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On p-Sylow Subgroups of Groups of Self Homotopy Equivalences of Sphere Bundles over Spheres

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Dedicated to Professor Nobuo Shimada on his 60th birthday

Introduction

The set $\mathscr{E}(X)$ of homotopy classes of homotopy equivalences of a space X to itself forms a group under composition of maps. This group $\mathscr{E}(X)$ has been investigated by several authors (e.g. [2], [7] and [12]).

In the case where X is an S^m-bundle over S^n, the group $\mathscr{E}(X)$ has been investigated for $X=V_{n,2}$ and $W_{n,2}$ by Y. Nomura [10] and for X with $3<m+1<n<2m-2$ by S. Sasao [13], where $V_{n,2}=O(n)/O(n-2)$ and $W_{n,2}=U(n)/U(n-2)$ are the real and complex Stiefel manifolds respectively.

In this note, we study the p-Sylow subgroup of $\mathscr{E}(X)$ for an S^m-bundle X over S^n with a mod p H-structure such that $i_{(p)}: S^m_{(p)}\to X_{(p)}$ is an H-map, where m and n are odd integers, $S^m_{(p)}$ and $X_{(p)}$ are localizations of S^m and X at $\{p\}$ respectively and $i_{(p)}$ is the localization of the inclusion $i: S^m\subset X$ at $\{p\}$. Our main result is as follows:

Theorem 4.5. *Let m and n be odd integers such that $3\leqq m<n-1$, and let $S^m\xrightarrow{i}X\xrightarrow{q}S^n$ be an S^m-bundle over S^n. Let p be an odd prime. If $S^m_{(p)}$ and $X_{(p)}$ are H-spaces such that $i_{(p)}: S^m_{(p)}\to X_{(p)}$ is an H-map, then the group $\mathscr{E}(X)$ is a finite group with a unique p-Sylow subgroup $\tilde{S}_p$ given by the semi direct product*

$$\tilde{S}_p\cong\pi_{m+n}(X;p)\underset{T}{\times}\pi_n(S^m;p),$$

where $\alpha T\beta=\alpha+i\circ\beta\circ q\circ\alpha$ for $\alpha\in\pi_{m+n}(X;p)$ and $\beta\in\pi_n(S^m;p)$.

In Section 1, we determine the p-Sylow subgroup of $\mathscr{E}(S^m\cup e^n)$ (Proposition 1.3). In Section 2, we define a homomorphism $j^1: \mathscr{E}(X)\to\mathscr{E}(K)$ and study the p-Sylow subgroup of Im j^1 (Lemma 2.7). In Section

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3, we prepare three lemmas and the above theorem is proved in Section 4. In the last section, Section 5, we calculate the p-Sylow subgroup of $\mathscr{E}(X)$ of some S^m-bundles X over S^n for any odd prime p and determine the group $\mathscr{E}(X)$ as a group extension of a certain group by a 2-group.

Throughout this note, all spaces have base points and all maps and homotopies preserve base points. For given spaces X and Y, we denote by $[X, Y]$ the set of (based) homotopy classes of maps of X to Y and by the same letter a map $f: X \to Y$ and its homotopy class $f \in [X, Y]$.

§ 1. The p-Sylow subgroup of $\mathscr{E}(S^m \cup e^n)$

Let $f \in \pi_{n-1}(S^m)$ $(2 \leqq m < n-1)$ be a given element and let $K = S^m \cup_f e^n$ denote the mapping cone of f. Let $\ell_1: K = S^m \cup e^n \to (S^m \cup e^n) \vee S^n = K \vee S^n$ be the coaction defined by shrinking the equator $S^{n-1} \times \{1/2\}$ of e^n in $S^m \cup e^n$ to the base point. Then we can define a map

$$\Lambda: \pi_n(K) \longrightarrow [K, K] \qquad \text{by } \Lambda(\alpha) = \nabla \circ (1 \vee \alpha) \circ \ell_1,$$

where ∇ is the folding map and 1 is the class of the identity map of K. Let $i': S^m \subset K$ be the inclusion. Then by composing $i'_*: \pi_n(S^m) \to \pi_n(K)$ with Λ, we obtain a homomorphism (cf. [11, Lemmas 1.4 and 1.8])

$$(1.1) \qquad \lambda_1: \pi_n(S^m) \longrightarrow \mathscr{E}(K) \qquad \text{by } \lambda_1(\alpha) = \nabla \circ (1 \vee i' \circ \alpha) \circ \ell_1.$$

We put

$$H = \pi_n(S^m) / (f_* \pi_n(S^{n-1}) + \gamma(f) \pi_{m+1}(S^m)).$$

Here $\gamma(f)(\eta) = \eta \circ Sf + [\iota_m, \eta] \circ Sh(f)$ for $\eta \in \pi_{m+1}(S^m)$ where ι_m is the class of the identity map of S^m, $[\iota_m, \eta] \in \pi_{2m}(S^m)$ is the Whitehead product of ι_m and η and $h(f) \in \pi_{n-1}(S^{2m-1})$ is the generalized Hopf invariant of f due to P. J. Hilton [3]. Then the homomorphism λ_1 induces a monomorphism: $H \to \mathscr{E}(K)$ and we have

(1.2) ([11, Theorem 3.15]) *For a two-cell complex $S^m \cup_f e^n$ $(2 \leqq m < n-1)$, we have the exact sequence*

$$0 \longrightarrow H_1 \longrightarrow \mathscr{E}(S^m \cup_f e^n) \longrightarrow G_1 \longrightarrow 1,$$

$$H_1 = H \textit{ if } 2f \neq 0, \quad H_1 = D(H) \textit{ if } 2f = 0,$$

$$G_1 = \begin{cases} Z_2 & \textit{if } 2f = a(f), \textit{ or } 2f \neq 0 \textit{ and } a(f) = 0, \\ 1 & \textit{otherwise,} \end{cases}$$

where $D(H)$ is the split extension $0 \to H \to D(H) \to Z_2 \to 1$ with action of

$\boldsymbol{Z}_2=\{1, -1\}$ *on* H *by* $(-1)\cdot\alpha=-\alpha$ *for* $\alpha\in H$ *and* $a(f)=f+(-1)\circ f=[\iota_m, \iota_m]\circ h(f)$.

