Advanced Studies in Pure Mathematics 9, 1986
Homotopy Theory and Related Topics
pp. 41–55

Unique Triangulation of the Orbit Space of a Differentiable Transformation Group and its Applications

Takao Matumoto and Masahiro Shiota

Dedicated to the memory of Shichirô Oka

Introduction

Let G be a compact Lie group throughout this paper. We consider a paracompact differentiable manifold M of class C^k and dimension m with a differentiable G-action $G\times M\to M$ of class C^k, which we call a C^k G-manifold.

We shall see that a differentiable (i.e., C^k with $1\leqq k\leqq\infty$) G-manifold M is equivariantly diffeomorphic to a real analytic (i.e., C^ω) G-manifold (Theorem 1.3). A C^ω equivariant smoothing is "uniquely" determined (Theorems 1.2–1.2′): unique up to C^ω equivariant diffeomorphism if M is compact or more generally M has only a finite number of orbit types and unique up to subanalytic C^1 equivariant diffeomorphism in general. We use here the equivariant embedding theorem for real analytic G-manifold with finite orbit types in a finite dimensional linear representation space (Theorem 1.1).

Reviewing the notion of subanalytic sets and maps defined by Hironaka [H1] in Section 2, we treat the real analytic G-manifolds in Section 3. A natural subanalytic set structure is introduced on the orbit space (Theorem 3.1) and the stratification filtered by orbit types is subanalytic (Lemma 3.2). So, we have a unique triangulation of the orbit space which is compatible with the subanalytic set structure and consequently with the orbit type decomposition in the sense that two such triangulations have a common subanalytic and combinatorial subdivision (Theorem 3.3) using the results of [SY].

Combining these results we get a unique triangulation of the orbit space also for any differentiable G-manifold M. Notice that the orbit space of a differentiable G-manifold M with boundary is nothing but that of the differentiable $G\times Z_2$-manifold DM, where DM is the double of M;

Received February 13, 1985.

our argument applies also to the G-manifold with boundary.

Lifting each simplex in the barycentric subdivision of a triangulation compatible with the orbit type decomposition in the manner that the isotropy subgroups are constant in the image of each open simplex, we get a G-CW complex structure on M (Theorem 4.1). (See [M1] and [I3].) The uniqueness of the triangulation of the orbit space implies a kind of uniqueness of G-CW complex structure on M in the sense of Theorem 4.2. This means in particular that a compact differentiable G-manifold (with or without boundary) has a well-defined equivariant simple homotopy type in the sense of Illman [I1] (Corollary 4.2). This seems to have applications in equivariant differential topology.

We add some history about the works related to (unique) triangulation. A C^1 manifold is uniquely (modulo isotopy) triangulated by Whitehead [Wh] supplementing the work of Cairns. Before this an analytic set and hence a C^ω manifold in $\boldsymbol{R}^n$ is uniquely triangulated in the sense that two analytic triangulations have a common analytic subdivision by Koopman-Brown and Lefschetz-Whitehead following the algebraic triangulation of an algebraic set by van der Waerden. (See [H2] and [S].) Any C^ω manifold is proved to be embeddable in $\boldsymbol{R}^n$ much later by Morrey (for the compact case) and Remmert-Grauert. (See [Sh] or [Hi].) The combinatorial uniqueness follows from the fact that analytic triangulation of a simplex is combinatorially equivalent to a simplex, which is recently proved by Shiota-Yokoi [SY]. Note also that the unique C^1 equivariant triangulation and hence the unique triangulation of the orbit space are given by Illman [I2] when G is a finite group. We see that it is much easier to get a C^ω equivariant triangulation in this case, because we have a stronger uniqueness property.

For a compact Lie group of positive dimension we know that the orbit space of a differentiable G-manifold has a Thom-Mather stratification defined by the orbit type decomposition and hence a triangulation compatible with it. But, since two triangulations compatible with a stratification neither have a common subdivision nor are combinatorially equivalent in general, this is not enough to our present purpose. Also triangulability of a Thom-Mather stratification was a more delicate problem than that of a subanalytic set. We introduce some references to it, for "Theorem" of C.T. Yang [Y] was considered to be remedied by them. It is a good way to quote Verona's paper [V] for the triangulability of a stratified set in the sense of Thom-Mather, because it is the first published paper with a rigorous and detailed proof of a reasonable length. We may recommend also Johnson [J] for the case of a compact stratified set, because this is a sophisticated version of his thesis (in 1972) which was the first written paper about it. The proof given in Kato [K] of the triangulability of a Whitney stratification has still worth to understand; he claimed that the proof follows from the weak homotopy equivalence $PL(\nu)\to PD(\nu)$, where ν is a system of tubular neighborhoods with fixed control data and $PL(\nu)$ (or $PD(\nu)$) is the complete semi-simplicial Kan group complex of PL (or PD) automorphisms of ν.

We hope that the state of existence of a G-CW complex structure on a smooth G-manifold becomes clear by this paper especially by this introduction.

§ 1. On C^k equivariant smoothing of C^r G-manifolds $(1\leqq r<k\leqq\omega)$

We use 'class C^ω' for an abbreviation of 'real analytic'. Recall that Lie group G has a unique C^ω structure for which the map $G\times G\to G$ taking $(g, h)\mapsto gh^{-1}$ is of class C^ω. A manifold M of class C^k with action $G\times M\to M$ of class C^k is called a C^k G-manifold. We can remark that C^0 G-manifold is a C^k G-manifold if each homeomorphism θ_g: $M\to M$ taking $x\mapsto gx$ is of class C^k (see [MZ]).

