-1. Therefore we have

$$0 = R^i f_* \mathcal{O}_Y(L) = R^i f_* \mathcal{O}_Y \quad \text{for } i > 0. \qquad \text{q.e.d.}$$

Chapter 2. Non-Vanishing Theorem

An important application of the Vanishing Theorem of Kawamata and Viehweg is the following Non-Vanishing Theorem, which Shokurov [S1] originally proved extending the technique developed by the first author to prove the finiteness of generators of a canonical ring in [Ka 5]. The Non-Vanishing Theorem, with the Vanishing Theorem itself, leads us to the Base Point Free Theorem in Chapter 3.

§ 2-1. The proof of the Non-Vanishing Theorem

Theorem 2-1-1 (Non-Vanishing Theorem, [S1]). *Let X be a non-singular complete variety with $D \in \mathrm{Div}(X)$ and $A \in \mathrm{Div}(X) \otimes \boldsymbol{Q}$ satisfying the following conditions:*

(i) *D is nef,*

(ii) *$pD + A - K_X$ is nef and big for some $p \in \boldsymbol{N}$,*

(iii) *$\ulcorner A \urcorner \geqq 0$ and $\langle A \rangle$ has support with only normal crossings.*

Then $H^0(X, \mathcal{O}_X(mD + \ulcorner A \urcorner)) \neq 0$ for any sufficiently large $m \in \boldsymbol{N}$.

Proof. By the same argument as in the proof of Theorem 1-2-3, we may assume that X is projective and that $pD + A - K_X$ is ample. We shall prove the theorem by induction on $d = \dim X$.

Case: $D \approx 0$. In this case for any $m \in \boldsymbol{Z}$,

$$h^0(X, \mathcal{O}_X(mD+\ulcorner A \urcorner))=\chi(\mathcal{O}_X(mD+\ulcorner A \urcorner)) \quad \text{by Theorem 1-2-3}$$
$$=\chi(\mathcal{O}_X(\ulcorner A \urcorner))=h^0(X, \mathcal{O}_X(\ulcorner A \urcorner))\neq 0 \quad \text{by the condition (iii).}$$

Thus we are done.

Case: $D \not\approx 0$. Our proof of this case is divided into 3 steps.

Step 1. Take an integer $a \in N$ with $aA \in \mathrm{Div}(X)$. Then for some fixed $q \in N$, we have

$$h^0(X, \mathcal{O}_X(ak(qD+A-K_X)))>a^d(d+1)^d k^d/d! \quad \text{for } k \gg 0.$$

Indeed, since $pD+A-K_X$ is ample and since $D \not\approx 0$, there exists a sufficiently large $q \in N$ such that $(qD+A-K_X)^d>(d+1)^d$, and we fix such q once and for all. Then for any sufficiently large $k \in N$,

$$\begin{aligned} &h^0(X, \mathcal{O}_X(ak(qD+A-K_X)))\\ &=\chi(\mathcal{O}_X(ak(qD+A-K_X)) \quad \text{by Serre's vanishing theorem}\\ &=a^d(qD+A-K_X)^d k^d/d!+(\text{lower terms in } k)\\ &>a^d(d+1)^d k^d/d!. \end{aligned}$$

Step 2. Fix a closed point x of X with $x \notin \mathrm{supp}\, A$. Then there exists a member $M \in |ak(qD+A-K_X)|$ such that the multiplicity of M at x is greater than or equal to $ak(d+1)$. Indeed, considering the power series expansion of the local defining equation for M at x, we have the following number of conditions for M to have multiplicity $\geqq ak(d+1)$ at x:

$$\begin{aligned} &\sharp\,\{\text{monomials of degree } <ak(d+1) \text{ in } d \text{ variables}\}\\ &=\binom{ak(d+1)-1+d}{d}\\ &=a^d(d+1)^d k^d/d!+(\text{lower terms in } k), \end{aligned}$$

which is less than $h^0(X, \mathcal{O}_X(ak(qD+A-K_X)))$ by Step 1. This guarantees the existence of such a member $M \in |ak(qD+A-K_X)|$.

Step 3. There exists a composite $f=f_1 \circ f_2$: $Y \to X$ of the blowing-up at x denoted by f_1 and a proper birational morphism f_2 from a nonsingular variety Y with a family of divisors $\{F_j\}$ with only simple normal crossings, containing the strict transform of the exceptional divisor of f_1 as F_1, which satisfies the following conditions:

(1) $K_Y=f^*K_X+\sum a_j F_j$ for some nonnegative integers a_j,

(2) $f^*(qD+A-K_X)-(d+1)\sum \delta_j F_j$ is ample for some $\delta_j \in Q$ with $0<\delta_j \ll 1$,

(3) $f^*A+\sum a_j F_j=\sum b_j F_j$ for $b_j \in Q$ with $b_j>-1$, the inequality holding because of the condition $\ulcorner A \urcorner \geqq 0$ and of the Logarithmic Ramifica-

tion Formula, and

(4) $f^*M=\sum r_jF_j$.

Then by Step 2, we have $r_1\geqq ak(d+1)$ and $b_1=a_1=d-1$, since $x\notin \operatorname{supp} A$. Therefore, defining

$$c:=\min_j(b_j+1-\delta_j)/r_j,$$

we have

$$0<c\leqq(b_1+1-\delta_1)/r_1<d/ak(d+1).$$

By changing δ_j's slightly if necessary, we may assume that the minimum c is attained only at a unique index $j=0$, which we are allowed to do since ampleness is an open condition as is clear from Kleiman's criterion for ampleness.

Set $A':=\sum_{j\neq 0}(-cr_j+b_j-\delta_j)F_j$ and $B:=F_0$. Then

$$\begin{aligned} N:&=mf^*D+A'-B-K_Y \\ &\sim_{Q}(m-q)f^*D+(1-cak)f^*(qD+A-K_X)-\sum \delta_jF_j \end{aligned}$$

is ample for $m\geqq q$, since $0<cak<d/(d+1)$ and since we have the condition (2). Thus by the Vanishing Theorem of Kawamata and Viehweg

$$0=H^1(Y, \mathcal{O}_Y(\ulcorner N\urcorner+K_Y))=H^1(Y, \mathcal{O}_Y(mf^*D+\ulcorner A'\urcorner-B)),$$

which implies that the homomorphism

$$H^0(Y, \mathcal{O}_Y(mf^*D+\ulcorner A'\urcorner))\longrightarrow H^0(B, \mathcal{O}_B(mf^*D+\ulcorner A'\urcorner|_B))$$

is surjective. By induction hypothesis,

$$H^0(B, \mathcal{O}_B(mf^*D+\ulcorner A'\urcorner|_B))\neq 0 \quad \text{for } m\gg 0.$$

Therefore

$$H^0(Y, \mathcal{O}_Y(mf^*D+\ulcorner A'\urcorner))\neq 0.$$

Note that

$$f^*\ulcorner A\urcorner+\sum a_jF_j-\ulcorner A'\urcorner\geqq 0.$$

Therefore

$$\begin{aligned} H^0(X, \mathcal{O}_X(mD+\ulcorner A\urcorner))&\cong H^0(Y, \mathcal{O}_Y(mf^*D+f^*\ulcorner A\urcorner+\sum a_jF_j)) \\ &\supset H^0(Y, \mathcal{O}_Y(mf^*D+\ulcorner A'\urcorner))\neq 0. \end{aligned}$$

q.e.d.

Chapter 3. Base Point Free Theorem

We can find in [Ka5] and [Be2] the prototype of the Base Point Free Theorem, which was later generalized in [Ka6], [R4], [An], [S1] and [K7]. In the first four papers, the Base Point Free Theorem was proved only in dimension $\leqq 3$, in the case where the Non-Vanishing Theorem follows immediately from the Riemann-Roch Theorem. Ando [An] extended this to the case of dimension 4 also by using the Riemann-Roch formula. Then Shokurov [S1] obtained the Base Point Free Theorem by proving the Non-Vanishing Theorem in dimension >3. The theorem formulated below is in the relative form and thus more general than those predecessors. The first author was most influenced by [Bo] and [A1] among preceding works.

§ 3-1. The proof of the Base Point Free Theorem

Theorem 3-1-1 (Base Point Free Theorem, cf. [Ka5], [Be2], [Ka6], [R4], [An], [S1], [Ka7]). *Let X be a normal variety of dimension d with $\Delta \in Z_{d-1}(X)\otimes \boldsymbol{Q}$ such that the pair (X, Δ) has only weak log-terminal singularities, and let $\pi: X \to S$ be a projective morphism onto a variety S. If $H \in \mathrm{Div}(X)$ is π-nef and $aH-(K_X+\Delta)$ is π-ample for some $a \in \boldsymbol{N}$, then mH is π-generated for $m \gg 0$, i.e., there exists a positive integer $m_0 \in \boldsymbol{N}$ such that for any $m \geqq m_0$ the natural homomorphism $\pi^*\pi_*\mathcal{O}_X(mH) \to \mathcal{O}_X(mH)$ is surjective.*

Proof. Since (X, Δ) has only weak log-terminal singularities, there exists a proper birational morphism $g: X' \to X$ from a nonsingular variety X' such that

$$g^*(aH-(K_X+\Delta))+\varepsilon g^{-1}(\Delta)-\sum \varepsilon_j E_j$$

is $\pi \circ g$-ample for some $\varepsilon, \varepsilon_j \in \boldsymbol{Q}$ with $0<\varepsilon \ll \min_{\varepsilon_j \neq 0} \varepsilon_j \ll 1$, where $\{E_j\}$ is a family of divisors with only normal crossings and $\mathrm{supp}(\sum \varepsilon_j E_j)$ is exceptional for g, and that

$$K_{X'}+\varepsilon g^{-1}(\Delta)=g^*(K_X+\Delta)+\sum c_j E_j \quad \text{for } c_j \in \boldsymbol{Q} \text{ with } c_j>-1.$$

Let $C:=\sum (c_j-\varepsilon_j)E_j$. Then η being the generic point of S, it follows that X'_η, g^*H_η and C_η satisfy the conditions (i) (ii) and (iii) of the Non-Vanishing Theorem. Therefore

$$0 \neq H^0(X'_\eta, \mathcal{O}_{X'_\eta}(mg^*H_\eta+\ulcorner C_\eta \urcorner))=((\pi \circ g)_*\mathcal{O}_{X'}(mg^*H+\ulcorner C \urcorner))_\eta$$

for $m \gg 0$. In particular,

$$0 \neq (\pi \circ g)_*\mathcal{O}_{X'}(mg^*H+\ulcorner C \urcorner) \cong \pi_*\mathcal{O}_X(mH),$$

since $\ulcorner C \urcorner$ is exceptional for g.

Now fix a prime number p. We claim that $p^n H$ is π-generated for $n \gg 0$. Take a sufficiently large $n_0 \in \boldsymbol{N}$ so that $\pi_* \mathcal{O}_X(p^{n_0}H) \neq 0$, as is guaranteed in the previous argument. If the natural homomorphism $\pi^* \pi_* \mathcal{O}_X(p^{n_0}H) \to \mathcal{O}_X(p^{n_0}H)$ is surjective, we have nothing more to prove. Thus we assume the contrary.

First, by the definition of weak log-terminal singularities, we can take a desingularization $f_1 \colon Y_1 \to X$ with a family of divisors $\{G_i\}$ having only normal crossings which satisfies the following conditions:

(1_1) $f_1^*(aH - (K_X + \Delta)) + \delta f_1^{-1}(\Delta) - \sum \delta_{1i} G_i$ is $\pi \circ f_1$-ample for some $\delta, \delta_{1i} \in \boldsymbol{Q}$ with $0 < \delta \ll \min_{\delta_{1i} \neq 0} \delta_{1i} \ll 1$ and $\operatorname{supp}(\sum \delta_{1i} G_i)$ is exceptional for f_1, and

(2_1) $K_{Y_1} + \delta f_1^{-1}(\Delta) = f_1^*(K_X + \Delta) + \sum b_i G_i$ for $b_i \in \boldsymbol{Q}$ with $b_i > -1$.

Secondly, by taking a succession of blowing-ups with nonsingular centers, we can find a proper birational morphism $f_2 \colon Y \to Y_1$ with a family of divisors $\{F_j\}$ with only *simple* normal crossings which satisfies the following conditions:

(1)
$$\begin{aligned} &f_2^*(f_1^*(aH - (K_X + \Delta)) + \delta f_1^{-1}(\Delta) - \sum \delta_{1i} G_i) - \delta' A_2 \\ &= f^*(aH - (K_X + \Delta)) + \delta f_2^* f_1^{-1}(\Delta) - \sum \delta_j F_j \end{aligned}$$

is $\pi \circ f$-ample for an f_2-exceptional divisor $A_2 \in \operatorname{Div}(Y) \otimes \boldsymbol{Q}$ with $0 < \delta' \ll \delta$, where $f := f_1 \circ f_2$,

(2) $K_Y + \delta f_2^* f_1^{-1}(\Delta) = f^*(K_X + \Delta) + \sum a_j F_j$ for $a_j \in \boldsymbol{Q}$ with $a_j > -1$, and

(3) $(\pi \circ f)^* (\pi \circ f)_* \mathcal{O}_Y(f^* p^{n_0} H) \twoheadrightarrow \mathcal{O}_Y(f^* p^{n_0} H - \sum r_j F_j) \subset \mathcal{O}_Y(f^* p^{n_0} H)$

for some nonnegative integers r_j, where $\sum r_j F_j$ is the $\pi \circ f$-fixed part of $f^* p^{n_0} H$.

Since $0 < \delta' \ll \delta \ll \min_{\delta_{1i} \neq 0} \delta_{1i} \ll 1$, the Logarithmic Ramification Formula implies that $a_j + 1 - \delta_j > 0$ for all j. Set

$$c := \min_j (a_j + 1 - \delta_j)/r_j.$$

By changing δ_j's slightly if necessary, we may assume that the minimum c is attained only at a unique index $j = 0$. Setting

$$A := \sum_{j \neq 0} (-c r_j + a_j - \delta_j) F_j \quad \text{and} \quad B := F_0,$$

the $\boldsymbol{Q}$-divisor

$$\begin{aligned} N &:= p^{n'} f^* H + A - B - K_Y \\ &\sim_{\boldsymbol{Q}} c(f^* p^{n_0} H - \sum r_j F_j) + f^*((p^{n'} - c p^{n_0}) H - (K_X + \Delta)) \\ &\quad + \delta f_2^* f_1^{-1}(\Delta) - \sum \delta_j F_j \end{aligned}$$

is $\pi \circ f$-ample for $n' \in \boldsymbol{N}$ with $p^{n'} \geqq cp^{n_0}+a$. Since

$$R^1(\pi \circ f)_* \mathcal{O}_Y(p^{n'}f^*H+\ulcorner A \urcorner - B) = R^1(\pi \circ f)_* \mathcal{O}_Y(\ulcorner N \urcorner + K_Y) = 0$$

by Theorem 1-2-3, the homomorphism

$$(\pi \circ f)_* \mathcal{O}_Y(p^{n'}f^*H+\ulcorner A \urcorner) \longrightarrow (\pi \circ f)_* \mathcal{O}_B(p^{n'}f^*H+\ulcorner A \urcorner|_B)$$

is surjective. By the Non-Vanishing Theorem again,

$$(\pi \circ f)_* \mathcal{O}_B(p^{n'}f^*H+\ulcorner A \urcorner|_B) \neq 0 \quad \text{for } n' \gg 0.$$

Noting that $(\pi \circ f)_* \mathcal{O}_Y(p^{n'}f^*H+\ulcorner A \urcorner) \cong \pi_* \mathcal{O}_X(p^{n'}H)$, we come to the conclusion that

$$f(B) \not\subset \operatorname{supp} \operatorname{Coker}(\pi^* \pi_* \mathcal{O}_X(p^{n'}H) \longrightarrow \mathcal{O}_X(p^{n'}H)).$$

Therefore

$$\operatorname{supp} \operatorname{Coker}(\pi^* \pi_* \mathcal{O}_X(p^{n'}H) \longrightarrow \mathcal{O}_X(p^{n'}H))$$
$$\subsetneqq \operatorname{supp} \operatorname{Coker}(\pi^* \pi_* \mathcal{O}_X(p^{n_0}H) \longrightarrow \mathcal{O}_X(p^{n_0}H)).$$

By noetherian induction,

$$\operatorname{supp} \operatorname{Coker}(\pi^* \pi_* \mathcal{O}_X(p^n H) \longrightarrow \mathcal{O}_X(p^n H)) = \phi \quad \text{for } n \gg 0,$$

which is the claim we wanted.

q being another prime number, $q^l H$ is also π-generated for $l \gg 0$. Take positive integers l_1, $n_1 \in \boldsymbol{N}$ such that $p^{n_1}H$ and $q^{l_1}H$ are π-generated. Then for any sufficiently large $m \in \boldsymbol{N}$, there exist nonnegative integers a and b such that $m = ap^{n_1} + bq^{l_1}$, which implies that mH is π-generated.

q.e.d.

