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A Generalization of the Adams Invariant and Applications to Homotopy of the Exceptional Lie Group G_2

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The original e-invariant of Adams [2] is a homotopy invariant of a map between spheres, which is defined in terms of a coefficient in Chern character equations for a cofibre of a given map, and takes values in rationals modulo one. Generalizations in several directions have been obtained including the case of a map between spaces with no torsion in homology and with suitable conditions. In this paper, we present and discuss a variant of the Adams invariant for a map between spaces whose comlex K-groups are isomorphic to those of spheres. Such a space must have torsion in homology unless it is a sphere, and we give, in (1.3), (1.3)′, (1.3)″ below, a class of three-cell complexes which are simplest examples of such spaces.

The theorem of Hodgkin [9] support that there is a class of Lie groups which have the property that their K-groups are free but they have indeed torsion in homology. The first example of our three-cell complexes is, in fact, closely related to the compact, simply connected exceptional Lie group G_2, an example of such Lie groups. Namely, if we put

$$X^n = S^n \cup_\eta e^{n+2} \cup_2 e^{n+3}, \qquad n \geqq 3,$$
$$Y^n = S^{n-3} \cup_2 e^{n-2} \cup_\eta e^n, \qquad n \geqq 6^\dagger,$$

then they have the K-groups isomorphic to those of S^n, and we may define the stable Adams invariant

$$e\colon \{Y^{n+7}, X^n\} \longrightarrow \boldsymbol{Q}/\tfrac{1}{2}\boldsymbol{Z}$$

($\{Y, X\}$ denotes the group of stable maps from Y to X), which takes values in rationals modulo integral multiples of $1/2$, the rational depending only on the source space Y. The relation between the spaces X^n, Y^n and the Lie group G_2 is that, up to homotopy equivalence, there is a CW decom-

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† It is known that Y^5 also exists by the computation of the unstable group $\pi_4(S^2 \cup e^3)$; however, we shall need to use Y^n only for $n \geqq 8$.

position of G_2 such that

$$G_2^{(6)} = X^3, \qquad G_2^{(11)}/G_2^{(6)} = Y^{11}, \qquad G_2/G_2^{(11)} = S^{14},$$

where $A^{(n)}$ denotes the n-skeleton of a CW complex A. As G_2 is of rank 2, the theorem of Hodgkin states that $K^*(G_2) = E(\tilde{\xi}, \tilde{\eta})$, the exterior algebra over Z, with $\tilde{\xi}, \tilde{\eta} \in \tilde{K}^{-1}(G_2)$ (Theorem 4.3). The spaces X^3, Y^{11} and S^{14} contribute, via above CW decomposition, to the factor Z in $K^*(G_2)$ generated respectively by $\tilde{\xi}$, $\tilde{\eta}$ and $\tilde{\xi}\tilde{\eta}$.

The following theorem shows the importance of our Adams invariant for $\{Y^{n+7}, X^n\}$.

Theorem 3.9. *$\{Y^{n+7}, X^n\} = Z/60$ with generator having e-invariant* $1/120 \bmod (1/2)Z$. *Hence e is monic.*

The determination of $\{Y^{n+7}, X^n\}$ together with generator is made by direct computation. In Corollary 2.5, we give the relation between original e-invariant of Adams and our e-invariant, which completes evaluating e-invariant on $\{Y^{n+7}, X^n\}$ as required.

Extending the cofibration $G_2^{(6)} \to G_2^{(11)} \to G_2^{(11)}/G_2^{(6)}$ to the right, we get a map

$$\phi \colon Y^{11} = G_2^{(11)}/G_2^{(6)} \longrightarrow \Sigma G_2^{(6)} = \Sigma K^3 = X^4,$$

where Σ denotes the suspension. To determine the stable class of ϕ, it is enough, by Theorem 3.9, to know the e-invariant of ϕ, which may be obtained from computing the Chern character in $K^*(G_2)$, namely, for suitable generators $\tilde{\xi}$, $\tilde{\eta}$ in $K^*(G_2)$, we have

$$\operatorname{ch} \tilde{\xi} = 2h_3 - (1/60)h_{11}, \qquad \operatorname{ch} \tilde{\eta} = (1/2)h_{11},$$

where h_3, h_{11} are generators of the free part of the integral cohomology of G_2 with $\deg h_i = i$. The computation of the Chern character will be given in Theorem 4.3′ and Lemmas 4.10, 4.11 with studying the image in $K^*(S^3)$ of our generators of $K^*(G_2)$ given by the above CW decomposition. Professor H. Minami kindly advised the author to pay attention to the book [23] as to the complex representation ring of G_2 and also told him more direct computation in spirit of Hodgkin [9] and Atiyah [4].

Now, the above Chern character equations immediately imply that the e-invariant of ϕ is $-1/60$, hence

Theorem 4.14. *In stable range, ϕ is twice generator, or of order* 30.

This recovers the recent result of F. Cohen-F.P.Peterson [8], namely,

Corollary 4.15. *$G_2^{(6)}$ can not be a stable* mod 2 *retract of G_2.*

As the mod 2 cohomology of G_2 has a simple system of generators x_3, x_5, x_6 with $\deg x_i = i$ and $G_2^{(6)}$ realizes the part of generator set $\{x_3, x_5, x_6\}$, the result is contrasted with James' result that stunted projective spaces inside Stiefel manifolds are stable retracts. Another consequence is concerned with desuspension of the generator σ' of $\pi_{14}(S^7)$ after composing suitable maps. Let $i: S^n \to X^n$ be the inclusion and $j: Y^n \to S^n$ the projection. The element σ' is known to be not a suspension, however we have

Corollary 4.17. *The composite $i\sigma' j: Y^{14} \to S^{14} \to S^7 \to X^7$ is a three times suspension.*

The element σ' is the Samelson square $\langle \iota_7, \iota_7 \rangle$, which is an analogue of the Blakers-Massey element $\omega = \langle \iota_3, \iota_3 \rangle$ in $\pi_6(S^3)$. It is well known that the image of ω in $\pi_6(\Sigma CP^2)$ is the suspension of the projection map $S^5 \to CP^2$ of the S^1-bundle.

For a self-map f of G_2, we define $d_i(f)$, the degree of f in dimension i, $i = 3, 11$, to be the integer in the equality $f^*(h_i) = d_i(f) h_i$ in $H^*(G_2)$. We then define the degree map

$$d = d_3 \times d_{11}: [G_2, G_2] \longrightarrow Z \oplus Z,$$

where $[X, Y]$ denotes the set of homotopy classes of based maps from X to Y, and the stable degree map

$$d^S: \{G_2, G_2\} \longrightarrow Z \oplus Z$$

with $d = d^S \Sigma^\infty$, where Σ^∞ is the stabilization. The Chern character equations for G_2 gives an upper bound of the image of d, that is, $d_3 - d_{11}$ is a multiple of 30. On the other hand, as the Hurewicz homomorphism $\pi_{11}(G_2) \to H_{11}(G_2) = Z$ has the image $120Z$, we may easily construct self-maps of G_2 with d_3 arbitrary integer and $d_3 - d_{11}$ arbitrary multiple of 120, giving a lower bound of the image of d. It is natural to ask which estimate is better. We prove that the bound given by K-theory is the best result, answering the question.

Theorem 5.7. Im $d =$ Im $d^S = \{(m, m+30n) \mid m, n \in Z\}$.

Mimura-Nishida-Toda [13] introduced H-spaces $G_{2,b}$ with $-2 \leqq b \leqq 5$, variants of G_2 with prototype $G_{2,0} = G_2$. For $p \geqq 7$ and for $p = 2$, they are p-equivalent to G_2, and for $p = 3, 5$, to $S^3 \times S^{11}$ or G_2 according as $b \equiv -2 \bmod p$ or not. Theorem 5.7 in the body of this paper also gives the result for $G_{2,b}$; we only need to replace the period 30 by $b_2=30/(15, 8b+1)$, given in the table (5.4). In particular, $b_2>2$ if and only if $-1\leqq b\leqq 5$.

The Chern character equations for G_2 (and for $G_{2,b}$ given in Theorem 5.3) have still more application. Let μ: $G_{2,b}\times G_{2,b}\to G_{2,b}$ be an H-structure (multiplication) on $G_{2,b}$ and denote by $[G_{2,b}, G_{2,b}]_\mu$ the set of homotopy classes of H-maps of $G_{2,b}$ to itself. We may estimate the image of d restricted to $[G_{2,b}, G_{2,b}]_\mu$.

Theorem 7.1. *For* $-2\leqq b\leqq 5$ *and for arbitrary multiplication* μ, $d[G_{2,b}, G_{2,b}]_\mu\subset\{(m, m+30n)\,|\,m, n\in Z, m\equiv 0, 1 \bmod 4$ *if n is even*$\}$.

The set $[G_{2,b}, G_{2,b}]_\mu$ becomes a semi-group with multiplication given by the composition of maps. We denote by $\mathscr{E}_H(G_{2,b}; \mu)$ the subset of all invertible elements which then becomes a group. In case $b_2>2$, that is, $-1\leqq b\leqq 5$ as above, we may completely determine the image of d restricted to $\mathscr{E}_H(G_{2,b}; \mu)$. Together with recent result of Sawashita [20] on the group $\mathscr{E}_H(G_{2,b}; \mu)$, we get

Theorem 7.7. *For* $-1\leqq b\leqq 5$ *and for arbitrary multiplication* μ, $\mathscr{E}_H(G_{2,b}; \mu)$ *is the unit group. In other words, if* f: $G_{2,b}\to G_{2,b}$ *is an H-map for some* μ *and simultaneously a homotopy equivalence, then it is homotopic to the identity map.*

The paper is organized as follows. In Section 1, we give examples of finite complexes, including all the case of three-cell complexes, whose K-groups are isomorphic to those of spheres. In Section 2, we introduce the Adams invariant for maps between spaces discussed in Section 1, and discuss the relation between the Adams invariants for the case of three-cell complexes and for spheres. In Section 3, we discuss the special case $\{Y^{n+7}, X^n\}$ and prove that the Adams invariant faithfully detects it, as in Theorem 3.9. In Section 4, we discuss the complex representation ring and the K-group of the compact, simply connected Lie group G_2. In consequence, we determine the Chern character on $K^*(G_2)$ and the stable class of the attaching map ϕ in G_2 as above, along with reproving the result of Cohen-Peterson [8] that the 6-skeleton $G_2^{(6)}$ inside G_2 is not a stable mod 2 retract. In Section 5, we determine the image of the degree map for self-maps of $G_{2,b}$ as in Theorem 5.7. To complete Theorem 5.7, we need to construct a self map giving the bound obtained from K-theory. Section 6 is devoted to the computation for the construction of such a map. In Section 7, we discuss the degree map restricted to $[G_{2,b}, G_{2,b}]_\mu$ and to $\mathscr{E}_H(G_{2,b}; \mu)$ and prove Theorems 7.1, 7.7 as stated above.

A part of this paper including Theorems 3.9 and 7.7 was obtained while the author visited Bonn in 1982/83. The author would like to thank

Professor F. Hirzebruch and Max-Planck-Institut für Mathematik for their hospitality and support. Many thanks also go to Professor H. Minami, who pointed out to the author to pay attention to the book [23] for enlightening circle of ideas and allowed the author to include his proof on the Chern character for G_2 in this paper.

The book [23], in particular Chapters 5 and 6, is an excellent guide book on the representation ring of the classical groups and the exceptional groups G_2 and F_4, which makes readers familiar to the exceptional groups G_2, F_4 like linear algebra. Unfortunately it is written in Japanese and the author can not find a similar kind of books in English. In fact, enlightening the paper with the book has been finished after the author came back to Japan.

Spaces in this paper are always assumed to be homotopy equivalent to CW complexes and to have base points. Maps and homotopies also preserve base points. $[X, Y]$ denotes the set of homotopy classes of maps from X to Y, Σ^n denotes the n-fold suspension, $\Sigma^1=\Sigma$, and $\{X, Y\}$ the additive group of stable maps from X to Y. For a map, we sometimes denote its homotopy class by the same symbol. Z and Z/n denote the additive groups isomorphic to the groups of integers and of integers modulo n, respectively. Inside { } just after Z or Z/n denotes the generator.

§ 1. Theree-cell complexes K-theoretically equivalent to spheres

We shall begin with recalling the definition of the (complex) Adams invariant [2]:

(1.1) $$e(=e_C);\ \pi^S_{2k-1}(S^0)\longrightarrow Q \bmod Z^{*)};$$

for f: $S^{2k-1+n}\to S^n$, n even, representing an element $\alpha\in\pi^S_{2k-1}(S^0)$, form a mapping cone $C_f=S^n\cup e^{n+2k}$ with attaching map f, then $\widetilde{K}(C_f)=Z\oplus Z$ with generators a, b, $\widetilde{H}^*(C_f, Q)=Q\oplus Q$ with generators x_n, x_{n+2k}, $\deg x_i=i$, which come from the integral cohomology, and the Chern character for C_f is given as follows:

(1.2) $$\mathrm{ch}\, a=x_n+\lambda x_{n+2k},\quad \lambda\in Q,\quad \mathrm{ch}\, b=x_{n+2k}.$$

The e-invariant (Adams invaraint) $e(\alpha)$ of α is then defined to be $\lambda \bmod Z$. It is independent of the choice of generators a, b satisfying (1.2), and depends only on the stable homotopy class α of f.

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*) For readable printing, we use an alternative notation $Q \bmod rZ$ for a factor group of Q, instead of the usual one: Q/rZ, where r is a rational written with fraction.

Suppose given an element $\alpha \in \pi_{k-1}^S(S^0)$ with condition

(1.3) $k=k(\alpha)$ is even, α is of order $q=q(\alpha)$, and $e(\alpha)=r/q$ with $(r, q)=1$.

Since $q\alpha=0$, we may form (not uniquely) a stable complex

(1.3)′ $$X_\alpha^n = S^n \cup_\alpha e^{n+k} \cup_q e^{n+k+1}$$

and its dual

(1.3)″ $$Y_\alpha^n = S^{n-k-1} \cup_q e^{n-k} \cup_\alpha e^n.$$

They are rationally equivalent to S^n with equivalences

i: $S^n \to X_\alpha^n$, the inclusion,

j: $Y_\alpha^n \to S^n$, the map which pinches the $(n-k)$-skeleton to a point.

