

On Polarized Manifolds Whose Adjoint Bundles Are Not Semipositive

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Introduction

Let L be an ample (not necessarily very ample) line bundle on a projective variety M with $\dim M = n$ having only rational normal Gorenstein singularities. Let ω be the dualizing sheaf and let K be the line bundle such that $\mathcal{O}_M(K) = \omega$. We will study the line bundles $K + tL$, where t is a positive integer. By the base point free theorem (cf. [K2; Theorem 2.6]), we have $\text{Bs}|m(K + tL)| = \emptyset$ for $m \gg 0$ if $K + tL$ is *numerically semipositive* (*= nef*, for short), which means $(K + tL)C \geq 0$ for any curve C in M . We do not know, however, how large m should be. Here we just pose the following:

Conjecture. $\text{Bs}|m(K + tL)| = \emptyset$ if $m > n + 1 - t$ and if $K + tL$ is nef.

In this paper we will study the case in which $K + tL$ is not nef. Our result is similar to those in [M1] and is based on the theory in [K2], [KMM]. We use also techniques in [M1] and [M2].

Basically we use the customary notation in algebraic geometry. Line bundles and the invertible sheaves of their sections are used interchangeably. Tensor products of them are denoted additively while we use multiplicative notation for intersection products in Chow rings. The pull-back of a line bundle B on V by a morphism $h: T \rightarrow V$ is denoted by B_T , or often just by B when confusion is impossible or harmless.

Acknowledgment

After the first version of this note was completed, I received an interesting preprint [Io] of P. Ionescu. There he independently obtained our results Theorems 1, 2 and 3' together with Corollaries 1 and 2 although he assumed that M is non-singular. His method is different from ours and is

Received November 7, 1985.

The author is partly supported by the Grants-in-Aid for Scientific as well as Co-operative Research, The Ministry of Education, Science and Culture, Japan.

based on a theorem which we use in (2.13) to prove Theorem 4. Without this technique, I would not have been able to complete the proof of Theorem 4 by ruling out the possibility of bad contractions. I thank him very much for sending this preprint. I should mention also that he gives various results on a threefold which contains a surface of non-general type as an ample divisor.

§ 1. Statement of main results

Definition. The Δ -genus of the polarized variety (M, L) is defined by $\Delta(M, L) = n + L^n - h^0(M, L)$.

A precise definition of the *sectional genus* $g(M, L)$ can be found in [F1], [F2; (1.2)]. Here we recall the formula $(K + (n-1)L)L^{n-1} = 2g(M, L) - 2$, which may be used to define $g(M, L)$.

The polarized variety (M, L) is said to be a *scroll* over a variety W if there exists a surjective morphism $f: M \rightarrow W$ such that $(F, L_F) \simeq (\mathbf{P}^r, \mathcal{O}(1))$ with $r = n - \dim W$ for every fiber F of f . This condition is equivalent to saying that $(M, L) \simeq (\mathbf{P}_W(\mathcal{E}), \mathcal{O}(1))$ for some ample vector bundle \mathcal{E} on W .

Theorem 1. $K + nL$ is nef unless $(M, L) \simeq (\mathbf{P}^n, \mathcal{O}(1))$. In particular, $K + (n+1)L$ is always nef.

Theorem 2. Suppose that $K + nL$ is nef. Then $K + (n-1)L$ is nef except in the following cases:

- (a) M is a hyperquadric in \mathbf{P}^{n+1} and $L = \mathcal{O}_M(1)$.
- (b) $(M, L) \simeq (\mathbf{P}^2, \mathcal{O}(2))$.
- (c) (M, L) is a scroll over a smooth curve.

Corollary 1. $g(M, L) = 0$ if and only if $\Delta(M, L) = 0$. Moreover, $g(M, L) \geq 0$ always.

Corollary 2. $g(M, L) = 1$ if and only if one of the following conditions is satisfied:

- (a) (M, L) is a del Pezzo variety, which means, $K + (n-1)L$ is linearly equivalent to zero (more precisely, see [F2; (5.6)]).
- (b) (M, L) is a scroll over a smooth elliptic curve.

