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Complete Intersections with Growth Conditions

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Introduction

Cornalba-Griffiths [1] posed the following problem: Let $X \subset \boldsymbol{C}^3$ be a smooth algebraic curve. Do there exist two holomorphic functions f, g of finite order on $\boldsymbol{C}^3$ such that X is the complete intersection of the surfaces $\{f=0\}$ and $\{g=0\}$, i.e., f and g generate the ideal of X? By Serre [6] one knows that it is not always possible to find two polynomials with this property. In fact, it follows from the solution of the Serre conjecture (cf. Quillen [5] and Suslin [8]) that X is an algebraic complete intersection if and only if the canonical bundle of X is algebraically trivial (the same is true more generally for two-codimensional algebraic submanifolds of $\boldsymbol{C}^n$). On the other hand, one knows [2] that any smooth analytic curve X in a Stein manifold M of dimension $\leqslant 3$ is analytically a complete intersection.

The purpose of the present paper is to solve the problem of Cornalba-Griffiths. In fact we prove a more general theorem: Let $X \subset \boldsymbol{C}^n$ be an algebraic submanifold of pure codimension two such that the canonical bundle of X is topologically trivial. Then the ideal of X is generated by two entire functions of finite order (cf. Corollary 3.2). Note that the condition on the canonical bundle is necessary, since the normal bundle of every complete intersection is trivial. If X is a curve, this condition is automatically fulfilled, since on an open Riemann surface every holomorphic vector bundle is analytically (a fortiori topologically) trivial.

The proof uses analytic and algebraic methods. As an analytic tool we prove an extension and division theorem with growth conditions (cf. §2). With the help of this theorem the problem is algebraically reduced to the application of a theorem of Quillen-Suslin on projective modules over a polynomial ring $B[T]$ and in this application B is the ring of certain functions of finite order.

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§ 1. Functions of finite order

In this section we define the notion of finite order for holomorphic functions and sections of algebraic vector bundles on affine algebraic manifolds and collect some elementary properties.

A holomorphic function $f\colon \boldsymbol{C}^n \to \boldsymbol{C}$ is said to be of finite order, if there exists a polynomial P with real coefficients such that

$$|f(z)| \leqslant e^{P(\|z\|)} \qquad \text{for all } z \in \boldsymbol{C}^n.$$

We denote the ring of all holomorphic functions of finite order on $\boldsymbol{C}^n$ by $\Gamma_{f\cdot o}(\boldsymbol{C}^n, \mathcal{O}_{\boldsymbol{C}^n})$, or briefly by R_n.

More generally, let X be an affine algebraic manifold over $\boldsymbol{C}$. A holomorphic function $f\colon X \to \boldsymbol{C}$ is said to be of finite order if it satisfies an estimate of the type $|f(x)| \leqslant e^{P(\|x\|)}$ for all $x \in X$, where the norm $\|x\|$ is taken with respect to an embedding $X \hookrightarrow \boldsymbol{C}^N$ of X as an algebraic submanifold of some $\boldsymbol{C}^N$. The definition is independent of the choice of the embedding. The ring of all holomorphic functions of finite order on X is denoted by $R_X = \Gamma_{f\cdot o}(X, \mathcal{O}_X)$.

We denote by $\Gamma_{\mathrm{alg}}(X, \mathcal{O}_X) \subset \Gamma_{f\cdot o}(X, \mathcal{O}_X)$ the ring of regular functions on X regarded as an algebraic variety. With respect to the embedding $X \hookrightarrow \boldsymbol{C}^N$, $\Gamma_{\mathrm{alg}}(X, \mathcal{O}_X)$ consists of the restrictions of polynomials $f \in \boldsymbol{C}[z_1, \cdots, z_N]$ to X.

An equivalent method of defining functions of finite order on an affine algebraic manifold X is the following. By the desingularization theorem of Hironaka there exists a completion $X \subset \bar{X}$ of X to a smooth projective algebraic manifold $\bar{X}$ such that $D = \bar{X} \backslash X$ is the union of smooth divisors with normal crossings. Then for every point $p \in D$ one can find a neighborhood $U \ni p$ such that $(U, U \cap X)$ is biholomorphic to

$$(\varDelta^n, \varDelta^n \backslash \{z_1 \cdots z_k = 0\}),$$

where $\varDelta^n = \{(z_1, \cdots, z_n) \in \boldsymbol{C}^n : |z_\nu| < 1\}$. A holomorphic function f on X is of finite order if it has finite order singularities along D. This means that f satisfies, with respect to local charts as above, an estimate of the type

$$|f(z)| \leqslant \mathrm{const} \cdot \exp\left(\sum_{i=1}^{k} |z_i|^{-r}\right)$$

for some $r \in \boldsymbol{N}$.