Let $s_n \in H^n(S^n)=Z$, $\sigma_n \in \tilde{K}(S^n)=Z$ be generators with ch $\sigma_n = s_n$, where n is even. Let $x_n \in H^n(X_\alpha^n)=Z$, $y_n \in H^n(Y_\alpha^n)=Z$ be generators with $i^* x_n = s_n$, $y_n = j^* s_n$. We write the rational classes of s_n, x_n, y_n with the same symbols.

Proposition 1.4. (a) $K^*(X_\alpha^n) \cong K^*(S^n)$, $K^*(Y_\alpha^n) \cong K^*(S^n)$† *as additive groups.*

(b) *Suppose n is even. For the generators $\xi_n \in \tilde{K}^0(X_\alpha^n)=Z$, $\eta_n \in \tilde{K}^0(Y_\alpha^n)=Z$, the following relations hold.*

$$\text{ch}\,\xi_n = q x_n, \qquad \text{ch}\,\eta_n = (1/q) y_n,$$
$$i^* \xi_n = q \sigma_n, \qquad j^* \sigma_n = q \eta_n,$$

Proof. Put $K = S^n \cup_\alpha e^{n+k}$, where we may assume n is even. Then X_α^n is a cofibre of a map $\tilde{q}$: $S^{n+k} \to K$ which is of degree q in dimension $n+k$. Choose generators a, b of $\tilde{K}^0(K)=Z \oplus Z$ and x_n, x_{n+k} of $\tilde{H}^*(K, Q) = Q \oplus Q$ as in (1.2). Then $\tilde{q}^*(b) = q\sigma_{n+k}$, $\tilde{q}^*(x_n)=0$, $\tilde{q}^*(x_{n+k}) = q x_{n+k}$, hence $\tilde{q}^*(a)$ has to be $r\sigma_{n+k}$ by the naturality of ch. Since $(r, q)=1$, the subgroup Ker $\tilde{q}^*$ ($\cong Z$) of $\tilde{K}^0(K)$ is generated by $qa - rb$, and $\tilde{K}^0(X_\alpha^n)=Z$ with generator ξ_n corresponding with $qa-rb$. Also $\tilde{K}^{-1}(X_\alpha^n)=0$ and the relations in (b) for ξ_n are easy. The case of Y_α^n is similar. Q.E.D

(1.4) (a) indicates that there are numbers of three-cell complexes W with

(1.5) $$K^*(W) \cong K^*(S^n) \qquad \textit{for some } n=n(W).$$

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† K-cohomology is understood to have $Z/2$-grading.

If a simply connected finite complex W satisfies (1.5), it must be a rational sphere, hence $n(W)$ is uniquely determined by

$$(1.5)' \qquad H^*(W)/\text{torsion} \cong H^*(S^{n(W)}),$$

although (1.5) determines only the parity of n. In particular, no two-cell complex can satisfy (1.5), (1.5)$'$ and, by observing the proof of (1.4), complexes (1.3)$'$, (1.3)$''$ are only three-cell complexes satisfying (1.5), namely, they are simplest examples which satisfy (1.5) but are not homotopy equivalent to spheres.

In (1.3), suppose further that $\alpha\eta$ can be halved in $\pi_k^S(S^0)$ if $q \equiv 2 \bmod 4$, where $\eta \in \pi_1^S(S^0) = Z/2$ is a generator. The extra condition is equivalent to the existence of a complex

$$V = S^n \cup_q e^{n+1} \cup_\alpha e^{n+k+1} \cup_q e^{n+k+2} \qquad \text{with } n \text{ large}$$

(in case $q \not\equiv 2 \bmod 4$, V exists with no extra condition) [21], and hence (1.4) (a) implies (cf. [2])

$$(1.6) \qquad \tilde{K}^*(V) = 0 \quad \text{i.e., } V \text{ is a } K\text{-null space.}$$

and vice versa, because Y_α^{n+k+1} is the skeleton of V with top cell removed and x_α^{n+1} is the quotient complex of V with bottom cell pinched to the base point.

In general, if V_1 and V_2 are K-null spaces, so is the cofibre of a map $V_1 \to V_2$. For example, the complex $V(1)$ of Adams [2] and Toda [22] is K-null, hence the complexes $V(2)$, $V(3)$ [22] are K-null. Each $V(n)$ then produce spaces satisfying (1.5) in the following two ways: the skeleton with top cell removed, and the quotient complex with bottom cell pinched to the base point. We know numbers of variants of $V(n)$ ([18, §§ 6–7], [19, Theorem 4.2], for instance); all of them are K-null, and hence, produce examples of W satisfying (1.5).

§ 2. A generalization of the Adams invariant

We denote, as usual, by $\{A, B\}$ the additive group consisting of stable maps from A to B, i.e., $\{A, B\} = \text{Dir lim}\,[\Sigma^l A, \Sigma^l B]$. Suppose that *finite complexes W and W' satisfy* (1.5), *and hence* (1.5)$'$, *with $n(W)$ odd and $n(W')$ even.* Then we may define an Adams invariant

$$e\colon \{W, W'\} \longrightarrow \boldsymbol{Q} \bmod s\boldsymbol{Z} \qquad \text{for some } s = s(W) \in \boldsymbol{Q}$$

in the same way as the original e-invariant (1.1). For the purpose, we should mention that, for generators σ of $\tilde{K}^*(W) = \boldsymbol{Z}$ and x of the free part of $\tilde{H}^*(W)$, which we regard as a generator of $\tilde{H}^*(W; \boldsymbol{Q})=\boldsymbol{Q}$, the rational $s(W)$ is defined to be the coefficient in

$$\text{ch } \sigma = s(W)x.$$

Then the Chern character equations (1.2) become

$$\begin{aligned} \text{ch } a &= s(W')x_{n(W')} + \lambda x_{n(W)+1}, \qquad \lambda \in \boldsymbol{Q}, \\ \text{ch } b &= s(W)x_{n(W)+1}, \end{aligned}$$

and the coefficient λ modulo $s(W)\boldsymbol{Z}$ is independent of the choice of generators a, b.

By (1.4) (b), the values of the rationals s for X^n_α, Y^n_α, S^n are as follows:

$$(2.1) \qquad s(X^n_\alpha)=q, \quad s(Y^n_\alpha)=1/q, \quad s(S^n)=1.$$

We have therefore defined the following e-invariants:

$$(2.2) \qquad \begin{aligned} &e\colon \{X^m_\alpha, X^n_\beta\} \longrightarrow \boldsymbol{Q} \bmod q\boldsymbol{Z}, && e\colon \{X^m_\alpha, Y^n_\beta\} \longrightarrow \boldsymbol{Q} \bmod q\boldsymbol{Z}, \\ &e\colon \{Y^m_\alpha, X^n_\beta\} \longrightarrow \boldsymbol{Q} \bmod (1/q)\boldsymbol{Z}, && e\colon \{Y^m_\alpha, Y^n_\beta\} \longrightarrow \boldsymbol{Q} \bmod (1/q)\boldsymbol{Z}, \\ &e\colon \{X^m_\alpha, S^n\} \longrightarrow \boldsymbol{Q} \bmod q\boldsymbol{Z}, && e\colon \{S^m, X^n_\beta\} \longrightarrow \boldsymbol{Q} \bmod \boldsymbol{Z}, \\ &e\colon \{Y^m_\alpha, S^n\} \longrightarrow \boldsymbol{Q} \bmod (1/q)\boldsymbol{Z}, && e\colon \{S^m, Y^n_\beta\} \longrightarrow \boldsymbol{Q} \bmod \boldsymbol{Z}, \end{aligned}$$

where m is odd, n is even and β also satisfies (1.3) *with* $k'=k(\beta)$, $q'=q(\beta)$ *and* $e(\beta)=r'/q'$, $(r', q')=1$.

Let W and W' be finite complexes which satisfy (1.5), (1.5)$'$ with $n(W)=n(W')$. Let σ, σ' be generators of $\tilde{K}^n(W)$, $\tilde{K}^n(W')$, and x, x' be those of $H^n(W)$/torsion, $H^n(W')$/torsion, where $n=n(W)=n(W')$. Then

$$\text{ch } \sigma = s(W)x, \qquad \text{ch } \sigma' = s(W')x'$$

as before. For a map $k\colon W \to W'$, define an integer $u(k)$ to be the coefficient in the relation $k^*\sigma' = u(k)\sigma$. Also k induces the multiplication by an integer $v(k)$ on the integral cohomology modulo torsion. By naturality ch $k^* = k^*$ ch,

(2.3) $u(k)s(W) = v(k)s(W')$, hence $u(k)s(W)/s(W')$ and $v(k)s(W')/s(W)$ are integers.

Theorem 2.4. *Suppose W, W' satisfy* (1.5), (1.5$'$) *with* $n(W)=n(W')$. *Let $k\colon W \to W'$ be a map with integers $u(k)$, $v(k)$ defined as above.*

(a) *Let W'' satisfy* (1.5), (1.5)$'$ *with $n(W'')$ odd, and suppose $n(W)=n(W')$ is even. Then the following diagram is commutative*:

$$\begin{array}{ccc} \{W'', W\} & \xrightarrow{e} & Q \bmod s(W'')Z \\ \downarrow k_* & & \downarrow \times u(k) \\ \{W'', W'\} & \xrightarrow{e} & Q \bmod s(W'')Z. \end{array}$$

(b) *Let W'' satisfy* (1.5), (1.5)$'$ *with $n(W'')$ even, and suppose $n(W)=n(W')$ is odd. Then the following diagram is commutative*:

$$\begin{array}{ccc} \{W, W''\} & \xrightarrow{e} & Q \bmod s(W)Z \\ \uparrow k^* & & \uparrow \times v(k) \\ \{W', W''\} & \xrightarrow{e} & Q \bmod s(W')Z. \end{array}$$

Proof. (a) Let f: $W''\to W$ be a map and put $g=kf$: $W''\to W'$. Then there is a commutative diagram

$$\begin{array}{ccccc} W & \longrightarrow & C_f & \longrightarrow & \Sigma W'' \\ \downarrow k & & \downarrow \bar{k} & & \| \\ W' & \longrightarrow & C_g & \longrightarrow & \Sigma W'', \end{array}$$

where each row is a cofibration. The diagram then induces the diagram of $\tilde{K}^0$:

$$\begin{array}{ccccccccc} 0 & \longleftarrow & Z\{\sigma\} & \longleftarrow & Z\oplus Z & \longleftarrow & Z\{\sigma''\} & \longleftarrow & 0 \\ & & \uparrow k^* & & \uparrow \bar{k}^* & & \| & & \\ 0 & \longleftarrow & Z\{\sigma'\} & \longleftarrow & Z\oplus Z & \longleftarrow & Z\{\sigma''\} & \longleftarrow & 0 \end{array}$$

Let a, b be generators of $\tilde{K}^0(C_f)=Z\oplus Z$ so that a goes to σ and b is the image of σ''. We may choose generators $\bar{a}$, $\bar{b}$ of $\tilde{K}^0(C_g)=Z\oplus Z$ in such a way that $\bar{k}^*\bar{a}=u(k)a$, $\bar{k}^*\bar{b}=b$, since $k^*\sigma'=u(k)\sigma''$. There is a similar diagram of $\tilde{H}^*(\ ;Q)$, and for appropriate generators $x_{n(W)}$, $x_{n(W'')+1}$ of $\tilde{H}^*(C_f;Q)$ and $\bar{x}_{n(W)}$, $\bar{x}_{n(W'')+1}$ of $\tilde{H}^*(C_g;Q)$ with $\deg x_i=\deg\bar{x}_i=i$, we have $\bar{k}^*\bar{x}_{n(W)}=v(k)x_{n(W)}$, $\bar{k}^*\bar{x}_{n(W'')+1}=x_{n(W'')+1}$, and

$$\mathrm{ch}\, a=s(W)x_{n(W)}+\lambda x_{n(W'')+1} \qquad \text{with } e(f)\equiv\lambda \bmod s(W'')Z,$$
$$\mathrm{ch}\, \bar{a}=s(W')\bar{x}_{n(W)}+\mu\bar{x}_{n(W'')+1} \qquad \text{with } e(g)\equiv\mu \bmod s(W'')Z.$$

The naturality $\mathrm{ch}\,\bar{k}^*=\bar{k}^*\,\mathrm{ch}$ then implies $\mu\equiv u(k)\lambda \bmod s(W'')Z$.

(b) Let f: $W'\to W''$ and put $g=fk$: $W\to W''$. With the commutative diagram of cofibrations

$$\begin{array}{ccccc} W'' & \longrightarrow & C_f & \longrightarrow & \Sigma W' \\ \| & & \uparrow{\scriptstyle \tilde{k}} & & \uparrow{\scriptstyle \Sigma k} \\ W'' & \longrightarrow & C_g & \longrightarrow & \Sigma W, \end{array}$$

we get generators of $\tilde{K}^0(C_f)$, $\tilde{K}^0(C_g)$, $\tilde{H}^*(C_f; \boldsymbol{Q})$, $\tilde{H}^*(C_g; \boldsymbol{Q})$ (in the same notations as in (a)) such that

$$\begin{aligned} &\mathrm{ch}\, a = s(W'')x_{n(W'')} + \lambda x_{n(W)+1}, \qquad e(f) \equiv \lambda \bmod s(W')\boldsymbol{Z}, \\ &\mathrm{ch}\, \bar{a} = s(W'')\bar{x}_{n(W'')} + \mu \bar{x}_{n(W)+1}, \qquad e(g) \equiv \mu \bmod s(W)\boldsymbol{Z}, \\ &\tilde{k}^*\bar{a} = a, \quad \tilde{k}^*\bar{b} = u(k)b, \quad \tilde{k}^*\bar{x}_{n(W'')} = x_{n(W'')}, \quad \tilde{k}^*\bar{x}_{n(W)+1} = v(k)x_{n(W)+1}. \end{aligned}$$

The naturality $\tilde{k}^*\, \mathrm{ch}\, \bar{a} = \mathrm{ch}\, \tilde{k}^*\bar{a}$ implies $\lambda \equiv v(k)\mu \bmod s(W)\boldsymbol{Z}$. Q.E.D.