Theorem 3. Suppose that $K + (n-1)L$ is nef. Then $K + (n-2)L$ is nef except in the following cases:

- (a) There exist a birational morphism $f: M \rightarrow W$ and an effective Weil divisor E on M such that $f(E)$ is a point and that $K + (n-1)L$ is trivial in $\text{Pic}(E)$.
- (b) There exists a surjective morphism f onto a normal projective

variety W with Picard number $\rho(W) = \rho(M) - 1$ and $\dim W \leq 2$. Moreover, any general fiber F of f is of one of the following types:

b0) If $\dim W = 0$, then $(F, L_F) = (M, L)$ is isomorphic to either $(\mathbf{P}^3, \mathcal{O}(j))$ with $j = 2$ or 3 , $(\mathbf{P}^4, \mathcal{O}(2))$, a hyperquadric in \mathbf{P}^4 with $L = \mathcal{O}_M(2)$, or a del Pezzo variety.

b1) If W is a curve, then (F, L_F) is either $(\mathbf{P}^2, \mathcal{O}(2))$ or a hyperquadric in \mathbf{P}^n with $L_F = \mathcal{O}_F(1)$.

b2) If $\dim W = 2$, then $(F, L_F) \simeq (\mathbf{P}^{n-2}, \mathcal{O}(1))$.

Theorem 3'. Let things be as in Theorem 3 and suppose in addition that M is non-singular. Then:

In the above case (a), $(E, L_E) \simeq (\mathbf{P}^{n-1}, \mathcal{O}(1))$ and f is the contraction of E to a smooth point.

In case b1) and if $(F, L_F) \simeq (\mathbf{P}^2, \mathcal{O}(2))$, then every fiber of f is $(\mathbf{P}^2, \mathcal{O}(2))$.

In case b2), W is smooth and f makes (M, L) a scroll over W .

Theorem 4. Suppose that M is non-singular, $n \geq 4$ and $K + (n-2)L$ is nef. Then $K + (n-3)L$ is nef except in the following cases:

(a) There is a birational morphism $f: M \rightarrow W$ onto a normal projective variety W with Picard number $\rho(W) = \rho(M) - 1$. Moreover, $X = \{x \in W \mid \dim f^{-1}(x) > 0\}$ is at most of dimension one and $E = f^{-1}(X)$ is a prime divisor on M . The type of E is classified further as follows:

a1) $\dim X = 1$ and $(E_x, L_x) \simeq (\mathbf{P}^{n-2}, \mathcal{O}(1))$ for any smooth point x on X , where E_x is the fiber of $E \rightarrow X$ over x and L_x is the restriction of L to E_x . In this case the restriction of $\mathcal{O}_M[E]$ to E_x is $\mathcal{O}(-1)$.

a2) X is a point and E is isomorphic to a (possibly singular) hyperquadric in \mathbf{P}^n . In this case $L_E = \mathcal{O}_E(1)$ and $\mathcal{O}[E]_E = \mathcal{O}_E(-1)$.

a3) X is a point and $(E, L_E) \simeq (\mathbf{P}^3, \mathcal{O}(2))$. In this case $\mathcal{O}[E]_E = \mathcal{O}(-1)$.

a4) X is a point, $(E, L_E) \simeq (\mathbf{P}^{n-1}, \mathcal{O}(1))$ and $\mathcal{O}[E]_E = \mathcal{O}(-2)$.

(b) There is a morphism $f: M \rightarrow W$ onto a normal projective variety W with $\dim W \leq 3$ and $\rho(W) = \rho(M) - 1$ such that any general fiber F of f is connected. Moreover, the type of the polarized manifold (F, L_F) is classified as follows:

b0) If $\dim W = 0$, then $(F, L_F) = (M, L)$ is isomorphic to either $(\mathbf{P}^6, \mathcal{O}(2))$, $(\mathbf{P}^5, \mathcal{O}(2))$, $(\mathbf{P}^4, \mathcal{O}(j))$ with $j = 3$ or 4 , a hyperquadric in \mathbf{P}^8 with $L = \mathcal{O}(2)$, a hyperquadric in \mathbf{P}^5 with $L = \mathcal{O}(2)$ or $\mathcal{O}(3)$, $(M, 2A)$ for some del Pezzo 4-fold (M, A) , or a Mukai manifold (this means $K = (2-n)L$).

b1) If $\dim W = 1$, then (F, L_F) is isomorphic to either $(\mathbf{P}^3, \mathcal{O}(j))$ with $j = 2$ or 3 , $(\mathbf{P}^4, \mathcal{O}(2))$, a hyperquadric in \mathbf{P}^4 with $L = \mathcal{O}(2)$, or a del Pezzo manifold.

b2) If $\dim W = 2$, then (F, L_F) is isomorphic to either $(\mathbf{P}^2, \mathcal{O}(2))$ or a hyperquadric in \mathbf{P}^{n-1} with $L = \mathcal{O}(1)$.

b3) If $\dim W = 3$, then $(F, L_F) \simeq (\mathbf{P}^{n-3}, \mathcal{O}(1))$.