From $\pi_k(S^{k-1})=\boldsymbol{Z}_2$ for $k\geqq 4$ and (1.2), we have immediately

Proposition 1.3. *Let* $3\leqq m<n-1$ *and let* p *be an odd prime. In the case where* $n\ m$ *is even, we assume* $n\neq 2m-1$. *Then, for the mapping cone* $K=S^m\cup_f e^n$ *of* $f\in\pi_{n-1}(S^m)$, *the group* $\mathscr{E}(K)$ *is a finite group with a unique* p*-Sylow subgroup* S_p *given by*

$$S_p=\lambda_1\pi_n(S^m;p)\cong\pi_n(S^m;p),$$

where $\pi_n(S^m;p)$ *denotes the* p*-primary component of* $\pi_n(S^m)$.

§ 2. The p-Sylow subgroup of Im $j^!$

Let X denote an S^m-bundle over S^n $(2\leqq m<n-1)$. Then by James-Whitehead [6], X has a cell structure given by

$$X=K\cup_g e^{m+n},\qquad K=S^m\cup_f e^n. \tag{2.1}$$

Since the inclusion j: $K\subset X$ induces a bijection j_*: $[K, K]\to[K, X]$, the homomorphism

$$j^!:\ \mathscr{E}(X)\longrightarrow\mathscr{E}(K) \tag{2.2}$$

can be defined by the restriction to $\mathscr{E}(X)$ of the composite

$$[X, X]\xrightarrow{j^*}[K, X]\underset{\cong}{\xrightarrow{j_*^{-1}}}[K, K].$$

We define the coaction

$$\ell_2:\ X=K\cup e^{m+n}\longrightarrow(K\cup e^{m+n})\vee S^{m+n}=X\vee S^{m+n}$$

by shrinking the equator $S^{m+n-1}\times\{1/2\}$ of e^{m+n} to the base point. Since $\pi_{m+n}(S^m)$ and $\pi_{m+n}(S^n)$ for $2\leqq m<n-1$ are finite, $\pi_{m+n}(X)$ is finite by the exact sequence associated with the S^m-bundle over S^n:

$$S^m\xrightarrow{i}X\xrightarrow{q}S^n. \tag{2.3}$$

Therefore, by the Blakers-Massey theorem and the exact sequence of the pair (X, K) we have

$$j_*:\ \pi_{m+n}(K)\longrightarrow\pi_{m+n}(X)\quad\text{is epimorphic.} \tag{2.4}$$

Hence, similarly to the way that we defined Λ in Section 1, we can define a homomorphism (cf. [11, Lemmas 1.4 and 1.8])

(2.5) $\lambda_2\colon \pi_{m+n}(X)=j_*\pi_{m+n}(K)\longrightarrow\mathscr{E}(X)$ by $\lambda_2(\alpha)=\nabla\circ(1\vee\alpha)\circ\ell_2$,

where $\alpha\in\pi_{m+n}(X)$, $\nabla\colon X\vee X\to X$ is the folding map and 1 is the class of the identity map of X. Also, since the attaching element $g\in\pi_{m+n-1}(K)$ of e^{m+n} in $X=K\cup e^{m+n}$ is of infinite order, by Barcus-Barratt [2, Theorem 6.1], J. W. Rutter [12, Theorem 3.1*] and (2.4) we have the following exact sequence:

$$0\longrightarrow\lambda_2(\pi_{m+n}(X))\longrightarrow\mathscr{E}(X)\xrightarrow{j!}G\longrightarrow 1, \tag{2.6}$$

where $G=\{h\in\mathscr{E}(K)\,|\,h\circ g=\varepsilon g\ (\varepsilon=\pm 1)\text{ in }\pi_{m+n-1}(K)\}\subset\mathscr{E}(K)$.

Lemma 2.7. (i) *For $\alpha\in\pi_n(S^m)$, $\lambda_1(\alpha)\in\mathscr{E}(K)$ given in* (1.1) *can be extended to an element of $\mathscr{E}(X)$ if and only if $i'_*[\alpha,\iota_m]=0$, where $i'\colon S^m\subset K$ is the inclusion and ι_m is the class of the identity map of S^m.*

(ii) *Let m be an odd integer. Then for any odd prime p the subgroup G of $\mathscr{E}(K)$ in the above sequence is a finite group with a unique p-Sylow subgroup $S_p\cong\pi_n(S^m;p)$ given in Proposition* 1.3.

Proof. (i) Let g be the attaching element of e^{m+n} in $X=K\cup e^{m+n}$ given in (2.1). Then we have $j_*g=\pm[\sigma,i']$, where $j_*\colon\pi_{m+n-1}(K)\to\pi_{m+n-1}(K,S^m)$ and $\sigma\in\pi_n(K,S^m)$ is an element such that $\partial\sigma=f$, the attaching element of e^n in $K=S^m\cup e^n$. So, by [4, Lemma 5.4], we have

$$\ell_1\circ g=k_*g\pm[k_n,k_m],$$

where $\ell_1\colon K\to K\vee S^n$ is the coaction given in Section 1, $k\colon K\to K\vee S^n$ and $k_r\colon S^r\to K\vee S^n$ $(r=m,n)$ are obvious inclusions. Therefore, for $\lambda_1(\alpha)$ $(\alpha\in\pi_n(S^m))$ given in (1.1), we have

$$\begin{aligned}\lambda_1(\alpha)\circ g&=\nabla\circ(1\vee(i'\circ\alpha))\circ\ell_1\circ g\\&=\nabla\circ(1\vee(i'\circ\alpha))\circ(k_*g\pm[k_n,k_m])\\&=g\pm[i'\circ\alpha,i']\\&=g\pm i'_*[\alpha,\iota_m].\end{aligned}$$

Since g is of infinite order and $[\alpha,\iota_m]$ is of finite order, the above equalities imply that $\lambda_1(\alpha)\circ g\neq -g$ for any $\alpha\in\pi_n(S^m)$ and that $\lambda_1(\alpha)\circ g=g$ if and only if $i'_*[\alpha,\iota_m]=0$.