As a consequence, we obtain the following relationship between the e-invariants in (2.2).

Corollary 2.5. (a) *Let W be either of S^m, X^m_α, Y^m_α and $s = 1, q, 1/q$ according as $W = S^m$, X^m_α, Y^m_α. Then the following diagrams are commutative:*

$$\begin{array}{ccc} \{W, S^n\} & \xrightarrow{e} & \boldsymbol{Q} \bmod s\boldsymbol{Z} \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle q'} \\ \{W, X^n_\beta\} & \xrightarrow{e} & \boldsymbol{Q} \bmod s\boldsymbol{Z}, \end{array} \qquad \begin{array}{ccc} \{W, Y^n_\beta\} & \xrightarrow{e} & \boldsymbol{Q} \bmod s\boldsymbol{Z} \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle q'} \\ \{W, S^n\} & \xrightarrow{e} & \boldsymbol{Q} \bmod s\boldsymbol{Z}, \end{array}$$

where q' denotes the multiplication by q', which is monic.

(b) *Let W' be either of S^n, X^n_β, Y^n_β. Then the following diagrams are commutative:*

$$\begin{array}{ccc} \{X^m_\alpha, W'\} & \xrightarrow{e} & \boldsymbol{Q} \bmod q\boldsymbol{Z} \\ \downarrow{\scriptstyle i^*} & & \downarrow \\ \{S^m, W'\} & \xrightarrow{e} & \boldsymbol{Q} \bmod \boldsymbol{Z}, \end{array} \qquad \begin{array}{ccc} \{S^m, W'\} & \xrightarrow{e} & \boldsymbol{Q} \bmod \boldsymbol{Z} \\ \downarrow{\scriptstyle j^*} & & \downarrow \\ \{Y^m_\alpha, W'\} & \xrightarrow{e} & \boldsymbol{Q} \bmod (1/q)\boldsymbol{Z}, \end{array}$$

where the right vertical arrows are the canonical projections induced by the identity map of $\boldsymbol{Q}$.

Proof. If we consider the cases $k = i$ and $k = j$ in (2.4), the corollary is immediate, because

$$\begin{aligned} u(i) &= q, \qquad u(j) = q, \\ v(i) &= 1, \qquad v(j) = 1 \end{aligned}$$

by (1.4) (b), (2.1) and (2.3). Q.E.D.

§ 3. Some computations

For our computation, we shall quote some results on $\pi_*^S(S^0)$, including Toda brackets $\langle\ ,\ ,\ \rangle$, from [21].

(3.1)
(a) $\pi_0^S(S_0)=Z\{\iota\}$, (b) $\pi_1^S(S^0)=Z/2\{\eta\}$, (c) $\pi_2^S(S^0)=Z/2\{\eta^2\}$,
(c′) $\langle 2\iota, \eta, 2\iota\rangle=\eta^2$, (d) $\pi_3^S(S^0)=Z/24\{\nu\}$, (d′) $\eta^3=12\nu$,
(d″) $\langle \eta, 2\iota, \eta\rangle\equiv 6\nu \bmod 12\nu$, (e) $\pi_4^S(S^0)=0$, (e′) $\eta\nu=\nu\eta=0$,
(f) $\pi_5^S(S^0)=0$, (g) $\pi_6^S(S^0)=Z/2\{\nu^2\}$, (g′) $\langle \eta, \nu, \eta\rangle=\nu^2$,
(h) $\pi_7^S(S^0)=Z/240\{\sigma\}$.*)

Denote the mod 2 Moore space by

$$M^n=S^n\cup_{2\iota}e^{n+1}$$

and the usual cofibration by

$$S^n\xrightarrow{i}M^n\xrightarrow{p}S^{n+1}.$$

Since $2\eta=0$, in stable range there are elements

$$\bar{\eta}: M^{n+1}\longrightarrow S^n, \qquad \tilde{\eta}: S^{n+2}\longrightarrow M^n$$

such that

$$(3.2)\qquad \text{(a)}\quad \bar{\eta}i=\eta, \qquad p\tilde{\eta}=\eta.$$

By (3.1) (c′), (d′), (d″) and (g′),

$$(3.2)\quad \text{(b)}\quad 2\bar{\eta}=\eta^2p,\quad 2\tilde{\eta}=i\eta^2, \quad \text{(c)}\quad \bar{\eta}\tilde{\eta}=6\nu, \quad \text{(d)}\quad \langle\bar{\eta}, i\nu p, \tilde{\eta}\rangle=\nu^2.$$

We notice that there are two choices of $\bar{\eta}$, $\tilde{\eta}$, if $\bar{\eta}$ (resp. $\tilde{\eta}$) is a choice, the other is $\bar{\eta}+\eta^2p=-\bar{\eta}$ (resp. $\tilde{\eta}+i\eta^2=-\tilde{\eta}$), and that (3.2) (c) must generally be $\pm 6\nu$.

From (3.1), (3.2), the universal coefficient exact sequences lead to the following results of Mukai [16] (we modify the description of several generators).

(3.3)
(a) $\{S^{n+4}, M^n\}=Z/2\{\tilde{\eta}\eta^2\}$, (b) $\{S^{n+5}, M^n\}=0$,
(c) $\{S^{n+6}, M^n\}=Z/2\{i\nu^2\}$.

(3.4)
(a) $\{M^{n+3}, S^n\}=Z/2\{\eta^2\bar{\eta}\}$, (b) $\{M^{n+4}, S^n\}=0$,
(c) $\{M^{n+5}, S^n\}=Z/2\{\nu^2p\}$.

*) The naming of generators ν, σ is modified from [21]. The ν and σ in [21] are the generators of the 2-primary parts.

$$(3.5)\quad \begin{aligned} &\text{(a)}\quad \{M^{n+2}, M^n\}=Z/2\{i\eta\bar{\eta}\}\oplus Z/2\{\tilde{\eta}\eta p\}\oplus Z/2\{i\nu p\},\\ &\text{(b)}\quad \{M^{n+3}, M^n\}=Z/4\{\tilde{\eta}\bar{\eta}\}\oplus Z/2\{\nu\wedge 1_M\},\qquad \text{(b}'\text{)}\quad 2\tilde{\eta}\bar{\eta}=i\eta^2\bar{\eta}=\tilde{\eta}\eta^2 p,\\ &\text{(c)}\quad \{M^{n+4}, M^n\}=Z/2\{\tilde{\eta}\eta\bar{\eta}\}.\end{aligned}$$

The element η of $\pi_1^S(S^0)$ has non-trivial e-invariant [2]:

$$e(\eta)=1/2 \in Q \bmod Z$$

and it satisfies (1.3) with $q=2$, $k=2$, $r=1$. The complexes

$$(3.6)\qquad X^n=S^n \cup_\eta e^{n+2} \cup_2 e^{n+3}, \qquad Y^n=S^{n-3} \cup_2 e^{n-2} \cup_\eta e^n$$

of (1.3)′, (1.3)″ for η with n large, where the subscript η for X, Y is omitted, are the mapping cones of $\bar{\eta}$, $\tilde{\eta}$, hence there are cofibration

$$(3.7)\qquad \begin{aligned} &M^{n+1}\xrightarrow{\bar{\eta}} S^n \xrightarrow{i} X^n \xrightarrow{j'} M^{n+2},\\ &S^{n-1}\xrightarrow{\tilde{\eta}} M^{n-3} \xrightarrow{i'} Y^n \xrightarrow{j} S^n.\end{aligned}$$

By the remark below (3.2), they are unique up to homotopy equivalence though the attaching maps $\bar{\eta}$, $\tilde{\eta}$ are not. The maps i and j are the rational equivalences in (1.4), which induce the homomorphism

$$i_* j^*: \pi_7^S(S^0)\longrightarrow\{Y^{n+7}, X^n\}.$$

It is known [2] that the e-invariant on $\pi_7^S(S^0)$ is monic, more precisely

$$(3.8)\qquad e(\sigma)=1/240 \in Q \bmod Z.$$

The purpose of this section is to show that the e-invariant (in our generalized sense) is still monic on $\{Y^{n+7}, X^n\}$, simultaneously to determine the group structure of $\{Y^{n+7}, X^n\}$:

Theorem 3.9. *The group* $\{Y^{n+7}, X^n\}$ *is the cyclic group of order* 60 *with generator* $i\sigma j$, *which has the e-invariant* $e(i\sigma j)=1/120 \in Q \bmod (1/2)Z$. *In particular, the e-invariant is monic on* $\{Y^{n+7}, X^n\}$.

The proof is divided into the following three steps:

(3.9a) $j^*: \{S^{n+7}, X^n\}\longrightarrow\{Y^{n+7}, X^n\}$ is epic.

(3.9b) $i_*: \pi_7^S(S^0)\longrightarrow\{S^{n+7}, X^n\}$ is isomorphic.

(3.9c) The kernel of $j^*: \{S^{n+7}, X^n\}\longrightarrow\{Y^{n+7}, X^n\}$ is of order 4.

Proof of (3.9), *assuming* (3.9a), (3.9b), (3.9c). By (3.1) (h), the group is the cyclic group of order $240/4=60$ with generator $i\sigma j$, whose e-invariant is easily computed from (2.5) and (3.8). Q.E.D.

Proof of (3.9a). Consider the commutative diagram with exact rows, which are induced by the first cofibration in (3.7):

$$(3.10)\qquad \begin{array}{ccccccc} \{M^{n+4}, S^n\} & \xrightarrow{i_*} & \{M^{n+4}, X^n\} & \xrightarrow{(j')_*} & \{M^{n+2}, M^n\} & \xrightarrow{\bar\eta_*} & \{M^{n+3}, S^n\} \\ \downarrow{\scriptstyle \tilde\eta^*} & & \downarrow{\scriptstyle \tilde\eta^*} & & \downarrow{\scriptstyle \tilde\eta^*} & & \\ \pi_6^S(S^0) & \xrightarrow{i_*} & \{S^{n+6}, X^n\} & \xrightarrow{(j')_*} & \{S^{n+4}, M^n\}. & & \end{array}$$

By (3.5) (a), (3.4) (a), (3.2) (a), (3.2) (c) and (3.1) (e′), the second and the third factors $Z/2$ in (3.5) (a) generate the kernel of $\bar\eta_*$ in (3.10), hence, by (3.4) (b), (3.1) (g) and (3.3) (a), the diagram (3.10) becomes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \{M^{n+4}, X^n\} & \xrightarrow{(j')_*} & Z/2\{\tilde\eta\eta p\} \oplus Z/2\{i\nu p\} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \tilde\eta^*} & & \downarrow{\scriptstyle \tilde\eta^*} & & \downarrow{\scriptstyle \tilde\eta^*} & & \\ Z/2\{\nu^2\} & \xrightarrow{i_*} & \{S^{n+6}, X^n\} & \xrightarrow{(j')_*} & Z/2\{\tilde\eta\eta^2\}. & & \end{array}$$

By (3.2) (a) and (3.1) (e′), the right $\tilde\eta^*$ is epic and its kernel is generated by $i\nu p$. By definition of Toda bracket, $\langle\bar\eta, i\nu p, \tilde\eta\rangle = i_*^{-1}\tilde\eta^*(j')_*^{-1}(i\nu p)$, which is non-trivial by (3.2) (d). Therefore the middle $\tilde\eta^*$ is isomorphic. The second cofibration in (3.7) induces the following exact sequence:

$$(3.11)\qquad \{M^{n+5}, X^n\} \xrightarrow{\tilde\eta^*} \{S^{n+7}, X^n\} \xrightarrow{j^*} \{Y^{n+7}, X^n\} \xrightarrow{(i')^*} \{M^{n+4}, X^n\} \xrightarrow{\tilde\eta^*} \{S^{n+6}, X^n\},$$

where the last $\tilde\eta^*$ is isomorphic as before. Then j^* is epic. Q.E.D.

Proof of (3.9b). Consider the exact sequence:

$$\{S^{n+6}, M^n\} \xrightarrow{\bar\eta_*} \pi_7^S(S^0) \xrightarrow{i_*} \{S^{n+7}, X^n\} \xrightarrow{(j')_*} \{S^{n+5}, M^n\} = 0,$$

where the last term vanishes by (3.3) (b). By (3.3) (c), (3.2) (a), (3.1) (e′), $\bar\eta_* = 0$, hence i_* is isomorphic. Q.E.D.

Proof of (3.9c). The element $\bar\eta\tilde\eta\bar\eta$ lies in $\{M^{n+4}, S^n\}$, which is trivial by (3.4) (b). Therefore $\bar\eta\tilde\eta\bar\eta = 0$. Similarly, $\tilde\eta\bar\eta\tilde\eta = 0$ by (3.3) (b). Then the Toda bracket $\langle\bar\eta, \tilde\eta\bar\eta, \tilde\eta\rangle$ is defined. Then,

$$(3.12)\qquad \langle\bar\eta, \tilde\eta\bar\eta, \tilde\eta\rangle = 60\sigma \quad \textit{or} \quad -60\sigma \text{ mod } \textit{zero}.$$

We will prove (3.12) after completing (3.9c). By (3.9b) and (3.1) (*h*),

$$\{S^{n+7}, X^n\} = Z/240\{i\sigma\}.$$

This is cyclic, hence, by the exact sequence (3.11), it is enough to show that the maximal order of elements in the image of the first $\tilde{\eta}^*$ in (3.11) is 4. By (3.12), there is an element γ in $\{M^{n+5}, X^n\}$ such that $(j')_*\gamma = \tilde{\eta}\bar{\eta}$ and $\tilde{\eta}^*(\gamma) = \pm 60 i\sigma$, an element of order 4. By (3.2) (b) and (3.1) (c), $4\tilde{\eta} = 0$, hence the image is a $Z/4$-module. Q.E.D.