§ 2. Proofs

(2.1) Let $\pi: M' \rightarrow M$ be a desingularization of M and let ω' (resp. K') be the canonical sheaf (resp. bundle) of M' . Then $\pi_*\omega' = \omega$ since M has only rational singularities. Hence, by [GR], [K1] and [V], we have $h^i(M, K+A) = h^i(M', K' + \pi^*A) = 0$ for any $i > 0$ and any nef line bundle A on M with $A^n > 0$.

(2.2) Using the natural homomorphism $\pi^*\pi_*\omega' \rightarrow \omega'$, we infer that $K' = \pi^*K + E$ for some effective divisor E on M' . Hence M has only canonical singularities. Now we apply the theory in [K2] (Cone Theorem and Contraction Theorem, see also [KMM]) to obtain:

Key Lemma. *If $K+tL$ is not nef for $t > 0$, there exist a curve R in M and a morphism $f: M \rightarrow W$ onto a normal projective variety W such that*

- 1) every fiber of f is connected,
- 2) a curve C in M is numerically proportional to R (this means $C \sim mR$ for some $m > 0$, where \sim denotes the numerical equivalence) if and only if $f(C)$ is a point,
- 3) $B \in \text{Pic}(M)$ comes from $\text{Pic}(W)$ if and only if $BR = 0$,
- 4) $B \in \text{Pic}(M)$ is relatively ample with respect to f if and only if $BR > 0$,
- 5) $(K+tL)R < 0$.

Such a morphism f will be called the *contraction of the extremal ray* spanned by R . Note that $\rho(W) = \rho(M) - 1$ by the condition 3).

(2.3) **Lemma.** *Let things be as in (2.2) and suppose in addition that f is birational. Let x be a point on W and let X be a subscheme of M with $\dim X > 0$ such that $\text{Supp}(X) \subset f^{-1}(x)$. Then $H^q(X, \mathcal{O}_X(K+B)) = 0$ for any $q \geq \dim f^{-1}(x)$ and any $B \in \text{Pic}(M)$ with $BR \geq 0$.*

Proof (due to the idea in [M2]). Clearly it suffices to consider the case $q = \dim f^{-1}(x) = \dim X$. Take a very ample line bundle H on W and let Λ be the linear subsystem of $|H|$ consisting of members containing x . Take general members D_1, \dots, D_{n-q} of $f^*\Lambda$ and let $V_j = dD_1 \cap \dots \cap dD_j$. Then $\text{codim } V_j = j$. Moreover, we take a sufficiently large d such that X is a subscheme of V_{n-q} . Setting $V_0 = M$, we claim $H^p(V_j, K+B+sH) = 0$ for any $p > 0, j \geq 0, s \gg 0$.

When $j=0$, the assertion follows from (2.1) because of the properties 3) and 4) in (2.2). So suppose $j > 0$. Since $\dim V_j = n-j$, the defining equations of D_i 's form a regular sequence at every point of V_j . Hence we have an exact sequence $0 \rightarrow \mathcal{O}_{V_{j-1}}(-dH) \rightarrow \mathcal{O}_{V_{j-1}} \rightarrow \mathcal{O}_{V_j} \rightarrow 0$. This gives an exact sequence $H^p(V_{j-1}, K+B+sH) \rightarrow H^p(V_j, K+B+sH) \rightarrow H^{p+1}(V_{j-1}, K+B+(s-d)H)$. So we easily finish the proof of the claim by induction on j .

Now we use $0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{V_{n-q}} \rightarrow \mathcal{O}_X \rightarrow 0$, where \mathcal{I} is the ideal of the subscheme X in V_{n-q} . Since $\dim(\text{Supp}(\mathcal{I})) \leq q$, we infer $h^q(X, K+B+sH) \leq h^q(V_{n-q}, K+B+sH) = 0$ for $s \gg 0$ from the above claim. On the other hand $H_X = 0$ since $f(X)$ is a point. So $H^q(X, K+B) = 0$.

(2.4) **Lemma.** *Let f, x be as in (2.3) and let X be an irreducible component of $f^{-1}(x)$ such that $\dim X = \dim f^{-1}(x) = q > 0$. Let $g: Y \rightarrow X$ be a desingularization of X . Then $H^q(Y, g^*(K+B)) = 0$ for any $B \in \text{Pic}(M)$ with $BR \geq 0$.*

Proof. Let \mathcal{C} be the cokernel of the natural homomorphism $\mathcal{O}_X \rightarrow g_*\mathcal{O}_Y$. Since $\text{Supp}(\mathcal{C})$ is the set of points at which X is not normal, its dimension is $< q$. So $h^q(X, g_*\mathcal{O}_Y(K+B)) \leq h^q(X, K+B) = 0$ by (2.3). Next we use the Leray spectral sequence with $E_2^{i,j} = H^i(X, R^jg_*\mathcal{O}_Y(K+B))$ converging to $H^{i+j}(Y, K+B)$. In order to prove the lemma, it is enough to show $E_2^{i,j} = 0$ if $i+j = q, j > 0$.