Proposition (1.1). *Let X be an affine algebraic manifold and let U $\{U_1, \cdots, U_m\}$ be a Zariski open affine covering of X. Then a holomorphic function f on X belongs to R_X if and only if*

$$f|U_i \in R_{U_i} \qquad \textit{for all } i.$$

Proposition (1.2). *Let X be an affine algebraic manifold and f a function of finite order on X without zeroes. Then $1/f$ is also a function of finite order.*

This follows from:

Lemma (1.3). *Let $\Delta_n^k := \Delta^n \backslash \{z_1 \cdots z_k = 0\}$ and let f be a holomorphic function on Δ_k^n which satisfies*

$$0 < |f(z)| \leqslant \exp\left(\sum_{i=1}^{k} |z_i|^{-r}\right)$$

for some $r \in \boldsymbol{N}$. Then

$$\left|\frac{1}{f(z)}\right| \leqslant \text{const}\cdot\exp\left(\sum_{i=1}^{k} |z_i|^{-r-1}\right)$$

for all $z \in \Delta_k^n$ with $|z_\nu| \leqslant 1/2$, $1 \leqslant \nu \leqslant n$.

This is classical for $n=k=1$. The proof in the general case is similar.

Let E be an algebraic vector bundle over the affine algebraic manifold X. The space $\Gamma_{f \cdot o}(X, E)$ of sections of finite order of E is defined in the following way: E can be embedded as an algebraic subbundle $E \subset \theta^r$, where θ^r denotes the trivial vector bundle of rank r. A holomorphic section $s \in \Gamma(X, E)$ is then represented by an r-tuple $(s_1, \cdots, s_r)$ of holomorphic functions. The section s is said to be of finite order, if all functions s_j are of finite order. This definition is independent of the choice of the (algebraic) embedding $E \subset \theta^r$. Let $s_1, \cdots, s_r$ be holomorphic functions of finite order on an affine algebraic manifold X without common zeroes. Then in general one cannot find holomorphic functions $f_1, \cdots, f_r$ of finite order satisfying $f_1 s_1 + \cdots + f_r s_r = 1$. For $r=1$ this is possible by Proposition (1.2). But for $r=2$ there is already a counterexample on $X = \boldsymbol{C}$.

However we have:

Proposition (1.4). *Let $L \subset \theta^r$ be an algebraic subline bundle of a trivial vector bundle over an affine algebraic manifold X. Let $s = (s_1, \cdots, s_r) \in \Gamma_{f \cdot o}(X, L) \subset \Gamma_{f \cdot o}(X, \mathcal{O}_X^r)$ be a section such that $s_1, \cdots, s_r$ have no common zeroes on X. Then there exist functions $f_1, \cdots, f_r \in \Gamma_{f \cdot o}(X, \mathcal{O}_X)$ with*

$$f_1 s_1 + \cdots + f_r s_r = 1.$$

Proof. Let $\alpha \colon L \hookrightarrow \theta^r$ be the embedding and $\alpha^* \colon \theta^r \to L^*$ be the dual homomorphism. Since X is affine, there exists an algebraic splitting of α^*, i.e., a homomorphism $\beta \colon L^* \to \theta^r$ such that $\alpha^* \circ \beta = id_{L^*}$. Since s has no zeroes we can consider the section s^{-1} of the bundle L^*. We prove now that $s^{-1} \in \Gamma_{f.o}(X, L^*)$. To see this, let $\{U_i\}$ be a finite Zariski open affine covering of X such that $L|U_i$ is algebraically trivial for every i. By Proposition (1.2), the restrictions $s^{-1}|U_i$ are of finite order and from Proposition (1.1) it follows that $s^{-1} \in \Gamma_{f.o}(X, L^*)$. We put

$$(f_1, \cdots, f_r) := \beta(s^{-1}) \in \Gamma_{f.o}(X, \mathcal{O}_X^r).$$

Since $\alpha^* \circ \beta(s^{-1}) = s^{-1}$, we have

$$1 = \langle \alpha^* \circ \gamma(s^{-1}), s \rangle = \langle \beta(s^{-1}), \alpha^*(s) \rangle = \sum f_i s_i.$$

Cornalba-Griffiths [1] have shown that topologically trivial algebraic line bundles on affine algebraic manifolds are trivial in the finite order category. This means in particular:

Theorem (1.5). *Let L be an algebraic line bundle on an affine algebraic manifold X. Suppose L is topologically trivial. Then there exists a finite order section $s \in \Gamma_{f.o}(X, L)$ without zeroes on X.*