Remark 3-1-2. (1) The above proof shows that the theorem holds also under the following assumptions: (X, Δ) has only log-terminal singularities, $\pi: X \to S$ is a proper surjective morphism, H is π-nef, and $aH - (K_X + \Delta)$ is π-nef and π-big. Moreover in this case, X need not be an algebraic variety over S, but we have only to assume $\pi: X \to S$ to be a Moishezon morphism from a complex analytic variety onto an algebraic variety, i.e., π is bimeromorphically equivalent to a projective morphism.

(2) If $[\Delta] \neq 0$, there exists a counterexample due to Zariski (cf. [Z] or [Mk1]) in which the assertion of the theorem above fails to hold even though $\pi: X \to S$ is projective and $aH - (K_X + \Delta)$ is π-nef and π-big (but not π-ample): Take a nonsingular elliptic curve C and a line L on $\boldsymbol{P}^2_{\boldsymbol{C}}$. Then fixing a positive integer h, choose $n = ((hL + C).C)$ distinct points p_1, $p_2, \cdots, p_n$ such that for any $m \in \boldsymbol{N}$,

$$m(p_1+p_2+\cdots+p_n) \notin |m(hL+C)|_C|.$$

Let S' be a surface obtained by blowing up $p_1, p_2, \cdots, p_n$. Let C' and Γ' be the strict transforms of C and a general member $\Gamma \in |hL|$, respectively. Now set $X:=S'$, $\Delta:=C'$ and $S=\operatorname{Spec} \boldsymbol{C}$. Then obviously the pair (X, Δ) has only weak log-terminal singularities. It is easy to see that $H:=\Gamma'+C'$ is nef and that $aH-(K_X+\Delta)$ is nef and big for $a>1$. Since the linear system

$$mH|_{C'} \sim m\{(hL+C)|_C-(p_1+p_2+\cdots+p_n)\}$$

never becomes effective for any $m \in \boldsymbol{N}$ by the choice of $p_1, p_2, \cdots, p_n$ (where we identify C and C' in the linear equivalence above), C' is always contained in the fixed component of $|mH|=|m(\Gamma'+C')|$.

In the following two sections we prove some of the direct consequences of Theorem 3-1-1, which are easy but the most important in our theory.

§ 3-2. Contractions of extremal faces

The following theorem is just another form of Theorem 3-1-1.

Theorem 3-2-1 (Contraction Theorem). *Let X be a normal variety of dimension d with $\Delta \in Z_{d-1}(X)\otimes \boldsymbol{Q}$ such that the pair (X, Δ) has only weak log-terminal singularities, and let $\pi\colon X\to S$ be a projective morphism onto a variety S. Let $H \in \operatorname{Div}(X)$ be a π-nef Cartier divisor such that $F:=H^{\perp}\cap \overline{NE}(X/S)-\{0\}$ is entirely contained in the set*

$$\{z \in N_1(X/S);\ ((K_X+\Delta).z)<0\},$$

where $H^{\perp}:=\{z \in N_1(X/S);\ (H.z)=0\}$. Then there exists a projective morphism $\varphi\colon X\to Y$ onto a normal variety Y projective over S which makes the diagram below commutative,

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & Y \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\sigma} & \\ & S & \end{array}$$

and is characterized by the following properties:

(i) *For any irreducible curve C on X with $\pi(C)$ being a point, $\varphi(C)$ is a point if and only if $(H.C)=0$, i.e., if and only if* $\operatorname{cl}(C) \in F$,

(ii) $\operatorname{Rat}(X)/\operatorname{Rat}(Y)$ *is an algebraically closed extension, which is equivalent to the condition that φ has only connected fibers,*

(iii) *$H=\varphi^*A$ for some σ-ample Cartier divisor $A \in$* Div(Y).

Proof (cf. [Mo2]). By Kleiman's criterion for ampleness, it is clear that there exists $a \in N$ such that $aH-(K_X+\Delta)$ is π-ample. Thus we can apply Theorem 3-1-1 to obtain the morphism $\pi:=\Phi_{|mH|}$ for $m\gg 0$ which satisfies conditions (i) and (ii). By Zariski's Main Theorem, φ is characterized by the properties (i) and (ii). In order to see the property (iii), observe that for $m\gg 0$, the morphisms

$$\Phi_{|mH|}\colon X\to Y\subset P_m:=P(\pi_*\mathcal{O}_X(mH)) \quad \text{and}$$
$$\Phi_{|(m+1)H|}\colon X\to Y\subset P_{m+1}:=P(\pi_*\mathcal{O}_X((m+1)H))$$

turn out to give the same contraction morphism φ. Therefore,

$$\mathcal{O}_X(mH)=(\Phi_{|mH|})^*\mathcal{O}_{P_m}(1) \quad \text{and} \quad \mathcal{O}_X((m+1)H)=(\Phi_{|(m+1)H|})^*\mathcal{O}_{P_{m+1}}(1).$$

Thus $H=\varphi^*A$ for some $A \in$ Div(Y), and $\mathcal{O}_Y(mA)\cong\mathcal{O}_{P_m}(1)\otimes\mathcal{O}_Y$ is σ-ample.
q.e.d.

Remark 3-2-2. It is obvious that the Contraction Theorem directly implies the Base Point Free Theorem.

Definition 3-2-3 ([Mo2], [R4]). Since the morphism φ is characterized by the properties which depend only on the face F of $\overline{NE}(X/S)$ and do not depend on the choice of $H \in$ Div(X), we may call φ the *contraction* of F, while any π-nef Cartier divisor $H \in$ Div(X) with $H^{\perp}\cap\overline{NE}(X/S)-\{0\}=F$ is called a *supporting function* of F. F itself is called an *extremal face of $\overline{NE}(X/S)$ for (X, Δ)* (or *for $K_X+\Delta$*). If $\dim_{\mathbf{R}} F=1$, an extremal face is called an *extremal ray*.

Lemma 3-2-4. *If $N_1(X/Y)$ is regarded as a subspace of $N_1(X/S)$ via the natural inclusion, then*

$$F=\overline{NE}(X/Y)-\{0\} \quad \textit{in } N_1(X/S).$$

Namely, F is spanned by curves on X which are mapped to points on Y. In particular, an extremal ray contains an effective curve in its class.

Proof. Let us suppose that $F\not\subset\overline{NE}(X/Y)-\{0\}$. (It is clear that $F\supset\overline{NE}(X/Y)-\{0\}$.) Then there is a separating function $J \in$ Div(X) such that $J>0$ on $\overline{NE}(X/Y)-\{0\}$ and $(J.z)<0$ for some $z \in F$. But since J is φ-ample by Kleiman's criterion for ampleness, $aH+J$ is π-ample for some $a \in N$. Thus $(J.z)=((aH+J).z)>0$, which is a contradiction. q.e.d.

We conclude this section by pointing out the following facts which are easily deduced from the Base Point Free Theorem.

Lemma 3-2-5 (cf. [Mo2], [R4], [Ka7]). *Let X be a normal variety of dimension d with $\Delta \in Z_{d-1}(X) \otimes \boldsymbol{Q}$ such that the pair (X, Δ) has only weak log-terminal singularities, and let $\pi: X \to S$ be a projective morphism onto a variety S. Let $f: X \to Z$ and $\tau: Z \to S$ be projective surjective morphisms with $\pi = \tau \circ f$, where Z is a normal variety. Assume that $-(K_X + \Delta)$ is f-ample and that f has only connected fibers (i.e.,* $\mathrm{Rat}(X)/\mathrm{Rat}(Z)$ *is an algebraically closed extension). Then*

(1) *there exists an extremal face F of $\overline{NE}(X/S)$ for (X, Δ) such that f is nothing but the contraction morphism of F,*

(2) *the image of f^*:* $\mathrm{Pic}(Z) \to \mathrm{Pic}(X)$ *coincides with* $\{D \in \mathrm{Pic}(X)$; $(D.z) = 0$ *for all $z \in F\}$, and*

(3) *the following mutually dual sequences*

$$0 \longrightarrow N_1(X/Z) \longrightarrow N_1(X/S) \longrightarrow N_1(Z/S) \longrightarrow 0$$
$$0 \longleftarrow N^1(X/Z) \longleftarrow N^1(X/S) \longleftarrow N^1(Z/S) \longleftarrow 0$$

are exact.

Proof. (1) Take $H \in \mathrm{Div}(Z)$ which is τ-ample. Then we have only to put $F = (f^*H)^{\perp} \cap \overline{NE}(X/S) - \{0\}$.

(2) Let D be a line bundle on X such that $(D.z) = 0$ for all $z \in F$. We have only to show that $D \in f^* \mathrm{Pic}(Z)$. (It is obvious that the former set in (2) is contained in the latter.) The Base Point Free Theorem tells us that $\mathcal{O}_X(mD)$ is f-generated for $m \gg 0$. The contraction morphism associated to the surjection $f^* f_* \mathcal{O}_X(mD) \twoheadrightarrow \mathcal{O}_X(mD)$ is nothing but the morphism f itself. Thus mD and $(m+1)D \in f^* \mathrm{Pic}(Z)$, hence $D \in f^* \mathrm{Pic}(Z)$.

(3) The exactness of the second sequence is a direct consequence of (2). Since the first sequence is dual to the second, the first is also exact.

q.e.d.

Remark 3-2-6. Let $f: X \to Z$ and $\tau: Z \to S$ be projective surjective morphisms between algebraic varieties. Then the sequence

$$0 \longrightarrow N_1(X/Z) \longrightarrow N_1(X/S) \longrightarrow N_1(Z/S) \longrightarrow 0$$

is not exact at the middle term in general as we shall see in the following example: Let $X = E \times E$ be the product of a nonsingular projective elliptic curve E with itself, let $f: X \to Z = E$ be the projection onto the first component, and let $\tau: Z \to \mathrm{Spec}\, k = S$. Now let Δ be the diagonal in $E \times E$, let $B := E \times \{p_2\}$ for a point p_2 of the second component, and let Γ be a fiber of f. Suppose that the sequence is exact. Then $0 \not\approx \Delta - B \approx a\Gamma$

for some $a \in \boldsymbol{R}$ with $a \neq 0$, since $f_*(\Delta - B) = 0$ and since $N_1(X/Z) = \boldsymbol{R}\Gamma$. But this gives $0 = \deg N_{\Delta/X} = \Delta^2 = (B + a\Gamma)^2 = 2a \neq 0$, a contradiction.

The properties of the contraction morphisms will be discussed in more detail in connection with the Flip Conjecture in Chapter 5.

§ 3-3. Canonical rings of varieties of general type

Theorem 3-3-1. *Let X be a normal variety of dimension d with $\Delta \in Z_{d-1}(X) \otimes \boldsymbol{Q}$ such that the pair (X, Δ) has only log-terminal singularities. Let $\pi: X \to S$ be a proper morphism onto a variety S. Assume that $K_X + \Delta$ is π-nef and π-big. Then $K_X + \Delta$ is π-semi-ample, and hence $R(X/S, K_X + \Delta) := \bigoplus_{m \geqq 0} \mathcal{O}_X([m(K_X + \Delta)])$ is finitely generated as an $\mathcal{O}_S$-algebra.*

Proof. By applying Theorem 3-1-1 with Remark 3-1-2 to $H := K_X + \Delta$, we deduce that $K_X + \Delta$ is π-semi-ample. The rest of the theorem is quite clear, since $\mathcal{O}_X(ma(K_X + \Delta)) \cong \psi^* \mathcal{O}_P(1)$, where ψ is the morphism $\psi: X \to P := P(\pi_* \mathcal{O}_X(ma(K_X + \Delta)))$ associated to the surjection

$$\pi^* \pi_* \mathcal{O}_X(ma(K_X + \Delta)) \longrightarrow \mathcal{O}_X(ma(K_X + \Delta)). \qquad \text{q.e.d.}$$

Corollary 3-3-2. *Let X be a normal complete variety with only canonical singularities such that the canonical divisor K_X is nef and big, i.e., let X be a minimal variety of general type. Then the canonical ring $R := \bigoplus_{m \geqq 0} H^0(X, \mathcal{O}_X(mK_X))$ is a finitely generated algebra over k. Thus the canonical model $X_{\mathrm{can}} := \operatorname{Proj} R$ of X exists. $\Phi_{|mK_X|}: X \to X_{\mathrm{can}}$ for $m \gg 0$ gives the canonical morphism onto X_{can}, which in the case of $\dim X = 2$ is nothing but the contraction of (-2)-curves on X.*

Corollary 3-3-2 in dimension $\leqq 3$ was proved in [Ka5] and [Be2], and it was the very beginning of our whole theory.

Chapter 4. Cone Theorem

In this chapter, we prove a structure theorem on the closed cone of curves of an algebraic variety (cf. [K1]). The theorem should be one of the key steps toward the theory of minimal models. It was first proved by Mori [Mo2] using the ingenious method of modulo p reduction in the case where the variety is nonsingular and $S = \operatorname{Spec} k$. The concept of extremal rays on Kleiman's cone was also introduced in this paper. After that, this "cone theorem" was generalized to the "log" category in [TM]. The case where the varieties have singularities was studied in [Ka6], [R4], [S2], [Ka7] and [Ko12]. We note here that Mori's method using the deformation theory cannot be applied to this case. In fact, there exists a singular 3-fold X which has an extremal ray of flipping type; an extremal ray which is generated by the class of an irreducible curve C such that $(K_X.C)<0$ and such that C does not "move" in an algebraic family, in contrast with the nonsingular case in [Mo2]. After [Ka6], Reid [R4] pointed out that the Cone Theorem can be derived from the combination of the Contraction Theorem and the following Rationality Theorem. This idea was fully developed in [Ka7], in which the Rationality Theorem was proved by the same technique as the one we used to prove the Base Point Free Theorem. Then [Kol2] proved the discreteness of extremal rays refining the argument in [Ka7].

§ 4-1. Rationality Theorem

Theorem 4-1-1 (Rationality Theorem, cf. [R4], [Ka7], [Kol2]). *Let X be a normal variety of dimension d with $\Delta \in Z_{d-1}(X)\otimes \boldsymbol{Q}$ such that the pair (X, Δ) has only weak log-terminal singularities, and let $\pi: X \to S$ be a projective morphism onto a variety S. Let $H \in \mathrm{Div}(X)$ be a π-ample Cartier divisor. If $K_X+\Delta$ is not π-nef, then*

$$r := \max\{t \in \boldsymbol{R};\ H+t(K_X+\Delta) \text{ is } \pi\text{-nef}\}$$

is a rational number. Furthermore, expressing $r/a=u/v$ with $u, v \in \boldsymbol{N}$ and $(u, v)=1$, we have $v \leq a(b+1)$, where

$$a := \min\{e \in \boldsymbol{N};\ e(K_X+\Delta) \in \mathrm{Div}(X)\} \quad \text{and}$$
$$b := \max_{s\in S}\{\dim_{k(s)}\pi^{-1}(s)\}.$$

Proof. We will derive a contradiction assuming that either one of the following two cases occurs,

(1) $r \notin \boldsymbol{Q}$, or

(2) $r \in \boldsymbol{Q}$ and $v>a(b+1)$.

Lemma 4-1-2. *Let X be a nonsingular projective variety, let p and q be positive real numbers, and let $D_1, D_2 \in \mathrm{Div}(X)$ and $A \in \mathrm{Div}(X)\otimes \boldsymbol{Q}$. Assume the following conditions:*

(i) *$\ulcorner A \urcorner \geq 0$ and $\langle A \rangle$ has support with only normal crossings.*

(ii) *$xD_1+yD_2+A-K_X$ is ample and*

$$H^0(X, \mathcal{O}_X(xD_1+yD_2+\ulcorner A\urcorner)) \cong H^0(X, \mathcal{O}_X(xD_1+yD_2)),$$

for $x, y \in \boldsymbol{N}$ with $y-px<q$.

(iii) *$P(x, y) := \chi(X, \mathcal{O}_X(xD_1+yD_2+\ulcorner A\urcorner))$ is a polynomial in x and y of degree at most d (or identically zero).*

Assume further that either one of the following two cases occurs:

(1) $p \notin \boldsymbol{Q}$, *or*

(2) $p \in \boldsymbol{Q}$, *and expressing* $p = u/v$ *with* $u, v \in \boldsymbol{N}$ *and* $(u, v) = 1$, *we have* $qv > d+1$.