Now, since $x \in \text{Supp}(R^jg_*\mathcal{O}_Y)$ implies $\dim g^{-1}(x) \geq j$, we infer

$$\dim(\text{Supp}(R^jg_*\mathcal{O}_Y)) \leq \dim E - j < q - j,$$

where E is the exceptional divisor of g . This implies $E_2^{i,j} = 0$ for $j = q - i > 0$.

(2.5) **Lemma.** *Let f be as in (2.3) (in particular birational) and suppose that $r = \dim f^{-1}(x) > 0$ for some point x on W . Then $(K+rA)R \geq 0$ for any ample line bundle A on M .*

Proof. We will derive a contradiction assuming $(K+rA)R < 0$. We may assume that there is X as in (2.4) such that $\dim X = \dim(f^{-1}(x)) = r$ for $x = f(X)$. Let $g: Y \rightarrow X$ be as above. By the property 4) in (2.2), we infer that $-(K+sA)$ is ample on X for $s \leq r$. So $H^i(Y, K+sA) = 0$ for $i < r, s \leq r$. Combining this with (2.4), we obtain $\chi(Y, K+sA) = 0$ for $0 \leq s \leq r$. This implies $\chi(Y, K+sA) \equiv 0$ because $\chi(Y, K+sA)$ is a polynomial in s of degree $\leq r$. This is impossible by the Riemann-Roch theorem because $A_Y > 0$.

(2.6) *Proof of Theorem 1.* Suppose that $K+nL$ is not nef. Take R and f as in (2.2) such that $(K+nL)R < 0$. By (2.5), f is not birational. Any general fiber F of f has only rational Gorenstein singularities, and its canonical bundle is the restriction of K . So $H^i(F, K+sL) = 0$ for any $s > 0, i > 0$.

On the other hand, $-(K+nL)$ is ample on F by the property 4) in (2.2). Hence $\chi(F, K+sL) = 0$ for $1 \leq s \leq n$. If $\dim F < n$, this would imply $\chi(F, K+sL) \equiv 0$, which is absurd. Thus we conclude $\dim F = n, M = F$

and W is a point.

Moreover we may set $\chi(M, K+sL) = c(s-1)(s-2)\cdots(s-n)$ for some constant c . On the other hand, by the Riemann-Roch theorem, we have $\chi(M, K+sL) = ds^n/n! + ks^{n-1}/(n-1)! + \mathcal{O}(s^{n-2})$ for $d=L^n$ and $k=KL^{n-1}/2$. Hence $d=c(n!) = -KL^{n-1}/(n+1)$. So $(K+(n+1)L)L^{n-1} = 0$ while $h^0(M, K+(n+1)L) = \chi(M, K+(n+1)L) = c(n!) = d > 0$. Therefore $K+(n+1)L = 0$ since L is ample. Moreover $d=1$. Now we have $h^0(M, L) = h^0(M, K+(n+2)L) = \chi(M, K+(n+2)L) = n+1$. So $\Delta(M, L) = 0$. This implies $(M, L) \simeq (\mathbf{P}^n, \mathcal{O}(1))$. Thus we complete the proof.

(2.7) *Proof of Theorem 2.* Assume that $K+(n-1)L$ is not nef. Take R and f as in (2.2) such that $(K+(n-1)L)R < 0$. Then f is not birational by (2.5).

Suppose that $\dim W > 0$. Since any general fiber F of f has only rational Gorenstein singularities, Theorem 1 applies to (F, L_F) . Thus we obtain $\dim W = 1$, $(F, L_F) \simeq (\mathbf{P}^{n-1}, \mathcal{O}(1))$. Now, for every fiber Z of f , we have $L^{n-1}Z = L^{n-1}F = 1$ since W is smooth. So Z is irreducible and reduced because L is ample. Moreover we have $\Delta(Z, L_Z) = 0$ by the lower-semicontinuity of the Δ -genus since f is flat. So $(Z, L_Z) \simeq (\mathbf{P}^{n-1}, \mathcal{O}(1))$. Thus we are in Case c).

