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On Some Stable Maps and Compact Lie Groups

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Dedicated to Professor Nobuo Shimada on his 60th birthday

§ 0. Introduction

In [5] Becker and Gottlieb defined the transfer which is a stable map from B_+ to E_+ where $E \to B$ is a differentiable fibre bundle whose fibre is a compact manifold. They used this map to solve some topological problems. In this paper, we define a transfer-type stable map between same dimensional manifolds with some conditions and study their properties. The typical example is related to the compact Lie group and its maximal rank subgroup. As an application, we can prove Becker-Segal type theorem for SU the infinite special unitary group.

This paper is constructed as follows:

In Section 1, we define a stable map associated with a differentiable map between same dimensional manifolds with a condition related to the tangent bundles and prove some pull back and evaluation formulas.

In Section 2, we construct the main example which is related to the compact Lie group G and its maximal rank subgroup H. We obtain a stable map

$$t(i): G_+ \longrightarrow G \times_H H_{c+}$$

where H_c is the space H whose H-action is given by the adjoint action, with some nice properties. (For details, see Section 2.)

In Section 3, we give two applications. One is related to the homotopy-normality of the maximal rank subgroup of a compact Lie group and the other is a Becker-Segal type theorem for SU.

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§ 1. Transfer

Let $f\colon M \to N$ be a smooth map between closed manifolds M and N

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with the same dimension. If we assume that there is an isomorphism $u: f^*\tau(N) \cong \tau(M)$ where $\tau(\ )$ is the tangent bundle, then we can construct a transfer-type stable map. (See Boardman [7].)

Let $\iota: M \to R^s$ be an embedding. Then $(f, \iota); M \to N \times R^s$ is also an embedding. The normal bundle $\nu(M, N \times R^s)$ is stably isomorphic to the trivial one $M \times R^s$ under the above assumption. In fact, there are isomorphisms

$$\begin{aligned}\nu(M, N \times R^s) \oplus M \times R^s &\cong \nu(M, N \times R^s) \oplus \tau(\iota(M)) \oplus \nu(\iota(M), R^s)\\ &\cong \nu(M, N \times R^s) \oplus \tau(M) \oplus \nu(\iota(M), R^s) \cong (f, \iota)^* \tau(N \times R^s) \oplus \nu(\iota(M), R^s)\\ &\cong f^*\tau(N) \oplus \nu(\iota(M), R^s) \oplus M \times R^s \cong \tau(M) \oplus \nu(\iota(M), R^s) \oplus M \times R^s\\ &\cong M \times R^s \oplus M \times R^s.\end{aligned}$$

So, if we identify the normal bundle and the tubular neighborhood, then the Pontrjagin-Thom construction gives the stable map

$$t(f): (N \times R^s)^c \longrightarrow (\nu(M, N \times R^s))^c \approx (M \times R^s)^c$$

where $(\ )^c$ represents one point compactification. We may consider $t(f)$ as a stable map from N_+ to M_+.

One can easily show that the stable class of $t(f)$ is independent of the choice of the embedding. (See Boardman [7] and Becker-Gottlieb [5].)

On the other hand, it depends on the choice of the isomorphism u. So we should write $t(f, u)$ for the above stable map. For convenience we use $t(f)$, since the choices are clear in the following sequel.

Unfortunately, there is no general pull-back formula, but in some special cases, we can get some useful formulas.

Let M and N be the closed manifolds with smooth S^1-actions and let $f: M \to N$ be a smooth S^1-map. We assume also that the isomorphism $u: f^*\tau(N) \cong \tau(M)$ as S^1-vector bundles. By restriction to fixed point sets, we obtain an isomorphism

$$u^{S^1}: (f|_{M^{S^1}}) * \tau(N^{S^1}) \cong \tau(M^{S^1}).$$

Then the following theorem holds. (See Becker [4] and Nagata-Nishida-Toda [14].)

Theorem 1.1. *Under the above situation, the diagram*

$$\begin{array}{ccc} N_+^{S^1} & \xrightarrow{t(f|_{M^{S^1}})} & M_+^{S^1} \\ \cap\downarrow & & \cap\downarrow \\ N_+ & \xrightarrow{t(f)} & M_+ \end{array}$$

commutes up to stable homotopy.

Proof. First we consider the composition

$$k\colon \nu(M^{S^1}, N^{S^1}\times \mathbf{R}^s)\oplus M^{S^1}\times \mathbf{R}^s \cong M^{S^1}\times \mathbf{R}^s\oplus M^{S^1}\times \mathbf{R}^s$$

$$\xrightarrow{\cong} M\times \mathbf{R}^s\oplus M\times \mathbf{R}^s \xleftarrow{\cong} \nu(M, N\times \mathbf{R}^s)\oplus M\times \mathbf{R}^s.$$

Let $x+y\in \nu(M^{S^1}, N^{S^1}\times \mathbf{R}^s)$ where $x\in \nu(M, N\times \mathbf{R}^s)|_{M^{S^1}}$ and $y\in \nu(M^{S^1}, M)$ and let $\alpha+\beta+\gamma\in M^{S^1}\times \mathbf{R}^s$ where $\alpha\in \tau(\iota(M^{S^1}))$, $\beta\in \nu(\iota(M^{S^1}), \iota(M))$ and $\gamma\in \nu(\iota(M), \mathbf{R}^s)|_{\iota(M^{S^1})}$. Denote the canonical isomorphism

$$\nu(M, N\times \mathbf{R}^s)\oplus\tau(M)\cong f^*\tau(N)\oplus M\times \mathbf{R}^s$$

by μ and the composite

$$f^*\tau(N)\xrightarrow{j_1} f^*\tau(N)\oplus M\times \mathbf{R}^s\cong \nu(M, N\times \mathbf{R}^s)\oplus\tau(M)\longrightarrow \nu(M, N\times \mathbf{R}^s)$$