§ 2. An extension and division theorem

Let $M \subset \boldsymbol{C}^n$ be an affine algebraic manifold of pure dimension m. We fix once for all the function

$$\varphi(z) := \log(1 + \|z\|^2).$$

φ is a strictly plurisubharmonic function on M and we have

$$\Gamma_{\mathrm{alg}}(M, \mathcal{O}_M) = \left\{ f \in \Gamma(M, \mathcal{O}_M) : \exists \nu \in \boldsymbol{N} \text{ such that } \int_M e^{-\nu\varphi} |f|^2 dv < \infty \right\}.$$

Here dv denotes the volume element of M induced by the Euclidean metric of $\boldsymbol{C}^n$. If we denote by dv_φ the volume element induced by $\partial\bar{\partial}\varphi$, the Fubini-Study metric of $\boldsymbol{P}^n$, we have the following comparison:

$$dv_\varphi \leqslant dv \leqslant \text{const } e^{(m+1)\varphi} dv_\varphi.$$

Definition. A $\mathscr{C}^\infty$ plurisubharmonic function $\eta: M \to \boldsymbol{R}$ is called a controlling function if there exists a $\nu \in \boldsymbol{N}$ such that

$$(1) \qquad |\eta(x) - \eta(y)| < 1, \text{ whenever } \|x - y\| < (2 + \|x\|)^{-\nu}.$$

We note that for any controlling function η on M and for any $\mu \in \boldsymbol{N}$, $\eta + \mu\varphi$ is again a controlling function.

Definition. For a controlling function η on M we put

a) $\Gamma_\eta(M, \mathcal{O}_M) := \{f \in \Gamma(M, \mathcal{O}_M) \colon \sup |f(x)|^2 e^{-\eta(x)} < \infty\}$,
b) $R_M(\eta) := \bigcup_{\nu \in N} \Gamma_{\eta+\nu\varphi}(M, \mathcal{O}_M)$,
c) $\Gamma_\eta(M, \omega_M) := \{\omega \in \Gamma(M, \omega_M) \colon \sup |\omega(x)|^2 e^{-\eta(x)} < \infty\}$.

Here ω_M denotes the sheaf of holomorphic sections of the canonical line bundle of M, i.e., the sheaf of holomorphic m-forms and the length of ω is measured with respect to the restriction of the Euclidean metric of $\boldsymbol{C}^n$ to M.

Remark. Since a controlling function always satisfies an estimate of the type $|\eta(z)| \leqslant \text{const}\,(1 + \|z\|^{2r})$, we have

$$R_M(\eta) \subset R_M.$$

(Recall that R_M denotes the ring of functions of finite order on M).

On the other hand, for the special controlling functions $\eta_r(z) := \|z\|^{2r}$, we have

$$R_M = \bigcup_{r \in N} R_M(\eta_r).$$

For the controlling functions

$$\gamma_r(z) := (|z_1|^2 + \cdots + |z_k|^2)^r$$

on $\boldsymbol{C}^n$ we get

$$\bigcup_{r \in N} R_{\boldsymbol{C}^n}(\gamma_r) = R_{\boldsymbol{C}^k}[z_{k+1}, \cdots, z_n].$$

If $X \subset \boldsymbol{C}^n$ is an algebraic submanifold such that the projection to the first k coordinates $p \colon X \to \boldsymbol{C}^k$ is proper, then we have with respect to the same controlling functions γ_r

$$\bigcup_{r \in N} R_X(\gamma_r) = R_X.$$

Using property (1) of controlling functions, by standard techniques we obtain the following:

Proposition (2.1). *For any controlling function η on an affine algebraic manifold M,*

$$R_M(\eta) = \left\{ f \in \Gamma(M, \mathcal{O}_M) \colon \exists \nu \in N\, s.t. \int_M |f|^2 e^{-\eta - \nu\varphi} dv < \infty \right\}.$$

Lemma (2.2). *Let X be an algebraic submanifold of M. Then, for any controlling function η on M, the following are equivalent.*

a) *There exists a $\nu \in \boldsymbol{N}$ such that for any $f \in \Gamma_\eta(X, \mathcal{O}_X)$ one can find an $F \in \Gamma_{\eta+\nu\varphi}(M, \mathcal{O}_M)$ with $F|X=f$.*

b) *There exist a finite system $\psi_1, \cdots, \psi_k \in \Gamma_{\mathrm{alg}}(M, \mathcal{O}_M)$ of functions without common zeroes and integers $\nu_1, \cdots, \nu_k \in \boldsymbol{N}$, such that for any $f \in \Gamma_\eta(X, \mathcal{O}_X)$ one can find functions $F_i \in \Gamma_{\eta+\nu_i\varphi}(M, \mathcal{O}_M)$ with $F_i|X=f\psi_i|X$ for every i.*

c) *For any $\omega \in \Gamma_{\mathrm{alg}}(M, \omega_M)$, one can find a $\nu \in \boldsymbol{N}$ such that for every $f \in \Gamma_\eta(X, \mathcal{O}_X)$ there exists a $\sigma \in \Gamma_{\eta+\nu\varphi}(M, \omega_M)$ with $\sigma|X=f\omega|X$.*