Suppose that $\dim W = 0$. Then every curve in M is numerically proportional to R , hence every line bundle on M is numerically proportional to L .

If $L = aA$ numerically for some $a > 1$ and $A \in \text{Pic}(M)$, then $K+(n-1)aA$ is not nef. So $(n-1)a \leq n$ by Theorem 1. This is possible only when $n = a = 2$. Moreover $(M, A) \simeq (\mathbf{P}^2, \mathcal{O}(1))$. Thus we are in Case b).

If the above is not the case, any line bundle on M is numerically an integral multiple of L . Since $K+nL$ is nef while $K+(n-1)L$ is not so, we have $K = -nL$ numerically. Moreover, by the property 3) in (2.2), we have $K+nL = 0$ in $\text{Pic}(M)$. Using (2.1) and the Serre duality, we obtain $0 = h^i(M, sL) = h^i(M, K+(n+s)L) = h^{n-i}(M, -(s+n)L)$ for $i > 0, s > -n$. So $\chi(M, sL) = 0$ for $-n < s < 0$ and $1 = \chi(M, \mathcal{O}_M) = (-1)^n \chi(M, -nL)$. From this we obtain $\chi(M, sL) = (s+1)\cdots(s+n-1)(2s+n)/n!$. So $L^n = 2$ and $h^0(M, L) = \chi(M, L) = n+2$. Hence $\Delta(M, L) = 0$ and M is a (possibly singular) hyperquadric in \mathbf{P}^{n+1} with $L = \mathcal{O}_M(1)$. Thus we are in Case a).

(2.8) *Proof of Corollary 1.* The "if" part is well-known. So it suffices to show $\Delta(M, L) = 0$ assuming $g(M, L) \leq 0$. Since $(K+(n-1)L)L^{n-1} = 2g(M, L) - 2 < 0$, $K+(n-1)L$ is not nef. In Case a) or b) of Theorem 2, we have $\Delta(M, L) = 0$. So we may assume (M, L) is a scroll over a smooth curve C . Then $(M, L) \simeq (\mathbf{P}_C(\mathcal{E}), \mathcal{O}(1))$ for some vector bundle \mathcal{E} on C . Hence $K = -nL + f^*(K_C + \det \mathcal{E})$. So $2g(M, L) - 2 = (f^*(K_C + \det \mathcal{E}) - L)L^{n-1} = \deg(K_C + \deg \mathcal{E}) - c_1(\mathcal{E}) = \deg K_C$. Thus the genus of C is $g(M, L)$ and C

is a rational curve. One easily sees $\Delta(M, L)=0$ for rational scrolls. Thus we complete the proof.

(2.9) *Proof of Corollary 2.* The “if” part is easy and well-known. We will prove the “only if” part. If $K+(n-1)L$ is not nef, we argue as in (2.8) to conclude that we are in Case b). If $K+(n-1)L$ is nef, we have $D \in |m(K+(n-1)L)|$ for some $m>0$. Since $L^{n-1}D=m(2g(M, L)-2)=0$, we infer $D=0$ because L is ample. So there is a cyclic étale covering $g: \tilde{M} \rightarrow M$ such that $g^*(K+(n-1)L)=0$. By (2.1) we have $\chi(M, \mathcal{O}_M)=1$ and similarly $\chi(\tilde{M}, \mathcal{O}_{\tilde{M}})=1$. Hence g must be an isomorphism. Thus $K+(n-1)L=0$ in $\text{Pic}(M)$ and we are in Case a).

(2.10) *Proof of Theorem 3.* Assume that $K+(n-2)L$ is not nef and let f, R be as in (2.2) such that $(K+(n-2)L)R<0$.

If $w=\dim W<n$, let F be a general fiber of f . By Theorem 1, $K+(n-w+1)L$ is nef on F . So $w \leq 2$. Moreover, if $w=2$, we have $(F, L_F) \simeq (\mathbb{P}^{n-2}, \mathcal{O}(1))$ and b2) is the case.

If $w=1$, $-K_F$ is numerically proportional to L_F . If (F, L_F) is a scroll over a curve, then we get $K_F=(1-n)L_F$ by restricting to fibers $\simeq \mathbb{P}^{n-2}$. This implies that (F, L_F) is a hyperquadric. If (F, L_F) is not a scroll, we apply Theorem 2 to show that b1) is valid.