Proof of (3.12). By (3.5) (b′) and (3.2) (a),

$$2\langle \bar{\eta}, \tilde{\eta}\bar{\eta}, \tilde{\eta}\rangle = \langle \bar{\eta}, \tilde{\eta}\eta, \eta^2\rangle.$$

It is enough to show

$$\langle \bar{\eta}, \tilde{\eta}\eta, \eta^2\rangle = 120\sigma \bmod 0.$$

By [21], the suspension $\Sigma^\infty\colon \pi_{12}(S^5) = Z/30 \to \pi_7^S(S^0)$ is monic and the Hopf invariant $H\colon \pi_{12}(S^5) \to \pi_{12}(S^9)$ $(\cong \pi_3^S(S^0) = Z/24)$ is a monomorphism of 2-primary component. It is therefore sufficient to show that $\langle \bar{\eta}, \tilde{\eta}\eta, \eta^2\rangle$ can be formed on S^5 with non-trivial Hopf invariant η^3. For $n \geqq 3$, $2\eta = 0$ holds on S^n. Therefore $\bar{\eta}$ on S^n and $\tilde{\eta}$ on M^n exist for $n \geqq 3$. Also, for $n \geqq 3$, η^3 on S^n is divisible by 4, $4\tilde{\eta} = 0$ on M^n and $\tilde{\eta}\eta^3 = 0$ on M^n. On S^3, $\bar{\eta}\tilde{\eta} = \nu'$ a generator of the 2-primary component of $\pi_6(S^3)$, and $\bar{\eta}(\tilde{\eta}\eta) = \nu'\eta$, which is non-zero on S^3, S^4 and becomes zero on S^5 [21]. Therefore the Toda bracket can be formed on S^5, We refer a part of the *EHP* exact sequence

$$Z/2\{\eta\} = \pi_{10}(S^9) \xrightarrow{P} \pi_8(S^4) \xrightarrow{\Sigma} \pi_9(S^5)$$

($P = \Delta$, $\Sigma = E$ in [21]). The element $\bar{\eta}\tilde{\eta}\eta$ in the middle group vanishes at the right end. Hence $P(\eta) = \bar{\eta}\tilde{\eta}\eta$. By the formula [21, Proposition 2.6], we conclude

$$H(\langle \bar{\eta}, \tilde{\eta}\eta, \eta^2\rangle \text{ on } S^5) = \eta\eta^2 = \eta^3 \in \pi_{12}(S^9). \qquad \text{Q.E.D.}$$

§ 4. The representation ring and the Chern character of G_2

We shall begin with quotations on compact, simply connected Lie group of exceptional type G_2 and its complex representation ring $R(G_2)$ from the book [23].

We denote by $\mathfrak{C}$ the (non-associative) field of Cayley numbers. Let e_i $(0 \leqq i \leqq 7)$ be the usual $\boldsymbol{R}$-basis for $\mathfrak{C}$ with multiplicative rule:

$$e_0 = 1 \text{ (the unit)}, \quad e_1 = i \text{ (the imaginary unit in } \boldsymbol{C}),$$

$$e_i^2 = -1 \ (i \geqq 1), \quad e_ie_j = -e_je_i \ (i \neq j,\ i, j \geqq 1),$$

$$e_ie_j=e_k,\quad e_je_k=e_i,\quad e_ke_i=e_j \quad \text{for } (i,j,k)=(1,2,3),\ (1,4,5),\ (1,6,7),\ (2,5,7),\ (2,6,4),\ (3,4,7),\ (3,5,6).$$

An $\boldsymbol{R}$-linear isomorphism $x\colon \mathfrak{C}\to\mathfrak{C}$ is said to be an automorphism of $\mathfrak{C}$, if it is multiplicative: $x(uv)=x(u)x(v)$, $u, v \in \mathfrak{C}$. The compact, simply connected Lie group of the exceptional type G_2 is realized as the automorphism group of $\mathfrak{C}$:

$$G_2=\mathrm{Aut}\,\mathfrak{C}=\{\text{automorphisms of } \mathfrak{C}\}.$$

Let $\mathfrak{C}_0=\{x \in \mathfrak{C} \mid \mathrm{Re}\,x=0\}=\sum_{i=1}^{7}\boldsymbol{R}e_i$. Any element $x \in G_2$ satisfies $x(1)=1$ and $|x(u)|=|u|$, $u \in \mathfrak{C}$. Therefore G_2 is a closed subgroup of $O(\mathfrak{C}_0)=O(7)$ [23, Lemma 5.1].

For $\theta \in \boldsymbol{R}$, let $r(\theta)$ be the 2×2 matrix over $\boldsymbol{R}$:

$$r(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

The maximal torus of G_2 is then given, as a subgroup of $O(7)$, as follows [23, Theorem 5.6]:

$$T^2=\left\{t(\theta_1,\theta_2,\theta_3)=\begin{pmatrix}1 & & & 0\\ & r(\theta_1) & & \\ & & r(\theta_2) & \\ 0 & & & r(\theta_3)\end{pmatrix}\,\middle|\, \theta_i \in \boldsymbol{R},\ \theta_1+\theta_2+\theta_3=0\right\}.$$

Let ρ be the 7-dimensional complex representation of G_2 given by the inclusion $G_2\subset O(7)\subset U(7)$, or by the $\boldsymbol{C}$-module $\mathfrak{C}_0^{\boldsymbol{C}}=\mathfrak{C}_0\otimes_{\boldsymbol{R}}\boldsymbol{C}$. Put $\rho'=\Lambda^2\rho$, the exterior power.

Theorem 4.1 [23, Theorem 5.9, 5.10]. *The complex representation ring of G_2 is the polynomial ring with generators ρ, ρ':*

$$R(G_2)=\boldsymbol{Z}[\rho,\rho'].$$

The ring homomorphism $j^*\colon R(G_2)\to R(T^2)$ induced by the inclusion $j\colon T^2\to G_2$ is a monomorphism given by

$$\begin{aligned}j^*(\rho)&=1+z_1+z_1^{-1}+z_2+z_2^{-1}+z_3+z_3^{-1},\\ j^*(\rho')&=3+2(z_1+z_1^{-1}+z_2+z_2^{-1}+z_3+z_3^{-1})\\ &\quad+z_1z_2^{-1}+z_1^{-1}z_2+z_2z_3^{-1}+z_2^{-1}z_3+z_3z_1^{-1}+z_3^{-1}z_1,\end{aligned}$$

where

$$R(T^2)=\boldsymbol{Z}[z_1,z_1^{-1},z_2,z_2^{-1},z_3,z_3^{-1}]/(z_1z_2z_3-1)$$

with $z_j(t(\theta_1, \theta_2, \theta_3))=\cos\theta_j+i\sin\theta_j$.

For the computation of $j^*(\rho)$, Yokota gave an explicit $\boldsymbol{C}$-basis of $\mathfrak{C}_0^{\boldsymbol{C}}$:

$$h=e_1\otimes 1,\quad u_j=e_{2j}\otimes 1-e_{2j+1}\otimes i,\quad \hat{u}_j=-e_{2j}\otimes i+e_{2j+1}\otimes 1$$

$(j=1, 2, 3)$, for which T^2 acts as follows:

$$t(\theta_1, \theta_2, \theta_3)h=h,\qquad t(\theta_1, \theta_2, \theta_3)u_j=u_j(\cos\theta_j+i\sin\theta_j),$$
$$t(\theta_1, \theta_2, \theta_3)\hat{u}_j=\hat{u}_j(\cos\theta_j-i\sin\theta_j).$$

Let H_1 and $H_{1,2}$ be the following closed subgroups of G_2:

$$H_1=\{x\in G_2\,|\,x(e_1)=e_1\},\qquad H_{1,2}=\{x\in G_2\,|\,x(e_1)=e_1,\ x(e_2)=e_2\},$$

and let S_1, $\hat{S}_1$, $S_{1,2}$, $\hat{S}_{1,2}$ be the following $\boldsymbol{C}$-linear subspaces of $\mathfrak{C}_0^{\boldsymbol{C}}$:

$$S_1=u_1\boldsymbol{C}\oplus u_2\boldsymbol{C}\oplus u_3\boldsymbol{C},\qquad \hat{S}_1=\hat{u}_1\boldsymbol{C}\oplus\hat{u}_2\boldsymbol{C}\oplus\hat{u}_3\boldsymbol{C},$$
$$S_{1,2}=u_2\boldsymbol{C}\oplus u_3\boldsymbol{C},\qquad \hat{S}_{1,2}=\hat{u}_2\boldsymbol{C}\oplus\hat{u}_3\boldsymbol{C}.$$

As in [23, Chap. 5, § 1, (6)], H_1 and $H_{1,2}$ are isomorphic to $SU(3)$ and $SU(2)=S^3$, respectively, and S_1, $\hat{S}_1$ (resp. $S_{1,2}$, $\hat{S}_{1,2}$) are H_1- (resp. $H_{1,2}$-) invariant subspaces; through the isomorphisms $H_1=SU(3)$, $H_{1,2}=SU(2)$, the $SU(3)$-action on S_1 coincides with the usual action on $\boldsymbol{C}^3$, the $SU(2)$-action on $S_{1,2}$ is its restriction to $\boldsymbol{C}^2$, and the $SU(2)$-action on $\hat{S}_{1,2}$ is its conjugate action. The S_1 and $\hat{S}_1$ (resp. $S_{1,2}$ and $\hat{S}_{1,2}$) determine the representations σ_1 and $\hat{\sigma}_1$ (resp. $\sigma_{1,2}$ and $\hat{\sigma}_{1,2}$) of $SU(3)$ (resp. $SU(2)$), which generate the representation ring [23, Theorem 5.11]:

$$R(SU(3))=\boldsymbol{Z}[\sigma_1, \hat{\sigma}_1],\qquad R(SU(2))=\boldsymbol{Z}[\sigma_{1,2}]$$

with $\hat{\sigma}_{1,2}=\sigma_{1,2}$ in the representation ring. Via inclusions $SU(2)\subset SU(3)\subset G_2$, ρ goes to $1+\sigma_1+\hat{\sigma}_1$ and σ_1, $\hat{\sigma}_1$ go to $1+\sigma_{1,2}=1+\hat{\sigma}_{1,2}$. Thus,

Theorem 4.2. *Let* i: $S^3=SU(2)\to G_2$ *be the inclusion. Then*

$$i^*\rho=2\sigma_{1,2}+3,\qquad i^*\rho'=(\sigma_{1,2})^2+6\sigma_{1,2}+5.$$

Considering a representation simply as a continuous map to the infinite unitary group defines the β-construction β: $R(G)\to\tilde{K}^{-1}(G)$, and we have, by [9],

Theorem 4.3. **(a)** $K^*(G_2)=E(\beta(\rho), \beta(\rho'))$, $K^*(S^3)=E(\beta(\sigma_{1,2}))$, *where* E *denotes the exterior algebra over* $\boldsymbol{Z}$.

(b) $i^*\beta(\rho)=2\beta(\sigma_{1,2})$, $i^*\beta(\rho')=10\beta(\sigma_{1,2})$.

The part (b) follows from (4.2) and [9, page 8, (1), (2), Lemma 4.1].

Borel [5, Theorem 17.2] determined the integral cohomology ring $H^*(G_2)$ of G_2, from which he [5, Theorem 17.3] also did the mod 2 cohomology.

Theorem 4.4 (Borel). *There are integral classes h_3 and h_{11} in $H^*(G_2)$, $\deg h_i = i$, which generate the ring $H^*(G_2)$ with relations $2h_3^2=0$, $h_3^4=0$, $h_3^2h_{11}=0$, $h_{11}^2=0$, Hence the additive group structure of $H^i = H^i(G_2)$ is given as follows*:

$$H^0=Z,\quad H^3=Z\{h_3\},\quad H^{3i}=Z/2\{h_3^i\}\quad (i=2,3),$$
$$H^{11}=Z\{h_{11}\},\quad H^{14}=Z\{h_3h_{11}\},\quad H^i=0 \text{ for other } i.$$

Let x_i be the mod 2 *reduction of h_i and x_5 be the* mod 2 *class whose Bockstein is h_3^2. Then the* mod 2 *cohomology $H^*(G_2; Z/2)$ has the following $Z/2$-basis*

$$1,\ x_3,\ x_5,\ x_3^2,\ x_3x_5,\ x_3^3,\ x_{11}=x_3^2x_5,\quad x_3x_{11}=x_3^3x_5$$

with

$$Sq^2x_3=x_5,\quad Sq^1x_5=x_3^2,\quad Sq^1x_3x_5=x_3^3,\quad Sq^2x_3^3=x_{11}.$$

From (4.4), since G_2 is simply connected, we may construct a (minimal) CW complex

$$A=S^3\cup e^5\cup e^6\cup e^8\cup e^9\cup e^{11}\cup e^{14}$$

homotopy equivalent to G_2. Considering the squaring operations in (4.4), we see that the 6-skeleton $A^{(6)}$ of A is homotopy equivalent to X^3 in (3.6), because η and 2ι are detected by Sq^2 and Sq^1, and the squaring operations determine the homotopy type of X (see the remark below (3.2)). Similarly, $A^{(11)}/A^{(6)}=Y^{11}$ (homotopy equivalent) in (3.6). We have therefore the following cofibrations:

(4.5) $$X^3\xrightarrow{k}A^{(11)}\xrightarrow{l}Y^{11},$$

(4.6) $$A^{(11)}\xrightarrow{k'}A\simeq G_2\xrightarrow{l'}S^{14}.$$

The second cofibration (4.6) is induced by the attaching map

(4.6)′ $$\psi: S^{13}\longrightarrow A^{(11)}$$

of the top cell in A, and extending the cofibration (4.5) gives a map

(4.5)′ $$\phi: Y^{11}\longrightarrow \Sigma X^3=X^4$$

whose cofibre is $\Sigma A^{(11)}$. By (1.4) (a), $\phi^*=0$ in K-cohomology and

$$\tilde{K}^0(A^{(11)})=0, \qquad \tilde{K}^{-1}(A^{(11)})=Z\{\bar{\xi}_3\}\oplus Z\{\bar{\eta}_{11}\}, \tag{4.7}$$

where $\bar{\eta}_{11}=l^*\eta_{11}$ and $\bar{\xi}_3$ is an element with $k^*\bar{\xi}_3=\xi_3$. As in [2], the Chern character may be defined on $\tilde{K}^{-1}$ via suspension isomorphisms:

$$\mathrm{ch}\colon \tilde{K}^{-1}(-)\cong\tilde{K}^0(\Sigma-)\xrightarrow{\mathrm{ch}}\tilde{H}^{\mathrm{even}}(\Sigma-;Q)\cong\tilde{H}^{\mathrm{odd}}(-;Q),$$

so that (1.4) (b) with $\tilde{K}^0$ replaced by $\tilde{K}^{-1}$ is also valid for odd n. Then

$$\begin{aligned}&\mathrm{ch}\,\bar{\xi}_3=2h_3+\lambda h_{11} \qquad \text{for some } \lambda\in Q,\\ &\mathrm{ch}\,\bar{\eta}_{11}=(1/2)h_{11},\end{aligned} \tag{4.8}$$

where, as before, we write again h_i for the rational class of h_i and we identify $H^*(A^{(11)};Q)$ with a subgroup of $H^*(A;Q)=H^*(G_2;Q)=E(h_3, h_{11})$, the exterior algebra over Q. The map ψ of (4.6)′ is stably trivial

$$\Sigma^\infty\psi=0 \tag{4.9}$$

by the self-duality of $A=G_2$ [7]. Therefore $\psi^*=0$ in K-cohomology, and by (4.6),

$$\begin{aligned}&(l')^*\colon \tilde{K}^0(S^{14})\longrightarrow\tilde{K}^0(A),\\ &(k')^*\colon \tilde{K}^{-1}(A)\longrightarrow\tilde{K}^{-1}(A^{(11)})\end{aligned}$$

are isomorphic. In particular, $K^*(A)$ has no torsion and the Chern character for A is monic. By (4.8), $\mathrm{ch}\,\bar{\xi}_3\bar{\eta}_{11}=h_3h_{11}$, which is the image of the generator of $H^{14}(S^{14})$. This gives an alternative form of the theorem of Hodgkin (4.3):

Theorem 4.3′. **(a)** *$K^*(A)=E(\tilde{\xi}, \tilde{\eta})$, the exterior algebra over Z, where $\tilde{\xi}, \tilde{\eta}\in\tilde{K}^{-1}(A)$ are the elements satisfying*

$$(k')^*\tilde{\xi}=\bar{\xi}_3, \qquad (k')^*\tilde{\eta}=\bar{\eta}_{11}, \qquad (l')^*\sigma_{14}=\tilde{\xi}\tilde{\eta}.$$

(b) $\mathrm{ch}\,\tilde{\xi}=2h_3+\lambda h_{11}$, $\mathrm{ch}\,\tilde{\eta}=(1/2)h_{11}$

with the same coefficient λ as in (4.8).

The element $\tilde{\xi}$ is determined modulo $\tilde{\eta}$, while $\tilde{\eta}$ is unique. Let $i^*\colon \tilde{K}^{-1}(A)\to\tilde{K}^{-1}(S^3)$ be induced by the inclusion $i\colon S^3\to A$. By (1.4) (b) and the definition of $\tilde{\xi}, \tilde{\eta}$, we see that $i^*\tilde{\xi}$ is twice generator and $\tilde{\eta}$ generates the kernel of i^*. By (4.3) (b), we know the relation between two generating systems, $\{\beta(\rho), \beta(\rho')\}$ in (4.3) (a) and $\{\tilde{\xi}, \tilde{\eta}\}$ in (4.3)′ (a), of $\tilde{K}^{-1}(G_2)$ $=Z\oplus Z$.

Lemma 4.10. *There is a choice of* $\tilde{\xi}$ *such that*

$$\beta(\rho)=\tilde{\xi}, \qquad \beta(\rho')=5\tilde{\xi}\pm\tilde{\eta}.$$

In (4.3)′ (b), the coefficient $\lambda \bmod (1/2)\boldsymbol{Z}$ is the e-invariant of the stable class of ϕ in (4.5)′:

$$e(\phi)=\lambda \bmod (1/2)\boldsymbol{Z}$$

for e: $\{Y^{11}, X^4\}\to \boldsymbol{Q} \bmod (1/2)\boldsymbol{Z}$.

Lemma 4.11. $e(\phi)=\pm 1/60 \bmod (1/2)\boldsymbol{Z}$.

Proof. We compute the Adams operation ψ^2 of $\tilde{\xi}$, $\tilde{\eta}$. Apply [1, Theorem 5.1, (vi)] to (4.3)′ (b) to get

$$\mathrm{ch}\,\psi^2\tilde{\xi}=8h_3+64\lambda h_{11}, \qquad \mathrm{ch}\,\psi^2\tilde{\eta}=32h_{11}.$$

Since ch is monic, again by (5.3)′ (b),

$$(*) \qquad \psi^2\tilde{\xi}=4\tilde{\xi}+120\lambda\tilde{\eta}.$$

On the other hand, $\psi^2\rho=\rho^2-2\Lambda^2\rho=\rho^2-2\rho'$ by definition of ψ^2. By (4.10),

$$\psi^2\tilde{\xi}=\psi^2\beta(\rho)=14\beta(\rho)-2\beta(\rho')=4\tilde{\xi}\mp 2\tilde{\eta}.$$

Comparing this with (*) leads to $\lambda=\pm(1/60)$. Q.E.D.

The Chern character on $K^*(G_2)$ must be computed directly from Theorems 4.1 and 4.3 without 4.3′. In fact, Professor H. Minami kindly told the author a direct proof. He used the adjoint representation Ad in place of $\Lambda^2\rho$, but, since the computation of j^*Ad leads to $\mathrm{Ad}=\Lambda^2\rho-\rho$, we shall give here his proof with $\rho'=\Lambda^2\rho$.

Alternative Proof (due to H. Minami). A representation of G, viewed as a linear G-action on $\boldsymbol{C}^n$, defines a vector bundle associated to the universal principal bundle over BG, and we have a ring homomorphism α: $R(G)\to K(BG)$. We denote the image $\alpha(\rho)$ of a representation ρ in $K(BG)$ by the same letter ρ. Let c_i be the i-th Chern class and $c=\sum c_i$ the total Chern class. Write

$$H^*(BT^2)=\boldsymbol{Z}\,[a, b, c]/(a+b+c)=\boldsymbol{Z}[a, b],$$

where $a=c_1(z_1)$, $b=c_1(z_2)$, $c=c_1(z_3)\in H^2(BT^2)$. We put

$$u_4=a^2+ab+b^2 \in H^4(BT^2), \qquad u_{12}=a^4b^2+2a^3b^3+a^2b^4 \in H^{12}(BT^2).$$

Let $w\colon BT^2 \to BG_2$ be the map induced by the inclusion $j\colon T^2 \to G_2$. By (4.1), $w^*\rho = 1 + z_1 + z_1^{-1} + z_2 + z_2^{-1} + z_3 + z_3^{-1} \in K(BT^2)$. Therefore,

$$\begin{aligned} c(w^*\rho) &= c(z_1)c(z_1^{-1})c(z_2)c(z_2^{-1})c(z_3)c(z_3^{-1}) \\ &= (1+a)(1-a)(1+b)(1-b)(1-(a+b))(1+(a+b)) \\ &= 1 - 2u_4 + u_4^2 - u_{12} \end{aligned}$$

and

(4.12) $$w^*c_2(\rho) = -2u_4, \quad w^*c_4(\rho) = u_4^2, \quad w^*c_6(\rho) = -u_{12}.$$

Next we have

$$\begin{aligned} w^* \operatorname{ch} \rho = \operatorname{ch}(w^*\rho) &= 1 + e^a + e^{-a} + e^b + e^{-b} + e^{-a-b} + e^{a+b} \\ &= 7 + 2\sum_{k=1}^{\infty} \frac{a^{2k} + b^{2k} + (a+b)^{2k}}{(2k)!}, \end{aligned}$$

where we have

$$\begin{aligned} &a^2 + b^2 + (a+b)^2 = 2u_4, \qquad a^4 + b^4 + (a+b)^4 = 2u_4^2, \\ &a^6 + b^6 + (a+b)^6 = 3u_{12} + 2u_4^3. \end{aligned}$$

Hence

$$w^* \operatorname{ch} \rho = 7 + 2u_4 + (1/6)u_4^2 + (1/120)u_{12} + (1/180)u_4^3 + \text{higher terms}.$$

To simplify the computation of $w^* \operatorname{ch} \rho'$, we do it for $\rho'' = \rho' - 2\rho$ instead of ρ'. Then $w^*\rho'' = 1 + z_1 z_2^{-1} + z_2 z_1^{-1} + z_2 z_3^{-1} + z_3 z_2^{-1} + z_3 z_1^{-1} + z_1 z_3^{-1}$, and we have

$$w^* \operatorname{ch} \rho'' = 1 + e^{a-b} + e^{b-a} + e^{a+2b} + e^{-(a+2b)} + e^{2a+b} + e^{-(2a+b)}.$$

From the relations

$$\begin{aligned} &(a-b)^2 + (a+2b)^2 + (2a+b)^2 = 6u_4, \\ &(a-b)^4 + (a+2b)^4 + (2a+b)^4 = 18u_4^2, \\ &(a-b)^6 + (a+2b)^6 + (2a+b)^6 = -81u_{12} + 66u_4^3, \end{aligned}$$

we get

$$w^* \operatorname{ch} \rho'' = 7 + 6u_4 + (3/2)u_4^2 - (9/40)u_{12} + (11/60)u_4^3 + \text{higher terms}.$$

Consider the commutative diagram

$$\begin{array}{ccc} H^4(BG_2) = Z\{y_4\} & \xrightarrow{w^*} & H^4(BT^2) = Z\{a^2\} \oplus Z\{ab\} \oplus Z\{b^2\} \\ \downarrow & & \downarrow \\ H^4(BS^2) = Z & \longrightarrow & H^4(BT^1) = Z\{d^2\} \end{array}$$

of integral cohomology groups, where the generator y_4 is the transgression image of h_3 and $H^*(BT^1)=Z[d]$. By (4.12) and the definition of u_4, $c_2(\rho)$ goes to $-2d^2$, while y_4 to d^2. Therefore

$$c_2(\rho)=-2y_4, \qquad w^*y_4=u_4.$$

Also u_{12} is the image of the integral class $-c_6(\rho)$, by (4.12). Hence

$$\text{(4.13)}\quad \begin{aligned} &\mathrm{ch}\,\rho=7+2y_4+(1/6)y_4^2-(1/120)c_6(\rho)+(1/180)y_4^3+\text{higher terms},\\ &\mathrm{ch}\,\rho''=7+6y_4+(3/2)y_4^2+(9/40)c_6(\rho)+(11/60)y_4^3+\text{higher terms}.\end{aligned}$$

Let $\iota\colon \Sigma G\rightarrow BG$ be the canonical map and σ_K be the composite:

$$\sigma_K\colon \tilde{K}(BG)\xrightarrow{\iota^*}\tilde{K}(\Sigma G)\cong\tilde{K}^{-1}(G)$$

Then the β-construction is, up to sign, the composite [9, Proposition 4.1]:

$$\beta\colon R(G)\xrightarrow{\alpha}K(BG)\longrightarrow\tilde{K}(BG)\xrightarrow{\sigma_K}\tilde{K}^{-1}(G)$$

Therefore, up to sign,

$$\mathrm{ch}\,\beta(\rho)=\sigma_{HQ}(\mathrm{ch}\,\rho-7), \qquad \mathrm{ch}\,(\beta(\rho')-2\beta(\rho))=\sigma_{HQ}(\mathrm{ch}\,\rho''-7),$$

where the suspension σ_{HQ} for the rational cohomology is similarly defined as σ_K. We also denote by σ_H the suspension for the integral cohomology. Then $\sigma_H(y_4)=h_3$, and $\sigma_H(c_6(\rho))=\bar{h}_{11}$ is an integral multiple of h_{11}. Since σ_{HQ} kills decomposable elements and higher terms in (4.13), we get

$$\mathrm{ch}\,\beta(\rho)=2h_3-(1/120)\bar{h}_{11}, \qquad \mathrm{ch}\,(\beta(\rho')-2\beta(\rho))=6h_3+(9/40)\bar{h}_{11}.$$

Since $\beta(\rho)^2=0$, $h_3^2=0$ (in rational cohomology), $\bar{h}_{11}h_3=-h_3\bar{h}_{11}$ and $\bar{h}_{11}^2=0$,

$$\mathrm{ch}\,(\beta(\rho)\beta(\rho'))=(1/2)h_3\bar{h}_{11}.$$

On the other hand, by [4, Proposition 1], $\mathrm{ch}\,(\beta(\rho)\beta(\rho'))=h_3h_{11}$, hence $\bar{h}_{11}=2h_{11}$. Since ch is monic, we may put $\tilde{\xi}=\beta(\rho)$, $\tilde{\eta}=\beta(\rho')-5\beta(\rho)$ to get

$$\mathrm{ch}\,\tilde{\xi}=2h_3-(1/60)h_{11}, \qquad \mathrm{ch}\,\tilde{\eta}=(1/2)h_{11}. \qquad \text{Q.E.D.}$$

Now we shall go back to Lemma 4.11. As in (3.9), the e-invariant faithfully determines the stable class of ϕ.

Theorem 4.14. *The stable class ϕ of the attaching map* (4.5)′ *in $\Sigma G_2^{(11)}$ is twice generator, i.e., $\phi=\pm 2i\sigma j$, hence it is of order* 30.

Corollary 4.15. *For p prime,* $G_2^{(6)}=X^3$ *is not* mod p *stable retract of* G_2 *if and only if* $p=2, 3, 5$.

Proof. By (4.9), $G_2^{(6)}$ is mod p stable retract of G_2, if and only if it is mod p stable retract of $G_2^{(11)}$, hence, if and only if ϕ localized at p vanishes. Q.E.D.