The implication b)$\Rightarrow$a) *follows from the fact that there exist functions $\alpha_1, \cdots, \alpha_k \in \Gamma_{\mathrm{alg}}(M, \mathcal{O}_M)$ with $\sum \alpha_i\psi_i=1$. We prove the implication* c)$\Rightarrow$b) *by choosing $\omega_1, \cdots, \omega_r \in \Gamma_{\mathrm{alg}}(M, \omega_M)$ without common zeroes.*

We prove c) in the course of the proof below of Theorem 2.4. For the proof of Theorem 2.4 we need the following:

Lemma (2.3). *Let $X \subset \boldsymbol{C}^n$ be an affine algebraic manifold. Then:*

a) *There exists a neighborhood $U \supset X$ in $\boldsymbol{C}^n$ and a holomorphic retraction $r_U\colon U \to X$ with the following properties:*

i) $\exists \nu_0 \in \boldsymbol{N}$ *such that*

$$U \supset \{z \in \boldsymbol{C}^n\colon \inf_{x \in X} \|z-x\| < (2+\|z\|)^{-\nu_0}\}.$$

ii) $\exists \nu_1 \in \boldsymbol{N}$ *such that*

$$\|r_U(z)-z\| < (2+\|z\|)^{-\nu_1} \qquad \textit{for every } z \in U.$$

b) *Let $g_1, \cdots, g_r$ be polynomials which generate the ideal I_X of X. Then, for any $\mu \in \boldsymbol{N}$, there exists a $\nu \in \boldsymbol{N}$ such that $\sum_{i=1}^r |g_i(z)|^2 > (2+\|z\|)^{-\nu}$ on $\{z \in \boldsymbol{C}^n\colon \inf_{x \in X}\|z-x\| > (2+\|z\|)^{-\mu}\}$.*

Proof of a). Let N_X be the normal bundle of X in $\boldsymbol{C}^n$. Then N_X is an affine algebraic manifold containing X as the zero section and we have an algebraic map from N_X to $\boldsymbol{C}^n$ which is the identity on X and maps every fiber of N_X isomorphically to an affine linear subspace of $\boldsymbol{C}^n$ intresecting X transversally. Therefore, for sufficiently large ν_0 and ν_1, we can choose a neighborhood $U \supset X$ such that the projection $r_U\colon U \to X$ along the fibers of N_X satisfies the requirement.

The proof of b) is left to the reader.

We are now ready to prove the following extension theorem.

Theorem (2.4). *Let $X \subset M$ be an algebraic submanifold. Then, for* *any controlling function η on M, one can find a $\nu \in \boldsymbol{N}$ such that for any $f \in \Gamma_{\eta}(X, \mathcal{O}_X)$ there exists an $F \in \Gamma_{\eta+\nu\varphi}(M, \mathcal{O}_M)$ with $F|X=f$.*

Proof. Let U, r_U and $g_1, \cdots, g_r$ be as in Lemma (2.3). Then we can find a $\nu_2 \in \boldsymbol{N}$ and a $\mathscr{C}^{\infty}$ function ρ: $M \to \boldsymbol{R}$ which satisfies the following properties:

$$\operatorname{Supp} \rho \subset \{z \in M: \inf_{x \in X} \|z-x\| < 2(2+\|z\|)^{-\nu_2}\} \subset U,$$

$$\operatorname{Supp}(1-\rho) \subset \{z \in M: \inf_{x \in X} \|z-x\| > (2+\|z\|)^{-\nu_2}\},$$

$$|\partial \rho| < \text{const } e^{\nu_2 \varphi}.$$

We need only to prove assertion c) of Lemma (2.2). Let $\omega \in \Gamma_{\text{alg}}(M, \omega_M)$ and put

$$u := \begin{cases} \rho(f \circ r_U)\omega & \text{on } U \cap M, \\ 0 & \text{on } M \setminus U. \end{cases}$$