If $w=0$, every line bundle on M is numerically a multiple of L . By Theorem 1, $(K+(n+1)A)R \geq 0$ for any ample A . So the number AR is bounded below. Let us take an ample line bundle A such that AR attains the minimum. Then, numerically, $K \sim kA$ and $L \sim lA$ for some integers k, l . By assumption we have $k+(n-1)l \geq 0$ and $k+(n-2)l < 0$. Moreover $k \geq -n-1$ by Theorem 1. From these inequalities we infer $l=1$ and $k=1-n$ unless $(n, -k, l)=(3, 4, 2), (3, 4, 3), (3, 3, 2)$ or $(4, 5, 2)$. Using preceding results we easily see that we are in Case b0).

If $w=n$, f is birational and we apply (2.5) to infer that $\dim f^{-1}(x)=n-1$ for some $x \in W$. If $(K+(n-1)L)R>0$, then we can find $A \in \text{Pic}(M)$ such that $LR=mAR$ for some integer $m>1$. This is impossible by (2.5). Hence $(K+(n-1)L)R=0$. So $K+(n-1)L$ comes from $\text{Pic}(M)$ by the property 3) in (2.2). Thus we are in Case a).

(2.11) *Proof of Theorem 3'.*

Case a): Take a prime divisor E as in Theorem 3, a). Taking hyperplane sections successively and applying the index theorem for surfaces, we infer that $EC<0$ for some curve C in E . So $ER<0$. Hence $EZ<0$ for every curve Z such that $f(Z)$ is a point. This implies $Z \subset E$. Thus, f contracts E to a point, but nothing else.

Now, using the exact sequence $0 \rightarrow \mathcal{O}_M[-E] \rightarrow \mathcal{O}_M \rightarrow \mathcal{O}_E \rightarrow 0$, we infer $H^i(E, K+sL)=0$ for any $i>0, s \geq 0$ similarly as in (2.3) because $H^i(M, K+$

$sL + tH = H^i(M, K + sL + tH - E) = 0$ for $t \gg 0$. Now, since $K + (n - 1)L = 0$ in $\text{Pic}(M)$, we infer $\chi(E, sL) = 0$ for $1 - n \leq s < 0$ and $\chi(E, \mathcal{O}) = 1$. Since this is a polynomial in s of degree $n - 1$, we infer that $\chi(E, sL) = (s + 1) \cdots (s + n - 1) / (n - 1)!$. This implies $L^{n-1}E = 1$ by the Riemann-Roch Theorem. Now we have $h^0(E, L) = \chi(E, L) = n$ and hence $\Delta(E, L) = 0$. So $(E, L) \simeq (\mathbf{P}^{n-1}, \mathcal{O}(1))$. By the adjunction formula we obtain $[E]_E = \mathcal{O}(-1)$. So $f(E)$ is a smooth point on W .

In Case b1) and $(F, L_F) \simeq (\mathbf{P}^2, \mathcal{O}(2))$, set $A = K + 2L$. Then, for every fiber X of f , we have $A^2X = A^2F = 1$. Similar as in (2.7; case c), we infer that f makes (M, A) a scroll over W . So $(X, L_X) \simeq (\mathbf{P}^2, \mathcal{O}(2))$.

In Case b2), similarly as above, we have $(X, L_X) \simeq (\mathbf{P}^{n-2}, \mathcal{O}(1))$ for the fiber over any smooth point of W . Moreover, $\dim Z = n - 2$ for every fiber Z of f . Indeed, if $f(D)$ is a point for some effective divisor D on M , we would have $DR < 0$ similarly as in Case a). So $DC < 0$ for any curve C such that $f(C)$ is a point, which is clearly absurd. Thus we reduce the problem to the following:

(2.12) **Lemma.** *Let $f: M \rightarrow S$ be a surjective morphism from a manifold M onto a normal projective variety S . Let L be an ample line bundle on M and suppose that $(F, L_F) \simeq (\mathbf{P}^r, \mathcal{O}(1))$ for every general fiber F of f . Suppose further that $\dim Z = r$ for every fiber Z of f . Then S is non-singular and f makes (M, L) a scroll over S .*

Proof. Let B be the singular locus of S . Then f is flat over $S - B$ and makes (M, L) a scroll there. So we will derive a contradiction assuming $B \neq \emptyset$. Cutting by hyperplane sections on S if necessary, we reduce the problem to the case $\dim B = 0$.

Take a point x on B . Note that, for any irreducible component X_i of $f^{-1}(x)$, we have $\dim X_i = r$ by assumption. Set $d_i = L^r X_i$ for each i .

Take a large integer a such that aL is very ample and let D_1, \dots, D_r be general members of $|aL|$. Then $N = D_1 \cap \dots \cap D_r$ is non-singular. Moreover, applying Bertini's theorem to the restriction of $|aL|$ to each X_i , we infer that $N \cap X_i$ is a non-singular subscheme consisting of $a^r d_i$ points.