Then there exists a positive integer y_1 *such that*

$$H^0(X, \mathcal{O}_X(xD_1 + yD_2 + \ulcorner A \urcorner)) \neq 0,$$

whenever $px + q > y \geqq y_1$.

Proof. Note that if $y - px < q$, we have

$$P(x, y) = h^0(X, \mathcal{O}_X(xD_1 + yD_2 + \ulcorner A \urcorner))$$

by the condition (ii) and by Theorem 1-2-3. Take $p_0 \in \boldsymbol{Q}$ with $0 < p_0 < p$. Then on the line L which is defined by the equation $y = p_0 x$, $P(x, y)$ is not identically zero. In fact, for $x_0, y_0 \in \boldsymbol{N}$ with (x_0, y_0) being on the line L,

$$x_0 D_1 + y_0 D_2 = \lim_{m \to \infty} (1/m)(mx_0 D_1 + my_0 D_2 + A - K_X)$$

is nef, since to be nef is a closed condition, and $x_0 D_1 + y_0 D_2 + A - K_X$ is ample. Thus the Non-Vanishing Theorem implies $P(mx_0, my_0) \neq 0$ for $m \gg 0$. Set

$$U := \{(x, y) \in \boldsymbol{N};\ 0 < y - px < q/(d+1)\}.$$

Then in both cases (1) and (2), we have $\sharp U = \infty$. For any member $(x_0, y_0) \in U$, let $L(x_0, y_0)$ be the line defined by the equation $y = (y_0/x_0)x$. If $P(jx_0, jy_0) = 0$ for $j \in \boldsymbol{N}$ with $1 \leqq j \leqq d+1$, then $P|_{L(x_0, y_0)} \equiv 0$, since the degree of $P(x, y)$ in x and y is at most d. Since $\sharp\{L(x, y);\ (x, y) \in U\} = \infty$, the fact $P(x, y) \not\equiv 0$ implies that there exists $(x_1, y_1) \in \boldsymbol{N}^2$ such that $0 < y_1 - px_1 < q$ and that $P(x_1, y_1) \neq 0$.

Now assume that such y_1 as stated in the lemma does not exist. Then there exist infinitely many $(x_i, y_i) \in \boldsymbol{N}^2$ such that $y_i - px_i < q$, $x_i - dx_1 > 0$, $y_i - dy_1 > 0$ and that $P(x_i, y_i) = 0$. For such (x_i, y_i)'s the polynomials $P(x_i - jx_1, y_i - jy_1)$ must be zero for $j \in \boldsymbol{N}$ with $0 \leqq j \leqq d$, since $P(x_1, y_1) = h^0(X, \mathcal{O}_X(x_1 D_1 + y_1 D_2)) \neq 0$. But this implies $P(x, y) \equiv 0$, a contradiction.

q.e.d.

Now we go back to the proof of the theorem. We may assume that H is π-generated. Indeed, take $c, n \in \boldsymbol{N}$ so large with $a < cr$ and $(nc, v) = 1$ that both $H' := n(cH + a(K_X + \Delta))$ and $H' - (K_X + \Delta)$ are π-very ample. Putting

$$r' := \max\{t \in \boldsymbol{R}:\ H' + t(K_X + \Delta) \text{ is } \pi\text{-nef}\},$$

we have $r'/a=ncr/a-n$. Therefore, the condition $r \in \boldsymbol{Q}$ is equivalent to $r' \in \boldsymbol{Q}$. In this case, expressing $r'/a=u'/v'$ with $u', v' \in \boldsymbol{N}$ and $(u', v')=1$, we have $v=v'$ by the choice of taking c and n. It follows that $v \leqq a(b+1)$ is equivalent to $v' \leqq a(b+1)$. Thus we may replace H by H'.

Set

$$M(x, y):=xH+ya(K_X+\Delta) \quad \text{and}$$
$$\Lambda(x, y):=\operatorname{supp}(\operatorname{Coker} \pi^*\pi_*\mathcal{O}_X(M(x, y)) \longrightarrow \mathcal{O}_X(M(x, y))).$$

Then it is sufficient to show that there exists $(x, y) \in \boldsymbol{N}^2$ such that $0<ya-xr<1$ and $\Lambda(x, y)=\phi$ to derive a contradiction. By taking some resolution of X_η, η being the generic point of S, we can show by Lemma 4-1-2 with $p=r/a$, $q=1/a-\varepsilon$ for a very small positive rational number ε, and D_1 (resp. D_2) being the pull-back of H (resp. $a(K_X+\Delta)$), that there exists $(x_0, y_0) \in \boldsymbol{N}^2$ such that $0<ay_0-rx_0<1$ and that $h^0(X_\eta, \mathcal{O}_X(M(x_0, y_0))|_{X_\eta}) \neq 0$, which implies that $\pi_*\mathcal{O}_X(M(x_0, y_0))$ is not a zero sheaf. Take such (x_0, y_0). If $\Lambda(x_0, y_0)=\phi$, we are done. Thus we assume $\Lambda(x_0, y_0) \neq \phi$.

Then as in the proof of the Base Point Free Theorem, we take a proper birational morphism $f=f_1 \circ f_2$: $Y \to X$ from a nonsingular variety Y and a family of divisors $\{F_j\}$ with only simple normal crossings which satisfy the following conditions:

(1) $f^*(x_0H+(y_0a-1)(K_X+\Delta))+\delta f_2^* f_1^{-1}(\Delta)-\sum \delta_j F_j$ is $\pi \circ f$-ample for some δ, $\delta_j \in \boldsymbol{Q}$, where $\operatorname{supp}(\sum \delta_j F_j)$ is exceptional for f,

(2) $K_Y+\delta f_2^* f_1^{-1}(\Delta)=f^*(K_X+\Delta)+\sum a_j F_j$ for $a_j \in \boldsymbol{Q}$ with $a_j+1-\delta_j>0$, and

(3) $(\pi \circ f)^*(\pi \circ f)_*\mathcal{O}_Y(f^*M(x_0, y_0)) \twoheadrightarrow \mathcal{O}_Y(f^*M(x_0, y_0)-\sum r_j F_j) \subset \mathcal{O}_Y(f^*M(x_0, y_0))$ for some nonnegative integers r_j, where $\sum r_j F_j$ is the $\pi \circ f$-fixed part of $M(x_0, y_0)$.

Set

$$c:=\min_j (a_j+1-\delta_j)/r_j.$$

By changing δ_j's slightly if necessary, we may assume that the minimum c is attained only at a unique index $j=0$. Set

$$A:=\sum_{j \neq 0} (-cr_j+a_j-\delta_j)F_j \quad \text{and} \quad B:=F_0.$$

Then

$$\begin{aligned} N:=&f^*(x'H+y'a(K_X+\Delta))+A-B-K_Y \\ &\sim_{\boldsymbol{Q}} c(f^*M(x_0, y_0)-\textstyle\sum r_j F_j) \\ &\quad +f^*\{(x'-(c+1)x_0)H+(y'-(c+1)y_0)a(K_X+\Delta)\} \\ &\quad +f^*\{x_0H+(y_0a-1)(K_X+\Delta)\}+\delta f_2^* f_1^{-1}(\Delta)-\textstyle\sum \delta_j F_j \end{aligned}$$

is $\pi\circ f$-ample for $(x', y') \in N^2$ such that

$$ay'-rx' \leqq (c+1)(ay_0-rx_0).$$

Then

$$0=R^1(\pi\circ f)_*\mathcal{O}_Y(\ulcorner N\urcorner+K_Y)=R^1(\pi\circ f)_*\mathcal{O}_Y(f^*M(x', y')+\ulcorner A\urcorner-B)$$

by Theorem 1-2-3, which implies that the homomorphism

$$(\pi\circ f)_*\mathcal{O}_Y(f^*M(x', y')+\ulcorner A\urcorner)\longrightarrow(\pi\circ f)_*\mathcal{O}_B(f^*M(x', y')+\ulcorner A\urcorner|_B)$$

is surjective. Then the commutativity of the following diagram implies that the injection on the right hand side column actually becomes an isomorphism.

$$\begin{array}{ccc} (\pi\circ f)_*\mathcal{O}_Y(f^*M(x', y')) & \longrightarrow & (\pi\circ f)_*\mathcal{O}_B(f^*M(x', y')|_B) \\ \downarrow\cong & & \downarrow \\ (\pi\circ f)_*\mathcal{O}_Y(f^*M(x', y')+\ulcorner A\urcorner) & \twoheadrightarrow & (\pi\circ f)_*\mathcal{O}_B(f^*M(x', y')+\ulcorner A\urcorner|_B) \end{array}$$

In case (1), set

$$x':=[lay_0/r]=lx_0+[l(ay_0/r-x_0)] \quad \text{and} \quad y':=ly_0,$$

where l is a sufficiently large integer such that

$$r\langle lay_0/r\rangle<\min\{1, (c+1)(ay_0-rx_0)\}.$$

In case (2), set

$$x':=x_0+lv \quad \text{and} \quad y':=y_0+lu \quad \text{with } l\gg 0.$$

Then in both cases, it is easy to see that

$$0<ay'-rx'<\min\{1, (c+1)(ay_0-rx_0)\}$$

and that

$$\Lambda(x', y')\subset\Lambda(x_0, y_0)$$

by the Base Point Free Theorem, since we did take H to be π-generated. By Lemma 4-1-2, we have

$$(\pi\circ f)_*\mathcal{O}_B(f^*M(x', y')+\ulcorner A\urcorner|_B)\neq 0.$$

which implies $\Lambda(x', y')\subsetneqq\Lambda(x_0, y_0)$. By noetherian induction, we finally conclude that there exists $(x, y)\in N^2$ such that $0<ay-rx<1$ and that $\Lambda(x, y)=\phi$, a contradiction. q.e.d.

§ 4-2. The proof of the Cone Theorem

The following theorem is a generalization of the one in [Ka7] and [Ko12]. The estimate for the denominator is better than that in [Ko12], though we do not need this fact to carry out the program for constructing minimal models.

Theorem 4-2-1 (Cone Theorem, [Mo2], [TM], [Ka6], [R4], [S2], [Ka7], [Ko12]). *Let X be a normal variety of dimension d with $\Delta \in Z_{d-1}(X)\otimes \boldsymbol{Q}$ such that the pair (X, Δ) has only weak log-terminal singularities, and let $\pi: X \to S$ be a projective morphism onto a variety S. Then*

$$\overline{NE}(X/S)=\overline{NE}_{K_X+\Delta}(X/S)+\sum R_j,$$

where R_j's are extremal rays of $\overline{NE}(X/S)$ for (X, Δ). Furthermore, if C_j is a reduced irreducible curve with $R_j=\boldsymbol{R}_+\mathrm{cl}(C_j)$, then for any π-ample divisor $A \in \mathrm{Div}(X)$ we have an inequality

$$v_j \leqq a(b+1)$$

about the denominator of the fraction

$$(A.C_j)/a((K_X+\Delta).C_j)=-u_j/v_j$$

with $u_j, v_j \in \boldsymbol{N}$ and $(u_j, v_j)=1$, where

$$a:=\min\{e \in \boldsymbol{N};\ e(K_X+\Delta) \in \mathrm{Div}(X)\} \quad \textit{and}$$
$$b:=\max_{s\in S}(\dim_{k(s)}\pi^{-1}(s)).$$

In particular, the R_j are discrete in the half space

$$\{z \in N_1(X/S);\ ((K_X+\Delta).z)<0\}.$$

Proof. First we note the following easy fact.

Lemma 4-2-2. *Let X be a normal variety of dimension d with $\Delta \in Z_{d-1}(X)\otimes\boldsymbol{Q}$ such that the pair (X, Δ) has only weak log-terminal singularities, and let $\pi: X\to S$ be a projective morphism onto a variety S. Let $f: X\to Z$ and $\tau: Z\to S$ be projective surjective morphisms, where Z is a variety. Then any extremal face F of $\overline{NE}(X/Z)$ for (X, Δ) is at the same time an extremal face of $\overline{NE}(X/S)$ for (X, Δ), if we regard $N_1(X/Z)$ as a subspace of $N_1(X/S)$.*

Proof. By the Contraction Theorem, we have the contraction morphism of F denoted by $\mathrm{cont}_F: X\to W$. Then $-(K_X+\Delta)$ is cont_F-ample and cont_F has connected fibers. Thus $F=\overline{NE}(X/W)-\{0\}$ (see Lemma 3-2-4) is an extremal face of $\overline{NE}(X/S)$ for (X, Δ) by Lemma 3-2-5. q.e.d.

Now the Cone Theorem follows directly from the Rationality Theorem as we shall see below.

Step 1. If $\dim_{\mathbf{R}} N_1(X/S) \geqq 2$, then

$$\overline{NE}(X/S)=\overline{NE}_{K_X+\Delta}(X/S)=(\sum_{L\neq 0} F_L)^{-},$$

where the L vary among all supporting functions which are not zero on $N^1(X/S)$ and $^{-}$ denotes the closure with respect to the real topology.

Proof. Let $B:=\overline{NE}_{K_X+\Delta}(X/S)+(\sum_{L\neq 0}F_L)^{-}$. It is clear that $\overline{NE}(X/S) \supset B$. Supposing $\overline{NE}(X/S)\neq B$, we shall derive a contradiction. Then there is a seperating function $M \in \mathrm{Div}(X)$ which is not a multiple of $K_X+\Delta$ in $N^1(X/S)$ such that $M>0$ on $B-\{0\}$ and $(M.z_0)<0$ for some $z_0 \in \overline{NE}(X/S)$. Let C be the dual cone of $\overline{NE}_{K_X+\Delta}(X/S)$, i.e.,

$$C:=\{D \in N^1(X/S);\ (D.z)\geqq 0 \quad \text{for } z \in \overline{NE}_{K_X+\Delta}(X/S)\}.$$

Then C is generated by π-nef divisors and $K_X+\Delta$. Since $M>0$ on $\overline{NE}_{K_X+\Delta}(X/S)-\{0\}$, M is in the interior of C, and hence there exists a π-ample $\mathbf{Q}$-Cartier divisor $A \in \mathrm{Div}(X)\otimes\mathbf{Q}$ such that $M-A=L'+p(K_X+\Delta)$ where $L' \in \mathrm{Div}(X)\otimes\mathbf{Q}$ is π-nef and p is a nonnegative rational number. Therefore, M is expressed in the form $M=H+p(K_X+\Delta)$ where $H:=A+L' \in \mathrm{Div}(X)\otimes\mathbf{Q}$ is π-ample. If $K_X+\Delta$ is π-nef, the claim is obvious. Thus we assume that $K_X+\Delta$ is not π-nef. Then the Rationality Theorem implies that there exists a positive rational number r such that $L:=H+r(K_X+\Delta) \in \mathrm{Div}(X)$ is π-nef but not π-ample. Note that $L\neq 0$ in $N^1(X/S)$, since M is not a multiple of $K_X+\Delta$. Thus the extremal face F_L associated to the supporting function L is contained in B, which implies $M>0$ on F_L. Therefore $p<r$. But this implies that M is π-ample, a contradiction. This completes the proof of our first claim.

Step 2. In the equality of Step 1, we may take such L's that has the extremal face F_L of dimension one.

Proof. Let L be a supporting function with $\dim F_L \geqq 2$, and let $\psi\colon X\to W$ be the contraction morphism associated to L. Since $-(K_X+\Delta)$ is ψ-ample, Step 1 applied to $\overline{NE}(X/W)$ gives $F_L=\overline{NE}(X/W)-\{0\}=(\sum_{M\neq 0}F_M)^{-}-\{0\}$, where the M vary among all supporting functions of $\overline{NE}(X/W)$ which are not zero. By Lemma 4-2-2, the F_M are also extremal faces of $\overline{NE}(X/S)$ for (X, Δ). Since $\dim F_M<\dim F_L$, the inductive procedure shows our claim. Therefore, we have the following formula

$$(*)\quad \overline{NE}(X/S)=\overline{NE}_{K_X+\Delta}(X/S)+(\sum R_j)^-,$$

where the R_j are extremal rays.

Step 3. The Contraction Theorem guarantees that for each extremal ray R_j there exists a reduced irreducible curve C_j on X such that $\mathrm{cl}(C_j) \in R_j$. Let $\psi_j: X \to W_j$ be the contraction morphism of R_j, and let A be a π-ample Cartier divisor. We set

$$r_j := -(A.C_j)/((K_X+\Delta).C_j).$$

Then

$$r_j = \max\{t \in \boldsymbol{R};\ A+t(K_X+\Delta) \text{ is } \psi_j\text{-nef}\}.$$

By the Rationality Theorem, expressing $r_j/a=u_j/v_j$ with $u_j, v_j \in \boldsymbol{N}$ and $(u_j, v_j)=1$, we have the inequality

$$v_j \leqq a(b_j+1) \leqq a(b+1),$$

where $b_j=\max_{w \in W_j}(\dim_{k(w)}\psi_j^{-1}(w))$.