$$(\text{resp. } f^*\tau(N)\xrightarrow{j_1} f^*\tau(N)\oplus M\times \mathbf{R}^s\cong \nu(M, N\times \mathbf{R}^s)\oplus\tau(M)\longrightarrow \tau(M))$$

by π_ν (resp. π_τ) where j_n represents the inclusion to the n-th summand. Then k agrees with the composition

$$\begin{aligned}
&\nu(M^{S^1}, N^{S^1}\times \mathbf{R}^s)\oplus\tau(\iota(M^{S^1}))\oplus\nu(\iota(M^{S^1}), \iota(M))\oplus\nu(\iota(M), \mathbf{R}^s)|_{\iota(M^{S^1})}\\
&\quad\cong \nu(M^{S^1}, N^{S^1}\times \mathbf{R}^s)\oplus\tau(M^{S^1})\oplus\nu(M^{S^1}, M)\oplus\nu(\iota(M), \mathbf{R}^s)|_{\iota(M^{S^1})}\\
&\quad\xrightarrow{\mu\oplus\mathrm{id}\oplus\mathrm{id}} f^*\tau(N^{S^1})\oplus M^{S^1}\times \mathbf{R}^s\oplus f^*\nu(N^{S^1}, N)\oplus\nu(\iota(M), \mathbf{R}^s)|_{\iota(M^{S^1})}\\
&\quad\cong f^*(\tau(N)|_{N^{S^1}})\oplus M^{S^1}\times \mathbf{R}^s\oplus\nu(\iota(M), \mathbf{R}^s)|_{\iota(M^{S^1})}\\
&\quad\xrightarrow{\text{incl.}} f^*\tau(N)\oplus M\times \mathbf{R}^s\oplus\nu(\iota(M), \mathbf{R}^s)\\
&\quad\xrightarrow{\mu^{-1}\oplus\mathrm{id}} \nu(M, N\times \mathbf{R}^s)\oplus\tau(M)\oplus\nu(\iota(M), \mathbf{R}^s)\\
&\quad\cong \nu(M, N\times \mathbf{R}^s)\oplus\tau(\iota(M))\oplus\tau(\iota(M), \mathbf{R}^s).
\end{aligned}$$

This gives the equation

$$k(x+y, \alpha+\beta+\gamma)=(x+\pi_\nu(u^{-1}h^{-1}\beta), \alpha+\tilde{y}+\gamma+\widetilde{\pi_\tau(u^{-1}h^{-1}\beta)})$$

where $h\colon \tau(M)\cong\tau(\iota(M))$ and $\tilde{a}=h(a)$ for $a\in\tau(M)$.

If we replace $\tilde{y}$ and β by $\cos(\theta)\cdot\tilde{y}+\sin(\theta)\cdot\beta$ and $-\sin(\theta)\cdot\tilde{y}+\cos(\theta)\cdot\beta$, then k is properly homotopic to k_1 given by

$$k_1(x+y, \alpha+\beta+\gamma)=(x-\pi_\nu(u^{-1}y), \alpha+\beta+\gamma-\widetilde{\pi_\tau(u^{-1}y)}).$$

Since $\pi_\nu\colon f^*\nu(N^{S^1}, N)\to\nu(M, N\times \mathbf{R}^s)$ is proper, k_1 is properly homotopic to $k_2(x+y, \alpha+\beta+\gamma)=(x-\pi_\nu(u^{-1}y), \alpha+\beta+\gamma)$.

Put $L=\nu(M^{S^1}, M)$. By the collaring theorem, we may assume that $L=L'\cup(\partial L\times I)$, $\partial L'=\partial L\times\{0\}$, $\partial(\overline{M-L})=\partial L\times\{1\}$ and $M=L'\cup(\partial L\times I)\cup\overline{M-L}$. Let $p\colon \nu(M^{S^1}, M)\to M^{S^1}$ and $p'\colon \partial L\times I\to I$ be the projections. We may also assume that $L'=\{x\in M;\ \|xp(x)\|<1\}$ where $\|\ \|$ is the norm in $\mathbf{R}^s$. The S^1-action of M gives the non-singular vector field X on $(\partial L\times I)\cup\overline{M-L}$ which is tangent to $\partial L\times\{t\}$ for any $t\in I$ and $\|X(x)\|=1$ for any $x\in(\partial L\times I)\cup\overline{M-L}$.

The element of $(N^{S^1}\times\mathbf{R}^s)\cap\nu(M, N\times\mathbf{R}^s)$ can be written

$$x+y \quad \text{where } x\in\nu(M, N\times\mathbf{R}^s)|_{M^{S^1}} \text{ and } y\in\nu(M^{S^1}, M).$$

Then we define a locally proper homotopy H by the following equation:

$$H(x+y, t)=\begin{cases} x+(1-t)\cdot y+t\cdot\pi_\nu(-u^{-1}(y)) & \text{if } y\in L' \\ x+(1-t)\cdot y+t\cdot\pi_\nu(-u^{-1}(X(y))) & \text{if } y\in M-L \\ x+(1-t)\cdot y+t\cdot(1-2s+2s\cdot\|\pi_\nu(u^{-1}(y))\|^{-1})\cdot\pi_\nu(-u^{-1}(y)) & \\ \qquad\qquad \text{if } y\in\partial L\times I,\ 0\leq s=p'(y)\leq 1/2 & \\ x+(1-t)\cdot y+t\cdot K(2s-1;\ \|\pi_\nu(u^{-1}(y))\|^{-1}\cdot\pi_\nu(-u^{-1}(y)), & \\ \qquad \pi_\nu(-u^{-1}(X(y)))) & \\ \qquad\qquad \text{if } y\in\partial L\times I,\ 1/2\leq s=p'(y)\leq 1 & \end{cases}$$

where $K(s;\ v, w)=\|(1-s)\cdot v+s\cdot w\|^{-1}\cdot\{(1-s)v+s\cdot w\}$ which is well defined when v and w are linearly independent and $X(y)$ is the value of X at the terminal point of the vector y. We may assume that the correspondence given by $y\mapsto\pi_\nu(u^{-1}(y))$ is isometric. Since y and $\pi_\nu(u^{-1}(y))$ (or y and $\pi_\nu(u^{-1}(X(y)))$) are linearly independent, H is a well-defined locally proper homotopy. Thus the composition

$$(N^{S^1}\times\mathbf{R}^s)^c\longrightarrow(N\times\mathbf{R}^s)^c\longrightarrow(\nu(M, N\times\mathbf{R}^s))^c$$

is deformed to the composition

$$(N^{S^1}\times\mathbf{R}^s)^c\longrightarrow(\nu(M^{S^1}, N^{S^1}\times\mathbf{R}^s))^c\xrightarrow{(k)^c}(\nu(M, N\times\mathbf{R}^s))^c.$$

Let z be a point of N and assume $f^{-1}(z)=F$ where F is the disjoint union of the closed submanifolds of M.