By Lemma (2.3), a), ii) and b), there exists a $\nu_3 \in \boldsymbol{N}$ independent of f such that

$$(1) \qquad \int_{M \setminus X} e^{-\eta-\nu_3\varphi} \left(\sum_{i=1}^{r} |g_i|^2 \right)^{-k} |\bar{\partial} u|^2 dv_{\varphi} < \infty,$$

where $k = \operatorname{codim}_M X$. Since $M \setminus X$ has a complete Kähler metric, there exists by Theorem 2.8 in [4] a locally square integrable $(m, 0)$-form h on $M \setminus X$ such that $\bar{\partial} h = \bar{\partial} u$ and

$$(2) \qquad \int_{M \setminus X} e^{-\eta-\nu_3\varphi} \left(\sum_{i=1}^{r} |g_i|^2 \right)^{-k} |h|^2 dv_{\varphi} \leqslant \infty.$$

Since $\omega \in \Gamma_{\text{alg}}(M, \omega_M)$, there exists a $\nu \in \boldsymbol{N}$ independent of f such that

$$(3) \qquad \int_{M \setminus X} e^{-\eta-\nu\varphi} |u-h|^2 dv < \infty.$$

Because $u-h$ is holomorphic on $M \setminus X$ and locally square integrable on M it extends to a holomorphic m-form σ on M. Since u is holomorphic on a neighborhood of X, h also extends to an $(m, 0)$-form $\tilde{h}$ on M which is holomorphic on a neighborhood of X. By (2) it follows that $\tilde{h}|X=0$. Therefore

$$\sigma | X = u | X = f\omega | X.$$

By (3), $\sigma \in \Gamma_{\eta+\nu\varphi}(M, \omega_M)$. q.e.d.

Corollary (2.5). *For any controlling function η on M, the restriction map*

$$\pi\colon R_M(\eta)\to R_X(\eta)$$

is surjective.

Next we investigate the kernel of π. We shall apply a special case of Skoda's division theorem on weakly 1-complete manifolds ([7], Théorème 4).

Theorem (2.6). *Let M be a Stein manifold and let $f_1, \cdots, f_p$ be holomorphic functions on M with $p<\dim M+1$. Let $\varphi\colon M\to\boldsymbol{R}$ be a $\mathscr{C}^\infty$ strictly plurisubharmonic function. Then, for any plurisubharmonic function ψ on M and any $u\in\Gamma(M, \omega_M)$ such that*

$$\varepsilon\int_M e^{-\psi-\varphi}\left(\sum_{i=1}^p |f_i|^2\right)^{-p} u\wedge\bar{u}<\infty,\quad \varepsilon=(-1)^{m(m-1)/2}\sqrt{-1}^m,\quad m=\dim X,$$

there exists a p-tuple $(h_1, \cdots, h_p)\in\Gamma(M, \omega_M)^p$ such that

$$u=\sum_{j=1}^p f_j h_j$$

and

$$\varepsilon\int_M e^{-\psi-\varphi}\left(\sum_{i=1}^p |f_i|^2\right)^{-p+1} h_j\wedge\bar{h}_j$$
$$\leqslant\varepsilon\int_M e^{-\psi-\varphi}\left(\sum_{i=1}^p |f_i|^2\right)^{-p} u\wedge\bar{u}.$$

Proposition (2.7). *Let $X\subset M\subset\boldsymbol{C}^n$ be algebraic submanifolds and $g_1, \cdots, g_r\in\Gamma_{\mathrm{alg}}(M, \mathcal{O}_M)$ generators of $I_X\subset\Gamma_{\mathrm{alg}}(M, \mathcal{O}_M)$. Then, for any controlling function η on M, the kernel of the restriction map*

$$\pi\colon R_M(\eta)\to R_X(\eta)$$

is $\sum_{i=1}^r R_M(\eta)g_i$.

Proof. Let $f\in\operatorname{Ker}\pi$ and let p be the codimension of X in M. We remark first that if

$$\operatorname{rank}(dg_{i_1}, \cdots, dg_{i_p})(x)=p\qquad\text{for some point } x\in X,$$

then

$$|f|^2\left(\sum_{k=1}^p |g_{i_k}|^2\right)^{-p}$$

is integrable on a neighborhood of x. Therefore we can find a finite system $\omega_1, \cdots, \omega_s \in \Gamma_{\mathrm{alg}}(M, \omega_M)$ without common zeroes such that for every j there is a p-tuple $(g_{i_1}, \cdots, g_{i_p})$ such that

$$|f|^2\left(\sum_{k=1}^{p}|g_{i_k}|^2\right)^{-p}\omega_j\wedge\bar{\omega}_j$$

is locally square integrable on M. We apply to $(f_1, \cdots, f_p)=(g_{i_1}, \cdots, g_{i_p})$ and $u=f\omega_j$ Skoda's theorem and get an r-tuple $(h_{j_1}, \cdots, h_{j_r}) \in \Gamma_{\eta+\nu\varphi}(M, \omega_M)^r$ with a suitable $\nu \in \boldsymbol{N}$ such that

$$f\omega_j=\sum_{i=1}^{r}h_{j_i}g_i.$$

By an argument similar to the proof of Lemma (2.2) we get the result.