Step 4. Now take π-ample Cartier divisors $H_1, H_2, \cdots, H_{\rho-1} \in \mathrm{Div}(X)$ such that $K_X+\Delta$ and the H_i form a basis of $N^1(X/S)$, where $\rho=\dim_{\boldsymbol{R}} N^1(X/S)$. By Step 3, the intersection of the extremal rays R_j with the hyperplane $\{z \in N_1(X/S);\ (a(K_X+\Delta).z)=-1\}$ in $N_1(X/S)$ lie on the lattice

$$\{z \in N_1(X/S);\ (a(K_X+\Delta).z)=-1,\ (H_i.z) \in (a(b+1)!)^{-1}\boldsymbol{Z}\}.$$

This implies that the extremal rays are discrete in the half space $\{z \in N_1(X/S);\ ((K_X+\Delta).z)<0\}$. Thus we can omit the closure sign $^-$ from the formula $(*)$ and this completes the proof of the theorem. q.e.d.

Example 4-2-3. In general, the number of the extremal rays on the closed cone of curves is infinite as in the following example (cf. [Nt]).

Take two nonsingular elliptic curves E_1, E_2 on $\boldsymbol{P}^2_{\boldsymbol{C}}$ so that p_1-p_2 is not of finite order on the abelian group E_1, where p_1 and p_2 are two of the nine intersection points of E_1 and E_2. Let G_1 and G_2 be the defining equations of E_1 and E_2, respectively. The rational map which maps $x \in \boldsymbol{P}^2_{\boldsymbol{C}}-(E_1 \cap E_2)$ to $(G_1(x): G_2(x)) \in \boldsymbol{P}^1_{\boldsymbol{C}}$ becomes a morphism from S which is obtained by blowing up $\boldsymbol{P}^2_{\boldsymbol{C}}$ at the nine intersection points of E_1 and E_2. It is easy to see that inverse images of p_1 and p_2 on S are sections of $\pi: S \to \boldsymbol{P}^1_{\boldsymbol{C}}$, and hence by the choice of p_1 and p_2, there exist infinitely many sections of π, which are exceptional curves of the first kind.

In contrast with the example above, the number of extremal rays are finite for a variety of general type.

Proposition 4-2-4. *Let X be a normal variety of dimension d with $\Delta \in Z_{d-1}(X)\otimes\boldsymbol{Q}$ such that the pair (X, Δ) has only log-terminal singularities, and let $\pi: X \to S$ be a projective morphism onto a variety S. Assume that $K_X+\Delta$ is π-big. Then the number of extremal rays of $\overline{NE}(X/S)$ for (X, Δ) is finite.*

Proof. By Kodaira's Lemma there exists an effective $\boldsymbol{Q}$-Cartier divisor $\Delta' \in \mathrm{Div}(X)\otimes\boldsymbol{Q}$ such that $(K_X+\Delta)-\Delta'$ is π-ample. For any sufficiently small $\varepsilon>0$, $(X, \Delta+\varepsilon\Delta')$ has also log-terminal singularities, and the extremal rays of $\overline{NE}(X/S)$ for $(X, \Delta+\varepsilon\Delta')$ are discrete in the half space $\{z \in N_1(X/S);\ ((K_X+\Delta+\varepsilon\Delta').z)<0\}$. Since the space $\{z \in N_1(X/S);\ ((K_X+\Delta).z)\leqq 0\}$ is entirely contained in the former space, it is clear that the extremal rays for (X, Δ) are finite in number. q.e.d.

Problem 4-2-5. Is the number of extremal rays finite whenever

$$\kappa(X_\eta, K_X+\Delta|_{X_\eta})\geqq 0?$$

Note that the answer to the above problem is affirmative if $\dim X=2$.

Remark 4-2-6. By the Cone Theorem $\overline{NE}(X/S)$ looks like a rational polyhedral cone in the half space $\{z \in N_1(X/S);\ ((K_X+\Delta).z)<0\}$. But this is not the case as for the shape of the cone in the other half space $\{z \in N_1(X/S);\ ((K_X+\Delta).z)\geqq 0\}$ (cf. [C]):

Let A be an abelian surface. Then the set $\{z \in N_1(A);\ z^2\geqq 0\}$ consists of two cones which intersect each other only at the origin. $\overline{NE}(A)$ is nothing but one of these cones, the one containing ample classes. This follows from the Riemann-Roch Theorem on A and the fact that any effective divisor on an abelian surface is nef. Therefore $\overline{NE}(A)$ is a quadric cone in $N_1(A)$, which is not polyhedral.

Chapter 5. Flip Conjecture

In this chapter, we discuss the Flip Conjecture which still remains to be proved in order for our flow chart to work.

The concept of a *flip*, which appeared in preceding papers (e.g., [At]), was first used by Kulikov [Ku] as an essential technique to construct a minimal model for a semi-stable degeneration of surfaces with trivial canonical bundles. The proof was later improved by Persson and

Pinkham [PP]. The flip considered by them is one obtained by contracting $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ on a nonsingular 3-fold in two different directions, which we may call "a flip of symmetric type" for the moment. An idea similar to theirs can be seen in [R2]. Tsunoda [Ts2] has succeeded in constructing a minimal model for a semi-stable degeneration of surfaces with nonnegtive Kodaira dimension, a generalization of the result of Kulikov, where he obtains "a flip of asymmetric type" as a composite of divisorial contractions, their inverses and flips of symmetric type (cf. also [Ka12], [Mo5], [S3]). Mori [Mo2] first constructed the contraction morphism of extremal rays on nonsingular 3-folds. A naive reader of his paper might have expected to obtain a minimal model by a simple repetition of contractions. But this is not the case; see [F] for an explicit counterexample. Then Reid [R3] proposed a program for constructing minimal models by introducing the concept of flips to overcome such obstructions as those observed in Francia's example. He proved the minimal model conjecture for toric morphisms as a practical and nice evidence for his program to work. In this paper, we step forward along the program of Reid, introducing the notion of *log-flips* to treat flips of symmetric and asymmetric types in a unified way.

§ 5-1. Types of contractions of extremal rays

Let X be a normal variety of dimension d with $\Delta \in Z_{d-1}(X) \otimes \boldsymbol{Q}$ such that the pair (X, Δ) has only weak log-terminal singularities, and let $\pi: X \to S$ be a projective morphism onto a variety S. Assume that X is $\boldsymbol{Q}$-factorial. If $K_X + \Delta$ is not π-nef, then there exist an extremal ray R by the Cone Theorem and the contraction morphism $\varphi: X \to X'$ of R by the Contraction Theorem. Let $\pi': X' \to S$ be the induced morphism. By Lemma 3-2-5 (3), we have $\rho(X'/S) = \rho(X/S) - 1$, where $\rho(X/S)$ denotes the Picard number of X relative to S. We set

$$A := \{x \in X;\ \varphi \text{ is not an isomorphism at } x\}.$$

We shall classify the extremal rays into three types according to $\delta :=$ dim A, where "dim" denotes as usual the maximum of the dimensions of irreducible components.

Remark 5-1-1. We note that $R^i \varphi_* \mathcal{O}_X = 0$ for $i > 0$. Indeed, since (X, Δ) has only weak log-terminal singularities and since $-(K_X + \Delta)$ is φ-ample, this assertion follows immediately from Theorem 1-2-5. As a consequence, X' has only rational singularities when $\delta \neq d$, i.e., when φ is birational. It is also true when $\delta = d$ by [Ko13, Corollary 7.4].

(A) Type: $\delta=d$ (Contraction of fiber type)

This is the case where our answer to the question in Introduction "Is dim $X'=d$?" is NO. Since $-(K_X+\Delta)$ is φ-ample, the generic fiber (X_η, Δ_η) is a pair having only weak log-terminal singularities with the anti-log-canonical divisor $-(K_{X_\eta}+\Delta_\eta)$ being ample. In this case $\kappa(X, K_X+\Delta) = -\infty$ by the Easy Addition Theorem of Iitaka [I3]. When $\Delta=0$ and X has only terminal singularities, the generic fiber of φ is a *Q-Fano variety*, which we define to be a projective variety Z having only terminal singularities with the ample anti-canonical divisor $-K_Z$.

Definition 5-1-2. A variety X is said to be *uniruled* if there exists a generically finite and generically surjective rational map ψ: $Y\times \boldsymbol{P}^1 \cdots\to X$ for some variety Y.

Now we state a characterization of uniruled varieties due to Miyaoka and Mori [MM].

Theorem 5-1-3. *Let X be a nonsingular projective variety over $\boldsymbol{C}$. Then X is uniruled if and only if there exists a nonempty Zariski open set U of X such that for every closed point $x \in U$ there is an irreducible curve C containing x with $(K_X.C)<0$.*

Corollary 5-1-4. *A variety X which has a contraction of fiber type is a uniruled variety.*

Proof. Take a subfield k_0 of k, which is finitely generated over $\boldsymbol{Q}$, and a variety X_0 over k_0 such that $X\cong X_0\times_{\operatorname{Spec} k_0}\operatorname{Spec} k$. Then by Theorem 5-1-3, $\bar{X}:=X_0\times_{\operatorname{Spec} k_0}\operatorname{Spec}\boldsymbol{C}$ is uniruled, where the morphism $\operatorname{Spec}\boldsymbol{C}\to\operatorname{Spec} k_0$ is given by some embedding $k_0\subset\boldsymbol{C}$. It follows that there exists an irreducible component Z of the Hilbert scheme of $\bar{X}$ and a closed subvariety Z_1 of Z such that any geometric fiber of the universal family $W_1\to Z_1$ consists of rational curves with W_1 irreducible and that the natural projection $W_1\to\bar{X}$ is generically surjective. Let Z' be the maximal closed algebraic subset of Z on which geometric fibers of the universal family all consist of rational curves. Then the irreducible component $\bar{Y}$ of Z' which contains Z_1 is defined over some field k_1 which is finite algebraic over k_0; $\bar{Y}\cong Y_1\times_{\operatorname{Spec} k_1}\operatorname{Spec}\boldsymbol{C}$ for some Y_1. Then we have only to put $Y=Y_1\times_{\operatorname{Spec} k_1}\operatorname{Spec} k$ in order to see that X is uniruled. q.e.d.

Lemma 5-1-5. *If the pair (X, Δ) has a contraction of fiber type φ: $X \to X'$, then X' is again $\boldsymbol{Q}$-factorial.*

Proof. Let D' be a prime divisor on X'. Take a prime divisor D on X which is mapped surjectively onto D' by φ. Since X is $\boldsymbol{Q}$-factorial, there is a positive integer $a\in\boldsymbol{N}$ such that aD is a Cartier divisor. There also exists a curve C on X such that $\varphi(C)$ is a point and that $\varphi(C) \notin D'$. Then since $\mathrm{cl}(C) \in R$ and since $(D.C)=0$, Lemma 3-2-5 implies that $aD=\pi^* D_0$ for a Cartier divisor D_0 on X'. Since $\mathrm{supp}\, D_0=\varphi(D)=D'$, D' is clearly a $\boldsymbol{Q}$-Cartier divisor. Thus X' is $\boldsymbol{Q}$-factorial. q.e.d.

(B) Type: $\delta=d-1$ (Contraction of divisorial type)

When $\delta=d-1$, which is always the case for surfaces unless φ is of fiber type, X' again satisfies the same conditions as X so that we have no trouble to proceed in the flow chart in Introduction.

Proposition 5-1-6 (cf. [Ka7]). *If φ is a contraction of divisorial type, then the exceptional locus A of φ consists of a unique prime divisor on X, the pair $(X', \varphi_*(\Delta))$ has only weak log-terminal singularities, and X' is $\boldsymbol{Q}$-factorial. Moreover, if $\Delta=0$ and X has only terminal singularities, then X' also has terminal singularities.*

Proof. First we note the following lemma, which is clear from the Hodge index theorem.

Lemma 5-1-7. *Let $f: G \to H$ be a generically finite proper surjective morphism from a normal surface, and let E be a nonzero $\boldsymbol{Q}$-Cartier divisor on G whose support is mapped to a point on H by f. Then $(E^2)<0$.*

Let A_1 be a prime divisor on X contained in the support of A. Set $b_1=\dim \varphi(A_1)$. Then $b_1 \leqq d-2$. Take some affine open subset S_0 of S with $\pi(A_1) \cap S_0 \neq \phi$. Let L (resp. M) be a very ample divisor on $\pi^{-1}(S_0)$ (resp. on $\pi'^{-1}(S_0)$). Take general members $L_i \in |L|$ for $1 \leqq i \leqq d-b_1-2$, and $M_j \in |M|$ for $1 \leqq j \leqq b_1$. M'_j being the strict transform of the M_j by φ, we set $G:=(\bigcap_i L_i) \cap (\bigcap_j M'_j)$ and $C:=A_1 \cap G$. Since X is $\boldsymbol{Q}$-factorial, A_1 is a $\boldsymbol{Q}$-Cartier divisor, which implies that C is also a $\boldsymbol{Q}$-Cartier divisor on a normal surface G. Since $\varphi(C)$ is a point, Lemma 5-1-7 gives $(A_1.C)=(A_1^2.G)<0$. Suppose there exists another irreducible component A_2 of A. Then we find a curve C' on A_2 such that $\mathrm{cl}(C') \in R$ and that $C' \not\subset A_1$. Thus we have $(A_1.C') \geqq 0$, which contradicts the fact $(A_1.C)<0$. Therefore A consists of a unique prime divisor.

Take an arbitrary prime divisor D' on X', and let D be its strict transform by φ. Since $(A.C)<0$ for any curve C with $\mathrm{cl}(C) \in R$, we can choose $q \in \boldsymbol{Q}$ so that $((D+qA).C)=0$. Then by Lemma 3-2-5 (2), there exists a Cartier divisor $D_0 \in \mathrm{Div}(X')$ such that $r(D+qA)=\varphi^* D_0$ for some $r \in \boldsymbol{N}$, and hence $rD'=D_0$. Thus D' is a $\boldsymbol{Q}$-Cartier divisor and X' is therefore $\boldsymbol{Q}$-factorial. Now we can write $K_X+\Delta=\varphi^*(K_{X'}+\varphi_*(\Delta))+pA$ for some $p \in \boldsymbol{Q}$. The inequality $((K_X+\Delta).C)<0$ with the equality $(\varphi^*(K_{X'}+\varphi_*(\Delta)).C)=0$ gives us $p>0$. Hence the pair $(X', \varphi_*(\Delta))$ has only weak log-terminal singularities. The rest of the proof is clear.

q.e.d.

Proposition 5-1-8. *When φ is a contraction of divisorial type, the exceptional locus A is a uniruled variety.*

Proof. We may assume that $k=\boldsymbol{C}$ as in the proof of Corollary 5-1-4. Then the proposition follows immediately from Theorem 5-1-3, once we have the following Lemma 5-1-9.

The phenomenon which we call the "subadjunction" was first observed by M. Reid. A simple example due to him is a generating line C on a quadric cone X in $\boldsymbol{P}^3$. In this case, $(K_X+C)|_C=-3/2.H$ but $K_C=-2H$, where H is a hyperplane section.

Lemma 5-1-9 (Subadjunction Lemma). *Let X be a normal Cohen-Macaulay variety with a prime divisor A, and let $\mu\colon \tilde{A}\to A$ be the normalization of A. Assume that $m(K_X+A)\in \operatorname{Div}(X)$ for some positive integer m. Then there is a natural injective homomorphism*

$$\omega_{\tilde{A}}^{[m]} \longrightarrow \mu^*\mathcal{O}_X(m(K_X+A)).$$

Proof. We shall construct the homomorphisms as in the following diagram:

$$\begin{array}{ccc}
(\mu^*\mu_*\omega_{\tilde{A}})^{\otimes m} & \xrightarrow{\ \beta\ } & (\mu^*\omega_A)^{\otimes m}=\mu^*(\omega_A^{\otimes m}) \\
\alpha\downarrow & & \downarrow\gamma \\
\omega_{\tilde{A}}^{\otimes m} & & \mu^*\mathcal{O}_X(m(K_X+A))
\end{array}$$

(i) α is obtained from the natural homomorphism $\mu^*\mu_*\omega_{\tilde{A}}\to\omega_{\tilde{A}}$, which is surjective since μ is finite.