The differential of f gives $df\colon \nu(F, M)\to f^*\tau(N)|_F$. Denote the composition

$$\nu(F, M)\xrightarrow{j_1}\nu(F, M)\oplus\tau(F)\cong\tau(M)|_F \text{ by } j_1.$$

We consider the following assumption:

(*) There is a locally proper homotopy between $u \circ df$ and j_1.

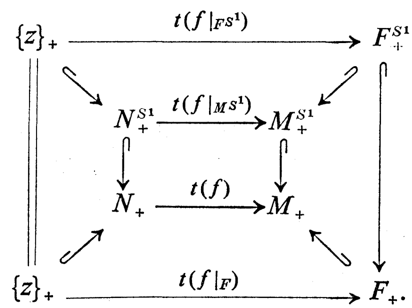
Then we have the following proposition.

Proposition 1.2. *Under the above assumptions, the diagram*

$$\begin{array}{ccc} \{z\}_+ & \xrightarrow{t(f|_F)} & F_+ \\ \cap\downarrow & & \cap\downarrow \\ N_+ & \xrightarrow{t(f)} & M_+ \end{array}$$

commutes up to stable homotopy where $t(f|_F)$ is the Becker-Gottlieb transfer of the fibre bundle $f|_F: F \to \{z\}$.

Moreover, in the situation of (1.1), *if we assume that $z \in N^{S^1}$ and the homotopy in* (*) *is S^1-equivariant, then the following is so:*



Once (1.2) proved, the proof of the following theorem is easy.

Let B be a subspace of N which is a finite complex and assume that $f|_{f^{-1}(B)}: f^{-1}(B) \to B$ is a fibre bundle whose fibre is a finite disjoint sum of compact manifolds, whose structure group is a compact Lie group acting smoothly on the fibre F. We put $E = f^{-1}(B)$ and assume:

$(*)_B$ There is a locally proper homotopy between $u \circ df$ and j_1 which is continuously parametrized by $z \in B$.

Theorem 1.3. *The diagram*

$$\begin{array}{ccc} B_+ & \xrightarrow{t(f|_E)} & E_+ \\ \cap\downarrow & & \cap\downarrow \\ N_+ & \xrightarrow{t(f)} & M_+ \end{array}$$

commutes up to stable homotopy where $t(f|_E)$ is the Becker-Gottlieb transfer of the fibre bundle $f|_E: E \to B$.

Proof of (1.2). Denote the composition

$$\tau(\iota(M))\xrightarrow{j_1}\tau(\iota(M))\oplus\nu(\iota(M),\,\mathbf{R}^s)\oplus f^*\tau(N)\cong(f,\iota)^*\tau(N\times\mathbf{R}^s)$$
$$\cong\nu(M,\,N\times\mathbf{R}^s)\oplus\tau(M)\xrightarrow{\text{proj.}}\nu(M,\,N\times\mathbf{R}^s)$$
$$\left(\begin{array}{l}\text{resp. }\tau(\iota(M))\xrightarrow{j_1}\tau(\iota(M))\oplus\nu(\iota(M),\,\mathbf{R}^s)\oplus f^*\tau(N)\cong(f,\,l^*)\tau(N\times\mathbf{R}^s)\\ \cong\nu(M,\,N\times\mathbf{R}^s)\oplus\tau(M)\xrightarrow{\text{proj.}}\tau(M)\end{array}\right)$$

by π_ν (resp. π_τ), and the composition

$$\nu(M,\,N\times\mathbf{R}^s)\xrightarrow{j_1}\nu(M,\,N\times\mathbf{R}^s)\oplus\tau(M)\cong(f,\iota)^*\tau(N\times\mathbf{R}^s)$$
$$\cong M\times\mathbf{R}^s\oplus f^*\tau(M)\xrightarrow{\text{proj.}}M\times\mathbf{R}^s$$
$$\left(\begin{array}{l}\text{resp. }\nu(M,\,N\times\mathbf{R}^s)\xrightarrow{j_1}\nu(M,\,N\times\mathbf{R}^s)\oplus\tau(M)\cong(f,\iota)^*\tau(N\times\mathbf{R}^s)\\ \cong M\times\mathbf{R}^s\oplus f^*\tau(N)\xrightarrow{\text{proj.}}f^*\tau(N)\end{array}\right)$$

by π_H (resp. π_V).

We identify the normal bundles and the tubular neighborhoods as usual. Since $(\{z\}\times\mathbf{R}^s)\cap\nu(M,\,N\times\mathbf{R}^s)$ can be identified with $\nu(F,\,\mathbf{R}^s)$, there is a map $g\colon\nu(F,\,\mathbf{R}^s)\to\nu(M,\,N\times\mathbf{R}^s)$ induced from the inclusion.