Now, take a small enough neighborhood (with respect to the metric topology) U of x such that any connected component U_λ of $f^{-1}(U) \cap N$ meets $f^{-1}(x)$ at only one point. This is possible because $f_N: N \rightarrow S$ is proper and finite over x . Let f_λ be the restriction of f to U_λ . We may assume that f_λ is a finite morphism of degree m_λ . Then $\deg(f_N) = \sum_\lambda m_\lambda \geq \sum_i a^r d_i$, because the number of U_λ 's are equal to $\#(N \cap f^{-1}(x)) = \sum_i a^r d_i$. On the other hand, we have $F \simeq \mathbf{P}^r$ and $D_j \in |\mathcal{O}(a)|$ for any general fiber F of f . So $\deg(f_N) = a^r$. Combining them we obtain $\sum_i d_i = 1$ and $m_\lambda = 1$ for every λ . In particular $f_\lambda: U_\lambda \rightarrow U$ is bimeromorphic. So, by the analytic version

of Zariski's Main Theorem, f_i is an isomorphism. Since $U_i \subset N$ is smooth, U must be non-singular, contradicting $x \in B$.

(2.13) In order to prove Theorem 4, we need the following:

Lemma. *Let things be as in (2.2) and suppose that M is non-singular. Let $X = \{x \in W \mid \dim f^{-1}(x) > 0\}$ and $E = f^{-1}(X)$. Then there is a curve C in E such that $f(C)$ is a point and that $KC \geq n - 1 - 2 \dim E$.*

This is just a reformulated version of [Io; (0.4)].

(2.14) *Outline of the proof of Theorem 4.* First we apply the Key Lemma (2.2) and consider $f: M \rightarrow W$. If f is not birational, similarly as in (2.10), we can show that (b) is the case. So, let us assume that f is birational.

Take X and E as in (2.13). If $\dim E < n - 1$, then $KC \geq 3 - n$ for some curve C with $f(C)$ being a point. Since $LC \geq 1$, we have $(K + (n - 3)L)C \geq 0$, contradicting the property 5) in (2.2). Thus we conclude $\dim E = n - 1$.

Similarly as in (2.11; a), we infer $E_i Z < 0$ for some component E_i of E and a curve Z in E_i such that $f(Z)$ is a point. Then $E_i R < 0$. So E_i contains every curve which is mapped to a point. This implies $E_i = E$. Hence E is a prime divisor.

By the method in (2.5), we conclude that $\dim E_x \geq n - 2$ for every $x \in X$, where E_x is the fiber of $E \rightarrow X$. Hence $\dim X \leq 1$.

When $\dim X = 1$, take a general hyperplane section H of W and let x be a point on $X \cap H$. Then E_x is the divisor $E \cap f^{-1}(H)$ on the manifold $f^{-1}(H)$. Similarly as in Case (a) in (2.11), we infer that $(E_x, L_x) \simeq (\mathbf{P}^{n-2}, \mathcal{O}(1))$ and $\mathcal{O}[E]_{E_x} = \mathcal{O}(-1)$. Since $E \rightarrow X$ is flat and the fiber is irreducible and reduced over every smooth point x of X , this is a scroll over the smooth part of X . Thus we are in case a1).

When $\dim X = 0$, take an ample line bundle A on M such that AR attains the minimum. Then we may set $LR = jAR$ and $KR = kAR$ for some integers j, k . From $k + (n - 2)j \geq 0$, $k + (n - 3)j < 0$ and $k + n - 1 \geq 0$ we infer $j = 1$ and $k = 2 - n$ unless $(n, k, j) = (4, -3, 2)$.

In the latter case we infer $(E, A_E) \simeq (\mathbf{P}^3, \mathcal{O}(1))$ similarly as in (2.11; a). So (E, L_E) is of the type a3).

In case $j = 1$ and $k = 2 - n$, we may assume $A = L$. We set $ER = -eLR$ for some positive integer e . Since $K + (n - 2)L = 0$ in $\text{Pic}(E)$ as before, the dualizing sheaf ω_E is $(2 - n - e)L_E$. Moreover, similarly as in (2.11; case a), we have $H^i(E, K + sL) = 0$ for any $i > 0, s \geq 0$. This implies $\chi(E, sL) = 0$ for $2 - n \leq s < 0$. Moreover $h^i(E, -sL) = h^{n-1-i}(E, \omega_E[sL]) = 0$ if $i < n - 1$ and $s \geq e$. Hence $\chi(E, -sL) = 0$ for $e \leq s < n - 2 + e$. Since $\chi(E, sL)$ has at most $(n - 1)$ zeros, $[2 - n, 0) \cup (2 - n - e, -e]$ contains at most $(n - 1)$

integers. Hence $e \leq 2$.