(ii) If m is an odd integer, then $[\alpha,\iota_m]=0$ for any $\alpha\in\pi_n(S^m;p)$ and so by (i) and Proposition 1.3, G has a unique p-Sylow subgroup $S_p\cong\pi_n(S^m;p)$. q.e.d.

§ 3. Some lemmas for an H-structure on $X_{(p)}$

Let m and n be odd integers such that $3 \leq m < n-1$ and $S^m \xrightarrow{i} X \xrightarrow{q} S^n$ be an S^m-bundle over S^n. In this section we assume that p is an odd prime such that the localized space $X_{(p)}$ at $\{p\}$ is an H-space. Then we have the following lemmas which will be used in the next section.

Lemma 3.1. *Let $\alpha \in \pi_n(X; p)$, $\beta \in \pi_n(X; p)$, $\gamma \in \pi_{m+n}(X; p)$ and $\xi \in \pi_n(S^m; p)$, and let $\pi: X \to X/K = S^{m+n}$ be the collapsing map, where K is the subcomplex of X given in* (2.1). *Then we have*

(i) $(1+\alpha_{(p)} \circ q_{(p)}) + \beta_{(p)} \circ q_{(p)} = 1 + (\alpha_{(p)} \circ q_{(p)} + \beta_{(p)} \circ q_{(p)})$,

(ii) $(1+\alpha_{(p)} \circ q_{(p)}) + \gamma_{(p)} \circ \pi_{(p)} = 1 + (\alpha_{(p)} \circ q_{(p)} + \gamma_{(p)} \circ \pi_{(p)})$,

(iii) $i_{(p)} \circ \xi_{(p)} \circ q_{(p)} + \gamma_{(p)} \circ \pi_{(p)} = \gamma_{(p)} \circ \pi_{(p)} + i_{(p)} \circ \xi_{(p)} \circ q_{(p)}$,

where $+$ denotes the multiplication induced from the H-structure on $X_{(p)}$.

Proof. Since $[Y, X_{(p)}]$ is an algebraic loop for any CW-complex Y by [5, Theorem 1.1], we can define an obstruction $\phi \in [X_{(p)} \times X_{(p)} \times X_{(p)}, X_{(p)}]$ for the multiplication to be homotopy associative by

$$(p_1 + p_2) + p_3 = (p_1 + (p_2 + p_3)) + \phi, \tag{3.2}$$

where $p_i: X_{(p)} \times X_{(p)} \times X_{(p)} \to X_{(p)}$ $(i=1, 2, 3)$ is the i-th projection. We put $L = (X_{(p)} \times X_{(p)} \times \{*\}) \cup (X_{(p)} \times \{*\} \times X_{(p)}) \cup (\{*\} \times X_{(p)} \times X_{(p)})$. Using the Puppe exact sequence associated with the cofibering $L \to X_{(p)} \times X_{(p)} \times X_{(p)} \xrightarrow{\pi'} X_{(p)} \wedge X_{(p)} \wedge X_{(p)}$, we see that there exists an element ϕ' such that

$$\phi = \phi' \circ \pi', \qquad \phi' \in [X_{(p)} \wedge X_{(p)} \wedge X_{(p)}, X_{(p)}], \tag{3.3}$$

because in (3.2) we have $((p_1 + p_2) + p_3)|L = (p_1 + (p_2 + p_3))|L$. Therefore, by (3.2) and (3.3),

$$\begin{aligned}(1+\alpha_{(p)} \circ q_{(p)}) + \beta_{(p)} \circ q_{(p)} &= ((p_1+p_2)+p_3) \circ (1 \times \alpha_{(p)} \circ q_{(p)} \times \beta_{(p)} \circ q_{(p)}) \circ d \\ &= ((p_1 + (p_2 + p_3)) + \phi) \circ (1 \times \alpha_{(p)} \circ q_{(p)} \times \beta_{(p)} \circ q_{(p)}) \circ d \\ &= (1 + (\alpha_{(p)} \circ q_{(p)} + \beta_{(p)} \circ q_{(p)})) + \phi' \circ (1 \wedge \alpha_{(p)} \wedge \beta_{(p)}) \circ (1 \wedge q_{(p)} \wedge q_{(p)}) \\ &\qquad \circ \pi' \circ d,\end{aligned}$$

where $d: X_{(p)} \to X_{(p)} \times X_{(p)} \times X_{(p)}$ is the diagonal map. Since, in the above equalities, $(1 \wedge q_{(p)} \wedge q_{(p)}) \circ \pi' \circ d: X_{(p)} \to X_{(p)} \wedge S^n_{(p)} \wedge S^n_{(p)}$ is homotopic to the constant map for dimensional reasons, we have the equality of (i).

The proof of (ii) is similar to that of (i) and so we omit it.