Noticing that C^r G-manifolds with finite orbit types (i.e., finite non-conjugate isotropy subgroups) are properly C^r equivariantly embeddable in some linear G-space (Theorem 1.1), we shall show first that such C^r G-manifolds are uniquely C^k equivariantly smoothable and then applied this result to the case of G-manifolds with infinite orbit types. Palais [P2] proved Theorems 1.1–1.3 for the case that M is compact and $k\leqq\infty$. We remark also that a C^1 G-manifold has locally finite orbit types and in particular, a compact G-manifold has only a finite number of inequivalent orbit types. (See, e.g. [B].)

Theorem 1.1. *Let G be a compact Lie group and M a C^k G-manifold with a finite number of orbit types $(1\leqq k\leqq\omega)$. Then, there exists a proper C^k equivariant embedding of M in some real linear representation space W of G.*

Theorem 1.2. *Let G be a compact Lie group and let M and N be C^k G-manifolds $(2\leqq k\leqq\omega)$. When $k=\omega$, we assume that N has a finite number of orbit types. If M and N are C^r equivariantly diffeomorphic $(1\leqq r<k\leqq\omega)$, then they are C^k equivariantly diffeomorphic. In fact, any C^r equivariant map f: $M\to N$ can be approximated arbitrarily well in the Whitney C^r topology by a C^k equivariant map.*

Theorem 1.2′. (Complement to the exceptional case of Theorem 1.2). *If $k=\omega$ and N has an infinite number of orbit types, Theorem* 1.2 *remains true by replacing "C^k equivariant" by "subanalytic C^l equivariant $(l<\infty$ and $l\leqq r)$".*

Theorem 1.3. *Let G be a compact Lie group and M a C^r G-manifold $(1 \leqq r \leqq \infty)$. Then, there is a C^k G-manifold $\tilde{M}$ which is C^r equivariantly diffeomorphic to M $(r < k \leqq \omega)$.*

Now let M be a C^k G-manifold and W a linear G-space. Using a normalized Haar measure on G, we define $A: C^k(M, W) \to C^k_G(M, W)$ $(1 \leqq k \leqq \infty)$ by

$$(Af)(x) = \int_G gf(g^{-1}x)d\mu(g).$$

We know that the operator A is continuous with respect to the Whitney C^k topology. Moreover,

Lemma 1.4. *If M is a C^ω G-manifold and $f \in C^\omega(M, W)$, then $Af \in C^\omega_G(M, W)$.*

Lemma 1.5. *$C^\omega_G(M, W)$ is dense in $C^r_G(M, W)$ with the Whitney C^r topology $(1 \leqq r \leqq \infty)$.*

Proof of Lemma 1.4. There is a small complexification $\tilde{M}$ of M such that we have a holomorphic extension $\tilde{f}: \tilde{M} \to W \otimes_{\boldsymbol{R}} \boldsymbol{C}$ and holomorphic extensions $\tilde{\theta}_g: \tilde{M} \to \tilde{M}$ of θ_g $(g \in G)$, since G is compact. Then, $A\tilde{f}(x) = \int_G g\tilde{f}(g^{-1}x)d\mu(g)$ is a holomorphic map and $A\tilde{f}|M = Af$. Indeed, $A\tilde{f}$ satisfies the Cauchy-Riemann equation provided that $\tilde{f}$ does. q.e.d.

Proof of Lemma 1.5. Since M is properly (non-equivariantly) C^ω embeddable in a Euclidean space, we can approximate $f \in C^r_G(M, W)$ with respect to the Whitney C^r topology by a C^ω map (see e.g. [Hi]). So, if we take a C^ω map $\tilde{f}$ sufficiently close to f, then $A\tilde{f}$ is a C^ω equivariant map which is close to $Af = f$ because A is continuous. q.e.d.

Proof of Theorem 1.1. Theorem is well known for $1 \leqq k \leqq \infty$ by Mostow [Mo] and Wasserman [Wa]. Let $1 \leqq r \leqq \infty$ and $f: M \to W$ be a proper C^r equivariant embedding into a linear G-space. Notice that all the proper C^r embeddings form an open subset in $C^r(M, W)$ with the Whitney topology (see e.g. [Hi, Corollary 2.16]). So, if we choose a C^ω equivariant map $\tilde{f}$ sufficiently close to f by Lemma 1.5, $\tilde{f}$ is at the same time a proper C^r embedding. q.e.d.