We consider the composition

$$k\colon\nu(F,\,\mathbf{R}^s)\oplus F\times\mathbf{R}^s\xrightarrow{g\oplus\text{incl.}}\nu(M,\,N\times\mathbf{R}^s)\oplus M\times\mathbf{R}^s\cong M\times\mathbf{R}^s\oplus M\times\mathbf{R}^s.$$

All we have to prove is that k is properly homotopic to the composition

$$\nu(F,\,\mathbf{R}^s)\oplus F\times\mathbf{R}^s\xrightarrow{j_1\oplus\text{id}}\nu(F,\,\mathbf{R}^s)\oplus\tau(F)\oplus F\times\mathbf{R}^s$$
$$\cong F\times\mathbf{R}^s\oplus F\times\mathbf{R}^s\xrightarrow{\text{incl.}}M\times\mathbf{R}^s\oplus M\times\mathbf{R}^s.$$

The first half of the result follows by the definition of the Becker-Gottlieb transfer $t(f|_F)$.

Let $x+y\in\nu(F,\,\mathbf{R}^s)$ where $x\in\nu(\iota(M),\,\mathbf{R}^s)|_F$ and $y\in\nu(F,\,\iota(M))$ and let $\alpha+\beta+\gamma\in F\times\mathbf{R}^s$ where $\alpha\in\tau(F)$, $\beta\in\nu(F,\,\iota(M))$ and $\gamma\in\nu(\iota(M),\,\mathbf{R}^s)|_F$. One easily obtains

$$k(x+y,\,\alpha+\beta+\gamma)=(x+\pi_H\pi_\nu(y)+\alpha+\pi_H(\beta),\;u(\pi_V\pi_\nu(y)+df\circ h(\beta))+\gamma).$$

Since $\pi_V\pi_\nu(y)=-df\circ\pi_\tau(y)$ and $\pi_\tau(y)\in\tau(M)$ is properly deformed to y, k is properly homotopic to k_1 given by

$$k_1(x+y,\,\alpha+\beta+\gamma)=(x+\alpha+\pi_H\pi_\nu(y)+\pi_H(\beta),\;-y+\beta+\gamma).$$

Clearly, there is a locally proper homotopy between k_1 and the identity given by the equation

$$H(x+y,\ \alpha+\beta+\gamma,\ t)$$
$$=(x+(1-t)\cdot(\pi_H\pi_\nu(y)+\pi_H(\beta))+\cos(i\pi t/2)\cdot\alpha+\sin(i\pi t/2)\cdot y,$$
$$\sin(i\pi t/2)\cdot\alpha-\cos(i\pi t/2)\cdot y+\beta+\gamma).$$

The last half of (1.2) is clear since the deformation in (1.1) corresponds to the homotopy between the compositions

$$\nu(F^{S^1}, \mathrm{R}^s)\xrightarrow{j_1}\nu(F^{S^1}, \mathrm{R}^s)\oplus\tau(F^{S^1})\cong F^{S^1}\times\mathrm{R}^s\longrightarrow F\times\mathrm{R}^s \quad \text{and}$$
$$\nu(F, \mathrm{R}^s)\xrightarrow{j_1}\nu(F, \mathrm{R}^s)\oplus\tau(F)\cong F\times\mathrm{R}^s$$

under the above deformation.

Now we state some properties of $t(f)$ obtained more easily.

Let $M=M_1\coprod M_2$ and $f=f_1\coprod f_2$ where $f_\varepsilon\colon M_\varepsilon\to N$ ($\varepsilon=1$ or 2). Put $u_\varepsilon\colon f_\varepsilon^*\tau(N)\cong\tau(M_\varepsilon)$ which is the restriction of $u\colon f^*\tau(N)\cong\tau(M)$ and construct $t(f_\varepsilon)$ by using the isomorphism u_ε.

Then the following proposition is clear.

Proposition 1.4.

$$t(f)=t(f_1)\vee t(f_2).$$

Let L be a closed manifold. Then

$$u\times\mathrm{id}\colon (f\times\mathrm{id})^*\tau(N\times L)\longrightarrow\tau(M\times L)$$

is also an isomorphism and we have $t(f\times\mathrm{id})=t(f)\wedge\mathrm{id}$.

Let Δ be the diagonal maps and $\mathrm{graph}(f)\colon M\to M\times N$ the composition $(\mathrm{id}\times f)\circ\Delta$. Then we have the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{\mathrm{graph}(f)} & M\times N \\ \downarrow f & & \downarrow f\times\mathrm{id} \\ N & \xrightarrow{\Delta} & N\times N. \end{array}$$

Let $\iota_M=\iota$ be the previous embedding of M into R^s and let ι_N be that of N into R^t. Then we have a new embedding $\tilde{\iota}\colon M\to\mathrm{R}^{s+t}$ defined by the equation $\tilde{\iota}(m)=\iota_M(m)\oplus\iota_N(f(m))$. By using $\tilde{\iota}$, we obtain a stable map $(N\times\mathrm{R}^{s+t})^c\to(M\times\mathrm{R}^{s+t})^c$ which is clearly stably homotopic to the t-th suspension of the previous $t(f)\colon(N\times\mathrm{R}^s)^c\to(M\times\mathrm{R}^s)^c$.

Consider the composition

$$\begin{aligned}
&\nu(M, N\times R^{s+t})\oplus M\times R^{s+t} \longrightarrow f^*\tau(N)\oplus\nu(M, R^{s+t})\oplus M\times R^{s+t}\\
&\xrightarrow{u\oplus \mathrm{id}} \tau(M)\oplus\nu(M, R^{s+t})\oplus M\times R^{s+t}\\
&\longrightarrow \tau(M\times N)\oplus\nu(M\times N, R^{s+t})\oplus M\times N\times R^{s+t}\\
&\xrightarrow{u^{-1}\times \mathrm{id}\oplus \mathrm{id}} (f\times \mathrm{id})^*\tau(M\times N)\oplus\nu(M\times N, R^{s+t})\oplus M\times N\times R^{s+t}\\
&\longrightarrow \nu(M\times N, N\times N\times R^{s+t})\oplus M\times N\times R^{s+t}
\end{aligned}$$

where the embedding of $M\times N$ into R^{s+t} is $\iota_M\times\iota_N$ and that into $N\times N \times R^{s+t}$ is given by $f\times\mathrm{id}$ and this embedding $M\times N\to R^{s+t}$. Clearly this composition agrees with

$$\nu(M, N\times R^{s+t})\oplus M\times R^{s+t} \xrightarrow{\delta\oplus \mathrm{incl.}} \nu(M\times N, N\times N\times R^{s+t})\oplus M\times N\times R^{s+t}$$

where δ is the natural correspondence given by graph(f) and Δ: $N\to N\times N$. Using this fact, the proof of the following proposition is easy.