Theorem (2.8). *Let X be an algebraic submanifold of $\boldsymbol{C}^n$ of pure dimension k such that the projection p: $X \to \boldsymbol{C}^k$ to the first k coordinates is proper. Let $g_1, \cdots, g_r \in \boldsymbol{C}[z_1, \cdots, z_n]$ be generators of I_X. Then the natural restriction map*

$$\pi\colon\ R_k[z_{k+1}, \cdots, z_n] \longrightarrow R_X$$

is surjective and its kernel is generated by $g_1, \cdots, g_r$.

The proof follows from Corollary (2.5) and Proposition (2.7).

§ 3. Complete intersections

In this section we prove our main theorems (Theorems 3.1 and 3.6).

Theorem (3.1). *Let $X \subset \boldsymbol{C}^n$ be an algebraic submanifold of pure codimension 2 such that the projection to the first $n-2$ coordinates $p\colon X \to \boldsymbol{C}^{n-2}$ is proper. Suppose that the canonical line bundle of X is topologically trivial. Then the ideal of X can be generated by two functions $F_1, F_2 \in R_{n-2}[z_{n-1}, z_n]$. Here R_{n-2} denotes the ring of holomorphic functions of finite order on $\boldsymbol{C}^{n-2}$.*

Corollary (3.2). *Let $X \subset \boldsymbol{C}^n$ be an algebraic submanifold of pure codimension 2 with topologically trivial canonical line bundle. Then the ideal of X can be generated by two entire functions of finite order.*

Proof of Theorem (3.1). Let $\mathscr{I}_X \subset \mathcal{O}_{\boldsymbol{C}^n}$ be the ideal sheaf of X. Since X is algebraic, $\mathscr{I}_X$ admits a finite set of polynomial generators $g_1, \cdots, g_{N+1} \in \boldsymbol{C}[z_1, \cdots, z_n]$. Let $g\colon \mathcal{O}_{\boldsymbol{C}^n}^{N+1} \to \mathscr{I}_X$ be the epimorphism defined by $(g_1, \cdots, g_{N+1})$. Since X is a locally complete intersection of codimension

2, the kernel of g is locally free, hence globally algebraically free by the theorem of Quillen-Suslin [5, 8]. Therefore there is an exact sequence

$$(4)\qquad 0 \longrightarrow \mathcal{O}_{C^n}^N \xrightarrow{P} \mathcal{O}_{C^n}^{N+1} \xrightarrow{g} \mathcal{I}_X \longrightarrow 0,$$

where

$$P=\begin{pmatrix} P_{1,1}\cdots\cdots P_{1,N} \\ \vdots \qquad\qquad \vdots \\ P_{N+1,1}\cdots P_{N+1,N} \end{pmatrix}$$

is an $(N+1)\times N$ matrix with coefficients in $C[z_1, \cdots, z_n]$. We restrict the sequence (4) analytically to X. We have

$$\mathcal{I}_X|X=\mathcal{I}_X\otimes(\mathcal{O}_{C^n}/\mathcal{I}_X)=\mathcal{I}_X/\mathcal{I}_N^2=:\nu_X.$$

This is an algebraic locally free sheaf of rank 2 on X, the conormal sheaf of X. Since the tensor product is right exact, we get an exact sequence

$$(5)\qquad 0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_X^N \xrightarrow{P} \mathcal{O}_X^{N+1} \longrightarrow \nu_X \longrightarrow 0,$$

where $\mathcal{K}$ is an algebraic line subbundle of $\mathcal{O}_X^N$, and $\bar{P}=P\,|\,X$. It follows that $\mathcal{K}\cong\det(\nu_X)$, which is isomorphic to the canonical line bundle of X. By hypothesis, this bundle is topologically trivial, hence by Theorem (1.5) there exists a finite order section

$$(\bar{\varphi}_1, \cdots, \bar{\varphi}_N)\in\Gamma_{f\cdot o}(X, \mathcal{K})\subset\Gamma_{f\cdot o}(X, \mathcal{O}_X^N)$$

without zeroes. By Proposition (1.4) there exists an N-tuple $(\bar{f}_1, \cdots, \bar{f}_N)$ $\in\Gamma_{f\cdot o}(X, \mathcal{O}_X^N)$ with

$$(6)\qquad \bar{\varphi}_1\bar{f}_1+\cdots+\bar{\varphi}_N\bar{f}_N=1.$$

Let $f_1, \cdots, f_N\in R_{n-2}[z_{n-1}, z_n]$ be functions with $f_j|X=\bar{f}_j$, which exist by Theorem (2.8). Define

$$f:=(f_1, \cdots, f_N),$$
$$Q:=\begin{pmatrix} P \\ f \end{pmatrix}\in M((N+2)\times N, A),$$

where $A:=R_{n-2}[z_{n-1}, z_n]$.