(iii) Let $\omega \in [X_{(p)} \times X_{(p)}, X_{(p)}]$ be an obstruction for the multiplication to be homotopy commutative defined by

$$p_1 + p_2 = (p_2 + p_1) + \omega, \tag{3.4}$$

where $p_i\colon X_{(p)}\times X_{(p)}\to X_{(p)}$ $(i=1, 2)$ is the i-th projection. Using the Puppe exact sequence associated with the cofibering

$$X_{(p)}\vee X_{(p)}\longrightarrow X_{(p)}\times X_{(p)}\xrightarrow{\pi''}X_{(p)}\wedge X_{(p)},$$

we see that there exists an element ω'' such that

$$\omega=\omega''\circ\pi'',\qquad \omega''\in[X_{(p)}\wedge X_{(p)}, X_{(p)}]. \tag{3.5}$$

Therefore, by (3.4) and (3.5),

$$\begin{aligned} i_{(p)}\circ\xi_{(p)}\circ q_{(p)}+\gamma_{(p)}\circ\pi_{(p)} &=(p_1+p_2)\circ(i_{(p)}\circ\xi_{(p)}\circ q_{(p)}\times\gamma_{(p)}\circ\pi_{(p)})\circ d\\ &=((p_2+p_1)+\omega)\circ(i_{(p)}\circ\xi_{(p)}\circ q_{(p)}\times\gamma_{(p)}\circ\pi_{(p)})\circ d\\ &=(\gamma_{(p)}\circ\pi_{(p)}+i_{(p)}\circ\xi_{(p)}\circ q_{(p)})+\omega''\circ((i_{(p)}\circ\xi_{(p)})\wedge\gamma_{(p)})\circ(q_{(p)}\wedge\pi_{(p)})\circ\pi''\circ d, \end{aligned}$$

where $d\colon X_{(p)}\to X_{(p)}\times X_{(p)}$ is the diagonal map. Since, in the above equalities, $(q_{(p)}\wedge\pi_{(p)})\circ\pi''\circ d\colon X_{(p)}\to S^n_{(p)}\wedge S^{m+n}_{(p)}$ is homotopic to the constant map for dimensional reasons, we have the equality of (iii). q.e.d.

Lemma 3.6. *Let $\alpha\in\pi_n(S^m; p)$ and $\lambda_1(\alpha)$ be an element of $\mathscr{E}(K)$ given in* (1.1), *and let $j\colon K\subset X$ be the inclusion and $\pi\colon X\to X/K=S^{m+n}$ be the collapsing map. Then we have*

(i) $(1+i_{(p)}\circ\alpha_{(p)}\circ q_{(p)})\circ j_{(p)}=j_{(p)}\circ\lambda_1(\alpha)_{(p)}$,

(ii) $\pi_{(p)}\circ(1+i_{(p)}\circ\alpha_{(p)}\circ q_{(p)})=\pi_{(p)}$.

Proof. (i) Let $m\colon X_{(p)}\times X_{(p)}\to X_{(p)}$ be the multiplication on $X_{(p)}$. Then we have the following homotopy commutative diagram:

$$\begin{tikzcd} X_{(p)} \arrow[r, "d"] & X_{(p)}\times X_{(p)} \arrow[r, "1\times q_{(p)}"] & X_{(p)}\times S^n_{(p)} \arrow[r, "1\times i_{(p)}\circ\alpha_{(p)}"] & X_{(p)}\times X_{(p)} \arrow[r, "m"] & X_{(p)} \\ K_{(p)} \arrow[u, "j_{(p)}"] \arrow[r, "d"] \arrow[d, equal] & K_{(p)}\times K_{(p)} \arrow[u, "j_{(p)}\times j_{(p)}"] \arrow[r, "1\times\pi_{1(p)}"] & K_{(p)}\times S^n_{(p)} \arrow[u, "j_{(p)}\times 1"] \arrow[r, "1\times i'_{(p)}\circ\alpha_{(p)}"] & K_{(p)}\times K_{(p)} \arrow[u, "j_{(p)}\times j_{(p)}"] & \\ K_{(p)} \arrow[rr, "\ell_{1(p)}"] & & K_{(p)}\vee S^n_{(p)} \arrow[u] \arrow[r, "1\vee i'_{(p)}\circ\alpha_{(p)}"] & K_{(p)}\vee K_{(p)} \arrow[u] \arrow[r, "\nabla"] & K_{(p)} \arrow[uu, "j_{(p)}"'] \end{tikzcd}$$

where $\pi_1\colon K\to K/S^m=S^n$ is the collapsing map. Therefore we have the equality of (i), since $\lambda_1(\alpha)_{(p)}=\nabla\circ(1\vee i'_{(p)}\circ\alpha_{(p)})\circ\ell_{1(p)}$.

(ii) Consider the following diagram:

$$\begin{tikzcd} K_{(p)} \arrow[r, "j_{(p)}"] \arrow[d, "\lambda_1(\alpha)_{(p)}"] & X_{(p)} \arrow[r, "\pi_{(p)}"] \arrow[d, "h"] & S^{m+n}_{(p)} \arrow[d, "\varepsilon"] \\ K_{(p)} \arrow[r, "j_{(p)}"] & X_{(p)} \arrow[r, "\pi_{(p)}"] & S^{m+n}_{(p)}, \end{tikzcd}$$

where $h=1+i_{(p)}\circ\alpha_{(p)}\circ q_{(p)}$. The left square of this diagram is homotopy commutative by (i) and so is the right square for some map $\varepsilon: S^{m+n}_{(p)}\to S^{m+n}_{(p)}$. On the other hand, since α induces the trivial homomorphism in reduced cohomology group, we have

$$(1+i_{(p)}\circ\alpha_{(p)}\circ q_{(p)})^*=1: H^*(X_{(p)}; Z_{(p)})\longrightarrow H^*(X_{(p)}; Z_{(p)}),$$

where $Z_{(p)}$ denotes the ring of integers localized at $\{p\}$. Hence the above map ε is homotopic to the identity. q.e.d.