If $e=2$, we infer $(E, L_E) \simeq (\mathbf{P}^{n-1}, \mathcal{O}(1))$ similarly as in (2.6). Thus a4) is the case. If $e=1$, similarly as in (2.7), we infer that a2) is the case.

§ 3. Comments

(3.1) Our Theorem 3' can be viewed as a polarized version of the following classical:

Theorem. *The canonical bundle of a smooth projective surface S is nef except in the following cases:*

- a) *There exists a curve E such that $E \simeq \mathbf{P}^1$ and $E^2 = -1$.*
- b0) *$S \simeq \mathbf{P}^2$.*
- b1) *S is isomorphic to a \mathbf{P}^1 -bundle over a smooth curve.*

(3.2) Our Theorem 4 can be viewed as a polarized version of Mori's theory in [M1], [M2].

(3.3) For general $t > 0$, we have the following:

Proposition. *Let L be a line bundle on a manifold M with $\dim M = n$ such that L is nef and big (so $L^n > 0$). Then $\kappa(K + tL, M) \leq n + 1 - t$ or $= n$.*

Proof. By virtue of [Ii], there is a birational morphism $\pi: M' \rightarrow M$ together with a surjective morphism $\Phi: M' \rightarrow W$ such that $\dim W = \kappa(K + tL)$, that every general fiber F of Φ is connected and that $\kappa(K + tL, F) = 0$. For the canonical bundle K' of M' we have $H^0(M', m(K' + t\pi^*L)) \simeq H^0(M', m\pi^*(K + tL))$ for any $m \geq 0$. So $\kappa(K' + tL, F) = 0$. Moreover, L_F is nef and big if F is general. Hence it suffices to show the following:

(3.4) **Lemma.** *Let L be a line bundle on a manifold F with $\dim F = f$ such that L is nef and big. Then $\kappa(K' + tL, F) \geq 0$ for $t > f$. So $\kappa(K + tL) = f$ if $t \geq f + 2$.*

Proof. If $\kappa(K + (f+1)L) < 0$, then $h^0(K + tL) = 0$ for $1 \leq t \leq f + 1$. Using Kawamata-Viehweg's vanishing theorem, we infer $\chi(K + tL) = 0$ for $1 \leq t \leq f + 1$. Hence $\chi(K + tL) \equiv 0$ because this is a polynomial in t of degree $\leq f$. This contradicts $L^f > 0$ by the Riemann-Roch theorem.

(3.5) **Remark.** In Case a) of Theorem 3, E is not necessarily a Cartier divisor. A simple example can be constructed as follows.

Take a manifold X whose canonical bundle is sufficiently ample. Let X_1 be the blow-up of X at a point and let E_1 be the exceptional divisor on it. Let X_2 be the blow-up of X_1 at a point p on E_1 . Let E_2 be the exceptional divisor over p and let E'_1 be the proper transform of E_1 . This is

isomorphic to the blow-up of $E_1 \simeq \mathbf{P}^{n-1}$ at p and is a \mathbf{P}^1 -bundle over \mathbf{P}^{n-2} . The restriction of the normal bundle of E'_1 to each fiber $\simeq \mathbf{P}^1$ is $\mathcal{O}(-2)$. Let $g: X_2 \rightarrow M$ be the contraction of E'_1 along this direction. So, $g(E'_1) \simeq \mathbf{P}^{n-2}$ and M has a hypersurface singularity of the type $x^2 = yz$ at any point on $g(E'_1)$. Let A be the pull-back of an ample line bundle on X . Then $sA - E_1 - E_2 = sA - E'_1 - 2E_2$ is the pull-back of an ample line bundle L on M for $s \gg 0$. One easily sees that $E = g(E_2)$ is the obstruction for $K + (n-2)L$ to be nef, and we are in Case a) of Theorem 3. E is not a Cartier divisor although so is $2E$.

(3.6) Sommese obtained similar results assuming $\text{Bs}|L| = \emptyset$. However, his assumption on the singularity of M is weaker than ours.

(3.7) If M is a smooth threefold, as [BP] pointed out, our results follow from Mori's theory [M1], [M2].

(3.8) Theorems 3 and 3' will be useful in the study of polarized manifolds with $g(M, L) = 2$. See [BP] and a forthcoming paper of the author.

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