Lemma (3.3). *The cokernel of the map*

$$Q: A^N \longrightarrow A^{N+2}$$

is a free A-module of rank 2.

We will prove this lemma later and show now how it implies Theorem (3.1).

The freeness of Coker Q means that there exists a matrix

$$h \in M((N+2)\times 2, A)$$

such that $\det(Q, h)=1$. Consider the diagram

$$\begin{tikzcd} & \mathcal{O}^2_{C^n} \arrow[d, "h"] & & \\ \mathcal{O}^N_{C^n} \arrow[r, "\binom{P}{f}"] \arrow[d, equal] & \mathcal{O}^{N+1}_{C^n}\oplus\mathcal{O}_{C^n} \arrow[d, "pr"] & & \\ \mathcal{O}^N_{C^n} \arrow[r, "P"] & \mathcal{O}^{N+1}_{C^n} \arrow[r, "g"] & \mathcal{I}_X \arrow[r] & 0 \end{tikzcd}$$

where pr is the projection to the first $N+1$ components. It follows that

$$F:=g\circ pr\circ h\colon \mathcal{O}^2_{C^n}\longrightarrow \mathcal{I}_X$$

is an epimorphism. By construction, the components of $F=(F_1, F_2)$ are elements of $A=R_{n-2}[z_{n-1}, z_n]$ and F_1, F_2 generate $\mathcal{I}_X$. q.e.d.

In order to prove Lemma (3.3), we prove first:

Lemma (3.4). *The cokernel of the map*

$$Q=\begin{pmatrix}P\\ f\end{pmatrix}\colon A^N\longrightarrow A^{N+1}\oplus A$$

is a projective A-module of rank 2.

Proof. We use the following criterion for the projectivity of the cokernel (cf. Bourbaki, Algèbre commutative, Chap. II, §5): Let M, N be finitely generated projective A-modules of constant ranks r and s, respectively. Let $\alpha\colon M\to N$ be a homomorphism such that rank $\alpha(x)=r$ for all $x\in \operatorname{Spec}(A)$. Then Coker (α) is a projective A-module of rank $s-r$.

Here $\alpha(x)$ denotes the (A_x/m_x)-vector space homomorphism

$$\alpha(x)\colon M_x/m_xM_x\longrightarrow N_x/m_xN_x$$

induced by α with m_x the maximal ideal of A_x.

To prove Lemma (3.4) we set $A_0:=C[z_1, \cdots, z_n]$ and denote by α_0 the ideal in A_0 generated by $(g_1, \cdots, g_{N+1})$. Analogously we denote by $\mathfrak{a}$ the ideal generated by $(g_1, \cdots, g_{N+1})$ in the ring $A=R_{n-2}[z_{n-1}, z_n]$. The inclusion $A_0 \hookrightarrow A$ defines a morphism of affine schemes

$$\Phi\colon \operatorname{Spec}(A) \longrightarrow \operatorname{Spec}(A_0).$$

We have $V(\mathfrak{a})=\Phi^{-1}(V(\mathfrak{a}_0))$.

We must show that for every $x \in \operatorname{Spec}(A)$ the $(N+2)\times N$ matrix $Q(x)$ with coefficients in the field $A_x/m_x=\operatorname{Quot}(A/x)$ induced by Q, has rank N. This is true over $\operatorname{Spec}(A)\setminus V(\mathfrak{a})$, since

$$P\colon \mathcal{O}^N_{\operatorname{Spec}(A_0)} \longrightarrow \mathcal{O}^{N+1}_{\operatorname{Spec}(A_0)}$$

has constant rank N over $\operatorname{Spec}(A_0)\setminus V(\mathfrak{a}_0)$. Over every point of $V(\mathfrak{a}_0)$, the matrix P is of rank $N-1$, hence also over every point of $V(\mathfrak{a})$. By construction, f is of rank one on the kernel of P over every point of $V(\mathfrak{a})$. Indeed, let $\varphi_1, \cdots, \varphi_N \in R_{n-2}[z_{n-1}, z_n]=A$ be functions with $\varphi_j|X=\bar{\varphi}_j$ and $\varphi:={}^t(\varphi_1, \cdots, \varphi_N) \in A^N$. Let x be a point of $V(\mathfrak{a})$, i.e. a prime ideal of A with $x\supset\mathfrak{a}$. For the function $f\varphi=\sum f_j\varphi_j$ we have $f\varphi|X=1$ (cf. formula (6)), hence by Theorem (2.8), $f\varphi-1 \in x$, i.e. $(f\varphi)(x)=1$. Similarly $P\varphi|X=0$, hence $(P\varphi)(x)=0$.