Proposition 1.5. *The diagram of stable maps*

$$\begin{array}{ccc}
N^+ & \xrightarrow{t(f)} & M_+\\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \mathrm{graph}(f)}\\
(N\times N)_+ & \xrightarrow{t(f\times \mathrm{id})} & (M\times N)_+
\end{array}$$

commutes up to stable homotopy.

Corollary 1.6. *Let A be a ring spectrum and B an A module spectrum. Then $t(f)^*(f^*x\cup y)=x\cup t(f)^*y$ where $x\in A^m(N_+)$ and $y\in B^n(M_+)$.*

Corollary 1.7. *Under the same assumptions as in* (1.2), *if N is connected, then the composite*

$$\tilde{H}^*(N_+; \Lambda)\xrightarrow{f^*}\tilde{H}^*(M_+; \Lambda)\xrightarrow{t(f)^*}\tilde{H}^*(N_+; \Lambda)$$

is multiplication by the Euler characteristic $\chi(F)$ where $\tilde{H}^*(\ ;\Lambda)$ is reduced singular cohomology with coefficients in Λ.

Since the proofs of the corollaries are quite similar to the proof of Theorem 5.5 in [5], we omit them.

Remark 1.8. Our results can be generalized to some extent. For example, the existence of the appropriate S^1-actions can be replaced by that of the appropriate vector fields in (1.1).

Of course, some results have equivariant versions as in the case of the Becker-Gottlieb transfer. (See Nishida [15] and Nagata-Nishida-Toda [14].)

§ 2. Examples

The trivial example is the covering map $p: \tilde{M} \to M$. Notice that we can take an isomorphism $u: p^*\tau(M) \to \tau(\tilde{M})$ satisfying $(*)$ for all $m \in M$.

The our main example is concerned with compact Lie groups.

Let G be a compact connected Lie group, H a closed connected subgroup of G.

Let $\phi: G \times G \to G$ be the adjoint action on G which is given by $\phi(g, g') = g \cdot g' \cdot g^{-1}$ and denote G_c as the space G with this action. Then we define a G-map $i: G \times_H H_c \to G_c$ by equation $i(g, h) = g \cdot h \cdot g^{-1}$.

The element of $i^*\tau(G_c)$ can be represented as (g, h, v) where $g \in G$, $h \in H$ and $v \in T_{ghg^{-1}}(G)$. Let $[\mathrm{Ad}_{g^{-1}}(\ )]: T_{ghg^{-1}}(G) \to T_{[e]}(G/H)$ be the composition of $\mathrm{Ad}_{g^{-1}}: T_{ghg^{-1}}(G) \to T_h(G)$ and $[\ ]: T_h(G) \to T_{[e]}(G/H)$ where the last one is induced from the canonical quotient map. Then we can define a G-vector bundle epimorphism $f: i^*\tau(G_c) \to G \times_H (H_c \times L(G)/L(H))$ by $f(g, h, v) = (g, h, [\mathrm{Ad}_{g^{-1}}(\ )])$ where $L(G)$ and $L(H)$ are Lie algebra of G and H, respectively.

Since $\mathrm{Ker}[\mathrm{Ad}_{g^{-1}}(\ )] = T_{ghg^{-1}}(gHg^{-1})$, the G-vector bundle $\mathrm{Ker}(f)$ is G-isomorphic to $G \times_H (H_c)$ by corresponding (g, h, v) to $(g, h, \mathrm{Ad}_{g^{-1}} v)$ where $v \in T_{ghg^{-1}}(gHg^{-1})$.

Let $\pi_1: G \times_H H_c \to G/H$ be the G-map induced from the first projection. Then there is a canonical isomorphism of G-vector bundles $\tau(G \times_H H_c) \cong \pi_1^*(\tau(G/H)) \oplus G \times_H \tau(H_c)$. Thus we obtain the following proposition.

Proposition 2.1. *There is an isomorphism* $u: i^*\tau(G_c) \cong \tau(G \times_H H_c)$ *as G-vector bundles.*

Suppose that H is the maximal rank subgroup of G and T is the maximal torus of H. We denote the centralizer of the element g in G by $Z_G(g)$. Let $A(g)$ be the elements of T which are conjugate to g in G. As is well-known, $t \in A(g)$ is conjugate in $N_G(T)$. So $A(g)$ is a finite set. Since $N_H(T)$ acts on $A(g)$ by the conjugate action, we can consider the orbit set $A(g)/N_H(T)$. Then one can easily prove the following lemma.

Lemma 2.2. $i^{-1}(g) \approx \coprod_{[t] \in A(g)/N_H(T)} Z_G(t)/Z_H(t)$.

Now we want to show:

Proposition 2.3. $i: G \times_H H_c \to G_c$ *has the property* $(*)$ *for all* $g \in G$.

Proof. Since u and i are G-maps, all we have to show is that the kernel of the composition

$$C\colon T_{[e]}(G/H)\oplus T_t(H)\cong T_{(e,t)}(G\times_H H_c)\xrightarrow{\text{di}} T_t(G)$$
$$\xrightarrow{[\ ]\oplus\text{proj.}} T_{[e]}(G/H)\oplus T_t(H),\quad \text{where } t\in T,$$

agrees with the tangent space of $F_{(e,t)}$, the component of $i^{-1}(t)$ including (e, t), and the above composition is deformable to the orthogonal projection to the orthogonal complement of this kernel.

