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On 3-dimensional Bounded Cohomology of Surfaces

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§ 1. Introduction

In [3], Gromov introduced the notion of the bounded cohomology $H_b^*(M, \boldsymbol{R})$ of a manifold M. This is the cohomology of the complex of singular cochains ϕ which have the property:

There exists a constant c such that $|\phi(\sigma)|<c$ for any singular simplex σ.

Let S be a closed oriented surface of genus $\geqq 2$. In [1] and [5], it is shown that $H_b^2(S, \boldsymbol{R})$ is infinitely generated.

In this paper, we shall show

Theorem 1. *$H_b^3(S, \boldsymbol{R})$ is infinitely generated.*

Our method is an application of Thurston's theory of pleated (uncrumpled) surfaces in hyperbolic 3-manifolds ([7]).

§ 2. A construction of elements of $H_b^3(S, \boldsymbol{R})$

For a convenience, we choose and fix a complete hyperbolic structure on S.

Let f be a pseudo Anosov diffeomorphism of S. Let M_f be the mapping torus of f. It is the identification space obtained from $S\times[0, 1]$ by equivalence relation $(x, 0)\sim(f(x), 1)$ $(x \in S)$. M_f admits a complete hyperbolic structure which is unique up to isometry ([6]). The projection onto the second factor $S\times[0, 1]\to[0, 1]$ induces a fibering $p: M_f\to S^1$. Let $\tilde{M}_f$ be the infinite cyclic regular covering space of M_f defined by the pull-back by p of $e: \boldsymbol{R}\to S^1$, where $e(t)=\exp 2\pi\sqrt{-1}\,t$, $t \in \boldsymbol{R}$. The hyperbolic structure on M_f can be lifted to the hyperbolic structure on $\tilde{M}_f$. There is a natural inclusion $S\times[0, 1)\subset\tilde{M}_f$ and let $j: S\to\tilde{M}_f$ be the embedding defined by $j(x)=(x, 0) \in S\times[0, 1)\subset\tilde{M}_f$.

Let $\varDelta$ be the standard 3-simplex in $\boldsymbol{R}^4$. Let $\sigma: \varDelta\to S$ be a singular 3-simplex of S. Then $j\sigma: \varDelta\to\tilde{M}_f$ is a singular 3-simplex of $\tilde{M}_f$. The universal covering space of $\tilde{M}_f$ is isometric to the hyperbolic 3-space H^3,

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and there is a covering projection $q: H^3 \to \tilde{M}_f$. There is a map $\widetilde{j\sigma}: \Delta \to H^3$ such that $q\widetilde{j\sigma} = j\sigma$. Let straight $(j\sigma)$ be the geodesic 3-simplex in H^3 with the same vertices as $\widetilde{j\sigma}$. The isometry class of straight $(j\sigma)$ depends only on $j\sigma$. We define a singular 3-cochain ϕ_f of S by

$$\phi_f(\sigma) = \varepsilon \operatorname{vol}(\text{straight}\,(j\sigma)),$$

for each 3-simplex σ, where vol denotes the hyperbolic volume and $\varepsilon = +1$ if $\widetilde{j\sigma}$ maps Δ into H^3 orientation preservingly and $\varepsilon = -1$ otherwise. Since the volume of geodesic 3-simplices in H^3 has a finite upper bound ([7]), ϕ_f defines a bounded 3-cocycle of S.

§ 3. Linear independence of ϕ_f

Let Λ be the space of all the geodesic laminations on S with geometric topology ([7] § 8). Λ is compact. Any homeomorphism of S induces a homeomorphism of Λ. For a pseudo Anosov diffeomorphism f of S, there are two mutually transverse geodesic laminations λ_f^s and λ_f^u such that they are invariant by f, and for each simple closed geodesic γ on S, $f^k(\gamma) \to \lambda_f^s$ and $f^{-k}(\gamma) \to \lambda_f^u$ as $k \to +\infty$ ([2] [7]). λ_f^s and λ_f^u are called as the stable and the unstable geodesic lamination of f respectively.

Let T be a (not simplicial) triangulation of S such that it contains a simple closed geodesic γ and it has only one vertex lying on γ. Let τ_γ be the Dehn twist along γ. Let $T_n = \tau_\gamma^n T$ be the triangulation of S which is the image of T by τ_γ^n for each non-negative integer n $(T_0 = T)$. Let T_∞ be the ideal traingulation of S which is the limit of T_n as $n \to \infty$.

Let c, $c_n = \tau_{\gamma *}^n c$ and $c_\infty = \lim c_n$ be the singular 2-chains of S associated to T, T_n and T_∞ respectively which represent the fundamental class of S.

Since $f_* c_n$ is homologous to c_n, there is a singular 3-chain d_n such that $\partial d_n = f_* c_n - c_n$. We define a sequence of singular 3-chains of S by

$$D_n(f)_k = \sum_{i=-k}^{k} f_* d_n,$$

for $k = 1, 2, \cdots$ and $n = 0, 1, \cdots, \infty$. Then $\partial D_n(f)_k = f_*^{k+1} c_n - f_*^{-k} c_n$.