Lemma 3.7. *Let $\alpha\in\pi_n(S^m; p)$ and $\gamma\in\pi_{m+n}(X; p)$, and let $\pi: X\to X/K=S^{m+n}$ be the collapsing map. If $S^m_{(p)}$ and $X_{(p)}$ are H-spaces such that $i_{(p)}: S^m_{(p)}\to X_{(p)}$ is an H-map, then we have*

$$q_{(p)}\circ((1+\gamma_{(p)}\circ\pi_{(p)})+i_{(p)}\circ\alpha_{(p)}\circ q_{(p)})=q_{(p)}+q_{(p)}\circ\gamma_{(p)}\circ\pi_{(p)}.$$

Proof. Using the Puppe exact sequence associated with the cofibering $X_{(p)}\vee S^m_{(p)}\to X_{(p)}\times S^m_{(p)}\xrightarrow{\pi_1}X_{(p)}\wedge S^m_{(p)}$, we see that there exists $\omega_1\in[X_{(p)}\wedge S^m_{(p)}, S^n_{(p)}]$ such that $q_{(p)}\circ(p_1+p_2)\circ(1\times i_{(p)})=q_{(p)}\circ p_1\circ(1\times i_{(p)})+\omega_1\circ\pi_1$, where $+$ denotes the multiplication induced from the H-structure on $X_{(p)}$ or $S^n_{(p)}$ and p_i $(i=1, 2)$ is the projection. Furthermore, since $i_{(p)}: S^m_{(p)}\to X_{(p)}$ is an H-map by the assumption, we see easily that there exists $\omega_2\in[(S^n_{(p)}\cup e^{m+n}_{(p)})\wedge S^m_{(p)}, S^n_{(p)}]$ such that $\omega_1=\omega_2\circ\pi_2$, where $\pi_2: X_{(p)}\wedge S^m_{(p)}\to(X_{(p)}/S^m_{(p)})\wedge S^m_{(p)}=(S^n_{(p)}\cup e^{m+n}_{(p)})\wedge S^m_{(p)}$ is the collapsing map. Consequently we have

$$q_{(p)}\circ(p_1+p_2)\circ(1\times i_{(p)})=q_{(p)}\circ p_1\circ(1\times i_{(p)})+\omega_2\circ\pi_2\circ\pi_1.$$

Using this equality, by the similar way to the proof of Lemma 3.1, we have

$$q_{(p)}\circ((1+\gamma_{(p)}\circ\pi_{(p)})+i_{(p)}\circ\alpha_{(p)}\circ q_{(p)})=q_{(p)}\circ(1+\gamma_{(p)}\circ\pi_{(p)}).$$

Furthermore, since $1+\gamma_{(p)}\circ\pi_{(p)}=\lambda_2(\gamma)_{(p)}$, where λ_2 was defined in (2.5),

$$\begin{aligned}q_{(p)}\circ(1+\gamma_{(p)}\circ\pi_{(p)})&=q_{(p)}\circ(\nabla\circ(1\vee\gamma)\circ\ell_2)_{(p)}=(\nabla\circ(q\vee q)\circ(1\vee\gamma)\ell_2)_{(p)}\\&=(p_1+p_2)\circ(q_{(p)}\times q_{(p)}\circ\gamma_{(p)}\circ\pi_{(p)})\circ d\\&=q_{(p)}+q_{(p)}\circ\gamma_{(p)}\circ\pi_{(p)}.\end{aligned}$$

Hence we have the equality of this lemma. q.e.d.

§ 4. Main result

In this section we study the p-Sylow subgroup of $\mathscr{E}(X)$ for an S^m-bundle X over S^n by using an H-structure on $X_{(p)}$ and the method of localization.

The following result due to Lieberman-Smallen will be used to obtain our main result.

(4.1) (Lieberman-Smallen [7, Theorem 1.3]) *Let P and $\bar{P}$ denote sets of primes such that $P \cup \bar{P} = \{all\ primes\}$ and $P \cap \bar{P} = \phi$. Let Y be a simple finite CW-complex. Then localization induces the isomorphism*

$$L: \mathscr{E}(Y) \cong \{(h, h') \in \mathscr{E}(Y_P) \times \mathscr{E}(Y_{\bar{P}}) \mid h_{(0)} = h'_{(0)}\},$$

where $h_{(0)}$ and $h'_{(0)}$ are localizations of h and h' at the rational numbers Q respectively.

Let m and n be odd integers such that $3 \leqq m < n-1$ and X denote an S^m-bundle over S^n in (2.3). Let p be an odd prime. If $S^m_{(p)}$ and $X_{(p)}$ are H-spaces such that $i_{(p)}: S^m_{(p)} \to X_{(p)}$ is an H-map, then we have

Proposition 4.2. *The p-Sylow subgroup $S_p \cong \pi_n(S^m; p)$ of G given in* (ii) *of Lemma* 2.7 *splits partly the exact sequence* (2.6), *that is, there exists a homomorphism*

$$\psi: S_p \cong \pi_n(S^m; p) \longrightarrow \mathscr{E}(X) \qquad \text{such that } j^1 \circ \psi = 1.$$