Clearly, C can be written as

$$\begin{pmatrix} [\text{id}-\text{Ad}(t)] & 0 \\ * & \text{id}_{T_t(H)} \end{pmatrix}$$

where $[\text{id}-\text{Ad}(t)]$ is the endomorphism of $T_{[e]}(G/H)=L(G)/L(H)$ which is well-defined since $L(H)$ is $\text{Ad}(t)$-invariant. Let $L(G)^{\text{Ad}(t)}\oplus V$ be the $\text{Ad}(t)$-invariant orthogonal decomposition of $L(G)$ where $L(G)^{\text{Ad}(t)}$ is the subspace of the $\text{Ad}(t)$-fixed vectors.

By (2.2) and the fact that $L(G)^{\text{Ad}(t)}=L(Z_G(t))$, $L(G)^{\text{Ad}(t)}/L(G)^{\text{Ad}(t)}\cap L(H)$ agrees with $T_{(e,t)}(F_{(e,t)})$.

Let $p\colon L(G)\to V$ be the orthogonal projection. Then the deformation defined by

$$C_s\begin{pmatrix}[X]\\ Y\end{pmatrix}=\begin{pmatrix}[p-s\cdot\text{Ad}(t)p] & 0\\ s\cdot * & \text{id}\end{pmatrix}\begin{pmatrix}[X]\\ Y\end{pmatrix}\quad (s\in I),$$

where $X\in L(G)$ and $Y\in T_t(H)$, gives a locally proper homotopy. Thus the result follows.

Remark 2.4. (i) In the case $i\colon G\times_H H_c\to G_c$, we clearly have a G-equivariant stable map $t(i)\colon G_{c+}\to(G\times_H H_c)_+$ since u is an isomorphism of the G-vector bundle and (1.8).

Moreover, if $\alpha\colon \Gamma\to\text{Aut}(G)$ is a group homomorphism where Γ is a finite group such that H is Γ-invariant under the Γ-action given by α and if we put the Γ-action on $G\times_H H_c$ by $(g,h)^\gamma=(g^\gamma,h^\gamma)$, then i and u are $G\times_\alpha\Gamma$-equivariant. So we obtain the $G\times_\alpha\Gamma$-equivariant stable map $t(i)$.

(ii) Let P be the total space of the principal G-bundle. Then we can construct a stable map

$$t(i)\colon P\times_G G_{c+}\longrightarrow P\times_H H_{c+}$$

as in Becker-Gottlieb [5].

(iii) If H' is the subgroup of G and contains H as the finite index normal subgroup, then our arguments can be extended to the stable map

$$t(i)\colon G_{c+} \longrightarrow G \times_{H'} H_{c+}.$$

§ 3. Applications

Let H be a subgroup of a topological group G. According to James [11], we say that H is homotopy-normal in G if the adjoint map $G \times H \to G$ given by $(g, h) \mapsto ghg^{-1}$ can be deformed into H. We say that H is strongly homotopy-normal in G if $i\colon G \times_H H_c \to G_c$ in Section 2 can be deformed into H_c.

For a compact manifold M, we denote the dimension of the Q-vector space $H_n(M; \mathrm{Q})$ by $b_n(M)$.

We have the following theorem.

Theorem 3.1. *Let G be the compact Lie group and H a closed subgroup containing a maximal torus of G. If $b_n(H) < b_n(G)$ for some integer $n > 0$, then H cannot be strongly homotopy-normal in G.*

Proof. Let G' and H' be the connected components of G and H containing the identity element. Then there exists a map $k\colon G' \times_{H'} H'_c \to G \times_H H_c$ induced by the inclusions. If $i\colon G \times_H H_c \to G_c$ can be deformed into H_c, then $i'\colon G' \times_{H'} H'_c \to G'_c$ can be deformed into H'_c since the diagram

$$\begin{array}{ccc} G' \times_{H'} H'_c & \xrightarrow{i'} & G' \\ \downarrow k & & \downarrow \\ G \times_H H_c & \xrightarrow{i} & G \end{array}$$

is commutative and $G' \times_{H'} H'_c$ is connected.

So, we may assume G and H to be connected.

Consider the composite

$$G_+ \xrightarrow{t(i)} G \times_H H_{c+} \xrightarrow{i} G_+.$$

If H is strongly homotopy-normal in G, then this composite factors through H_+. So, $(i \circ t(i))^*$ factors through $H^n(H_+; \mathrm{Q})$.

On the other hand, since $\chi(G/H) \neq 0$, (1.7) and (2.3) give that $(i \circ t(i))^*$ is the automorphism of $H^n(G_+; \mathrm{Q})$. Since $b_n(H) < b_n(G)$, that is impossible.

The next application needs more precise evaluations.

Let CP^n be the n-th complex projective space and $U(n)$ (resp. $SU(n)$) the n-th unitary (resp. special unitary) group. Then there is a natural inclusion j: $\Sigma(CP^{n-1}_+)\to U(n)$ (resp. j: $\Sigma(CP^{n-1})\to SU(n)$). (See James [11] and Yokota [19].)

James has shown in [11], $\Sigma(CP^{n-1})$ is a stable retract of $SU(n)$. In fact, he constructed the map t: $SU(n)\to Q\Sigma(CP^{n-1})$ such that $t\circ j\simeq$ stabilization by using a geometrical construction and the mod k-Dold theorem, where $Q=\lim_n \Omega^n\Sigma^n$ is the stabilization functor. (He has proved also that RP^{n-1} is a stable retract of $SO(n)$ and that the $(n-1)$-th quaternionic quasiprojective space is a stable retract of $Sp(n)$.)

A short proof of the following theorem is given by F. Cohen and F. Peterson. This theorem has also announced by Crabb [10] and Mann-Miller-Miller [13].

Let U (resp. SU) be the infinite unitary (resp. special unitary) group.