Proposition 1. *Let f and g be two pseudo Anosov diffeomorphisms of S. Let λ_f^s, λ_f^u, λ_g^s and λ_g^u be the stable and the unstable geodesic laminations of f and g respectively. If none of λ_f^s and λ_f^u coincides with any of λ_g^s and λ_g^u, then ϕ_f and ϕ_g are linearly independent in $H_b^3(S, \boldsymbol{R})$.*

Proof. Let $j_f: S \to \tilde{M}_f$ and $j_g: S \to \tilde{M}_g$ be the embeddings given in Section 2. For each n and k, the image of the 3-chain $j_f(D_n(f)_k)$ under the projection $\tilde{M}_f \to M_f$ gives a singular 3-chain of M_f representing $(2k+1)$-times the fundamental class of M_f. Hence we have $\phi_f(D_n(f)_k)$

$=(2k+1)$ vol (M_f) by definition of ϕ_f. In particular,

$$\lim_{k\to\infty}\frac{1}{2k+1}\phi_f(D_\infty(f)_k)=\mathrm{vol}\,(M_f).$$

Next we consider $\phi_f(D_\infty(g)_k)$. Projecting the chain of the ideal geodesic simplices, straight $(j_f(D_\infty(g)_k))$, from H^3 to $\tilde{M}_f$, we may consider straight $(j_f(D_\infty(g)_k))$ as an ideal singular 3-chain of $\tilde{M}_f$. The boundary, ∂ straight $(j_f(D_\infty(g)_k))$, consists of two pleated surfaces S_k and S_{-k} which are the straightenings of the ideal triangulations $g^{k+1}T_\infty$ and $g^{-k}T_\infty$ of S in $\tilde{M}_f$ respectively. The bending locus $b(S_k)$ (resp. $b(S_{-k})$) of S_k (resp. S_{-k}) is the geodesic lamination which is the straightening of the ideal 1-simplices of $g^{k+1}T_\infty$ (resp. $g^{-k}T_\infty$). Since T_∞ contains a simple closed geodesic γ, $b(S_k)$ (resp. $b(S_{-k})$) converges in Λ to a geodesic lamination λ_+ (resp. λ_-) which contains λ_g^s (resp. λ_g^u) as $k\to\infty$. By assumption, none of λ_+ and λ_- contains any of λ_f^s and λ_f^u. By Thurston's realization theorem of geodesic laminations in $\tilde{M}_f$ ([7] 9.3.10), there exist two pleated surfaces S_+ and S_- in $\tilde{M}_f$ whose bending laminations are λ_+ and λ_- respectively. Since T_∞ is an ideal triangulation of S, both of $S-\lambda_+$ and $S-\lambda_-$ consist of finite ideal triangles. Hence S_+ and S_- are uniquely determined, and the pleated surfaces S_k and S_{-k} converge to S_+ and S_- respectively as $k\to\infty$ ([7] 9.5.6, 7). Therefore $\phi_f(D_\infty(g)_k)$ converges to the volume of the compact region bounded by S_+ and S_- in $\tilde{M}_f$ as $k\to\infty$, and it is bounded. Hence,

$$\lim_{k\to\infty}\frac{1}{2k+1}\phi_f(D_\infty(g)_k)=0.$$

Exchanging f and g, we have

$$\lim_{k\to\infty}\frac{1}{2k+1}\phi_g(D_\infty(f)_k)=0 \quad \text{and}$$

$$\lim_{k\to\infty}\frac{1}{2k+1}\phi_g(D_\infty(g)_k)=\mathrm{vol}\,(M_g).$$

Now suppose that $a\phi_f+b\phi_g=0$ in $H_b^3(S, \boldsymbol{R})$ for some $a, b\in\boldsymbol{R}$ and $ab\neq 0$. Then $a\phi_f+b\phi_g=\delta\omega$ for some bounded 2-cochain ω of S. For each $0\leqq n<+\infty$,

$$(a\phi_f+b\phi_g)(D_n(f)_k)=(\delta\omega)(D_n(f)_k)=\omega(f_*^{k+1}c_n)-\omega(f_*^{-k}c_n).$$

As ω is bounded and both of $f_*^{k+1}c_n$ and $f_*^{-k}c_n$ are sums of a constant number of simplices for each k, it follows

$$\lim_{k\to\infty}\frac{1}{2k+1}(a\phi_f+b\phi_g)(D_n(f)_k)=0.$$

Since ϕ_f and ϕ_g are continuous cochains, we have

$$\lim_{k\to\infty}\frac{1}{2k+1}(a\phi_f+b\phi_g)(D_\infty(f)_k)=0.$$

Replacing $D_\infty(f)_k$ by $D_\infty(g)_k$, the same equality holds. However this contradicts to the above facts. q.e.d.

The above proposition can be generalized in straightforward way as follows,

Proposition 2. *Let $f_1, \cdots, f_m$ be pseudo Anosov diffeomorphisms of S. If the stable and the unstable geodesic laminations of $f_1, \cdots, f_m$ are all distinct from each other, then $\phi_{f_1}, \cdots, \phi_{f_m}$ are linearly independent in $H_b^3(S, \boldsymbol{R})$.*

Now let α and β be two simple closed curves on S such that $S-(\alpha\cup\beta)$ is a disjoint union of open 2-discs. Then $f_m=\tau_\alpha^m\tau_\beta^{-m}$ is a pseudo Anosov diffeomorphism of S for each positive integer m ([8]). In [4], Masur calculates the stable and unstable geodesic laminations λ_m^s and λ_m^u of f_m (in terms of measured foliations and quadratic differentials), and it is shown that $\lambda_m^s\to\alpha$ and $\lambda_m^u\to\beta$ as $m\to\infty$. Hence we may choose an infinite family $\{f_m\}$ of pseudo Anosov diffeomorphisms such that each finite subset of $\{f_m\}$ satisfies the condition of Proposition 2. This proves Theorem 1.

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