Therefore $Q=\begin{pmatrix} P \\ f \end{pmatrix}$ is of rank N at every point of $\operatorname{Spec}(A)$. This concludes the proof of Lemma (3.4).

In order to deduce Lemma (3.3) from Lemma (3.4), we use the following theorem of Quillen-Suslin [5, 8] (cf. [3], Chap. IV, Prop. 3.14).

Theorem (3.5). *Let B be a commutative ring and $A:=B[T]$ the polynomial ring in one indeterminate T over B. Let E be a finitely generated projective A-module. If there exists a monic polynomial $g \in B[T]=A$ such that the localized module E_g is free over A_g, then E is free over A.*

Proof of Lemma (3.4). We apply Theorem (3.5) to $B=R_{n-2}[z_{n-1}]$ $A=R_{n-2}[z_{n-1}, z_n]$, $E=\operatorname{Coker} Q$. Since $p\colon X\to \boldsymbol{C}^{n-2}$ is proper, there exists a polynomial $g \in \boldsymbol{C}[z_1, \cdots, z_n]\subset A$, monic with respect to z_n, which vanishes on X. On the scheme $U:=\operatorname{Spec}(\boldsymbol{C}[z_1, \cdots, z_n]_g)=\operatorname{Spec}(\boldsymbol{C}[z_1, \cdots, z_n])\setminus V(g)$ we have an exact sequence

$$0 \longrightarrow \mathcal{O}^N_U \xrightarrow{P} \mathcal{O}^{N+1}_U \longrightarrow \mathcal{O}_U \longrightarrow 0,$$

which splits; hence there exist elements $\alpha_1, \cdots, \alpha_{N+1} \in \Gamma(U, \mathcal{O}_U)=\boldsymbol{C}[z_1, \cdots, z_n]_g$ such that

$$\det\left(P \left| \begin{matrix} \alpha_1 \\ \vdots \\ \alpha_{N+1} \end{matrix} \right.\right) = 1.$$

Therefore also

$$\det\left[\begin{array}{c|c|c} P & \begin{matrix} \alpha_1 \\ \vdots \\ \alpha_{N+1} \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline f_1, \cdots, f_N & 0 & 1 \end{array}\right] = 1.$$

The latter matrix has coefficients in A_g. This implies that E_g is a free A_g-module of rank 2. By Theorem (3.5), E is free over A. q.e.d.

Theorem (3.6). *Let $X \subset C^n$ be a smooth algebraic curve such that the projection p: $X \to C$ to the first coordinate is proper. Then the ideal of X can be generated by $n-1$ elements $F_1, \cdots, F_{n-1} \in R_1[z_2, \cdots, z_n]$.*

Proof. This is trivial for $n<3$. For $n=3$ it follows from Theorem (3.1), since on a one-dimensional Stein manifold every vector bundle is trivial.

Suppose now $n>3$. Since p: $X \to C$ is proper, there exists a linear function φ_1: $C^n \to C$ such that (p, φ_1): $X \to C^2$ is a proper immersion and generically injective. Now one can choose a linear function φ_2: $C^2 \to C$ such that $(p, \varphi_1, \varphi_2)$: $X \to C^3$ is an embedding. So we may assume that p factors as

$$\begin{tikzcd} X \arrow[r, hook] \arrow[drr, "p"'] & C^n \arrow[r, "p_2"] & C^3 \arrow[d, "p_1"] \\ & & C \end{tikzcd}$$

where p_1, p_2 are projections and p_2 maps X isomorphically onto the smooth algebraic curve $Y := p_2(X)$.

By Theorem (3.1) there exist functions $F_1, F_2 \in R_1[z_2, z_3]$ which generate the ideal of $Y \subset C^3$. Since $X \subset Y \times C^{n-3}$ is a graph over Y, there exist polynomials $G_1, \cdots, G_{n-3} \in C[z_1, \cdots, z_n]$ such that the restrictions $G_j | Y \times C^{n-3}$ generate the ideal of X in $Y \times C^{n-3}$. Then $F_1, F_2, G_1, \cdots, G_{n-3}$ generate the ideal of X in C^n. q.e.d.

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