Proof. In Proposition 1.3 we identify $\pi_n(S^m; p)$ with S_p by means of the homomorphism λ_1. We put $h = 1 + i_{(p)} \circ \alpha_{(p)} \circ q_{(p)}$ for $\alpha \in \pi_n(S^m; p)$, where $+$ is the multiplication induced from the H-structure on $X_{(p)}$. Then $h \in \mathscr{E}(X_{(p)})$ by Lemma 3.6, and $h_{(0)} = 1$ since α is of finite order. Therefore, using the isomorphism L in (4.1), we can define a map

$$(4.3) \quad \psi: S_p \cong \pi_n(S^m; p) \longrightarrow \mathscr{E}(X) \text{ by } \psi(\lambda_1(\alpha)) = L^{-1}(1 + i_{(p)} \circ \alpha_{(p)} \circ q_{(p)}, 1),$$

where $\alpha \in \pi_n(S^m; p)$ and $\lambda_1(\alpha) \in S_p$. First we show that ψ is an homomorphism. For $\alpha \in \pi_n(S^m; p)$ and $\beta \in \pi_n(S^m; p)$,

$$\begin{aligned}
&(1 + i_{(p)} \circ \alpha_{(p)} \circ q_{(p)}) \circ (1 + i_{(p)} \circ \beta_{(p)} \circ q_{(p)}) \\
&\quad = (1 + i_{(p)} \circ \beta_{(p)} \circ q_{(p)}) + i_{(p)} \circ \alpha_{(p)} \circ q_{(p)} \circ (1 + i_{(p)} \circ \beta_{(p)} \circ q_{(p)}) \\
&\quad = (1 + i_{(p)} \circ \beta_{(p)} \circ q_{(p)}) + i_{(p)} \circ \alpha_{(p)} \circ q_{(p)} \qquad \text{by Lemma 3.7} \\
&\quad = 1 + (i_{(p)} \circ \beta_{(p)} + i_{(p)} \circ \alpha_{(p)}) \circ q_{(p)} \qquad \text{by (i) of Lemma 3.1} \\
&\quad = 1 + i_{(p)} \circ (\alpha + \beta)_{(p)} \circ q_{(p)}.
\end{aligned}$$

Hence $\psi(\lambda_1(\alpha)) \circ \psi(\lambda_1(\beta)) = \psi(\lambda_1(\alpha + \beta))$. Next we show $j^1 \circ \psi = 1$. The naturality of localization gives the following commutative diagram:

$$(4.4)\qquad \begin{array}{ccc} \mathscr{E}(X) & \xrightarrow[\cong]{L} & \{(h, h') \in \mathscr{E}(X_{(p)}) \times \mathscr{E}(X_{(\bar{p})}) \mid h_{(0)} = h'_{(0)}\} \\ \downarrow j^! & & \downarrow j^!_{(p)} \times j^!_{(\bar{p})} \\ \mathscr{E}(K) & \xrightarrow[\cong]{L} & \{(h, h') \in \mathscr{E}(K_{(p)}) \times \mathscr{E}(K_{(\bar{p})}) \mid h_{(0)} = h'_{(0)}\}, \end{array}$$

where both L's are isomorphisms in (4.1), and $j^!_{(p)}$ and $j^!_{(\bar{p})}$ are defined by the same way as $j^!$ in (2.2). For $\alpha \in \pi_n(S^m; p)$ we have

$$\begin{aligned} j^! \circ (\psi(\lambda_1(\alpha))) &= L^{-1} \circ (j^!_{(p)} \times j^!_{(\bar{p})}) \circ L(\psi(\lambda_1(\alpha))) \quad \text{by (4.4)} \\ &= L^{-1} \circ (j^!_{(p)} \times j^!_{(\bar{p})}) \circ (1 + i_{(p)} \circ \alpha_{(p)} \circ q_{(p)}, 1) \quad \text{by (4.3)} \\ &= L^{-1}(\lambda_1(\alpha)_{(p)}, 1) \quad \text{by (i) of Lemma 3.6} \\ &= \lambda_1(\alpha), \end{aligned}$$

because $\lambda_1(\alpha)_{(\bar{p})} = 1$. Hence $j^! \circ \psi = 1$. q.e.d.

Now we consider the p-Sylow subgroup of $\mathscr{E}(X)$.

Theorem 4.5. *Let m and n be odd integers such that $3 \leqq m < n-1$, and let X denote an S^m-bundle over S^n in (2.3). Let p be an odd prime. If $S^m_{(p)}$ and $X_{(p)}$ are H-spaces such that $i_{(p)}: S^m_{(p)} \to X_{(p)}$ is an H-map, then the group $\mathscr{E}(X)$ is a finite group with a unique p-Sylow subgroup $\tilde{S}_p$ given by the semi direct product*

$$\tilde{S}_p \cong \pi_{m+n}(X; p) \underset{T}{\times} \pi_n(S^m; p),$$

where $\alpha T \beta = \alpha + i \circ \beta \circ q \circ \alpha$ for $\alpha \in \pi_{m+n}(X; p)$ and $\beta \in \pi_n(S^m; p)$.

Proof. By (2.6), Lemma 2.7 and Proposition 4.2, we have the exact sequence

$$(4.6)\qquad 0 \longrightarrow \lambda_2(\pi_{m+n}(X)) \underset{T}{\times} \psi(S_p) \longrightarrow \mathscr{E}(X) \longrightarrow G/S_p \longrightarrow 1,$$

where $S_p \cong \pi_n(S^m; p)$, G is given in (2.6) and G/S_p has the order prime to p. Let p' be a prime with $(p, p') = 1$. Then, using the isomorphism L in (4.1), for $\gamma \in \pi_{m+n}(X; p')$ and $\beta \in \pi_n(S^m; p)$ we have

$$(4.7)\qquad \lambda_2(\gamma) T \psi(\lambda_1(\beta)) = \psi(\lambda_1(\beta)) \circ \lambda_2(\gamma) \circ \psi(\lambda_1(\beta))^{-1} = \lambda_2(\gamma).$$