Proof of Theorem 1.2. *Case of finite orbit types*: We may assume that there is a proper C^k equivariant embedding $j: N \to W$ into a linear G-space. Let $\nu(N)$ be an equivariant tubular neighborhood of N in W

and $\pi: \nu(N) \to N$ its equivariant projection. (π is originally of class C^{k-1} but it is not difficult to deform it to a pair $(\nu(N), \pi)$ with π of class C^k.) We take a C^k equivariant approximation $\tilde{f}$ sufficiently close to the proper map $j \circ f: M \to W$. Then, $\pi \circ \tilde{f}$ is defined and $j^{-1} \circ \pi \circ \tilde{f}$ is a C^k equivariant approximation. If f is a C^r equivariant diffeomorphism, $\pi \circ \tilde{f}$ is a proper C^k equivariant embedding of M into W such that $\pi \circ \tilde{f}(M) \subset j(N)$. Since this is a properly embedded submanifold of the same dimension, $\pi \circ \tilde{f}(M) = j(N)$. Thus, $j^{-1} \circ \pi \circ \tilde{f}$ is a required C^k equivariant diffeomorphism. Note also that we can choose $\tilde{f}$ so that $j^{-1} \circ \pi \circ \tilde{f}$ is C^r equivariantly isotopic to f.

Case of infinite orbit types and $1 \leqq k \leqq \infty$: Let $\psi: M \to R$ be a positive C^k G-invariant proper function. Let $0 = a_0 < a_1 < a_2 < \cdots \to \infty$ be regular values of ψ. Put $M_i = \psi^{-1}((a_{i-1}, a_{i+1}))$, $f_i = f \mid M_i$ and $N_i = f(M_i)$ where $f: M \to N$ is a given C^r equivariant map. Since $f(\psi^{-1}((0, a_{i+1}]))$ is compact and contains N_i, N_i is of finite orbit types. Then, there exists C^k equivariant approximation $\tilde{f}_i: M_i \to N_i$ of f_i. Assume that each N_i is equivariantly embedded in some linear representation space W_i of G. Let $\pi_i: \nu_i(N_i) \to N_i$ be a C^k equivariant tubular neighborhood. Take C^k G-invariant functions $\alpha_i: M_i \cap M_{i+1} \to [0, 1]$ such that $\alpha_i = 1$ on $\psi^{-1}((a_i, a_i + \varepsilon))$ and $\alpha_i = 0$ on $\psi^{-1}((a_{i+1} - \varepsilon, a_{i+1}))$ for a small $\varepsilon > 0$. Define $\tilde{f}: M \to N$ by

$$\tilde{f}(x) = \begin{cases} \tilde{f}_1(x) & \text{if } \psi(x) \in (a_0, a_1], \\ \pi_i \circ (\alpha_i(x)\tilde{f}_i(x) + (1 - \alpha_i(x))\tilde{f}_{i+1}(x)) & \\ & \text{if } \psi(x) \in (a_i, a_{i+1}] \quad (i \geqq 1). \end{cases}$$

Here the summation is carried out in W_i. Then, $\tilde{f}$ is a C^k equivariant approximation of f. q.e.d.

Proof of Theorem 1.2′. It suffices to note the following. We can find only subanalytic C^ℓ ($\ell < \infty$) G-invariant functions $\alpha_i: M_i \cap M_{i+1} \to [0, 1]$ with $\alpha_i = 1$ on $\psi^{-1}((a_i, a_i + \varepsilon))$ and $\alpha_i = 0$ on $\psi^{-1}((a_{i+1} - \varepsilon, a_{i+1}))$ in the proof of Theorem 1.2 when $k = \omega$. The sum and the compositions of subanalytic maps defined on compact subanalytic sets are subanalytic. See Section 2 for the notion of subanalytic map and also [S]. q.e.d.

Proof of Theorem 1.3. *Case of finite orbit types*: Let $j: M \to W$ be a proper C^r equivariant embedding in a linear G-space and $\pi: \nu(M) \to M$ the equivariant projection of its equivariant tubular neighborhood. (We may assume π is of class C^r.) We have a following commutative diagram:

$$\begin{array}{ccc} \nu(M) & \xrightarrow{F} & \xi \\ \downarrow{\scriptstyle\pi} & & \downarrow \\ M & \xrightarrow{f} & \mathrm{Gr}_n(W) \end{array}$$

where ξ is a universal vector bundle and (F, f) is the classifying map. Since ξ and $\nu(M)$ are C^k G-manifolds, we have a C^k equivariant approximation $\tilde{F}$ of F so as to be fiberwise transverse, that is, $\pi|\tilde{M}: \tilde{M} \to M$ is a C^r covering map, where $\tilde{M} = \tilde{F}^{-1}$ (zero section) (cf. Palais [P2, Proposition 4.2]). Noticing that $\tilde{M}$ is a C^k G-manifold by the implicit function theorem and that $\pi|\tilde{M}$ is 1:1, we see that $\pi|\tilde{M}$ is a required C^r equivariant diffeomorphism.

Case of infinite orbit types: Let $\psi: M \to \boldsymbol{R}$ be a positive C^r G-invariant proper function. Let $0 = a_0 < a_1 < \cdots \to \infty$ be regular values of ψ. Put $M_i = \psi^{-1}((a_{i-1}, a_{i+2}))$. Denote $M_i^- = M_{i-1} \cap M_i$ and $M_i^+ = M_i \cap M_{i+1}$. Then, there are C^k G-manifolds $\tilde{M}_i$ and C^r equivariant diffeomorphisms $\varphi_i: \tilde{M}_i \to M_i$ for $i = 1, 2, \cdots$, because M_i has finite orbit types as a subset of a compact G-space $\psi^{-1}((0, a_{i+2}])$. Put $\tilde{M}_i^- = \varphi_i^{-1}(M_i^-)$ and $\tilde{M}_i^+ = \varphi_i^{-1}(M_i^+)$. We have C^r equivariant diffeomorphisms $\rho_i: \tilde{M}_i^+ \to \tilde{M}_{i+1}^-$ defined by $\varphi_{i+1}^{-1} \circ \varphi_i | \tilde{M}_i^+$. Let $\tilde{\rho}_i$ be a C^k equivariant approximation of ρ_i. We can choose $\tilde{\rho}_i$ so close to ρ_i that the map $\tilde{\Phi}_{i+1}: \tilde{M}_{i+1} \to M_{i+1}$, defined by