Remark 4.16. For $p=2$, (4.15) is recently obtained by F. Cohen and F. Peterson [8] in a different method. For $p=3$, (4.14) asserts that ϕ localized at 3 is detected by a secondary operation Φ: $H^4(\Sigma G_2; Z/3)\to H^{12}(\Sigma G_2; Z/3)$ which is equivalent to $\pi_{10}(G_2)_{(3)}=0$, recovering the old result of Bott-Samelson [6].

Similarly, (4.14) localized at 5 is equivalent to the non-triviality of $\mathscr{P}^1$: $H^3(G_2; Z/5)\to H^{11}(G_2; Z/5)$, originally obtained by Bott [6].

By [21], $\pi_{14}(S^7)=Z/120\{\sigma'\}$ and the generator σ' can not be a suspension. Since $\sigma'=2\sigma$ in stable range, we have, however,

Corollary 4.17. *The composite*

$$Y^{14}\xrightarrow{j}S^{14}\xrightarrow{\sigma'}S^7\xrightarrow{i}X^7$$

is a three times suspension.

Proof. Parts of the proofs of (3.9a) and (3.9b) are still valid in an unstable range to show that

$$j^*: \pi_{14}(X^7)\longrightarrow[Y^{14}, X^7] \quad \text{is epic,}$$
$$i_*: \pi_{14}(S^7)\longrightarrow\pi_{14}(X^7) \quad \text{is epic.}$$

Therefore $[Y^{14}, X^7]$ is a cyclic group with generator $i\sigma'j$, hence the kernel K of $\Sigma^\infty: [Y^{14}, X^7]\to\{Y^{n+7}, X^n\}$ is a cyclic group of order $d=1, 2$ or 4, because $120i\sigma'j=0$ and $\Sigma^\infty(i\sigma'j)=2i\sigma j$ is of order 30. Evaluating e-invariant gives the relation $i\sigma'j\equiv\Sigma^3\phi$ mod K, hence $\Sigma^3\phi=(1+120x/d)i\sigma'j$, $x\in Z$. The coefficient is a unit in any case. Q.E.D.

§ 5. The degree of self-maps of G_2

Mimura, Nishida and Toda [13] constructed simply connected H-spaces $G_{2,b}$ of rank 2 with homology torsion, whose prototype is $G_2=G_{2,0}$.

The $G_{2,b}$ is p-equivalent to G_2 for $p=2$ and for $p\geqq 7$, and to $S^3\times S^{11}$ or G_2 according as $b\equiv -2$ mod p or not, for $p=3, 5$. They proved that the set of distinct homotopy types of simply connected H-spaces of rank 2 with homology torsion is $\{G_{2,b} \mid -2 \leqq b \leqq 5\}$ [13, Theorem 5.1]. For the homotopy type of $G_{2,b}$, they proved the following

Lemma 5.1. **(a)** [13, above Lemma 5.2, Lemma 5.2] *$G_{2,b}$ is homotopy equivalent to a CW complex*

$$A_b = S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}, \qquad A_0 = A,$$

which coincides, up to 9-skeleton, with A in (4.6), *homotopy equivalent to G_2:*

$$A_b^{(8)} = A^{(9)}.$$

(b) [13, Lemma 4.3 (ii), Lemma 5.3] *The attaching map $\omega: S^{10} \to A^{(9)}$ of the top cell in $A^{(11)}$ is a generator of $\pi_{10}(A^{(9)}) = Z/120$ and the attaching map $\omega_b: S^{10} \to A_b^{(9)} = A^{(9)}$ of the top cell in $A_b^{(11)}$ is $(8b+1)\omega$.*

(c) [13, Lemma 4.3 (iii)] *The image of ω in $\pi_{10}(A^{(9)}/A^{(6)}) = \pi_{10}(M^8) = Z/4\{\tilde{\eta}\}$ is the generator $\pm\tilde{\eta}$, hence, so is the image of ω_b.*

(d) $A_b^{(6)} = X^3$, $A_b^{(11)}/A_b^{(6)} = Y^{11}$.

Since $(8b+1)\tilde{\eta} = \tilde{\eta}$, the latter halves of the parts (c) and (d) are clear. From (d), we get a map

$$\phi_b: Y^{11} \longrightarrow X^4, \qquad \phi_0 = \phi \quad \text{in (4.5)}',$$

extending the cofibration $A_b^{(6)} \to A_b^{(11)} \to A_b^{(11)}/A_b^{(6)}$.

Lemma 5.2. *In stable range, $\phi_b = (8b+1)\phi$.*

Proof. Consider the diagram

$$\begin{array}{ccccc} & & Y^{11} & \xrightarrow{j} & S^{11} \\ & & \downarrow \phi_b & & \downarrow \Sigma\omega_b = (8b+1)\Sigma\omega \\ M^8 & \longrightarrow & X^4 & \longrightarrow & \Sigma A^{(9)}, \end{array}$$

where the lower sequence is a cofibration, and the square is commutative in stable range. Then the difference $\phi_b - (8b+1)\phi$ factors through M^8. By definition of our e-invariant, any stable composite $Y^{11} \to M^8 \to X^4$ has trivial e-invariant because M^8 is rationally contractible, hence it is trivial by (3.9). Q.E.D.

The cohomology ring of $G_{2,b}$ is isomorphic to those of G_2, cf. [13, Theorem 2.2], so we use again h_3, h_{11} for the multiplicative generators of $H^*(G_{2,b})$ with deg $h_i = i$.

Theorem 5.3. *There are elements* $\tilde{\xi}$, $\tilde{\eta} \in \tilde{K}^{-1}(G_{2,b})$ *such that*

$$K^*(G_{2,b})=E(\tilde{\xi}, \tilde{\eta}),$$

the exterior algebra over $\boldsymbol{Z}$, *and that*

$$\mathrm{ch}\,\tilde{\xi}=2h_3-((8b+1)/60)h_{11}, \qquad \mathrm{ch}\,\tilde{\eta}=(1/2)h_{11}.$$

Proof. By virtue of (5.1), the computation for (4.3)′ may be exactly applied to the case of A_b. The coefficient of h_{11} in ch $\tilde{\xi}$ is the e-invariant of ϕ_b, which is easily computed by (5.2) and (4.11). Here we follow the computation above (4.14) as to the sign of $e(\phi)$, although it depends on the orientation of the generators. Q.E.D.

For given spaces X, Y, suppose $H^n(X)=\boldsymbol{Z}$, $H^n(Y)=\boldsymbol{Z}$ for some n. For a map $f: X\to Y$, we then define an integer $d_n(f)$, the degree of f in dimension n, when $f^*: H^n(Y)\to H^n(X)$ is the multiplication by $d_n(f)$. To unify the notation, we also define $d_n(f)=0$ in case $H^n(X)=0$ or $H^n(Y)=0$. The d_n is clearly a homotopy invariant.

We shall study the image of the degree map for $G_{2,b}$:

$$d=d_3\times d_{11}: [G_{2,b}, G_{2,b}]\longrightarrow \boldsymbol{Z}\oplus \boldsymbol{Z},$$

which factors through the stable degree map

$$d^S: \{G_{2,b}, G_{2,b}\}\longrightarrow \boldsymbol{Z}\oplus\boldsymbol{Z}, \qquad d^S\Sigma^\infty=d.$$

Both of d and d^S preserve the addition, given by an H-structure on $G_{2,b}$ in case $[G_{2,b}, G_{2,b}]$ and by the usual track addition in stable case, and the multiplication given by the composition of maps.

Let $b'=(15, 8b+1)$, the greatest common divisor, and put $b_1=(8b+1)/b'$, $b_2=30/b'$. For $-2\leqq b\leqq 5$, b_i are given as follows:

(5.4)

b	-2	-1	0	1	2	3	4	5
b_1	-1	-7	1	3	17	5	11	41
b_2	2	30	30	10	30	6	10	30

Proposition 5.5. Im $d\subset$ Im $d^S\subset\{(m, m+nb_2)\,|\,m, n\in \boldsymbol{Z}\}$.

Proof. The first inclusion is obvious. For given $f: \Sigma^t G_{2,b}\to\Sigma^t G_{2,b}$, let $m=d_3^S(f)=d_{3+t}(f)$, $m'=d_{11}^S(f)=d_{11+t}(f)$ so that

$$f^*(\sigma^t h_3)=m\sigma^t h_3, \qquad f^*(\sigma^t h_{11})=m'\sigma^t h_{11},$$

where σ^t is the t-times suspension. We shall determine

$$f^*: \tilde{K}^{-1+t}(\Sigma^t G_{2,b}) = Z\{\sigma^t\tilde{\xi}\} \oplus Z\{\sigma^t\tilde{\eta}\} \longrightarrow \tilde{K}^{-1+t}(\Sigma^t G_{2,b}).$$

We may put $f^*\sigma^t\tilde{\xi} = a\sigma^t\tilde{\xi} + a'\sigma^t\tilde{\eta}$, $f^*\sigma^t\tilde{\eta} = c\sigma^t\tilde{\xi} + c'\sigma^t\tilde{\eta}$. By (5.3), the naturality $\mathrm{ch} f^* = f^* \mathrm{ch}$ implies

$$2a = 2m, \qquad a'/2 - (8b+1)a/60 = -(8b+1)m'/60,$$
$$2c = 0, \qquad c'/2 - (8b+1)c/60 = m'/2.$$

Hence

$$a = m, \quad 30a' = (8b+1)(m-m'), \quad c = 0, \quad c' = m'.$$

The second equality implies the congruence $m \equiv m' \bmod b^2$. Q.E.D.

In the above proof, if we put $m' = m + nb_2$, then $a' = -nb_1$. We have therefore

(5.6) *If* $d^S(f) = (m, m')$ with $m' = m + nb_2$, then
$$f^*(\sigma^t\tilde{\xi}) = m\sigma^t\tilde{\xi} - nb_1\sigma^t\tilde{\eta}, \qquad f^*(\sigma^t\tilde{\eta}) = (m+nb_2)\sigma^t\tilde{\eta}.$$

We shall prove that the K-theoretic estimate for $\mathrm{Im}\, d$ given in (5.5) is the best result, namely.

Theorem 5.7. $\mathrm{Im}\, d = \mathrm{Im}\, d^S = \{(m, m+nb_2) \mid m, n \in Z\}$.

Since d is a homomorphism and $d(\mathrm{id}) = (1, 1)$, the theorem is equivalent to the existence of a self-map f of $G_{2,b}$ with $d_3(f) = 0$, $d_{11}(f) = b_2$. To construct such a map, we need the following lemmas.

Lemma 5.8. $\pi_{11}(G_{2,b}) = Z\{\gamma\} \oplus Z/2\{\tau_1\}$ *and the image of the Hurewicz homomorphism* $\pi_{11}(G_{2,b}) \rightarrow H_{11}(G_{2,b})$ *has index* $4b_2$.

Lemma 5.9. *Let* j: $Y^{11} \rightarrow S^{11}$ *be the map collapsing* $M^8 = (Y^{11})^{(9)}$ *to the base point. Then*

$$[Y^{11}, G_{2,b}] = Z\{\gamma'\} \oplus Z/2\{j^*\tau_1\} \quad \textit{with } j^*\gamma = 4\gamma'.$$

As we need a number of computations to prove (5.9), we shall delay the proof of (5.9) to the next section, and, in this section, first give a proof of (5.8) then a proof of (5.7) assuming (5.9).

Proof of Lemma 5.8. The homotopy group is computed in [14, Lemma 3.3]. Since $\pi_{11}(G_{2,b}) = \pi_{11}(A_b^{(11)})$, the index equals the order of ω_b, the attaching map of the top cell in $A_b^{(11)}$, which is $4b_2$ by (5.1) (b). Q.E.D.

Proof of Theorem 5.7. By (5.8), we have

$$d_3(\gamma)=0, \qquad d_{11}(\gamma)=4b_2.$$

Since $j^*: H^{11}(S^{11})\to H^{11}(Y^{11})$ is isomorphic, we then have

$$d_3(\gamma')=0, \qquad d_{11}(\gamma')=b_2.$$

Now, since $\pi_{13}(Y^{11})$ is already in the stable range,

$$G_{2,b}/G_{2,b}^{(6)}=Y^{11}\vee S^{14}$$

by a $G_{2,b}$ analogue of (4.9). Then there is a map $g: G_{2,b}\to Y^{11}$ with $d_3(g)=0$, $d_{11}(g)=1$. The composite

$$f=\gamma' g: G_{2,b}\longrightarrow G_{2,b}$$

is a solution to our construction explained below (5.7). Q.E.D.

§ 6. Proof of Lemma 5.9.

We recall the following cofibrations

$$S^n \xrightarrow{2\iota} S^n \xrightarrow{i} M^n \xrightarrow{p} S^{n+1} \qquad (n\geqq 1),$$

$$S^{n-1} \xrightarrow{\bar{\eta}} M^{n-3} \xrightarrow{i'} Y^n \xrightarrow{j} S^n \quad (n\geqq 6).$$

The second cofibration induces the exact sequence

$$[M^9, G_{2,b}]\xrightarrow{\bar{\eta}^*}\pi_{11}(G_{2,b})\xrightarrow{j^*}[Y^{11}, G_{2,b}]\xrightarrow{(i')^*}[M^8, G_{2,b}]\xrightarrow{\bar{\eta}^*}\pi_{10}(G_{2,b}).$$

The above groups except the middle one are computed by Mimura and Sawashita [14, Lemmas 3.3, 3.5]. Here we mention that the notation M^n in [13], [14] is different from ours; we write $M^n=S^n\cup_{2\iota}e^{n+1}$, while $M^n=S^{n-1}\cup_{2\iota}e^n$ in [13], [14]. From their results (with our notation for M^n),

$$\pi_{11}(G_{2,b})=Z\{\gamma\}\oplus Z/2\{\tau_1\},$$
$$[M^9, G_{2,b}]=Z/2\{\tau_2'\bar{\eta}\},$$
$$[M^8, G_{2,b}]=Z/4\{\tau_2\},$$
$$\pi_{10}(G_{2,b}) \text{ is an odd torsion group,}$$

where γ and τ_1 are denoted by $\langle 2\Delta\iota_{13}\rangle$ and $i_*[\nu_5^2]$, $\bar{\eta}$ is the element in (3.2) with suitable n, τ_2', denoted by $\langle\eta_6^2\rangle$ in [14], is a generator of $\pi_8(G_{2,b})=Z/2$ and τ_2 is an extension of τ_2', i.e., $\tau_2' i=\tau_2'$.