Noticing that $\lambda_2(\alpha)_{(p)} = 1 + \alpha_{(p)} \circ \pi_{(p)}$ for $\alpha \in \pi_{m+n}(X; p)$, by Lemmas 3.6 and 3.7 and the similar way to the proof of Lemma 3.1, we have

$$\begin{aligned} (\psi(\lambda_1(\beta)) \circ \lambda_2(\alpha) \circ \psi(\lambda_1(\beta))^{-1})_{(p)} &= ((1 + i_{(p)} \circ (-\beta_{(p)}) \circ q_{(p)}) + \alpha_{(p)} \circ \pi_{(p)}) \\ &\quad + (i_{(p)} \circ \beta_{(p)} \circ q_{(p)} + i_{(p)} \circ \beta_{(p)} \circ q_{(p)} \circ \alpha_{(p)} \circ \pi_{(p)}) = \lambda_2(\alpha + i \circ \beta \circ q \circ \alpha)_{(p)}. \end{aligned}$$

Also, obviously we have

$$(\psi(\lambda_1(\beta))\circ\lambda_2(\alpha)\circ\psi(\lambda_1(\beta))^{-1})_{(\bar{p})}=1=(\lambda_2(\alpha+i\circ\beta\circ q\circ\alpha))_{(\bar{p})}.$$

Hence, by (4.1) we have

$$(4.8)\quad \lambda_2(\alpha)T\psi(\lambda_1(\beta))=\psi(\lambda_1(\beta))\circ\lambda_2(\alpha)\circ\psi(\lambda_1(\beta))^{-1}=\lambda_2(\alpha+i\circ\beta\circ q\circ\alpha)$$

for $\alpha\in\pi_{m+n}(X;p)$ and $\beta\in\pi_n(S^m;p)$. Next we show that λ_2 is monomorphic on $\pi_{m+n}(X;p)$. Let L be the isomorphism in (4.1). Then $L\lambda_2(\gamma)=(1+\gamma_{(p)}\circ\pi_{(p)},1)$ for $\gamma\in\pi_{m+n}(X;p)$. Therefore, we have

$$\begin{aligned}\mathrm{Ker}\,\lambda_2\cap\pi_{m+n}(X;p)&\cong\mathrm{Ker}\,\{\pi_{(p)}^*:[S_{(p)}^{m+n},X_{(p)}]\longrightarrow[X_{(p)},X_{(p)}]\}\\&=\mathrm{Im}\,\{(Sg)_{(p)}^*:[SK_{(p)},X_{(p)}]\longrightarrow[S_{(p)}^{m+n},X_{(p)}]\}\\&=0,\end{aligned}$$

where g is the attaching element of e^{m+n} in $X=K\cup e^{m+n}$, because the middle equality is obtained by the Puppe exact sequence and the next one is obtained by the fact due to C. A. McGibbon [8, Theorem 1] that Sg has order at most 2. Hence, by (4.6), (4.7) and (4.8) we have the desired result. q.e.d.

Let P be a set of odd primes p such that $S_{(p)}^m$ and $X_{(p)}$ are H-spaces and $i_{(p)}:S_{(p)}^m\to X_{(p)}$ is an H-map, and we put $S_P=\sum_{p\in P}S_p$ for the p-Sylow subgroup S_p given in Proposition 1.3. Then S_P is the normal subgroup of G by Lemma 2.7 and (1.2) (see also e.g. [11, Theorem 2.11]), and the splitting homomorphism ψ in Proposition 4.2 can be extended to S_P. So by the same way as in (4.6) we have the exact sequence

$$0\longrightarrow\lambda_2(\pi_{m+n}(X))\underset{T}{\times}\psi(S_P)\longrightarrow\mathscr{E}(X)\longrightarrow G/S_P\longrightarrow1.$$

By Theorem 4.5 and (4.7), we have immediately

$$\lambda_2(\pi_{m+n}(X))\underset{T}{\times}\psi(S_P)\cong(\sum_{p\in P}\tilde{S}_p)\oplus\sum_{r\in\bar{P}}\lambda_2(\pi_{m+n}(X;r)),$$

where $\tilde{S}_p$ is given in Theorem 4.5. By (1.2) (see also e.g. [11, Theorem 2.11]), we have the exact sequence

$$0\longrightarrow H\longrightarrow G/S_P\longrightarrow \boldsymbol{Z}_2\oplus\boldsymbol{Z}_2.$$

Here

$$H=\sum_{p\in P_1}\pi_n(S^m;p)\oplus H_1/(f_*\pi_n(S^{n-1})+\gamma(f)\pi_{m+1}(S^m))\cap H_1,$$

where $P_1=\overline{P\cup\{2\}}$, $H_1=\{\alpha\in\pi_n(S^m;2)\,|\,i_*[\alpha,\iota_m]=0\}$ and $\gamma(f)$ is given in Section 1. Hence we have

Theorem 4.9. *Let m and n be odd integers such that $3\leqq m<n-1$ and X denote an S^m-bundle over S^n in (2.3). Let P be a set of odd primes p such that $S^m_{(p)}$ and $X_{(p)}$ are H-spaces and $i_{(p)}\colon S^m_{(p)}\to X_{(p)}$ is an H-map. Then we have the following exact sequences:*

$$0\longrightarrow\Big(\sum_{p\in P}\tilde{S}_p\Big)\oplus\sum_{r\in\bar{P}}\lambda_2(\pi_{m+n}(X;r))\longrightarrow\mathscr{E}(X)\longrightarrow G/S_P\longrightarrow 1,$$
$$0\longrightarrow H\longrightarrow G/S_P\longrightarrow Z_2\oplus Z_2.$$

Here $\tilde{S}_p$ and G are given in Theorem 4.5 and (2.6) respectively, $S_P\cong\sum_{p\in P}\pi_n(S^m;p)$ and H is given as above.

§ 5. Two examples

In this section, we give the following two examples in which the group $\mathscr{E}(X)$ is determined as a group extension of a certain group by a 2-group.