$$\tilde{\Phi}_{i+1}(x) = \begin{cases} \varphi_i \circ \tilde{\rho}_i^{-1}(x) & \text{if } x \in \tilde{M}_{i+1}^-, \\ \varphi_{i+1}(x) & \text{if } x \notin \tilde{M}_{i+1}^-, \end{cases}$$

is of class C^r. Then, $\tilde{M} = \bigcup \tilde{M}_i$, unioned by $\tilde{\rho}_i: \tilde{M}_i^+ \to \tilde{M}_{i+1}^-$, is a C^k G-manifold and $\tilde{\Phi}: \tilde{M} \to M$ defined by $\bigcup \tilde{\Phi}_i$ is a C^r equivariant diffeomorphism. q.e.d.

§ 2. Preliminaries about subanalytic sets

We review the facts about subanalytic sets that will be used in the next section.

Definition. A *subanalytic set* in a real analytic manifold M is a subset of M of the form

$$\bigcup_i (\mathrm{Im} f_{i1} - \mathrm{Im} f_{i2})$$

where f_{ij} are a finite number of proper real analytic maps of real analytic manifolds into M. A *subanalytic map* between subanalytic sets is a continuous map whose graph is subanalytic. A *subanalytic homotopy* $f_t: M \to Y$ is one such that $F: X \times I \to Y$ taking (x, t) to $f_t(x)$ is subanalytic.

For example a polyhedron PL embedded in $\boldsymbol{R}^n$ as a closed subset is subanalytic and a PL map between such polyhedra is subanalytic. So, a subanalytic structure on a polyhedron is thus uniquely determined by any closed PL embedding up to subanalytic homeomorphism. We list here the properties which will be used later.

Lemma 2.1 (Hironaka [H1]). *A (semi-)analytic set, the closure of a subanalytic set and the image of a subanalytic set by a proper (sub-)analytic map into a real analytic manifold M are all subanalytic. A subset X in M is subanalytic if for every point x of M there exists a neighborhood U of x in M such that $X \cap U$ is subanalytic in U.*

Lemma 2.2 (Hironaka [H1]). *Let X be a subanalytic set in a real analytic manifold M. Then, there exists a subanalytic stratification $\{X_i\}$ of X, i.e., X is the disjoint union of X_i, each X_i is subanalytic, connected and at the same time a real analytic submanifold of M, $\bar{X}_i \cap X_j \neq \phi$ implies $\bar{X}_i \supset X_j$ and $\{X_i\}$ is locally finite in M.*

Lemma 2.3 (Hironaka [H2], Hardt [Ha]). *Let $\{X_i\}$ be a locally finite family of subanalytic sets in $\boldsymbol{R}^n$ which are contained in a subanalytic closed set X in $\boldsymbol{R}^n$. Then, we have a subanalytic triangulation of X compatible with $\{X_i\}$, i.e., a locally finite simplicial complex K and a subanalytic homeomorphism $\tau: |K| \to X$ such that X_i is a union of some $\tau(\mathrm{Int}\,\sigma)$, $\sigma \in K$.*

The following is a refinement of Theorem 4.1 of [SY].

Lemma 2.4. *Let X, $\{X_i\}$ and (K, τ) be as in Lemma* 2.3. *Let (K', τ') be another subanalytic triangulation of X compatible with $\{X_i\}$. Then, there exist subanalytic isotopies $\tau_t: |K| \to X$ and $\tau'_t: |K'| \to X$ $(t \in I)$ which satisfy the following four conditions:* (i) $\tau_0 = \tau$ *and* $\tau'_0 = \tau'$, (ii) (K, τ_t) *and* (K', τ'_t) *are subanalytic triangulations of M for each $t \in I$,* (iii) $\tau_t(\sigma) = \tau(\sigma)$ *and* $\tau'_t(\sigma') = \tau'(\sigma')$ *for each $\sigma \in K$, $\sigma' \in K'$ and $t \in I$, and* (iv) $(\tau'_1)^{-1} \circ \tau_1: |K| \to |K'|$ *is a PL map.*

Proof. By the assumption of Lemma,

$$\Lambda = \{\tau(\sigma), \tau'(\sigma'); \sigma \in K, \sigma' \in K'\}$$

is a locally finite family of subanalytic sets in $\boldsymbol{R}^n$. Applying Lemma 2.3 to Λ, we have a 3rd subanalytic triangulation (K'', τ'') of X compatible with Λ. Put $\pi = \tau^{-1} \circ \tau'': |K''| \to |K|$. Then, Lemma follows from the following assertion.