(ii) Pulling back the trace map $\mu_*\omega_{\tilde{A}}\to\omega_A$ by μ and tensoring it m times, we have β.

(iii) Note the following diagram:

$$\begin{array}{ccccccccc}
 & & \mathcal{O}_X(K_X+A)\otimes\mathcal{O}_X(-A) & \longrightarrow & \mathcal{O}_X(K_X+A) & \longrightarrow & \mathcal{O}_X(K_X+A)\otimes\mathcal{O}_A & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
 & & \mathcal{O}_X(K_X) & & \| & & \omega_A & & \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X,\omega_X) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-A),\omega_X) & \longrightarrow & \mathcal{E}xt^1(\mathcal{O}_A,\omega_X) & \longrightarrow & 0,
\end{array}$$

where $\mathcal{E}xt^1(\mathcal{O}_A,\omega_X)=\omega_A$ since X is Cohen-Macaulay. Thus we have a surjection $\mu^*(\mathcal{O}_X(K_X+A)^{\otimes m})\twoheadrightarrow\mu^*(\omega_A^{\otimes m})$. There is a homomorphism

$\mu^*(\mathcal{O}_X(K_X+A)^{\otimes m})\to\mu^*\mathcal{O}_X(m(K_X+A))$, which is isomorphic at the generic point of $\tilde{A}$, since $(\mathcal{O}_X(K_X+A)^{\otimes m})^\wedge=\mathcal{O}_X(m(K_X+A))$. Now taking the double dual of the homomorphisms above combined with the natural homomorphism

$$\mu^*(\omega_A^{\otimes m})\longrightarrow(\mu^*(\omega_A^{\otimes m}))^\wedge=(\mu^*(\mathcal{O}_X(K_X+A)^{\otimes m}))^\wedge,$$

we obtain γ: $\mu^*(\omega_A^{\otimes m})\to\mu^*\mathcal{O}_X(m(K_X+A))$.

Note that α, β and γ are all isomorphic at the generic point of $\tilde{A}$. Therefore by taking the double dual, we obtain the required injection

$$\begin{aligned}\omega_{\tilde{A}}^{[m]}=(\omega_{\tilde{A}}^{\otimes m})^\wedge&\longrightarrow((\mu^*\mu_*\omega_{\tilde{A}})^{\otimes m})^\wedge\\&\longrightarrow(\mu^*\mathcal{O}_X(m(K_X+A)))^\wedge=\mu^*\mathcal{O}_X(m(K_X+A)).\end{aligned}$$ q.e.d.

We go back to the proof of the proposition. Set $b_1=\dim\varphi(A)$. Let L (resp. M) be a very ample divisor on $\pi^{-1}(S_0)$ (resp. on $\pi'^{-1}(S_0)$), S_0 being an affine open subset of S. Take general members $L_i\in|L|$ for $1\leqq i\leqq d-b_1-2$ and $M_j\in|M|$ for $1\leqq j\leqq b_1$. L'_i (resp. M'_j) being the pull-back of L_i (resp. M_j) by μ (resp. $\varphi\circ\mu$), we set $C'=(\bigcap_i L'_i)\cap(\bigcap_j M'_j)$. Then since codim Sing $\tilde{A}\geqq 2$, any irreducible component C of C' lies entirely in the nonsingular locus of $\tilde{A}$. By the Subadjunction Lemma, there exists some effective divisor E on the nonsingular locus of $\tilde{A}$ such that $mK_{\tilde{A}_{\mathrm{reg}}}=\mu^*(m(K_X+A))|_{\tilde{A}_{\mathrm{reg}}}-E$. We may further assume that $C\not\subset E$. Then

$$\begin{aligned}(K_{\tilde{A}_{\mathrm{reg}}}.C)&=((1/m)\{\mu^*(m(K_X+A))-E\}.C)\\&=((K_X+\Delta-\sum_{j\neq 0}d_jD_j+(1-d_0)A).\mu_*C)-(1/m)(E.C)\end{aligned}$$

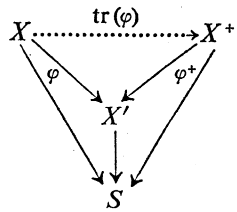
where $\Delta=\sum d_jD_j$ with $D_0=A$. Note that d_0 may be zero. Since $\varphi(\mu_*(C))$ is a point, $((K_X+\Delta).\mu_*(C))<0$ and $((1-d_0)A.\mu_*C)\leqq 0$, noting that $1-d_0\geqq 0$. We may further assume that $\mu_*(C)\not\subset\operatorname{supp}\sum_{j\neq 0}d_jD_j$, which implies that the value of the equation above is negative. Since such C's are dense in $\tilde{A}$, Theorem 5-1-3 implies that $\tilde{A}$ is uniruled and hence so is A. q.e.d.

(C) Type: $\delta<d-1$ (Contraction of flipping type)

This is the case where the major difficulty arises in carrying out our program for constructing the minimal models for higher dimensional varieties. Note again that this case is peculiar to the varieties of dimension $\geqq 3$. It is the existence of contractions of flipping type that enriches the geometry in dimension $\geqq 3$.

Now we propose the (Log-)Flip Conjecture, which consists of the following two parts (cf. [R2], [Ka7]).

Conjecture 5-1-10 ((Log-)Flip Conjecture I: The existence of a (log-) flip). *Let* $\varphi\colon X \to X'$ *be a contraction of flipping type. Then there should exist the following diagram*



which satisfies the following conditions:

(i) X^+ *is a normal variety projective over* X',

(ii) φ^+ *is a birational morphism isomorphic in codimension* 1,

(iii) $K_{X^+}+\Delta^+$ *is* φ^+*-ample, where* Δ^+ *is the strict transform of* Δ.

If this diagram with the properties above exists, we call it simply a *log-flip*. We denote $(\varphi^+)^{-1}\circ\varphi$ by $\mathrm{tr}(\varphi)$, which is a birational map isomorphic in codimension one. When $\Delta=0$ and X has only terminal singularities, the Log-Flip Conejcture is called the *Flip Conjecture* and a log-flip is called a *flip*.

Proposition 5-1-11 (cf. [S1], [Ka7]). *We have the following properties of a log-flip.*

(1) X^+ *is* $\boldsymbol{Q}$*-factorial. Moreover* $\rho(X^+/X')=\rho(X/X')=1$, *or more explicitly* $\mathrm{Div}(X^+)\otimes\boldsymbol{Q}=(\varphi^+)^*(\mathrm{Div}(X')\otimes\boldsymbol{Q})\oplus\boldsymbol{Q}(K_{X^+}+\Delta^+)$.

(2) *The conditions below are equivalent to each other.*

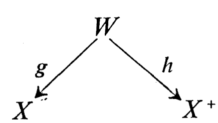
(a) *There exists a log-flip for* φ.

(b) $R(X/X', K_X+\Delta):=\bigoplus_{m\geq 0}\varphi_*\mathcal{O}_X([m(K_X+\Delta)])$ *is finitely generated as an* $\mathcal{O}_{X'}$*-algebra.*

In particular, the log-flip for φ *is unique if it exists, and* X^+ *should be of the form*:

$$X^+\cong \mathrm{Proj}\, R(X/X', K_X+\Delta).$$

(3) *For any common desingularization* W *of* X *and* X^+,



writing

$$K_W = g^*(K_X + \Delta) + \sum a_i F_i = h^*(K_{X^+} + \Delta^+) + \sum a_i^+ F_i,$$

we have $a_i^+ \geq a_i$, *where* $a_i^+ > a_i$ *if and only if* $g(F_i) \subset A$. *Recall that* A *is defined to be the exceptional locus for* φ. *In particular, the pair* (X^+, Δ^+) *has only weak log-terminal singularities.*

Proof. (1) There is an isomorphism of $\boldsymbol{Q}$-divisors

$$\mathrm{tr}(\varphi)_*: Z_{d-1}(X) \otimes \boldsymbol{Q} \xrightarrow{\cong} Z_{d-1}(X^+) \otimes \boldsymbol{Q}.$$

On the other hand, by Lemma 3-2-5 (2), we have

$$\mathrm{Div}(X) \otimes \boldsymbol{Q} = \varphi^*(\mathrm{Div}(X') \otimes \boldsymbol{Q}) \oplus \boldsymbol{Q}(K_X + \Delta).$$

Since $\mathrm{tr}(\varphi)_*(K_X + \Delta) = K_{X^+} + \Delta^+$, which is a $\boldsymbol{Q}$-Cartier divisor by the condition (iii) of a log-flip, $\mathrm{tr}(\varphi)_*$ induces a map

$$\mathrm{tr}(\varphi)_*: \mathrm{Div}(X) \otimes \boldsymbol{Q} \longrightarrow \mathrm{Div}(X^+) \otimes \boldsymbol{Q}.$$

Then the surjectivity of the natural homomorphism $\mathrm{Div}(X) \otimes \boldsymbol{Q} \to Z_{d-1}(X) \otimes \boldsymbol{Q}$ immediately implies the surjectivity of $\mathrm{Div}(X^+) \otimes \boldsymbol{Q} \to Z_{d-1}(X^+) \otimes \boldsymbol{Q}$. Therefore X^+ is $\boldsymbol{Q}$-factorial, and the equality stated in (1) is now obvious.

(2) The implication (a)$\Rightarrow$(b) is clear. In the following we shall prove that (b) implies (a). Set $X^+ := \mathrm{Proj}\, R(X/X', K_X + \Delta)$, and let φ^+ be the natural morphism $\varphi^+: X^+ \to X'$. Let r be a positive integer such that $r(K_X + \Delta) \in \mathrm{Div}(X)$ and that $\bigoplus_{m \geq 0} \varphi_* \mathcal{O}_X(mr(K_X + \Delta))$ is generated by $\varphi_* \mathcal{O}_X(r(K_X + \Delta))$, and let $\mathcal{O}_{X^+}(1)$ be the corresponding φ^+-very ample line bundle on X^+, i.e., $\varphi_*^+ \mathcal{O}_{X^+}(l) \cong \varphi_* \mathcal{O}_X(lr(K_X + \Delta))$ for all $l \in \boldsymbol{N}$. First we show that φ^+ is isomorphic in codimension one. Suppose there exists an exceptional divisor E for φ^+. Consider the following exact sequence

$$0 \longrightarrow \varphi_*^+ \mathcal{O}_{X^+}(l) \longrightarrow \varphi_*^+(\mathcal{O}_{X^+}(l) \otimes \mathcal{O}_{X^+}(E)) \longrightarrow \varphi_*^+(\mathrm{Coker}) \longrightarrow R^1 \varphi_*^+ \mathcal{O}_{X^+}(l),$$

where Coker is the cokernel of the natural injection $\mathcal{O}_{X^+}(l) \to \mathcal{O}_{X^+}(l) \otimes \mathcal{O}_{X^+}(E)$. Take l so large that $R^1 \varphi_*^+ \mathcal{O}_{X^+}(l) = 0$. Then since $\varphi_*^+ \mathcal{O}_{X^+}(l)$ is a reflexive sheaf and since codim supp $\varphi_*^+(\mathrm{Coker}) \geq 2$, we have $\varphi_*^+ \mathcal{O}_{X^+}(l) \cong \varphi_*^+(\mathcal{O}_{X^+}(l) \otimes \mathcal{O}_{X^+}(E))$, which contradicts $\varphi_*^+(\mathrm{Coker}) \neq 0$. By construction, $K_{X^+} + \Delta^+$ is φ^+-ample.

(3) Take a positive integer r such that $r(K_{X^+} + \Delta^+) \in \mathrm{Div}(X^+)$ is φ^+-free. Then

$$g^*(r(K_X + \Delta)) = h^*(r(K_{X^+} + \Delta^+)) + \sum r_i F_i,$$

where $\sum r_i F_i$ is the $\varphi \circ g$-fixed part of $g^*(r(K_X + \Delta))$. Note that $g(\mathrm{supp} \bigcup_{r_i \neq 0} F_i) = A$. Thus $a_i^+ = a_i + (r_i/r)$, which proves the assertion. Now we

take a desingularization h of X^+ which coincides, when restricted over $X^+ - (\varphi^+)^{-1}(\varphi(A)) \cong X - A$, with a suitable desingularization of X in the definition of weak log-terminal singularities. Then since $a_i^+ > a_i$ over $\varphi(A)$, (X^+, Δ^+) has only weak log-terminal singularities (cf. Lemma 0-2-12).
q.e.d.

Remark 5-1-12. (1) The variety which satisfies the conditions (i) and (ii) of the log-flip (not necessarily satisfying the condition (iii)) is either X^+, X' or X itself. This is clear from Proposition 5-1-11 (1).

(2) When $\Delta = 0$ and X has only terminal singularities, a flip has all the properties in Proposition 5-1-11, with $K_X + \Delta$, $K_{X^+} + \Delta^+$ and "weak log-terminal" replaced by K_X, K_{X^+} and by "terminal", respectively.

Suppose φ is a contraction of flipping type. If we can have the log-flip for φ, we go back to the flow chart with another variety X^+ instead of the original X. The second part of the (Log-)Flip Conjecture asserts the following:

Conjecture 5-1-13 ((Log-)Flip Conjecture II: The termination of a sequence of (log-)flips). *A sequence of* (log-)*flips terminates after finitely many steps. Namely there does not exist an infinite sequence of (log-)flips.*

We shall give a proof of the Flip Conjecture II when ($\Delta = 0$ and) X has only terminal singularities and is of dimension 3 or 4.

Definition 5-1-14 ([S1]). Let X be a normal variety with only terminal singularities, and let f: $Y \to X$ be a desingularization of X. Then writing $K_Y = f^*K_X + \sum a_i F_i$, where the F_i range over all the exceptional divisors for f, we define the *difficulty* $d(X)$ of X by $d(X) := \sharp\{i;\ a_i < 1\}$. Note that $d(X)$ does not depend on the choice of the desingularization, and hence well-defined.

Theorem 5-1-15. *The Flip Conjecture* II *holds when* $\dim X = 3$ *or* 4.

Proof (The case $\dim X = 3$ is due to Shokurov [S1]). First we note the following lemma.

Lemma 5-1-16 ([S1]). *Let $\varphi^+ : X^+ \to X'$ be the flip of a flipping contraction φ: $X \to X'$ of a variety X over S which has only terminal singularities. Then $d(X) \geq d(X^+)$. If in addition* codim $A^+ = 2$, *where A^+ is the exceptional locus of φ^+, then $d(X) > d(X^+)$.*

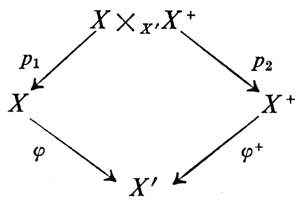
Proof. The first part follows immediately from Proposition 5-1-11 (3). It follows also that X^+ has only terminal singularities, and hence codim Sing $X^+ \geq 3$. Then on some common resolution of X and X^+, some exceptional component F_i over Γ has $a_i^+=1$, where Γ is a component of A^+ with codim $\Gamma=2$, since the generic point of Γ lies in the nonsingular locus of X^+. Then $a_i^+>a_i$ by Proposition 5-1-11 (3), which implies $d(X)>d(X^+)$. q.e.d.

When dim $X=3$, codim A^+ is always equal to 2 and we are in the situation to apply Lemma 5-1-16. Then the assertion is quite clear, since the difficulty is a nonnegative integer.

When dim $X=4$, we need the following lemma.