Theorem 3.2. *$Q\Sigma(CP^\infty)\simeq SU\times X$ where $\pi_i(X)$ is finite.*

Proof. (Due to F. Cohen and F. Peterson.) Let g: $Q\Sigma(CP^\infty)\to SU$ be the extension of j obtained by using the infinite loop structure of SU. Consider the composite $g\circ t\circ j$. Since $t\circ j\simeq$ stabilization, $g\circ t\circ j\simeq j$. Since $H^*(SU; Z)$ is an exterior algebra on elements which pull back epimorphically to $H^*(\Sigma(CP^\infty); Z)$, it follows that $(g\circ t)^*$ is an isomorphism. Thus $Q\Sigma(CP^\infty)\simeq SU\times$(homotopy fibre of g), by virtue of the theorem of J.H.C. Whitehead.

Put $X=$homotopy fibre of g. Then $\pi_i(X)$ is finite since

$$H_*(Q\Sigma(CP^\infty); Q)\cong H_*(SU; Q).$$

As an application of our transfer, we can construct a map t: $U\to Q\Sigma(CP^\infty_+)$ such that the composition $t\circ j$ is homotopic to the stabilization, merely using geometrical constructions.

Put $G_n=U(n)$ and $H_n=U(n-1)\times U(1)$. Let $S^1=U(1)\to U(n)$ be the inclusion given by $s\mapsto\begin{pmatrix}1&&&&\\&1&&&\\&&\ddots&&\\&&&1&\\&&&&s\end{pmatrix}$. Under this inclusion, the adjoint action defines an S^1-action on $(G_n)_c$, and left multiplication defines an action on $G_n\times_{H_n}(H_n)_c$.

First, we investigate the fixed point sets of these actions.

Denote the S^1-fixed point set of the S^1-space X as X^{S^1}. Then the proof of the following proposition is easy.

Proposition 3.3.

(i) $(G_n)_c^{S^1}=H_n$

(ii) $(G_n\times_{H_n}(H_n)_c)^{S^1}$

$$=H_n\times_{H_n}(H_n)_c\coprod H_n\cdot\sigma\times_{nU(1)\times H_{n-1}}(U(1)\times H_{n-1})_c$$

where $\sigma=\begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\\ 1 & & & & \end{pmatrix}\in G_n$.

Let $i_n\colon G_n\times_{H_n}(H_n)_c\to(G_n)_c$ be the map in Section 2. Then i_n is an S^1-equivariant map. Let $p_n\colon G_n\times_{H_n}(H_n)_c\to U(n)/U(n-1)\times_{U(1)}(U(1))_c$ be the map induced from the projection. By virtue of (1.2), we obtain the following homotopy commutative diagram of stable maps

$$\begin{array}{ccccc}\{e\}_+ & \longrightarrow & G_n\times_{H_n}\{e\}_+ & \longrightarrow & U(n)/U(n-1)\times_{U(1)}\{e\}_+\\ \downarrow & & \downarrow & & \downarrow\\ (G_n)_{c+} & \xrightarrow{t(i_n)} & G_n\times_{H_n}(H_n)_{c+} & \longrightarrow & U(n)/U(n-1)\times_{U(1)}U(1)_{c+}.\end{array}$$

If we take the cofibres of the vertical arrows of the above diagram, we have the map

$$t_n\colon U(n)=G_n\longrightarrow Q((U(n)/U(n-1)\times_{U(1)}U(1)_c)/(U(n)/U(n-1)\times_{U(1)}\{e\}))$$
$$=Q\Sigma(CP_+^{n-1}).$$

We need the following lemma. Let $k_n\colon U(n-1)\to U(n)$ be the inclusion given by $k_n(A)=\begin{pmatrix}A & \\ & 1\end{pmatrix}$ and let

$$k_n'\colon U(n-1)/U(n-2)\times_{U(1)}U(1)_{c+}\longrightarrow U(n)/U(n-1)\times_{U(1)}U(1)_{c+}$$

be the inclusion given by $k_n(\ )\cdot\sigma$. Clearly, k_n' induces the inclusion $k_n'\colon\Sigma(CP_+^{n-2})\to\Sigma(CP_+^{n-1})$.

Lemma 3.4. *The following diagram is homotopy commutative:*

$$\begin{array}{ccc}U(n-1) & \xrightarrow{t_{n-1}} & Q\Sigma(CP_+^{n-2})\\ \downarrow{\scriptstyle k_n} & & \downarrow{\scriptstyle k_n'}\\ U(n) & \xrightarrow{t_n} & Q\Sigma(CP_+^{n-1}).\end{array}$$

Proof. By virtue of (1.1), we have the diagram

$$\begin{array}{ccccc} (G_n)^{S^1}_{c+} & \xrightarrow{t(i_n^{S^1})} & (G_n\times_{H_n}(H_n)_c)^{S^1}_+ & & \\ \downarrow & & \downarrow & & \\ (G_n)_{c+} & \xrightarrow{t(i_n)} & G_n\times_{H_n}(H_n)_{c+} & \xrightarrow{p_n} & U(n)/U(n-1)\times_{U(1)}U(1)_{c+} \end{array}$$

which is stably homotopy commutative. By (1.4) and (3.2), $t(i_n^{S^1})$ is the wedge sum of

$$t_1\colon (H_n)_{c+}\longrightarrow H_n\times_{H_n}(H_n)_{c+} \quad \text{and}$$
$$t_2\colon (H_n)_{c+}\longrightarrow H_n\cdot\sigma\times_{U(1)\times H_{n-1}}(U(1)\times H_{n-1})_{c+}.$$

Clearly, the composite

$$U(n-1)_+\xrightarrow{k_n}U(n)_+=G_{n+}\xrightarrow{p_n\circ t_1}U(n)/U(n-1)\times_{U(1)}U(1)_{c+}$$

is trivial since t_1 is homotopic to identity.