Assertion. *There exists a subanalytic isotopy* $\pi_t: |K''| \to |K|$ $(t \in I)$ *with* $\pi_0 = \pi$ *such that* $\pi_t(\sigma) = \pi(\sigma)$ *for each* $\sigma \in K$ *and* $t \in I$ *and* $\{\pi_1(\sigma''); \sigma'' \in K''\}$ *is a linear subdivision of* K.

Proof of Assertion. Let K^k denote the k-skeleton of K and $K''(K^k)$ be defined by

$$K''(K^k) = \{\sigma'' \in K''; \tau(\sigma'') \subset |K^k|\}.$$

Then, since π'' is compatible with Λ, we have $\pi(|K''(K^k)|) = |K^k|$. We shall construct π_t on $K''(K^k)$ by induction of k. Put $\pi_t(\sigma'') = \pi(\sigma'')$ for each $\sigma'' \in K''(K^0)$ and $t \in I$, and as a hypothesis of induction we assume that π_t is already defined on $K''(K^k)$. It suffices to extend π_t on $\pi^{-1}(\sigma)$ for each $\sigma \in K^{k+1}$. As $\partial\sigma \subset |K^k|$, π_t is already defined on $\pi^{-1}(\partial\sigma)$. Now Theorem 4.4 of [SY] tells us that $\pi^{-1}(\sigma)$ is a PL ball, since it is subanalytically homeomorphic to σ. Hence, by the Alexander trick we can extend π_t over $\pi^{-1}(\sigma)$ as C^0 isotopy and moreover, we can do so as a subanalytic isotopy (cf. [SY, 3.3]). This completes the proof of Lemma. q.e.d.

§ 3. Subanalytic triangulation of the orbit space of a real analytic G-manifold

Let M^m be a C^ω G-manifold, that is, a real analytic manifold of dimension m with a real analytic action $G \times M \to M$ of a compact Lie group G. Collapsing each G-orbit to one point, we get a quotient map $q: M \to M/G$ onto the orbit space. The purpose of this section is to give a unique subanalytic triangulation of the orbit space M/G.

Theorem 3.1. *There exists a proper G-invariant (real) analytic map* $f: M^m \to \boldsymbol{R}^n$ *such that the induced map* $\bar{f}: M/G \to f(M)$ *is a homeomorphism for some* n. *(We can take* $n = 2m+1$.*) Moreover, if another subanalytic set structure on* M/G *is given by an inclusion* $j: M/G \to \boldsymbol{R}^N$ *such that* $j \circ q: M \to \boldsymbol{R}^N$ *is a proper subanalytic map, then* $j(M/G)$ *and* $\bar{f}(M/G) = f(M)$ *are subanalytically homeomorphic.*

Proof. The 2nd statement is trivial; in fact, $(\bar{f} \times j) \circ \Delta_2 \circ q$ is also a proper subanalytic map and the graphs of the projections are $(\bar{f} \times j \times j) \circ \Delta_3$ and $(\bar{f} \times j \times \bar{f}) \circ \Delta_3$ which are subanalytic in $\boldsymbol{R}^{2n+N}$ and $\boldsymbol{R}^{n+2N}$ respectively. Here, Δ_i denotes the diagonal map of M/G into the i-th product.

If M is of finite orbit types, then we have a proper C^ω equivariant embedding $h: M \to W$ into a linear representation space. By a classical invariant theory we know that the set of finite generators $\{p_1, \cdots, p_n\}$ of G-invariant polynomials gives a proper analytic map $p = (p_1, \cdots, p_n)$: $W \to \boldsymbol{R}^n$ which induces a homeomorphism of W/G into $\boldsymbol{R}^n$. (See Weyl

[W, Theorem 8.14A].) So, $p \circ h: M \to \boldsymbol{R}^n$ is a proper analytic map such that the induced map $(p \circ h)^-: M/G \to p \circ h(M)$ is a homeomorphism.

In the case with no restriction we put

$$X = \{(x, y) \in M \times M;\ q(x) = q(y)\} \subset M \times M.$$

Then, X is the image of the projection of the graph of the C^ω action $G \times M \to M$ to $M \times M$. Hence, by Lemma 2.1 X is subanalytic in $M \times M$. For any G-invariant analytic map F: $M \to \boldsymbol{R}^\ell$ we put

$$X_F = \{(x, y) \in M \times M - X;\ F(x) = F(y)\} \subset M \times M.$$

As an analytic set is subanalytic (Lemma 2.1), X_F is also subanalytic in $M \times M$. We remark that X_F is $G \times G$-invariant. We will define inductively proper G-invariant analytic maps F_k: $M \to \boldsymbol{R}^k$, $1 \leq k \leq 2m+1$, so that $\dim X_{F_k} = 2m - k$. Then, $f = F_{2m+1}$: $M \to \boldsymbol{R}^{2m+1}$ will satisfy the requirement of Theorem.