By (3.2) (c), $\tilde{\eta}^*[M^9, G_{2,b}]=0$. Hence we get a short exact sequence

$$0 \longrightarrow Z\{\gamma\} \oplus Z/2\{\tau_1\} \longrightarrow [Y^{11}, G_{2,b}] \longrightarrow Z/4\{\tau_2\} \longrightarrow 0.$$

The group extension at $[Y^{11}, G_{2,b}]$ is a 2-local problem because no odd torsion group is involved in the short exact sequence. Since $G_{2,b}$ is 2-equivalent to G_2, the above sequence is equivalent to

$$(6.1) \qquad \begin{array}{ccccccc} 0 \longrightarrow & \pi_{11}(A) & \xrightarrow{j^*} & [Y^{11}, A] & \xrightarrow{(i')^*} & [M^8, A] & \longrightarrow 0. \\ & \| & & & & \| & \\ & Z\{\gamma\} \oplus Z/2\{\tau_1\} & & & & Z/4\{\tau_2\} & \end{array}$$

For a CW complex W, we denote the exact sequence

$$\longrightarrow \pi_n(W) \xrightarrow{j^*} [Y^n, W] \xrightarrow{(i')^*} [M^{n-3}, W] \longrightarrow$$

by $[n, W]$, in particular, $(6.1)=[11, A]$.

Since A is homotopy equivalent to G_2, there is a well known fibration

$$SU(3) \xrightarrow{i''} A \xrightarrow{p''} S^6$$

classified by the generator $[2\iota] \in \pi_5(SU(3))=Z$ with $d_5([2\iota])=2$. To determine the group extension of (6.1), we examine the exact sequences $[11, SU(3)]$, $[11, S^6]$, $[10, SU(3)]$ in connection with (6.1) via the fibration.

Lemma 6.2. **(a)** *$\pi_{11}(SU(3)) = Z/4\{\tau_1'\}$, and $j^*: \pi_{11}(SU(3)) \rightarrow [Y^{11}, SU(3)]$ is epic, where $\tau_1=(i'')_*\tau_1'$.*

(b) *$\pi_{10}(SU(3))=Z/30\{\tau_3\}$, and the element $j^*\tau_3$ in $[Y^{10}, SU(3)]$ is of order* 15.

(c) *The image of $(i'')_*: [Y^{11}, SU(3)] \rightarrow [Y^{11}, A]$ is $Z/2$, generated by $j^*\tau_1=(i'')_*j^*\tau_1'$.*

Proof. The results on $\pi_i(SU(3))$ are obtained in [15], where τ_1' is denoted by $[\nu_5^2]$ and the 2-primary part of τ_3 by $[\nu_5\eta_8^2]$. By [12, Theorem 6.1], the generator $\tau_1 \in \pi_{11}(G_2)=\pi_{11}(A)$ in (6.1) is in the image of $(i'')_*$, that is, $\tau_1'=(i'')_*\tau_1'$. Then the part (c) is immediate from (a), (b). The sequences $[11, SU(3)]$ and $[10, SU(3)]$ are connected with $\tilde{\eta}^*: [M^8, SU(3)] \rightarrow \pi_{10}(SU(3))$. To show (a), (b), it is enough to show that the $\tilde{\eta}^*$ is an isomorphism of 2-primary part, because $[M^8, SU(3)]$ is a 2-group. Let

$$S^3 \xrightarrow{i_1} SU(3) \xrightarrow{p_1} S^5$$

be the usual S^3-bundle with characteristic element $\eta \in \pi_4(S^3)$. Consider the exact sequence

$$\xrightarrow{\partial}[M^8, S^3]\xrightarrow{(i_1)_*}[M^8, SU(3)]\xrightarrow{(p_1)_*}[M^8, S^5]\xrightarrow{\partial'},$$

where the boundary homomorphisms ∂, ∂' satisfy $\partial(\Sigma\alpha)=\eta\alpha$, $\alpha \in [M^8, S^4]$, $\partial'(\Sigma\alpha')=\eta\alpha'$, $\alpha' \in [M^7, S^4]$. We have, from the results on $\pi_i(S^3)$, $\pi_i(S^5)$, $i=8.\ 9$, in [21], that

$$[M^8, S^3]=Z/2\{\nu'\eta\bar{\eta}\},$$
$$[M^8, S^5]=Z/2\{\nu\eta p\}\oplus Z/2\{\eta^2\bar{\eta}\},$$

where ν' is a generator of the 2-primary part $Z/4$ of $\pi_6(S^3)$, and satisfies

(6.3) (a) $2\nu'=\eta^3$ *in* $\pi_6(S^3)$, (b) $\eta\nu=\nu'\eta$ *in* $\pi_7(S^3)$.

Then we have

$$\nu'\eta\bar{\eta}=\eta\nu\bar{\eta}=\partial(\nu\bar{\eta}) \in \partial[M^9, S^5],$$

hence $(i_1)_*=0$, and

$$\partial'(\nu\eta p)=\eta\nu\eta p=\nu'\eta^2 p, \qquad \partial'(\eta^2\bar{\eta})=\eta^3\bar{\eta}=\nu'(2\bar{\eta})=\nu'\eta^2 p$$

by unstable version of (3.2) (b). Since $\nu'\eta^2 \in \pi_8(S^3)$ can not be halved [21], $\nu'\eta^2 p\neq 0$, hence

$$\text{Ker }\partial'=Z/2\{\nu\eta p+\eta^2\bar{\eta}\},$$
$$(p_1)_*: [M^8, SU(3)]\longrightarrow \text{Ker }\partial' \quad \text{is isomorphic.}$$

We then get the commutative diagram

$$\begin{array}{ccc} [M^8, SU(3)] & \xrightarrow{\tilde{\eta}^*} & \pi_{10}(SU(3))=Z/30 \\ \cong\downarrow (p_1)_* & & \downarrow (p_1)_* \\ \text{Ker }\partial' & \xrightarrow{\tilde{\eta}^*} & \pi_{10}(S^5)=Z/2\{\nu\eta^2\}, \end{array}$$

where the upper $\tilde{\eta}^*$ is the one we are investigating. The lower $\tilde{\eta}^*$ is isomorphic, because

$$\begin{aligned}\tilde{\eta}^*(\nu\eta p+\eta^2\bar{\eta})&=\nu\eta p\tilde{\eta}+\eta^2\bar{\eta}\tilde{\eta}\\ &=\nu\eta^2+\eta^2(6\nu)=\nu\eta^2\end{aligned}$$

by (3.2) (a), (c). As in [15, Theorem 4.1], the right $(p_1)_*$ is an isomorphism of 2-primary part, hence, so is the upper $\tilde{\eta}^*$. Q.E.D.

Lemma 6.4. **(a)** $(i')^*\colon [Y^{10}, SU(3)] \to [M^7, SU(3)]$ *is an isomorphism of* 2*-primary component.*

(b) $[Y^{10}, SU(3)] = Z/30\{\tau_4\}$, $[M^7, SU(3)] = Z/2\{\tau_5\}$, *where*

$$j^*\tau_3 = 2\tau_4, \qquad (i')^*\tau_4 = \tau_5 = [2\iota]\nu p = i_1\nu'\bar{\eta}.$$

Proof. (a) By [15], $\pi_9(SU(3))$ is an odd torsion group, while $[M^7, SU(3)]$ is a 2-group. Hence $\tilde{\eta}^* = 0\colon [M^7, SU(3)] \to \pi_9(SU(3))$ and $(i')^*$ is epic. The result is then immediate from (6.2) (b).

(b) We compute $[M^7, SU(3)]$ in the different ways; one with the fibration of $SU(3)$ and the other with the cofibration of M^7. By [15], $\pi_7(SU(3)) = 0$ and $\pi_8(SU(3)) = Z/12\{[2\iota]\nu\}$. The exact sequence induced from the cofibration of M^7 then leads to $[M^7, SU(3)] = Z/2\{[2\iota]\nu p\}$. We next consider the exact sequence induced from the fibration:

$$[M^8, S^5] \xrightarrow{\partial'} [M^7, S^3] \xrightarrow{(i_1)_*} [M^7, SU(3)] \xrightarrow{(p_1)_*} [M^7, S^5] \xrightarrow{\partial''} [M^6, S^3],$$

where ∂', ∂'' satisfy $\partial'(\Sigma\alpha') = \eta\alpha'$, $\partial''(\Sigma\alpha') = \eta\alpha''$ for $\alpha' \in [M^7, S^4]$, $\alpha'' \in [M^6, S^4]$ with $\eta \in \pi_4(S^3)$. From the results on $\pi_i(S^3)$, $\pi_i(S^5)$, $i =$ 6, 7, 8, 9, in [21], we obtain

$$[M^8, S^5] = Z/2\{\nu\eta p\} \oplus Z/2\{\eta^2\bar{\eta}\}, \qquad [M^7, S^3] = Z/4\{\nu'\bar{\eta}\},$$
$$[M^7, S^5] = Z/2\{\nu p\} \oplus Z/2\{\eta\bar{\eta}\}, \qquad [M^6, S^3] = Z/2\{\nu'\eta p\} \oplus Z/2\{\eta^2\bar{\eta}\}.$$

In particular, by (3.2) (b), (6.3) (a), the following relations hold.

$$(6.5) \qquad 2\nu'\bar{\eta} = \eta^3\bar{\eta} = \nu'\eta^2 p \qquad \textit{in } [M^7, S^3].$$

We then have

$$\partial'(\nu\eta p) = \eta\nu\eta p = \nu'\eta^2 p = 2\nu'\bar{\eta}, \qquad \partial'(\eta^2\bar{\eta}) = \eta^3\bar{\eta} = 2\nu'\bar{\eta},$$
$$\partial''(\nu p) = \eta\nu p = \nu'\eta p, \qquad \partial''(\eta\bar{\eta}) = \eta^2\bar{\eta},$$

by (6.3) (b) and (6.5). Therefore $\operatorname{Coker}\partial' = Z/2\{\nu'\bar{\eta}\}$ and $\operatorname{Ker}\partial'' = 0$, proving $[M^7, SU(3)] = Z/2\{i_1\nu'\bar{\eta}\}$.

The odd primary part of $[Y^{10}, SU(3)]$ is isomorphic to the one of $\pi_{10}(SU(3))$ via j^*, while the 2-primary part is $Z/2$. Hence $[Y^{10}, SU(3)] = Z/30\{\tau_4\}$ with $j^*\tau_3 = 2\tau_4$. The last relation $(i')^*\tau_4 = \tau_5$ is immediate from (a). Q.E.D.

We next compute the exact sequence $[11, S^6]$.

Lemma 6.6. **(a)** *The sequence* $[11, S^6]$ *is short exact, where the marginal terms are*

$$\pi_{11}(S^6)=Z\{[\iota_6, \iota_6]\}, \qquad [M^8, S^6]=Z/2\{\eta\bar{\eta}\}\oplus Z/2\{\nu p\}.$$

($[\iota_6, \iota_6]$ *denotes the Whitehead square*).

(b) $[Y^{11}, S^6]=Z\{\gamma''\}\oplus Z/2\{\Sigma\tau_6\}$,

$$j^*[\iota_6, \iota_6]=2\gamma'', \qquad (i')^*\gamma''=\nu p, \qquad (i')^*(\Sigma\tau_3)=\eta\bar{\eta}.$$

Proof. (a) This is clear from the table of $\pi_i(S^6)$ in [21], since $\pi_{10}(S^6)=0$ and $[M^9, S^6]$ is finite.

(b) We first compute $[10, S^5]$, $[9, S^5]$. Extending the lower $\tilde{\eta}^*$ to the right, in the commutative diagram in the proof of (6.2), we see that

$$(i')^*: [Y^{10}, S^5]\longrightarrow[M^7, S^5]$$

is monic. As

$$[M^7, S^5]=Z/2\{\eta\bar{\eta}\}\oplus Z/2\{\nu p\}, \qquad \pi_9(S^5)=Z/2\{\nu\eta\},$$

$$\eta\bar{\eta}\tilde{\eta}=0 \quad \text{and} \quad \nu p\tilde{\eta}=\nu\eta \quad \text{in } \pi_9(S^5),$$

the image of $(i')^*$ is $Z/2\{\eta\bar{\eta}\}$. Hence

$$[Y^{10}, S^5]=Z/2\{\tau_6\}, \qquad (i')^*\tau_6=\eta\bar{\eta}.$$

In a similar computation, we also have

$$[Y^9, S^5]=0.$$

We next study the *EHP* exact sequence

$$[Y^{10}, S^5]\xrightarrow{\Sigma}[Y^{11}, S^6]\xrightarrow{H}[Y^{11}, S^{11}]\xrightarrow{P}[Y^9, S^5]=0$$

to know $[Y^{11}, S^6]$. Since $\eta\bar{\eta}$ is still non-trivial in $[M^8, S^5]$, $\Sigma\tau_6$ is non-trivial and generates $\operatorname{Im}\Sigma=Z/2$. Clearly, $[Y^{11}, S^{11}]=Z\{j\}$. Therefore there is an element γ'' with $H(\gamma'')=j$ for which

$$[Y^{11}, S^6]=Z\{\gamma''\}\oplus Z/2\{\Sigma\tau_6\}.$$

The Hopf invariant of $[\iota_6, \iota_6]$ is known to be $2\iota_{11}$, hence

$$H(j^*[\iota_6, \iota_6])=2j,$$
$$j^*[\iota_6, \iota_6]\equiv 2\gamma'' \bmod \Sigma\tau_6.$$

Clearly $(i')^*j^*[\iota_6, \iota_6]=(ji')^*[\iota_6, \iota_6]=0$. Since $[M^8, S^6]$ is a $Z/2$-module, $(i')^*(2\gamma'')=0$. Therefore $j^*[\iota_6, \iota_6]=2\gamma''$ because $(i')^*(\Sigma\tau_6)=\eta\bar{\eta}\neq 0$. As $(i')^*: [Y^{11}, S^6]\rightarrow[M^8, S^6]$ is epic, there is a choice of γ'' which satisfies $(i')^*\gamma''=\nu p$, keeping the other relations. Q.E.D.