Example 5.1. *Let m and n be odd integers such that $3\leqq m<n-1$ and let $X=S^m\times S^n$. Then, for any odd prime p, $\mathscr{E}(X)$ is a finite group with a unique p-Sylow subgroup $\tilde{S}_p$ given by the semi direct product*

$$\tilde{S}_p\cong(\pi_{m+n}(S^m;p)\oplus\pi_{m+n}(S^n;p))\underset{T_1}{\times}\pi_n(S^m;p),$$

where $(\alpha',\alpha'')T_1\beta=(\alpha'+\beta\circ\alpha'',\alpha'')$ for $(\alpha',\alpha'')\in\pi_{m+n}(S^m;p)\oplus\pi_{m+n}(S^n;p)$ and $\beta\in\pi_n(S^m;p)$. Furthermore, let P be the set of all odd primes. Then we have the following exact sequence:

$$0\longrightarrow\Big(\sum_{p\in P}\tilde{S}_p\Big)\oplus\tilde{H}\longrightarrow\mathscr{E}(X)\longrightarrow\tilde{G}\longrightarrow 1.$$

Here $\tilde{H}=\pi_{m+n}(S^m;2)/\{[\iota_m,\pi_{n+1}(S^m)]\}\oplus\pi_{m+n}(S^n;2)$ *and*

$$\tilde{G}\cong\{\alpha\in\pi_n(S^m;2)\,|\,[\iota_m,\alpha]=0\}\underset{T_2}{\times}(Z_2\oplus Z_2),$$

where $\alpha T_2(\varepsilon',\varepsilon'')=\varepsilon'\circ\alpha\circ\varepsilon''$ for $(\varepsilon',\varepsilon'')\in Z_2\oplus Z_2=\{\pm\iota_m\}\oplus\{\pm\iota_n\}$.

In fact, since $S^k_{(p)}$ $(k=m,n)$ is an H-space for any odd prime p by J. F. Adams [1] and $i_{(p)}\colon S^m_{(p)}\to S^m_{(p)}\times S^n_{(p)}$ is an H-map, the first half of this example is obtained from Theorem 4.5. Also, by [14, Theorem 2.6] we have the exact sequence

$$0\longrightarrow\{[\iota_m,\pi_{n+1}(S^m)]\}\longrightarrow\pi_{m+n}(S^m\times S^n)\xrightarrow{\lambda_2}\mathscr{E}(S^m\times S^n)\xrightarrow{j!}G\longrightarrow 1,$$

where $G\cong\{\alpha \in \pi_n(S^m) \mid [\iota_m, \alpha]=0\}\times_{T_2}(Z_2\oplus Z_2)$. Therefore, by noticing that $[\iota_m, \pi_{n+1}(S^m; p)]=0$, the latter half of this example is obtained from Theorem 4.9 and the above exact sequence.

Let r be an odd prime and let α_1 be a generator of $\pi_{2r}(S^3; r)\cong Z_r$. Let

$$S^3 \xrightarrow{i} B(r) \xrightarrow{q} S^{2r+1}$$

be the principal S^3-bundle over S^{2r+1} with a characteristic element α_1. Then we have the following

Example 5.2. *For any odd prime p, $\mathscr{E}(B(r))$ is a finite group with a unique p-Sylow subgroup $\tilde{S}_p$ given by the semi direct product*

$$\tilde{S}_p\cong\pi_{2r+4}(S^3; p)\oplus\pi_{2r+1}(S^3; p) \quad (p\neq 3 \text{ or } r=3),$$
$$\tilde{S}_3\cong(\pi_{2r+4}(S^3; 3)\oplus\pi_{2r+4}(S^{2r+1}; 3))\underset{T_1}{\times}\pi_{2r+1}(S^3; 3) \quad (r\neq 3),$$

where $\pi_{2r+4}(S^{2r+1}; 3)\cong Z_3$, $(\alpha', \alpha'')T_1\beta=(\alpha'+\beta\circ\alpha'', \alpha'')$ for $(\alpha', \alpha'')\in \pi_{2r+4}(S^3; 3)\oplus\pi_{2r+4}(S^{2r+1}; 3)$ and $\beta\in\pi_{2r+1}(S^3; 3)$. Furthermore, we have the following exact sequences:

$$0\longrightarrow(\sum_{p\in P}\tilde{S}_p)\oplus\pi_{2r+4}(S^3; 2)\oplus\pi_{2r+4}(S^{2r+1}; 2)\longrightarrow\mathscr{E}(B(r))\longrightarrow\tilde{G}\longrightarrow 1,$$
$$0\longrightarrow\pi_{2r+1}(S^3; 2)\longrightarrow\tilde{\tilde{G}}\longrightarrow Z_2\longrightarrow 1,$$

where P is the set of all odd primes and $\pi_{2r+4}(S^{2r+1}; 2)\cong Z_8$.

In fact, since $(S^5\times\cdots\times S^{2r-1}\times B(r))_{(r)}\simeq SU(r+1)_{(r)}$ by H. Toda [15] and $B(r)_{(p)}\simeq(S^3\times S^{2r+1})_{(p)}$ for any prime $p\neq r$, $B(r)_{(p)}$ is an H-space for any odd prime p and $i_{(p)}\colon S^3_{(p)}\to B(r)_{(p)}$ is an H-map for dimensional reasons. Therefore we can apply Theorem 4.5 to $B(r)$ for any odd prime p, and by the same way as in [9, Example 3.3] the homotopy group $\pi_{2r+4}(B(r))$ is calculated and we have the first half of this example. Also, by [9, Theorem 3.1] we have the exact sequence

$$0\longrightarrow\pi_{2r+4}(B(r))\xrightarrow{\lambda_2}\mathscr{E}(B(r))\xrightarrow{j!}G\longrightarrow 1,$$

where $G=\mathscr{E}(S^3\cup_{\alpha_1}e^{2r+1})$ is given in (1.2). Therefore the latter half of this example is obtained from Theorem 4.9 and the above exact sequence.

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