Let F_0: $M \to R^0$ be the constant map. Assume that we have already constructed F_k. We want to define F_{k+1}. Applying Lemma 2.2, we have a subanalytic stratification of X_{F_k}. Collecting the dimension $2\text{m} - k$ strata, we get a locally finite family $\{Y_\alpha\}_{\alpha \in A}$ (may be empty) in $M \times M$ such that

$$X_{F_k} \supset \bigcup Y_\alpha \quad \text{and} \quad \dim (X_{F_k} - \bigcup Y_\alpha) < 2m - k.$$

On each Y_α we pick up one point $z_\alpha = (x_\alpha, y_\alpha)$. Now we note that $M = \bigcup M_i$ is the union of the compact G-invariant manifolds M_i with $M_i \subset \operatorname{Int} M_{i+1}$. Then, $\{M_i \times M_i \cap Y_\alpha\}$ is a finite family for each i by Lemma 2.2. Since $q(x_\alpha) \neq q(y_\alpha)$ we can choose a proper G-invariant C^∞ function h on M such that $h(M_{i+1} - M_i) \subset (i - 1/3,\ i + 1 + 1/3)$ and $h(x_\alpha) \neq h(y_\alpha)$ if both x_α and y_α are contained in $M_{i+2} - M_{i-1}$ for some i. This implies $h(x_\alpha) \neq h(y_\alpha)$ for each α. Take a C^ω equivariant approximation $\tilde{h}$ sufficiently close to h such that $\tilde{h}(x_\alpha) \neq \tilde{h}(y_\alpha)$ for each α. Put $F_{k+1} = (F_k, \tilde{h})$ and $\psi(x, y) = \tilde{h}(x) - \tilde{h}(y)$. Then, F_{k+1} is the required map for the following reason. Trivially $X_{F_{k+1}} = X_{F_k} \cap \psi^{-1}(0)$. As $\psi(z_\alpha) \neq 0$ and Y_α is connected, $Y_\alpha \cap \psi^{-1}(0)$ has no inner point, which shows $\dim Y_\alpha \cap \psi^{-1}(0) < 2m - k$. Hence, $\dim X_{F_{k+1}} < 2m - k$. Thus, Theorem is proved by induction on k. q.e.d.

Lemma 3.2. *Let M be a C^ω G-manifold. The stratification of M defined by the orbit type decomposition is subanalytic. In particular, it induces a stratification on M/G which is subanalytic with respect to the subanalytic set structure given in Theorem* 3.1.

Proof. We want to prove that each stratum $M_{(H)}=\{x \in M; G_x=gHg^{-1}$ for some $g \in G\}$ is subanalytic in M and a C^ω submanifold of M. Take a point x with $G_x=H$. Then, there is a G-invariant neighborhood U of x which is identified with $G\times_H W$ by a C^ω equivariant diffeomorphism, where W is an isotropic representation space of H at x. By Lemma 2.1 it suffices to show that $M_{(H)}\cap U$ is subanalytic in U and $M_{(H)}\cap U$ is a C^ω submanifold. Notice that $M_{(H)}\cap U=G/H\times W^H$. Since W^H is a linear subspace of W, we see that $M_{(H)}\cap U$ is subanalytic in U and a C^ω submanifold of U. q.e.d.

In view of Theorem 3.1 and Lemma 3.2, the subanalytic triangulation of the orbit space is characterized as follows.

Definition. A *subanalytic triangulation* of the orbit space M/G of a C^ω G-manifold M is a pair of simplicial complex K and a homeomorphism $\tau: |K|\to M/G$ such that $\tau^{-1}\circ q: M\to |K|$ is subanalytic. A *subanalytic triangulation* isotopy of M/G is the pair of K and an isotopy $\tau_t: |K|\to M/G$ $(t \in I)$ such that (i) (K, τ_t) is a subanalytic triangulation of M/G for each $t \in I$ and (ii) we have $\tau_t(\sigma)=\tau_{t'}(\sigma)$ for each $\sigma \in K$ and $t, t' \in I$, and (iii) $M\times I\to |K|$, taking $(x, t)\mapsto \tau_t^{-1}(q(x))$, is subanalytic.

Theorem 3.3. *Let G be a compact Lie group and M a C^ω G-manifold. Then, there exists a subanalytic triangulation of M/G uniquely in the following sense. If there are two subanalytic triangulations (K, τ) and (K', τ'), we have subanalytic triangulation isotopies (K, τ_t) and (K', τ'_t) of M/G such that $\tau_0=\tau$, $\tau'_0=\tau'$ and $(\tau'_1)^{-1}\circ\tau_1: |K|\to|K'|$ is a PL map.*

Remark. By Lemma 3.2 we can consider only the subanalytic triangulations compatible with the orbit type stratification.

Proof of Theorem 3.3. *and Remark.* Clear by Lemmas 2.3–2.4 and Theorem 3.1.

§ 4. *G-CW* complex structure on a differentiable *G*-manifold and its equivariant simple homotopy type

In [M1] we have proved that there is a *G-CW* complex structure on a differentiable G-manifold M by lifting each simplex in the barycentric subdivision of a triangulation of the orbit space compatible with the orbit type decomposition (Theorem 4.1). Since two such liftings are concordant (Lemma 4.4), we get a uniqueness theorem for such *G-CW* complex structures. (See Theorem 4.2 for the precise meaning.) This defines the equivariant simple homotopy type of M at least when M is compact (Corollary 4.3). When M is non-compact, we may define its equivariant infinite simple homotopy type by Theorem 4.2.