Now, we are ready to prove Lemma 5.9.

Proof of Lemma 5.9. The commutative diagram of exact sequences

$$[11,\ SU(3)]\xrightarrow{(i'')_*}[11,\ A]\xrightarrow{(p'')_*}[11,\ S^6]\xrightarrow{\Delta}[10,\ SU(3)],$$

where the boundary homomorphism Δ satisfies $\Delta\Sigma=[2\iota]_*$, becomes, by previous computations, as follows:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & Z/4\{\tau_1'\} & \xrightarrow{j^*} & Z/2\{j^*\tau_1'\} & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle (i'')_*} & & \downarrow{\scriptstyle (i'')_*} & & & & \\
0 & \longrightarrow & Z\{\gamma\}\oplus Z/2\{\tau_1\} & \xrightarrow{j^*} & [Y^{11},\ A] & \xrightarrow{(i')^*} & Z/4\{\tau_2\} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (p'')_*} & & \downarrow{\scriptstyle (p'')_*} & & \downarrow{\scriptstyle (p'')_*} & & \\
0 & \longrightarrow & Z\{[\iota_6,\ \iota_6]\} & \xrightarrow{j^*} & Z\{\gamma''\}\oplus Z/2\{\Sigma\tau_6\} & \xrightarrow{(i')^*} & Z/2\{\eta\bar{\eta}\}\oplus Z/2\{\nu p\} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \\
 & & Z/30\{\tau_3\} & \xrightarrow{j^*} & Z/30\{\tau_4\} & \xrightarrow{(i')^*} & Z/2\{\tau_5\} & \longrightarrow & 0.
\end{array}$$

Here we have also proved

$$(i'')_*\tau_1'=\tau_1,$$
$$j^*[\iota_6,\ \iota_6]=2\gamma'',\qquad (i')^*\gamma''=\nu p,\qquad (i')^*(\Sigma\tau_6)=\eta\bar{\eta},$$
$$j^*\tau_3=2\tau_4,\qquad (i')^*\tau_4=\tau_5=[2\iota]\nu p=i_1\nu'\bar{\eta}.$$

Since $\pi_{10}(A)=\pi_{10}(G_2)=0$ as before, the left Δ is epic, hence

$$(p'')_*\gamma=30[\iota_6,\ \iota_6].$$

The right Δ is computed as follows:

$$\Delta(\eta\bar{\eta})=[2\iota]\eta\bar{\eta}=i_1\nu'\bar{\eta}=\tau_5,\qquad \Delta(\nu p)=[2\iota]\nu p=\tau_5,$$

because of the formula $\Delta(\Sigma\alpha)=[2\iota]\alpha$ and the relation $\Delta\eta=i_1\nu'$ in [12, Proposition 6.3 with $\alpha=\eta_6$]. Therefore

$$(p'')_*(\tau_2)=\eta\bar{\eta}+\nu p,\qquad \Delta(\gamma'')=\tau_4,\qquad \Delta(\Sigma\tau_6)=15\tau_4.$$

Then we can find an element $\gamma'\in[Y^{11},\ A]$ with

$$(p'')_*\gamma'=15\gamma''+\Sigma\tau_6,\qquad (i')^*\gamma'=\tau_2,$$
$$[Y^{11},\ A]=Z\{\gamma'\}\oplus Z/2\{j^*\tau_1\},$$

where we may replace τ_2 by $-\tau_2$ if necessary. Then

$$(p'')_* j^* \gamma = 60\gamma'' = 4(p'')_* \gamma' \quad \text{and} \quad j^* \gamma \equiv 4\gamma' \bmod \mathrm{Im}\,(i'')_*.$$

We may replace γ by $\gamma + \tau_1$ to get the exact relation $j^* \gamma = 4\gamma'$. Q.E.D.

§ 7. Self-H-maps of $G_{2,b}$

Let μ: $G_{2,b} \times G_{2,b} \to G_{2,b}$, $-2 \leqq b \leqq 5$, be a continuous multiplication on $G_{2,b}$ as H-space, where, of course, only the existence of the unit is assumed for μ. Let $[G_{2,b}, G_{2,b}]_\mu$ (resp. $\mathscr{E}(G_{2,b})$) denote the set of homotopy classes of those maps f: $G_{2,b} \to G_{2,b}$ such that $\mu(f \times f)$ is homotopic to $f\mu$, that is, f is an H-map with respect to μ (resp. f is a homotopy equivalence). We put

$$\mathscr{E}_H(G_{2,b}; \mu) = [G_{2,b}, G_{2,b}]_\mu \cap \mathscr{E}(G_{2,b}).$$

They are closed under the composition of maps, and both of $\mathscr{E}(G_{2,b})$, $\mathscr{E}_H(G_{2,b}; \mu)$ become groups.

The group $\mathscr{E}(G_{2,b})$ was determined, up to extension, by Mimura and Sawashita [14], and, in case $-1 \leqq b \leqq 5$, we settled in [17] the group extension. The recent work of Sawashita [20] states that, for $-2 \leqq b \leqq 5$ and for arbitrary multiplication μ, $\mathscr{E}_H(G_{2,b}; \mu)$ is at most the cyclic group of order 2, where the case of cyclic group of order 2 occurs only if there exists $f \in \mathscr{E}_H(G_{2,b}; \mu)$ with $d_3(f) = -1$, $d_{11}(f) = \pm 1$, $f^2 = \mathrm{id}$.

The purpose of this section is to eliminate such a case in case $-1 \leqq b \leqq 5$ with estimating the image of the degree map on $[G_{2,b}, G_{2,b}]_\mu$ in the same method used in [10], [11]. Our result is stated as follows:

Theorem 7.1. *For $-2 \leqq b \leqq 5$ and for arbitrary multiplication μ,*

$$d[G_{2,b}, G_{2,b}]_\mu \subset \{(m, m+nb_2) \mid m, n \in \boldsymbol{Z},\ m \equiv 0, 1 \bmod 4 \text{ if } n \text{ is even}\}.$$

Proof. Let P be the projective plane of the H-space $G = G_{2,b}$ with multiplication μ. P is a cofibre of the Hopf construction H: $\Sigma G \wedge G \simeq G * G \to \Sigma G$ of μ and we have the cofibration

$$\Sigma G \wedge G \xrightarrow{H} \Sigma G \xrightarrow{i} P \xrightarrow{j} \Sigma^2 G \wedge G.$$

We mention that the Künneth formula holds for the K-group [3], $K^*(G)$ then becomes a primitively generated Hopf algebra and that H is the reduced comultiplication map for $K^*(G)$ via suspension isomorphism. We may therefore conclude that $\tilde{K}(P)$ is a free $\boldsymbol{Z}$-module with basis $\{\alpha, \beta, \alpha^2, \alpha\beta, \beta^2, \gamma\}$, where

$$i^*\alpha=\tilde{\xi},\quad i^*\beta=\tilde{\eta},\quad \gamma=j^*(\sigma^2(\tilde{\xi}\tilde{\eta}\otimes\tilde{\xi}\tilde{\eta})),\quad \beta\alpha=\alpha\beta,$$
$$\alpha^3=0,\quad \alpha^2\beta=0,\quad \alpha\beta^2=0,\quad \beta^3=0,\quad \alpha\gamma=\beta\gamma=\gamma\alpha=\gamma\beta=\gamma^2=0.$$

From the Chern character formula in (5.3),

$$\psi^2\tilde{\xi}=4\tilde{\xi}-2(8b+1)\tilde{\eta},\qquad \psi^2\tilde{\eta}=64\tilde{\eta}.$$

Therefore we may put

$$\text{(7.2)}\qquad \begin{aligned}\psi^2\alpha&\equiv 2\beta+u\alpha^2+v\alpha\beta+w\beta^2+x\gamma &&\mod 4,\\ \psi^2\beta&\equiv u'\alpha^2+v'\alpha\beta+w'\beta^2+x'\gamma &&\mod 4,\end{aligned}$$

for some integers u, v, w, x, u', v', w', x'. We also have

$$\text{(7.2)}'\qquad \psi^2\alpha^2\equiv 0,\quad \psi^2\alpha\beta\equiv 0,\quad \psi^2\beta^2\equiv 0,\quad \psi^2\gamma\equiv 0\quad \mod 4.$$

Because of $\psi^2\alpha\equiv\alpha^2 \mod 2$, $\psi^2\beta\equiv\beta^2 \mod 2$,

(7.3) *u and w' are odd, and the other coefficients in* (7.2) *are all even.*

Now let $f\colon G\to G$ be an H-map with $d_3(f)=m$, $d_{11}(f)=m+nb_2$, $m, n\in Z$. By (5.6),

$$f^*\tilde{\xi}=m\tilde{\xi}-nb_1\tilde{\eta},\qquad f^*\tilde{\eta}=(m+nb_2)\tilde{\eta}.$$

Since f is an H-map, it defines a map $g\colon P\to P$ with commutative diagram

$$\begin{array}{ccccc}\Sigma G & \xrightarrow{i} & P & \xrightarrow{j} & \Sigma^2G\wedge G\\ \downarrow{\scriptstyle \Sigma f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \Sigma^2 f\wedge f}\\ \Sigma G & \xrightarrow{i} & P & \xrightarrow{j} & \Sigma^2G\wedge G.\end{array}$$

Then,

$$\text{(7.4)}\qquad \begin{aligned}&g^*\alpha\equiv m\alpha-nb_1\beta \quad \mod \alpha^2,\ \alpha\beta,\ \beta^2,\ \gamma,\\ &g^*\beta\equiv(m+nb_2)\beta \quad \mod \alpha^2,\ \alpha\beta,\ \beta^2,\ \gamma,\\ &g^*\alpha^2=m^2\alpha^2-2mnb_1\alpha\beta+n^2b_1^2\beta^2,\\ &g^*\alpha\beta=m(m+nb_2)\alpha\beta-n(m+nb_2)b_1\beta^2,\\ &g^*\beta^2=(m+nb_2)^2\beta^2,\quad g^*\gamma=m^2(m+nb_2)^2\gamma.\end{aligned}$$

We compare the coefficients modulo 4 of β^2 in $\psi^2g^*\beta$ and in $g^*\psi^2\beta$. Notice that, by (5.4), $b_2\equiv 2 \mod 4$, $b_1\equiv\pm 1 \mod 4$. Let D be the subgroup generated by α, β, α^2, $\alpha\beta$, $4\beta^2$, γ. Then

$$\psi^2g^*\beta\equiv(m+nb_2)\psi^2\beta\equiv(m+2n)w'\beta^2 \quad \mod D,$$

$$g^*\psi^2\beta \equiv u'g^*\alpha^2+v'g^*\alpha\beta+w'g^*\beta^2+x'g^*\gamma \equiv (n^2u'-mnv'+m^2w')\beta^2 \bmod D,$$

by (7.4), (7.2)′, (7.2) and (7.3). Hence

$$(m^2-m)w' \equiv 2nw'+n(-nu'+mv') \quad \bmod 4.$$

By (7.3),

$$m^2-m \equiv 2nt \bmod 4 \qquad \text{for some } t \in Z.$$

If n is even, then $m^2-m\equiv 0 \bmod 4$ and $m\equiv 0, 1 \bmod 4$. Q.E.D.

Remark 7.4. When $b=0$, i.e., $G_{2,b}=G_2$, and μ is the Lie group multiplication, the Chern character formula for P is the image of the one for BG_2 with D^3 vanishing, where D^3 is the subgroup of decomposable elements of three or more factors in $H^*(BG_2; Q)$. We may continue the computation of the Chern character given above (4.14), to get ch for BG_2 modulo D^3. We may consequently determine ψ^2 for P in this case. The result is

$$\psi^2\alpha=4\alpha-2\beta+\alpha^2, \qquad \psi^2\beta=64\beta-12\alpha^2+12\alpha\beta+\beta^2,$$

from which we can obtain more estimate; the result is complicated and we omit the details.

Corollary 7.5. *For* $-1\leqq b\leqq 5$ *and for arbitrary multiplication* μ,

$$d(\mathscr{E}_H(G_{2,b};\mu))=\{(1, 1)\}.$$

Proof. Since a homotopy equivalence induces an isomorphism of the cohomology,

$$(7.6) \qquad d(\mathscr{E}(G_{2,b}))\subset\{(1, 1), (1, -1), (-1, 1), (-1, -1)\}.$$

Let $f\in\mathscr{E}_H(G_{2,b};\mu)$ with $d_3(f)=m$, $d_{11}(f)=m+nb_2$. By (7.6),

$$nb_2=d_{11}(f)-d_3(f)=0 \qquad or\ \pm 2.$$

If $b_2>2$, which is equivalent to $-1\leqq b\leqq 5$ by (5.4), then $n=0$, hence by (7.1), $m\equiv 0, 1 \bmod 4$. Again by (7.6), $m=1$. Q.E.D.

The recent result of Sawashita [20, Proposition 5.6] states that, for $-2\leqq b\leqq 5$ and for any μ, the map $G_{2,b}\rightarrow K(Z, 3)$, which kills all the homotopy groups except π_3, induces a monomorphism

$$\mathscr{E}_H(G_{2,b};\mu)\longrightarrow\mathscr{E}_H(K(Z, 3);\mu_K).$$

The multiplication μ_K on the Eilenberg-Maclane space $K(Z, 3)$ is unique and $\mathscr{E}_H(K(Z, 3); \mu_K)$ is a cyclic group of order 2 with generator g acting non-trivially on π_3, H_3 and H^3. If there were a lift h of g to $\mathscr{E}_H(G_{2,b}; \mu)$, the action of h on H^3 must be non-trivial, which is impossible in case $-1 \leqq b \leqq 5$ by (7.5). In consequence, we have obtained

Theorem 7.7. *For* $-1 \leqq b \leqq 5$ *and for arbitrary multiplication* μ,

$$\mathscr{E}_H(G_{2,b}; \mu) = \{\mathrm{id}\}.$$

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