On the other hand, there is a diffeomorphism

$$\ell\colon (G_{n-1}\times_{H_{n-1}}(H_{n-1})_c)\times U(1)_c\longrightarrow H_n\cdot\sigma\times_{U(1)\times H_{n-1}}(U(1)\times H_{n-1})$$

given by $\ell((A, B), s)=(k_n(A)\cdot\sigma, (s, B))$ where $A\in G_{n-1}$, $B\in H_{n-1}$ and $s\in U(1)$. Then the following diagram is commutative:

$$\begin{array}{ccccc} (H_n)_c & \xrightarrow{i_n} & H_n\cdot\sigma\times_{U(1)\times H_{n-1}}(U(1)\times H_{n-1})_c & \xrightarrow{p_n} & U(n)/U(n-1)\times_{U(1)}U(1) \\ \| & & \uparrow \ell & & \uparrow k_1' \\ (H_n)_c & \xrightarrow{i_{n-1}\times\mathrm{id}} & (G_{n-1}\times_{H_{n-1}}(H_{n-1})_c)\times U(1)_c & \xrightarrow{p'_{n-1}} & U(n-1)/U(n-2)\times_{U(1)}U(1)_c \end{array}$$

where the map p'_{n-1} is the composition of p_{n-1} with the projection.

Since $t_2=\ell\circ t(i_{n-1}\times\mathrm{id})=\ell\circ(t(i_{n-1})\wedge\mathrm{id})$, we obtain the following equation

$$p_n\circ t(i_n)\circ k_n=p_n\circ t_2\circ k_n=k'_n\circ p_{n-1}\circ t(i_{n-1})$$

and, by collapsing the suitable parts, the second part of (1.2) gives the desired result.

Then we can put $t=\varinjlim_n t_n\colon U\to\Sigma(CP^\infty_+)$. We only have to prove that the composite

$$\Sigma(CP^{n-1}_+)\xrightarrow{j}U(n)\xrightarrow{t_n}\Sigma(CP^{n-1}_+)$$

is stably homotopic to the identity map.

The image of j in $U(n)$ agrees with the subspace

$$\left\{ A;\ A \in U(n) \text{ and } A \text{ is conjugate to } \begin{pmatrix} s & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \text{ where } s \in U(1) \right\}$$

which is denoted by B_n. Then fixed point set of the S^1-action of $i_n^-(e) \approx U(n)/(U(n-1)\times U(1)) = CP^{n-1}$ given by the multiplication for the last coordinate is $CP^{n-2} \coprod pt$. One can easily show that $E_n = (i_n^{-1}(e))^{S^1} \cup i_n^{-1}(B_n - e)$ is the smooth fibre bundle over B_n whose fibre is $CP^{n-2} \coprod pt$. We put $E_n = E_{n,1} \coprod E_{n,2}$ where $E_{n,1}$ corresponds to the fibre CP^{n-2} and $E_{n,2}$ to the fibre $\{pt\}$. Then one can easily check that $p_n\colon E_{n,1} \to \Sigma(CP_+^{n-1})$ is trivial, $E_{n,2}$ is identified with $\Sigma(CP_+^{n-1})$ by p_n, and under this identification $i_n | E_{n,2}$ is the identity map. So we only have to prove the following proposition.

Proposition 3.5. *The diagram*

$$\begin{array}{ccc} B_{n+} & \xrightarrow{t(i_n | E_n)} & E_{n+} \\ \downarrow & & \downarrow \\ (G_n)_{c+} & \xrightarrow{t(i_n)} & G_n \times_{H_n} (H_n)_{c+} \end{array}$$

is commutative up to stable homotopy where $t(i_n | E_n)$ is the Becker-Gottlieb transfer of the fibre bundle $i_n | E_n\colon E_n \to B_n$.

Proof. We denote the homotopy $u \circ df \simeq j_1$ in the proof of (2.3) on the fibre of $i_n^{-1}(g)$ by $h_{g,s} (s \in I)$. It is clear that $h_{e,s} = j_1$ and so $u \circ df = j_1\colon \nu(i_n^{-1}(e), G_n \times_{H_n} (H_n)_c) \to \tau(G_n \times_{H_n} (H_n)_c) | i_n^{-1}(e)$. Put

$$e_\theta = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & e^{i\theta} \end{pmatrix} \quad \text{for } 0 \leq \theta < 2\pi.$$

Let p be the bundle projection of $\nu(i_n^{-1}(e), G_n \times_{H_n} (H_n)_c)$. Then, by the construction in (2.3), there is a vector field X on $i_n^{-1}(e)$ such that

$$s \cdot X \circ p = \lim_{\theta \to 0} h_{e_\theta, s} - h_{e,s}.$$

Since one can easily show that X agrees with the vector field given by the S^1-action on $i_n^{-1}(e) \approx CP^{n-1}$, it follows that the composition

$$\{e\}_+ \longrightarrow (G_n)_{c+} \xrightarrow{t(i_n)} G_n \times_{H_n} (H_n)_{b+}$$

can be deformed into a map factoring through $(i_n^{-1}(e))_+^{S^1}$. (See (1.2) and Brumfiel-Madsen [6].) Thus we obtain the deformations

$$\{b\}_+ \longrightarrow (G_n)_{c+} \xrightarrow{t(i_n)} G_n \times_{H_n} (H_n)_{c+},$$

factoring through $(i_n | E_n)^{-1}(b)_+$ and these deformations are continuous for $b \in B_n$. Thus the proposition follows.

Remark 3.6. (i) If we consider the composition

$$SU \longrightarrow U \xrightarrow{t} Q\Sigma(CP_+^\infty) \longrightarrow Q\Sigma(CP^\infty),$$

then we get a splitting of $Q\Sigma(CP^\infty)$.

(ii) Let $\Gamma = Z/2$ and α: $Z/2 \rightarrow \mathrm{Aut}\, U(n)$ be the action of the complex conjugation. Then we obtain the $Z/2$-equivariant stable map t_n: $U(n) \rightarrow \Sigma^{1,0}(CP_+^{n-1})$ where the $Z/2$-actions on $U(n)$ and CP^{n-1} are given by complex conjugation and $\Sigma^{1,0}$ is the one point compactification of $R^{1,0}$ which is the unique non-trivial $Z/2$-representation on R^1. Then $t_n \circ j_n$ is $Z/2$-equivariantly homotopic to the stabilization.

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