Definition. A G-CW complex structure on a G-space M is a pair of G-CW complex X and a G-homeomorphism $\xi: X \to M$. It is said that (X, ξ) induces a triangulation on M/G if X/G is a simplicial complex and each characteristic G-map of a G-n-cell $G_\sigma: G/H_\sigma \times \Delta^n \to X$ induces a linear characteristic map $(G/H_\sigma \times \Delta^n)/G = \Delta^n \to X/G$ of some simplex in X/G. Moreover, if M is a C^k G-manifold $(1 \leqq k \leqq \infty)$ and the induced map $\bar{\xi}: X/G \to M/G$ is subanalytic with respect to the subanalytic set structure of M/G for some C^ω equivariant smoothing of M, (X, ξ) is said to induce a "subanalytic" triangulation on the orbit space M/G. If M has a non-empty boundary, we consider the double DM with a C^k $G \times Z_2$-action such that $M/G = DM/(G \times Z_2)$. So, a "subanalytic" triangulation is also meaningful in this case by using C^ω equivariant smoothing of DM.

Theorem 4.1. *A C^k G-manifold M (with or without boundary) admits a G-CW complex structure (X, ξ) which induces a "subanalytic" triangulation on the orbit space M/G $(1 \leqq k \leqq \omega)$.*

Theorem 4.2. *In Theorem 4.1 let (Y, η) be another such G-CW complex structure on M. Then, there exist such G-CW complex structures (X_i, ξ_i), $0 \leqq i \leqq n$, with $(X_0, \xi_0) = (X, \xi)$, $(X_n, \xi_n) = (Y, \eta)$ and G-homeomorphisms $f_i: X_i \to X_{i+1}$ which satisfy one of the following conditions:*

(1) *$X_{i-1} = X_i$, $f_i = \mathrm{id}$ and $\xi_{i+1} \circ \xi_i^{-1}$ is equivariantly isotopic to the identity.*

(2) *$\xi_i = \xi_{i+1} \circ f_i$ and $f_i: X_i \to X_{i+1}$ (or $f_i^{-1}: X_{i+1} \to X_i$) is a subdivision, that is, the characteristic G-maps of G-cells of X_{i+1} are the restrictions of the characteristic G-maps $G_\sigma: G/H_\sigma \times \Delta^n \to X_i$ of G-cells of X_i on $G/H_\sigma \times \Delta^{n,k}$ composed with f_i, where $\Delta^{n,k}$ are simplexes in a linear subdivision of Δ^n.*

(3) *$\xi_i = \xi_{i+1} \circ f_i$ and the induced map $\bar{f}_i: X_i/G \to X_{i+1}/G$ is a simplicial isomorphism. Moreover, there exists a G-CW complex structure (Z, ζ) on $M \times I$ which gives a G-cell-wise concordance between $\xi_i: X_i \to M \times 0$ and $\xi_{i+1}: X_{i+1} \to M \times 1$; that is, $Z/G = X_i/G \times I = X_{i+1}/G \times I$, and the G-cells of Z consist of the G-cells of X_i and X_{i+1} together with the G-cells having the characteristic G-maps $G_\sigma: G/H_\sigma \times \Delta^n \times I \to Z$ such that $G_\sigma | G/H_\sigma \times \Delta^n \times 0$ and $G_\sigma | G/H_\sigma \times \Delta^n \times 1$ are the characteristic G-maps for the corresponding G-cells of X_i and X_{i+1}.*

Corollary 4.3. *Let G be a compact Lie group. Then, any compact C^k G-manifold M (with or without boundary) has a well-defined equivariant simple homotopy type in the sense of Illman* [I1].

Proof. We define an equivariant simple homotopy type of M by that of X where (X, ξ) is a finite G-CW complex structure which induces a "subanalytic" triangulation on the orbit space given in Theorem 4.1.

It suffices to check that (X_i, ξ_i) and (X_{i+1}, ξ_{i+1}) in each of the conditions (1)–(3) in Theorem 4.2 define the same equivariant simple homotopy type. Since (1) does not change the G-CW complex X, there is no problem. It is easy and standard to find an equivariant expansion $X_i \nearrow Z$ and an equivariant collapsing $Z \searrow X_{i+1}$ in the case (3). The remaining case (2) is not difficult and a proof is given in Theorem 12.2 of Illman [I4] including also the general case that the subdivision of Δ^n is not necessarily linear.

q.e.d.

We prepare a lemma to prove Theorems 4.1–4.2.

Lemma 4.4. *Let X be a Hausdorff G-space such that there is a homeomorphism $\tau: \Delta^n \to X/G$ and suppose that orbit type is constant in each of the set $\tau(\Delta^m - \Delta^{m-1})$, where $\Delta^m = v_0 * \cdots * v_m$ $(0 \leq m \leq n)$. Then, there is a continuous section $s: X/G \to X$ such that any point of $s \circ \tau$ $(\Delta^m - \Delta^{m-1})$ has a constant isotropy subgroup H_m and consequently X has a G-CW complex structure $Gs \circ \tau(\Delta^n)$ $(=\Delta_n(G; H_0, \cdots, H_n)$ in the notation of Illman* [I3]). *Moreover, if two such sections s_0 and s_1 are given, there are an element $g \in G$ and a continuous section $S: X/G \times I \to X \times I$ commuting with the projection on I such that $S|X/G \times 0 = s_0$, $S|X/G \times I = gs_1$ and $S \circ (\tau \times \mathrm{id})$ $((\Delta^m - \Delta^{m-1}) \times I)$ has the constant isotropy subgroup H_m.*

Proof. Since the first part is proved in [M1] and [I3], we give only a sketch of the proof of the second part which is a relative version. Denote $x_0 = s_0(v_0)$ and $H_0 = G_{x_0}$. Then, there is a $g_0 \in G$ such that $g_0 s_1(v_0) = x_0$. We see that $S_0 = H_0 s_0(\Delta^n)$ and $S'_0 = H_0 g_0 s_1(\Delta^n)$ are two slices at x_0. We identify Δ^n with X/G by τ.