Lemma 5-1-17. *Let $\varphi^+: X^+\to X'$ be the log-flip of a flipping contraction $\varphi: X\to X'$. Then* dim $A+$dim $A^+\geqq d-1$, *where A and A^+ are the exceptional loci of φ and φ^+, respectively.*

Proof. Consider the following diagram



where p_1 and p_2 are the projections. Take a positive integer l such that $l(K_X+\Delta)$ becomes a Cartier divisor and that $\varphi_*\mathcal{O}_X(l(K_X+\Delta))$ generates $\bigoplus_{m\geqq 0}\varphi_*\mathcal{O}_X(ml(K_X+\Delta))$ as an $\mathcal{O}_{X'}$-algebra. Let I be the ideal of $\mathcal{O}_X$ defined by

$$\varphi^*\varphi_*\mathcal{O}_X(l(K_X+\Delta))\otimes\mathcal{O}_X(-l(K_X+\Delta))\twoheadrightarrow I\subset\mathcal{O}_X.$$

Then we have the closed immersion

$$\mathrm{Proj}\,(\bigoplus_{m\geqq 0} I^m)\longrightarrow\mathrm{Proj}\,(\bigoplus_{m\geqq 0}\varphi_*\mathcal{O}_X(ml(K_X+\Delta)))\times_{X'}X=X^+\times_{X'}X.$$

Since codim supp $(\mathcal{O}_X/I)\geqq 2$ and since Proj $(\bigoplus_{m\geqq 0} I^m)$ is nothing but the blowing-up of the ideal I, it follows that the exceptional locus of p_1, which is $A\times_{X'}A^+$, contains a variety of dimension $d-1$. Thus

$$d-1\leqq\dim(A\times_{X'}A^+)\leqq\dim A+\dim A^+. \qquad \text{q.e.d.}$$

We go back to the proof of Theorem 5-1-15 in case dim $X=4$. We may assume that $k=\boldsymbol{C}$. Lemma 5-1-17 implies that the pair (dim A, dim A^+) is either (1, 2), (2, 2) or (2, 1). When it is (1, 2) or (2, 2), Lemma 5-1-16 gives $d(X)>d(X^+)$. When it is (2, 1), we study the dimension of

$H_4(X_h; \boldsymbol{Q})_{\mathrm{alg}}$, the group of homology classes generated by compact algebraic 2-cycles on the underlying analytic variety X_h. Since φ is a proper surjective morphism, we have the surjective homomorphism

$$H_4(X_h; \boldsymbol{Q})_{\mathrm{alg}} \twoheadrightarrow H_4(X'_h; \boldsymbol{Q})_{\mathrm{alg}}.$$

For any compact algebraic subvariety B on X of complex dimension 2, $\mathrm{cl}(B)$ is not equal to zero in $H_4(X_h; \boldsymbol{Q})_{\mathrm{alg}}$, since X is projective. Thus the kernel of the surjection above is not zero in the case (2, 1). On the other hand, by the excision theorem, we have

$$H_4(X_h^+; \boldsymbol{Q}) \cong H_4(X_h^+, A_h^+; \boldsymbol{Q}) \cong H_4(X'_h, \varphi^+(A_h^+); \boldsymbol{Q}) \cong H_4(X'_h; \boldsymbol{Q}).$$

Therefore, since $\dim H_4(X_h; \boldsymbol{Q})_{\mathrm{alg}}$ is finite, we finally have the result

$$\dim H_4(X_h; \boldsymbol{Q})_{\mathrm{alg}} > \dim H_4(X'_h; \boldsymbol{Q})_{\mathrm{alg}} = \dim H_4(X_h^+; \boldsymbol{Q})_{\mathrm{alg}}.$$

By lexicographic induction on the pair $(d(X), \dim H_4(X_h; \boldsymbol{Q})_{\mathrm{alg}})$, we know that the Flip Conjecture II holds also when $\dim X = 4$. q.e.d.

§ 5-2. Flips of toric morphisms

Reid [R3] showed that the Flip Conjecture holds for toric morphisms, which is a prototype and a source of our Flip Conjecture itself. In this section we present his results not only as a good evidence for the Flip Conjecture in general but also as good examples of flipping contractions and their flips. For the details of the toric geometry and its terminology, we refer the reader to [KKMS] or [O], [D1].

In the following, T denotes the torus embedded equivariantly in a toric variety X with N representing the group of 1-parameter subgroups of T and M the group of characters of T.

The structure of the cone of curves for a toric morphism is given by the following.

Proposition 5-2-1 (cf. [R3, Corollary 1.7]). *Let $f: X \to S$ be a toric morphism of complete toric varieties. Then*

$$\overline{NE}(X/S) = \sum \boldsymbol{R}_+[l_i]$$

where the l_i run through the 1-dimensional strata of X in the fibers of f.

Now we give a criterion for a toric variety X to have only $\boldsymbol{Q}$-factorial, terminal or canonical singularities in terms of the corresponding geometry of its cones.

Proposition 5-2-2 (cf. [R3]). *Let X be the toric variety corresponding to a fan F in $N_{\mathbf{R}}$. Then*

(i) *X is $\mathbf{Q}$-factorial if and only if the fan F is simplicial, i.e., if and only if F is made up of simplicial cones. In this case, X has at most quotient singularities.*

(ii) *X has only terminal (resp. canonical) singularities if and only if for every $\sigma \in F$ the following two conditions* (a) *and* (b) *(resp.* (a) *and* (b′)) *hold*:

(a) *For the primitive vectors $e_1, \cdots, e_r$ of the 1-faces of σ, there exists an element $m(\sigma) \in M_{\mathbf{Q}}$ such that $\langle m(\sigma), e_i \rangle = 1$ for every i.*

(b) $\sigma \cap \{n \in N; \langle m(\sigma), n \rangle \leqq 1\} = \{0, e_1, \cdots, e_r\}$.

(b′) $\sigma \cap \{n \in N; \langle m(\sigma), n \rangle < 1\} = \{0\}$.

Proof. We may assume that X is affine. Let $\sigma = \langle e_1, \cdots, e_r \rangle$ be the corresponding cone in $N_{\mathbf{R}}$.

(i) Let D_i be the T-stable prime divisor on X corresponding to e_i. Then X is $\mathbf{Q}$-factorial if and only if the D_i are $\mathbf{Q}$-Cartier divisors for all i. This condition is equivalent to the existence of elements $m_i \in M_{\mathbf{Q}}$ such that $\langle m_i, e_j \rangle = \delta_{ij}$, i.e., σ is simplicial.

(ii) Noting that $K_X + \sum_{i=1}^r D_i \sim 0$, we know that $\mathcal{O}_X(rK_X)$ for $r \in \mathbf{Z}$ is generated by those $m \in M$ with $\langle m, e_i \rangle \geqq r$. Therefore, for X to be an r-Gorenstein variety, it is necessary and sufficient that there exists $m_0 \in M$ such that $\langle m_0, e_i \rangle = r$ for all i. Thus the condition (a) holds if and only if X is a $\mathbf{Q}$-Gorenstein variety. A prime divisor D on an equivariant desingularization of X corresponds to a primitive vector v of N in σ. The discrepancy at D is given by $\langle m_0/r, v \rangle - 1$. Thus we have the conditions for X to have terminal or canonical singularities. q.e.d.

In the following, we shall discuss the contraction of extremal rays for toric morphisms. Let $f: X \to S$ be a toric morphism of complete toric varieties with $d = \dim X$. Assume that X has only $\mathbf{Q}$-factorial terminal singularities. If R is an extremal ray of $\overline{NE}(X/S)$, we have the contraction morphism $\varphi: X \to Z$ of the extremal ray R by the Contraction Theorem. We know that, for any curve C in a fiber of f, $\varphi(C) =$ a point if and only if $\mathrm{cl}(C) \in R$. Since R is T-invariant, so is the exceptional locus of φ. Therefore φ is T-equivariant, which implies that Z and φ are toric. Let F_X and F_Z be the fans (which may be degenerated) corresponding to X and Z, respectively. Then we obtain the set of walls $F_Z^{(d-1)}$ by taking away the walls corresponding to R from $F_X^{(d-1)}$, i.e.,

$$F_Z^{(d-1)} = F_X^{(d-1)} - \{w; \mathrm{cl}(w) \in R\}.$$

As a matter of fact, for toric morphisms of complete toric varieties, we have the Contraction Theorem for any extremal ray of $\overline{NE}(X/S)$ in terms of the geometry of the corresponding fans without any assumption on the singularities of X (cf. [R3], see also Theorem 7-3-7).

By Proposition 5-2-1, we can put $R=\boldsymbol{R}_+[l]$ for some 1-stratum l of X in a fiber of f. Since l is a complete 1-dimensional normal toric variety, we get $l\cong\boldsymbol{P}^1$. Let w be a wall, i.e., a $(d-1)$-dimensional cone of F_X which corresponds to l. Then w separates two d-cones Δ_d and Δ_{d+1}. Let $e_1, \cdots, e_{d-1}$ be the primitive vectors of the 1-faces of w and let e_d, e_{d+1} be the primitive vectors of the opposite 1-faces of Δ_d, Δ_{d+1}, respectively. Since the $e_1, \cdots, e_d$ form a basis of $N_{\boldsymbol{R}}$, we obtain a linear relation

$$(*)\qquad \sum_{i=1}^{d+1} a_i e_i = 0$$

with $a_{d+1}=1$. Since e_d and e_{d+1} lie on opposite sides of w, we know that $a_d>0$. By reordering the e_i, we may assume that

$$a_i\begin{cases} <0 & \text{for } 1\leqq i\leqq\alpha \\ =0 & \text{for } \alpha+1\leqq i\leqq\beta \\ >0 & \text{for } \beta+1\leqq i\leqq d+1 \end{cases}$$

where $0\leqq\alpha\leqq\beta\leqq d-1$. Set

$$\Delta=\Delta(w)=\langle e_1, \cdots, e_{d+1}\rangle.$$

Then Reid observes that there exist the following two polyhedral decompositions of $\Delta(w)$:

$$(**)\qquad \Delta(w)=\bigcup\nolimits_{\beta+1\leqq j\leqq d+1}\Delta_j=\bigcup\nolimits_{1\leqq j\leqq\alpha}\Delta_j,$$

where

$$\Delta_j=\langle e_1, \cdots, \hat{e}_j, \cdots, e_{d+1}\rangle \quad (\text{for } j\leqq\alpha \text{ or } j\geqq\beta+1).$$

Let A be the exceptional locus of φ. Since A corresponds to the cone $\langle e_1, \cdots, e_\alpha\rangle$, we have $\dim A=d-\alpha$, while $\varphi(A)$ corresponds to the cone $\langle e_1, \cdots, e_\alpha, e_{\beta+1}, \cdots, e_{d+1}\rangle$, hence $\dim\varphi(A)=\beta-\alpha$. In particular, φ is of flipping type if and only if $\alpha\geqq 2$. Note that we obtain $F_Z^{(d)}$ from $F_X^{(d)}$ by replacing the Δ_j for $j=\beta+1, \cdots, d+1$ by $\Delta(w)$.

Now we shall prove the Flip Conjecture for toric morphisms. Assume that the contraction φ of the extremal ray R is of flipping type, i.e., $\alpha\geqq 2$. The first thing to show is the existence of the flip for φ (the Flip Conjecture I).

Recall the two distinct simplicial subdivisions (∗∗) of $\Delta(w)$. The first subdivision gives us a relation between $F_X^{(d)}$ and $F_Z^{(d)}$. Consider the second subdivision. Let F_1 be the simplicial subdivision of F_Z defined by

$$F_1^{(d)}=F_Z^{(d)}-(\{\Delta(w);\ w\in \mathrm{cl}(R)\})\cup\{\Delta_j;\ w\in \mathrm{cl}(R),\ 1\leqq j\leqq\alpha\},$$

or alternatively by

$$F_1^{(d-1)}=F_Z^{(d-1)}\cup\{w_{jk};\ w\in \mathrm{cl}(R),\ 1\leqq j<k\leqq\alpha\},$$

where

$$w_{jk}=\langle e_1,\ \cdots,\ \hat{e}_j,\ \cdots,\ \hat{e}_k,\ \cdots,\ e_{d+1}\rangle.$$

Let $\varphi_1\colon X_1=X_{F_1}\to Z$ be the toric morphism corresponding to the subdivision F_1 of F_Z.

Theorem 5-2-3 (cf. [R3, (3.4), (4.5)′]). *φ_1 gives the flip for φ.*

For toric morphisms, the Flip Conjecture II can be proved by means of the notion of the shed: Let σ be a cone in $N_{\boldsymbol{R}}$ and let $e_1, \cdots, e_s$ be the primitive vectors of 1-faces of σ. We define

$$\text{shed}\ \sigma:=\text{the convex hull of the set } \{0,\ e_1,\ \cdots,\ e_s\} \text{ in } N_{\boldsymbol{R}}.$$

For a fan F of $N_{\boldsymbol{R}}$, we define

$$\text{shed}\ F:=\textstyle\bigcup_{\sigma\in F^{(d)}}\text{shed}\ \sigma.$$

Then we have

$$\mathrm{vol}(\text{shed}\ F_1)<\mathrm{vol}(\text{shed}\ F_X).$$

Since $d!\,\mathrm{vol}(\text{shed}\ F)\in\boldsymbol{Z}$, it is now obvious that any sequence of flips must terminate (cf. [R3, (4.4)]). In this manner, Reid proved the Minimal Model Conjecture for toric morphisms.

Here are some examples of flipping contractions and their flips:

Example 5-2-4 ([R3]). The examples of flips for toric morphisms are given by the formula (∗) with $\alpha\geqq 2$. This first example is given by

$$ae_1+(r-a)e_2=re_3+e_4$$

where e_1, e_2, e_3 form a basis of N with rank $N=3$. The flip in this case is represented by the following diagram in Figure 3. The exceptional loci l and l_1 of φ and φ_1, respectively, are isomorphic to $\boldsymbol{P}^1$. By direct calculation we have

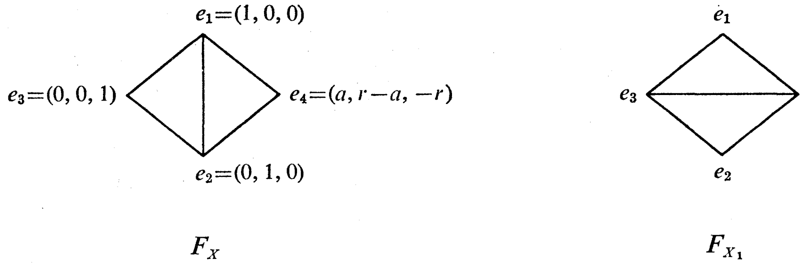


Figure 3

$$(K_X.l) = -1/r \quad \text{and} \quad (K_{X_1}.l_1) = 1/a(r-a).$$

Example 5-2-5. The second example of flips for 3-dimensional toric varieties is given by the following formula

$$ae_1 + e_2 = re_3 + e_4.$$

The flip in this case is represented by the diagram in Figure 4. The exceptional loci l and l_1 of φ and φ_1, respectively, are again isomorphic to $\boldsymbol{P}^1$, and we have

$$(K_X.l) = -(r-a)/r \quad \text{and} \quad (K_{X_1}.l_1) = (r-a)/a.$$

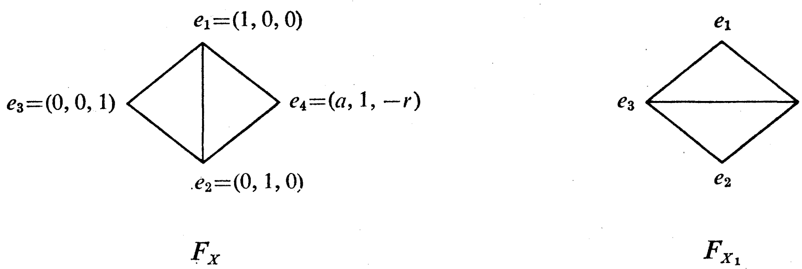


Figure 4

The morphisms given in Examples 5-2-4 and 5-2-5 are the only contractions of flipping type from $\boldsymbol{Q}$-factorial terminal toric varieties of dimension 3 by the theorem of White-Frumkin in [D2].

Example 5-2-6 (Hironaka [Hi1]). Let C_1, C_2 and C_3 be three non-singular curves on a nonsingular 3-fold X intersecting transversally at a point $p \in X$. Let X_1 be the blow up of X at p and let $E \cong \boldsymbol{P}^2$ be the exceptional locus on X_1. In the following, we denote the strict transforms by $'$. We take such lines l_1, l_2 and l_3 on $\boldsymbol{P}^2 \cong E$ that join two of $C_1' \cap E$, $C_2' \cap E$ and $C_3' \cap E$. Blow up X_1 with centers C_1', C_2' and C_3' to obtain X_2. Then the l_i' are $(-1, -1)$-curves, i.e., $N_{l_i'/X_2} = \mathcal{O}_{\boldsymbol{P}^1}(-1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-1)$. Blow up X_2 with centers l_1', l_2' and l_3' to obtain X_3. Let E_i be the inverse image of l_i', which turns out to be isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. It is well-known that the E_i can be contracted in the other direction to give X_4. Let $\tilde{C}_i$ be the image of E_i on X_4. Note that $E' \cong \boldsymbol{P}^2$ on X_4 and $N_{E'/X_4} \cong \mathcal{O}_{\boldsymbol{P}^2}(-2)$. Finally the contraction of E' on X_4 to a point p' gives X'. Then we have the following:

(1) (X', p') is a terminal singularity: in fact, it is a quotient singularity of type (1/2) (1, 1, 1).

(2) The $\tilde{C}_i'$ are extremal curves of $\overline{NE}(X'/X)$ with $(K_{X'}.\tilde{C}_i') = -1/2$, and the contractions of the $\tilde{C}_i'$ are as given in Example 5-2-4 setting $r=2$ and $a=1$.

The following example due to Mori shows that there exists a contraction of flipping type for nonsingular 4-folds, while the classification by Mori [Mo2] tells us that there is no contraction of flipping type on nonsingular 3-folds.

Example 5-2-7 (cf. [R3, (3.9)]). The formula $(*)$ in this example is given by

$$e_1 + e_2 = e_3 + e_4 + e_5,$$

where e_1, e_2, e_3 and e_4 form a basis of N with rank $N=4$. Then X and X_1 are both nonsingular. The exceptional locus A of φ is isomorphic to $\boldsymbol{P}^2$ with $N_{A/X} \cong \mathcal{O}_{\boldsymbol{P}^2}(-1) \oplus \mathcal{O}_{\boldsymbol{P}^2}(-1)$, while the exceptional locus A_1 of φ_1 is isomorphic to $\boldsymbol{P}^1$ with $N_{A_1/X} \cong \mathcal{O}_{\boldsymbol{P}^1}(-1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-1)$. The flip for φ can also be described in the following way: First, blow up X with the center A. Then the exceptional locus for this blowing-up is isomorphic to $\boldsymbol{P}^2 \times \boldsymbol{P}^1$ and it can be contracted in the other direction to give X_1.

We conclude this section with an example due to Mukai.

Example 5-2-8 (cf. [Mu, Theorem 0.7]). Let X be a nonsingular projective variety with a symplectic structure, that is, there is a nowhere vanishing holomorphic 2-form ω. Assume that there exists a nonsingular subvariety $P \subset X$ of $\operatorname{codim}_X P \geqq 2$ which has a structure of a $\boldsymbol{P}^n$-bundle $\sigma: P \to B$ for some n. Assume further that there is a $\boldsymbol{Q}$-divisor D on X such that (X, D) is weak log-terminal, and that a line l on a fiber of σ is an extremal ray of $\overline{NE}(X)$ for (X, D) whose contraction $\varphi: X \to X'$ is of flipping type with exceptional locus coinciding with P which, restricted to P, is nothing but σ. Then we can construct the log-flip for φ as follows: Blow up X with the center P. Then the exceptional divisor for this blowing-up can be contracted in the other direction to give a nonsingular variety X_1 and hence the morphism $\varphi_1: X_1 \to X'$. (X_1 is again symplectic.)

We claim that φ_1 gives the log-flip for φ. In fact

(i) $\varphi_1: X_1 \to X'$ is isomorphic in codimension one by construction.

(ii) $K_X \approx 0$ in $N^1(X/X')$ and $K_{X_1} \approx 0$ in $N^1(X_1/X')$.

(iii) Since $\dim_{\boldsymbol{R}} N^1(X_1/X') \leq 1$, letting D_1 be the strict transform of D, we know that either

(1) $D_1 \approx 0$ in $N^1(X_1/X')$,

(2) $-D_1$ is φ_1-ample, or

(3) D_1 is φ_1-ample.

If (1) holds, then the Contraction Theorem tells us that $D_1 = \varphi_1^* D'$ for some $D' \in \mathrm{Div}(X') \otimes \boldsymbol{Q}$ and hence $D = \varphi^* D'$, which contradicts the fact that $-D$ is φ-ample. If (2) holds, then $X_1 \cong \mathrm{Proj}\,(\bigoplus_{m \geqq 0} \varphi_{1*} \mathcal{O}_{X_1}(-mD_1)) \cong \mathrm{Proj}\,(\bigoplus_{m \geqq 0} \varphi_* \mathcal{O}_X(-mD)) \cong X$ over X', a contradiction. Therefore the case that actually occurs is (3), hence φ_1 is the log-flip for φ.

In the above example, K_X is numerically trivial. The log-flip is geometrically symmetric, while it is asymmetric when one considers the pairs (X, D) and (X_1, D_1).

Chapter 6. Abundance Conjecture

In this chapter, we give generalizations of the Non-Vanishing Theorem and the Base Point Free Theorem, replacing the ample (or nef and big divisors) in the original statements by nef and abundant divisors, which we shall define in the following section.

§ 6-1. Nef and abundant divisors

Definition 6-1-1 (cf. [Ka8]). Let X be a complete normal variety of dimension d, and let $D \in \mathrm{Div}(X) \otimes \boldsymbol{Q}$ be a nef divisor on X. We define the *numerical Iitaka dimension* to be

$$\nu(X, D) := \max\,\{e;\ D^e \not\approx 0\}.$$

Then it is easy to prove that $\kappa(X, D) \leqq \nu(X, D)$ (cf. [Ka8]). D is said to be *abundant* if the equality $\kappa(X, D) = \nu(X, D)$ holds. Let $\pi: X \to S$ be a proper surjective morphism of normal varieties and let D be a $\boldsymbol{Q}$-Cartier divisor on X. Then D is said to be *π-abundant* if $D|_{X_\eta}$ is abundant, where X_η is the generic fiber of π.

Remark 6-1-2. It is easy to see that the following three conditions are equivalent (cf. [Ka4]):

(a) $\nu(X, D) = d = \dim X$,

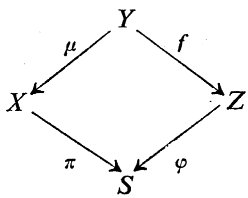
(b) $\kappa(X, D) = d$,

(c) $(D^d) > 0$.

It follows immediately that D is abundant if $\kappa(X, D) = d$ or $d - 1$.

The following proposition shows that nef and abundant divisors are birationally the pull-back of nef and big divisors.

Proposition 6-1-3 (cf. [Ka8, Proposition 2.1]). *Let X be a normal variety with a proper morphism $\pi: X \to S$ onto a variety S, and let $D \in \mathrm{Div}(X)\otimes \boldsymbol{Q}$ be a π-nef and π-abundant $\boldsymbol{Q}$-Cartier divisor. Then there exists a diagram*



which satisfies the following conditions:

(i) *μ, f and φ are projective morphisms,*

(ii) *Y and Z are nonsingular varieties,*

(iii) *μ is a birational morphism and f is a surjective morphism having connected fibers, and*

(iv) *There exists a φ-nef and φ-big $\boldsymbol{Q}$-Cartier divisor $D_0 \in \mathrm{Div}(Z)\otimes\boldsymbol{Q}$ such that $\mu^*D = f^*D_0$.*

Remark 6-1-4. In the situation of the proposition above, let $D' \in \mathrm{Div}(X)\otimes\boldsymbol{Q}$ be another $\boldsymbol{Q}$-Cartier divisor on X which is nef. Assume that $\nu(X_\eta, (D+D')|_{X_\eta}) = \nu(X, D)$ and that $\kappa(X, (D+D')) \geq 0$. Then there exists a φ-nef divisor D_0' on Z in the diagram of the proposition such that $\mu^*D' = f^*D_0'$ (cf. [Ka8]).

In order to carry out the proofs of the generalized versions of theorems for nef and abundant divisors, we introduce the notion of a generalized normal crossing variety.

Definition 6-1-5 (cf. [Ka8]). A reduced scheme X is called *a generalized normal crossing variety* if the complete local rings of X at the closed points x on X are isomorphic to

$$k[[x_{01}, \cdots, x_{0r_0}]]\hat{\otimes}(\hat{\otimes}_{i=1}^{t} k[[x_{i1}, \cdots, x_{ir_i}]]/(x_{i1}, \cdots, x_{ir_i}))$$

where integers t and r_i depend on x. Let X_0 be the normalization of X and let $\varepsilon_n: X_n = X_0 \times_X \cdots \times_X X_0$ ($n+1$ factors)$\to X$ be the projection onto X. A Cartier divisor D on X is called *permissible* if it induces Cartier divisors D_n on each X_n when pulled back by ε_n. We denote by $\mathrm{Div}_0(X)$ the group of permissible Cartier divisors. A generalized normal crossing divisor D on X is a permissible Cartier divisor such that the unions $D_n \cup B_n$ are reduced divisors with only normal crossings on each X_n (note that each X_n is nonsingular), where B_n is the union of the images on X_n of lower dimensional irreducible components of the $X_{n'}$, with $n' > n$ (B_n turns out to be a divisor with only normal crossings on X_n). Let D be an element of $\mathrm{Div}_0(X) \otimes \boldsymbol{Q}$ whose support is a generalized normal crossing divisor. Then we can define the permissible Cartier divisor $\ulcorner D \urcorner$ by the system of divisors $\ulcorner D_n \urcorner$ on the X_n.

Now we state some vanishing theorems for generalized normal crossing varieties. For the proofs and details, we refer the reader to [Ka8].

Theorem 6-1-6. *Let X be a complete generalized normal crossing variety, $L \in \mathrm{Div}_0(X) \otimes \boldsymbol{Q}$ and $D \in \mathrm{Div}_0(X)$. Assume that L is semi-ample and that the support of L is a generalized normal crossing divisor. Assume further that D is effective and that there exists an effective $D' \in \mathrm{Div}_0(X)$ such that $D + D' \in |mL|$ for some positive integer m with $mL \in \mathrm{Div}_0(X)$. Then the homomorphisms*

$$\varphi_D^i \colon H^i(X, \mathcal{O}_X(\ulcorner L \urcorner + K_X)) \longrightarrow H^i(X, \mathcal{O}_X(\ulcorner L \urcorner + D + K_X))$$

induced by the natural injection of sheaves $\mathcal{O}_X \to \mathcal{O}_X(D)$ are injective for all i.

Remark 6-1-7. A generalized normal crossing variety is locally complete intersection, and hence has an invertible dualizing sheaf ω_X. We denote by K_X here the corresponding line bundle.

Theorem 6-1-8. *Let X be a generalized normal crossing variety, let $\pi \colon X \to S$ be a proper morphism from X onto a variety S, and let $L \in \mathrm{Div}_0(X) \otimes \boldsymbol{Q}$ and $D \in \mathrm{Div}_0(X)$. Assume that L is π-semi-ample, and that the support of L is a generalized normal crossing divisor. Assume further that D is effective and that there exists an effective $D' \in \mathrm{Div}_0(X)$ such that $\mathcal{O}_X(D + D') \cong \mathcal{O}_X(mL)$ for some positive integer m with $mL \in \mathrm{Div}_0(X)$. Let $f \colon X \to Z$ be the morphism associated to the surjection $\pi^* \pi_* \mathcal{O}_X(nL) \to \mathcal{O}_X(nL)$ for some positive integer n such that $nL \in \mathrm{Div}_0(X)$ and that the natural homomorphism above becomes surjective. Assume that Z is irreducible the pull back by f induces a homomorphism $\mathrm{Div}(Z) \to \mathrm{Div}_0(X)$, i.e., if the morphisms $f \circ \varepsilon_n$ induce a surjective morphism from any irreducible component of each X_n onto Z. Then $R^i f_* \mathcal{O}_X(\llcorner L \lrcorner + K_X)$ are torsion free for all i.*

The Non-Vanishing Theorem for a generalized normal crossing variety with a nef and big divisor is of the following form.

Theorem 6-1-9 (cf. [Ka8]). *Let $f \colon X \to Z$ be a morphism from a complete generalized normal crossing variety X onto a complete variety Z, let*

$A \in \mathrm{Div}_0(X)\otimes \boldsymbol{Q}$, $H \in \mathrm{Div}_0(X)$, and let $q \in \boldsymbol{N}$ with $qA \in \mathrm{Div}_0(X)$, which satisfy the following conditions:

(i) *f induces a surjective morphism from any irreducible component of each X_n onto Z,*

(ii) *The support of A is a generalized normal crossing divisor on X and $\ulcorner A \urcorner$ is effective,*

(iii) *There is a nef Cartier divisor H_0 on Z such that $\mathcal{O}_X(qH)=f^*\mathcal{O}_Z(H_0)$, and*

(iv) *There is a nef and big Cartier divisor L_0 on Z such that $\mathcal{O}_X(q(H+A-K_X))=f^*\mathcal{O}_Z(L_0)$.*

Then there exists a positive integer p such that

$$H^0(X, \mathcal{O}_X(ptH+\ulcorner A \urcorner))\neq 0$$

for any sufficiently large integer t.

Remark 6-1-10. In contrast with the assertion of Theorem 2–1–1, we cannot say that $H^0(X, \mathcal{O}_X(tH+\ulcorner A \urcorner))\neq 0$ for all sufficiently large integer t, and actually we have to take a suitable positive integer p as above. For example, let X be an Enriques surface with f: $X \to Z = \mathrm{Spec}\, k$. Then $H = K_X$ satisfies the conditions of Theorem 6–1–9 with $A=0$ and $q=2$, while $H^0(X, \mathcal{O}_X(mK_X))\neq 0$ if and only if $m \equiv 0 \bmod 2$.

Finally we are in a position to give the generalized version of the Base Point Free Theorem.

Theorem 6-1-11 (cf. [Ka8], [Ny1]). *Let X be a normal variety of dimension d with $\Delta \in Z_{d-1}(X)\otimes \boldsymbol{Q}$ such that the pair (X, Δ) has only log-terminal singularities, and let π: $X \to S$ be a proper morphism onto a variety S. Assume that $H \in \mathrm{Div}(X)\otimes \boldsymbol{Q}$ satisfies the following conditions:*

(i) *H is π-nef,*

(ii) *$(H-(K_X+\Delta))$ is π-nef and π-abundant, and*

(iii) *$\nu(X_\eta, (aH-(K_X+\Delta))_\eta)=\nu(X_\eta, (H-(K_X+\Delta))_\eta)$ and $\kappa(X_\eta, (aH-(K_X+\Delta))_\eta)\geqq 0$ for some $a \in \boldsymbol{Q}$ with $a>1$, where η is the generic point of S.*

Then H is π-semi-ample.

Remark 6-1-12. The proof of Theorem 6–1–11 is almost the same as that of [Ka8, Theorem 6.1]. We note here that the condition on the singularities of the pair (X, Δ) cannot be replaced by the condition that (X, Δ) has only *weak* log-terminal singularities, as we saw in Remark 3-1-2 (2).

Corollary 6-1-13. *Let X be a normal variety of dimension d with $\Delta \in Z_{d-1}(X)\otimes \boldsymbol{Q}$ such that the pair (X, Δ) has only log-terminal singularities,* *and let π: $X \to S$ be a proper morphism onto a variety S. Assume that $K_X + \Delta$ is π-nef and π-abundant. Then $K_X + \Delta$ is π-semi-ample, and hence*

$$R(X/S, K_X + \Delta) := \bigoplus_{m \geq 0} \pi_* \mathcal{O}_X([m(K_X + \Delta)])$$

is finitely generated as an $\mathcal{O}_S$-algebra.

We conclude this chapter by stating the Abundance Conjecture.

Conjecture 6-1-14 (Abundance Conjecture). *If X is a minimal variety, then K_X is abundant.*

Remark 6-1-15. (1) If K_X is abundant, we call X a *good* minimal variety. In this terminology, the Abundance Conjecture is restated as follows: All minimal varieties are good.

(2) When $\nu(X, K_X) = \dim X$, the conjecture is automatically true. When $\nu(X, K_X) = 0$, the conjecture also holds, i.e., there exists a positive integer m such that $mK_X \sim 0$ (cf. [Ka9, Theorem 8.2]).

(3) We have only to show the following statement in order to prove the Abundance Conjecture (cf. [Ka8, Theorem 7.3]):

X being minimal, $K_X \not\approx 0$ implies $\kappa(X) > 0$.

(4) There are some works related to the Abundance Conjecture; see [W3], [Mi2], [Mi3], [Mi4] and [Mk2].

Chapter 7. Some applications and related problems

§ 7-1. Addition conjecture for the Kodaira dimension of an algebraic fiber space

In this section, we present the result of [Ka9], which claims that the Iitaka conjecture on the Kodaira dimension of algebraic fiber spaces follows from the Minimal Model Conjecture and the Abundance Conjecture.

Definition 7-1-1. An *algebraic fiber space* f: $X \to S$ is a proper surjective morphism between algebraic varieties such that the rational function field $\mathrm{Rat}(S)$ is algebraically closed in $\mathrm{Rat}(X)$. Note that the generic fiber X_η of an algebraic fiber space is geometrically irreducible. The *geometric generic fiber* $X_{\bar{\eta}}$ is defined by

$$X_{\bar{\eta}} := X_\eta \times_{\mathrm{Rat}(S)} \overline{\mathrm{Rat}(S)}$$

where $\overline{}$ denotes the algebraic closure.

Definition 7-1-2. Let f: $X \to S$ be an algebraic fiber space. A *minimal closed field of definition* of f is a minimal element with respect to the inclusion relation in the set of all the algebraically closed fields K contained in $\overline{\mathrm{Rat}(S)}$ which satisfy (one of) the following equivalent conditions:

(i) there is a finitely generated extension L of K such that

$$Q(L \otimes_K \overline{\mathrm{Rat}(S)}) \cong Q(\mathrm{Rat}(X) \otimes_{\mathrm{Rat}(S)} \overline{\mathrm{Rat}(S)}) \quad \text{over} \quad \overline{\mathrm{Rat}(S)},$$

where Q denotes the fraction field.

(ii) there exist an algebraic fiber space f': $X' \to S'$ with $\overline{\mathrm{Rat}(S')} = K$, a variety $\bar{S}$, a generically finite morphism π: $\bar{S} \to S$ and a surjective morphism ρ: $\bar{S} \to S'$ such that the main components of $X \times_S \bar{S}$ and $X' \times_{S'} \bar{S}$ are birationally equivalent over $\bar{S}$.

The *variation* $\mathrm{Var}(f)$ of f is the minimum of the transendental degrees over k of all the minimal closed fields of definition of f.

Theorem 7-1-3 ([Ka8, Theorem 7.2]). *Let f: $X \to S$ be an algebraic fiber space. Assume that there exists a good minimal model $X_{\bar{\eta},\min}$ of the geometric generic fiber $X_{\bar{\eta}}$ of f defined over $\overline{\mathrm{Rat}(S)}$. Then the minimal closed field of definition $K \subset \overline{\mathrm{Rat}(S)}$ of f is unique, and there exists a good minimal algebraic variety $X'_{\bar{\eta}',\min}$ defined over K such that there is an isomorphism over $\overline{\mathrm{Rat}(S)}$*

$$X'_{\bar{\eta}',\min} \times_K \overline{\mathrm{Rat}(S)} \cong X_{\bar{\eta},\min}.$$

Theorem 7-1-4 ([Ka8, Theorem 1.1]). *Let f: $X \to S$ be an algebraic fiber space with X and S being projective and nonsingular. Assume that there exists a good minimal model of the geometric generic fiber of f defined over $\overline{\mathrm{Rat}(S)}$. Then the following assertions hold*:

(i) *There exists a positive integer n such that*

$$\kappa(S, \hat{\det}(f_* \omega^n_{X/S})) \geq \mathrm{Var}(f)$$

where $\hat{\det}(f_* \omega^n_{X/S})$ is the double dual of $\wedge^r(f_* \omega^n_{X/S})$ with $r = \mathrm{rank}\, f_* \omega^n_{X/S}$.

(ii) *If L is a line bundle on S with $\kappa(S, L) \geq 0$, then*

$$\kappa(X, \omega_{X/S} \otimes f^* L) \geq \kappa(X_\eta) + \mathrm{Max}\,\{\kappa(L), \mathrm{Var}(f)\}.$$

In particular, we have the following corollary.

Corollary 7-1-5. *In the situation of Theorem 7-1-4, we have the following assertions*;

(i) $\kappa(X, \omega_{X/S}) \geq \kappa(X_\eta) + \mathrm{Var}(f)$.

(ii) *If $\kappa(S) \geq 0$, then $\kappa(X) \geq \kappa(X_\eta) + \mathrm{Max}\{\kappa(S), \mathrm{Var}(f)\}$.*

(This is the so called *Iitaka-Viehweg conjecture.*)

As an application of Theorem 7-1-4, we obtain the following theorem.

Theorem 7-1-6 ([Ka8, Theorem 8.2]). *Let X be a normal projective variety with only canonical singularities. Assume that the canonical divisor K_X is numerically equivalent to zero. Then the following assertions hold*:

(i) $\kappa(X)=0$, *that is to say, there is a positive integer m such that* $mK_X \sim 0$.

(ii) *Albanese map* $\alpha_X: X \to A = \mathrm{Alb}(X)$ *is an etale fiber bundle, i.e., there is an etale covering* $\pi: B \to A$ *such that* $X \times_A B \cong F \times B$ *for some projective variety F.*

§ 7-2. Invariance of plurigenera

In this section, we explain the outline of Nakayama's work [Ny1], which asserts that the invariance of plurigenera of algebraic varieties under smooth projective deformation follows from the Minimal Model Conjecture and the Abundance Conjecture. The invariance of plurigenera was first proved by Iitaka [I2] in case of surfaces.

Theorem 7-2-1 ([I2]). *Plurigenera of compact complex surfaces are invariant under smooth holomorphic deformations.*

Theorem 7-2-2 ([Ny1, Theorem 8]). *Let* $f: X \to C$ *be a proper surjective morphism from an algebraic variety X with only canonical singularities onto a smooth curve C such that K_X is f-semiample. Fix a point* $0 \in C$. *Let* $X_0 = f^{-1}(0) = \sum_i a_i \Gamma_i$ *be the irreducible decomposition of the fiber and put* $I = \{i;\, a_i = 1\}$. *Then,*

$$\sum_{i \in I} P_m(\Gamma_i) \leqq P_m(X_\eta) \qquad \text{for any } m \geqq 1,$$

where X_η is the generic fiber and P_m denotes the m-genus.

The following is the main theorem of [Ny1].

Theorem 7-2-3 ([Ny1, Theorem 11]). *Let* $f: X \to C$ *be a projective surjective morphism from a nonsingular algebraic variety X to a nonsingular curve C. Assume that f has semistable fibers and satisfies the following two conditions*:

(i) *The Flip Conjecture holds for all morphisms over C which are birationally equivalent to f.*

(ii) *The generic fiber of f has a good minimal model.*

Then we have $\sum_{\Gamma_i \subset X_0} P_m(\Gamma_i) \leqq P_m(X_\eta)$.

We present a theorem concerning deformations of canonical singularities. By using Theorem 7-2-3, Kollár obtained the following result: Let f: $X \to C$ be a projective surjective morphism from an algebraic variety X onto a nonsingular curve C with a point t_0. Assume that the fiber $X_0 := f^{-1}(t_0)$ has only canonical singularities. Assume further the Minimal Model Conjecture and the Abundance Conjecture. Then there exists a neighborhood C_0 of t_0 in C such that the fibers $X_t := f^{-1}(t)$ have only canonical singularities for all $t \in C_0$.

In the following, we generalize his result by taking a different approach.

Theorem 7-2-4. *Let X be an affine variety and let x be a closed point of X. Assume that there exists an effective Cartier divisor X_0 of X such that $x \in X_0$ and (X_0, x) is a canonical singularity. Assume that the Minimal Model Conjecture holds for a desingularization of X. Then (X, x) is a canonical singularity.*

Proof. Let μ: $Y \to X$ be the relative canonical model of a desingularization of X. Let X_0' be the strict transform of X_0 by μ and let Y_0 be the pull-back $\mu^* X_0$. Taking the normalization σ: $Z \to X_0'$, we have the trace map

$$\sigma_* \omega_Z \longrightarrow \omega_{Y_0} = \omega_Y \otimes \mathcal{O}_{Y_0}.$$

Let r be a common multiple of the indices of Y and X_0. Taking the double dual of the homomorphism $(\sigma^* \sigma_* \omega_Z)^{\otimes r} \to (\sigma^* \omega_{Y_0})^{\otimes r}$, we have the map

$$(1) \qquad \omega_Z^{[r]} \longrightarrow \sigma^* \omega_{Y_0}^{[r]} = \sigma^* \omega_Y^{[r]}.$$

Let ν be the composite of σ and μ. Since X_0 is canonical, we have a natural map

$$(2) \qquad \nu^* \omega_{X_0}^{[r]} \longrightarrow \omega_Z^{[r]}.$$

By (1) and (2), we obtain a homomorphism $\nu^* \omega_{X_0}^{[r]} \to \sigma^* \omega_Y^{[r]}$. Hence

$$(3) \qquad \sigma^* K_Y = \nu^* K_{X_0} + (\text{some effective exceptional divisor}).$$

Since K_Y is μ-ample and σ is finite, $\sigma^* K_Y$ is ν-ample.

Now we claim that ν is an isomorphism and that $K_Y|_Z = K_X|_{X_0}$. Suppose that ν is not an isomorphism. Then by taking the hyperplane sections, we can construct a curve which is in a fiber of ν and has the nonpositive intersection number with $\sigma^* K_Y$ by the Hodge index theorem, which contradicts the fact that $\sigma^* K_Y$ is ν-ample.

Next we claim that X is a $\boldsymbol{Q}$-Gorenstein variety. Indeed, by a generalization of a result of Schlessinger [Sch], the canonical cover $\tilde{X}_0 \to X_0$ extends to a covering deformation $\tilde{X} \to X$ (see [A2, §§ 9-10], [Kol 1]). Since $\tilde{X}$ is a Gorenstein variety, X is a $\boldsymbol{Q}$-Gorenstein variety; in fact, $rK_X \in \mathrm{Div}(X)$. Hence

$$K_Y = \mu^* K_X + \Delta,$$

where Δ denotes a $\boldsymbol{Q}$-linear combination of exceptional divisors.

Since K_Y is μ-ample, Δ is μ-ample. This implies that $-\Delta$ is effective and its support coincides with the exceptional locus of μ. Now we have

$$\begin{aligned} rK_X|_{X_0} &= r\mu^* K_X|_Z \quad \text{(identifying } Z \text{ with } X_0) \\ &\geqq r(\mu^* K_X + \Delta)|_Z = rK_Y|_Z = rK_X|_{X_0}. \end{aligned}$$

Hence $\Delta = 0$. Thus $K_Y = \mu^* K_X$. This proves that X has only canonical singularities. q.e.d.

§ 7-3. Zariski decomposition in higher dimensions

In the following, we discuss the problem of the Zariski decomposition for higher dimensional varieties, a problem which should be closely related to the minimal model problem.

Theorem 7-3-1 ([Z], [Ft1]). *Let S be a smooth projective surface over k and let $D \in \mathrm{Div}(S) \otimes \boldsymbol{R}$ be a pseudo-effective divisor on S, i.e., the numerical class of D is in the closure of the cone in $N^1(S)$ generated by the classes of effective divisors. Then we have a unique effective $\boldsymbol{R}$-divisor $N = \sum_{i \in I} a_i E_i \in \mathrm{Div}(S) \otimes \boldsymbol{R}$, where the right hand side is the decomposition into irreducible components, satisfying the following conditions:*

(1) *$N = 0$ or the matrix $[(E_i . E_j)]_{(i,j) \in I \times I}$ is negative definite.*

(2) *$P := D - N$ is nef.*

(3) *$(P . E_i) = 0$ for every $i \in I$.*

Furthermore, if D is a $\boldsymbol{Q}$-divisor, then N is a $\boldsymbol{Q}$-divisor.

P and N are said to be the *positive* and the *negative part* of D, respectively. If the pair of a nonsingular surface S and a $\boldsymbol{Q}$-divisor $\Delta \in Z_{d-1}(S) \otimes \boldsymbol{Q}$ has only weak log-terminal singularities and when $\kappa(S, K_S + \Delta) \geqq 0$, we have a morphism $f \colon S \to S'$ from S onto its log-minimal model $(S', f_*(\Delta))$ (cf. [Ka1], [TM], [Ft2]). Then the positive part of the Zariski decomposition of $K_S + \Delta$ is the pull-back $f^*(K_{S'} + f_*(\Delta))$, and the support of the negative part coincides with the exceptional locus of f.

There are some works (cf. [Be1], [Ft3], [C], [Mw2], [Ka10]) which try to obtain the concept of the Zariski decomposition for higher dimensional varieties.

(A) In the situation of Theorem 7-3-1, N has the following characterization:

$N = \min\{F;$ an effective $\boldsymbol{R}$-divisor such that $D-F$ is nef$\}$.

Fujita's idea is to use this characterization to define the Zariski decomposition in higher dimension.

Definition 7-3-2. Let D be a $\boldsymbol{Q}$-Cartier divisor on a normal projective variety X over k. An effective $\boldsymbol{Q}$-Cartier divisor F on X is said to be *numerically fixed* by D if the following condition is satisfied: For any birational morphism $f\colon Y \to X$ from a normal projective variety Y to X and for any effective $\boldsymbol{Q}$-Cartier divisor E on Y which makes f^*D-E nef, $E-f^*F$ is effective.

A decomposition $D=P+N$ in $\mathrm{Div}(X)\otimes\boldsymbol{Q}$ is called the *Zariski decomposition of D in Fujita's sense* if the following conditions are satisfied:

(1) P is nef.

(2) N is effective and numerically fixed by D.

P and N are said to be the *positive* and the *negative part* of D, respectively. The Zariski decomposition in Fujita's sense is unique if it exists.

By using the notion of the Zariski decomposition in Fujita's sense, we can prove the following.

Proposition 7-3-3. *Let X_1 and X_2 be minimal varieties birationally equivalent to each other. Then the following equalities hold*:

$$\nu(X_1, K_{X_1}) = \nu(X_2, K_{X_2}) \quad \text{and} \quad \mathrm{index}(X_1) = \mathrm{index}(X_2).$$

Fujita proved the following theorem, which was generalized by Moriwaki [Mw2].

Theorem 7-3-4 (cf. [Ft3, Theorem 3.2]). *Let $\pi\colon X \to S$ be an elliptic 3-fold over $\boldsymbol{C}$, i.e., X and S are nonsingular projective varieties of dimensions 3 and 1, respectively, and the geometric generic fiber of π is an elliptic curve. Assume that the canonical divisor K_X is pseudo-effective. Then there exists a proper birational morphism $f\colon Y \to X$ such that f^*K_X (and hence K_Y) admits the Zariski decomposition in Fujita's sense, and its positive part is semi-ample. In particular, the canonical ring of X is finitely generated over $\boldsymbol{C}$.*

(B) In the situation of Theorem 7-3-1, the natural homomorphism $H^0(S, \mathcal{O}_S([mP])) \to H^0(S, \mathcal{O}_S([mD]))$ is bijective for any $m \in \boldsymbol{N}$. Cutkosky, Moriwaki and the first author use this property to define the Zariski decomposition for higher dimensional varieties:

Definition 7-3-5. Let $f\colon X \to S$ be a proper surjective morphism of normal varieties. An expression $D = P + N$ of $\boldsymbol{R}$-Cartier divisors D, P and N is called the *Zariski decomposition of D relative to f in C-K-M's sense* if the following conditions are satisfied:

(1) P is f-nef,

(2) N is effective, and

(3) the natural homomorphisms $f_*\mathcal{O}_X([mP]) \to f_*\mathcal{O}_X([mD])$ are bijective for all $m \in \boldsymbol{N}$.

P and N are said to be the *positive* and the *negative part* of D, respectively. Note that the Zariski decomposition in Fujita's sense is the one in C-K-M's sense. Cutkosky pointed out the following important:

Remark 7-3-6 (cf. [C, Example 1.6]). There exists an example of a big divisor D on a smooth projective 3-fold X such that $f^*(D)$ does not have a Zariski decomposition in C-K-M's sense for any birational morphism $f\colon Y \to X$ from a smooth projective 3-fold Y to X, if we replace the field of coefficients $\boldsymbol{R}$ by $\boldsymbol{Q}$.

But this example has the Zariski decomposition in C-K-M's sense if the field of coefficients is $\boldsymbol{R}$, which shows that it is necessary to consider real coefficients even in the case of decompositions of $\boldsymbol{Q}$-divisors.

The Zariski decomposition of an f-big $\boldsymbol{R}$-Cartier divisor D in C-K-M's sense is unique if it exists (cf. [Ka10, Proposition 4]). For any proper surjective toric morphism $f\colon X \to S$ between toric varieties, an effective Cartier divisor D on X always has the Zariski decomposition in C-K-M's sense (cf. [Ka10, Proposition 5]).

By extending the techniques which are used in the previous chapters, one can also prove:

Theorem 7-3-7 (cf. [Ka10, Theorem 1]). *Let $f\colon X \to S$ be a proper surjective morphism of normal algebraic varieties, and let Δ be a $\boldsymbol{Q}$-divisor on X such that the pair (X, Δ) has only log-terminal singularities. Assume that $K_X + \Delta$ is f-big and that the Zariski decomposition in C-K-M's sense exists in* $\mathrm{Div}(X) \otimes \boldsymbol{R}$:

$$K_X + \Delta = P + N.$$

Then the positive part P is f-semi-ample, and hence the relative log-canonical ring

$$R(X/S, K_X + \Delta) = \bigoplus_{m \geq 0} f_*\mathcal{O}_X([m(K_X + \Delta)])$$

is finitely generated as an $\mathcal{O}_S$-algebra.

Theorem 7-3-7 tells us in the situation above that the positive part of the decomposition is the pull-back of the log-canonical divisor of the log-minimal model, a result similar to that in dimension 2.

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