Assertion (m). *Let S_m and S'_m be maximal slices at $x_m \times 0$ and $x_m \times 1$ in H_{m-1}-spaces $X_m \cap X \times 0$ and $X_m \cap X \times 1$ respectively for a Hausdorff H_{m-1}-space X_m over $d^{n-m} \times I$, where $H_{-1} = G$, $X_0 = X \times I$ and $d^{n-m} = v_m * \cdots * v_n$. Then, there is a tube T_m* (i.e., *T_m is an H_m-subspace and $H_{m-1} \times_{H_m} T_m \to H_{m-1} T_m$ is an H_{m-1}-homeomorphism where $H_m = G_{x_m}$*) *about $x_m \times I$ in X_m such that $T_m \cap X \times 0 = S_m$ and $T_m \cap X \times 1 = S'_m$.*

Proof of Assertion (m). We have an H_{m-1}-map $X_m \cap (X \times 0 \cup X \times 1) \to H_{m-1}/H_m$ given by S_m and S'_m. Embed H_{m-1}/H_m equivariantly into a linear H_{m-1}-space W and let $\pi: \nu \to H_{m-1}/H_m$ be an equivariant projection of its equivariant tubular neighborhood. Since X_m is a compact Hausdorff and hence normal H_{m-1}-space, there is an equivariant extension $f: U \to W$ over some neighborhood U of $X_m \cap (X \times 0 \cup X \times 1)$ in X_m such that $f(U) \subset \nu$ and $f(x_m, t) = H_m/H_m$. Then $f^{-1}(\pi^{-1}(H_m/H_m))$ is a tube about $x_m \times I$ in X_m. Let $d^{n-m-1}_{[t,t']} = \{\sum t_i v_{n-i};\ \sum t_i = 1,\ t_i \geq 0,\ t_{n-m} \in [t, t']\} \subset d^{n-m}$. Since

$d_{[0,t]}^{n-m-1}\times I$ is the product $\Sigma(H_{m-1})$-space $(d_{[0,t]}^{n-m-1}\times 0\cup d_t^{n-m-1}\times I\cup d_{[0,t]}^{n-m-1}\times 1)\times I$ for $t>0$, we can get a maximal tube T_m which is an extension of S_m and S'_m by the covering homotopy theorem of Palais [P1]. (See also [B, II.7.3].)

Applying Assertion (0) we can define for $m=0$ an H_m-space $X_{m+1}=T_m\cap q^{-1}(d^{n-m-1}\times I)$ where $d^{n-m-1}=v_{m+1}*\cdots*v_n$ and let $x_{m+1}=s_0(v_{m+1})$ and $H_{m+1}=G_{x_{m+1}}$. Then, we have a $g_{m+1}\in H_{m+1}$ such that $g_{m+1}\cdots g_1g_0s_1(v_{m+1})=x_{m+1}$ and $S_{m+1}=H_{m+1}s_0(d^{n-m-1})$ and $S'_{m+1}=H_{m+1}g_{m+1}\cdots g_0s_1(d^{n-m-1})$ are maximal slices at $x_{m+1}\times 0$ and $x_{m+1}\times 1$ in $X_{m+1}\cap X\times 0$ and $X_{m+1}\cap X\times 1$ respectively. By applying Assertion (1) to Assertion (n) inductively we get the concordances of slices:

$$T_0\supset T_1\supset\cdots\supset T_n=x_n\times I.$$

Since $T_n=x_n\times I$ we define $T|v_n\times I$ by $T(v_n,t)=(x_n,t)$. Notice that $x_n=gs_1(v_n)$ if we define $g=g_n\cdots g_0$. Assume as an induction hyperthesis that we have already defined a section T over $d^k\times I$ into T_{n-k} so that all the isotropy subgroups at points of $T(d^k\cap(\Delta^m-\Delta^{m-1})\times I)$ are constant and equal to H_m ($m\geqq n-k$), $T|d^k\times 0=s_0|d^k$ and $T|d^k\times 1=gs_1|d^k$. Regarding the section over $d^k\times I$ into T_{n-k-1} we may get an extension over $(d^{k+1}-v_{n-k-1})\times I$ by Palais' covering homotopy theorem such that all the isotopy subgroups at points of $T(d^{k+1}\cap(\Delta^m-\Delta^{m-1})\times I)$ are H_m ($m\geqq n-k$), $T|(d^{k+1}-v_{n-k-1})\times 0=s_0|(d^{k+1}-v_{n-k-1})$ and $T|(d^{k+1}-v_{n-k-1})\times 1=gs_1|(d^{k+1}-v_{n-k-1})$. Then, since $T_{n-k-1}\cap q^{-1}(v_{n-k-1}\times I)=x_{n-k-1}\times I$ we have a continuous extension of section over $d^{k+1}\times I$ with $T((d^{k+1}\cap\Delta^{n-k-1})\times I)=x_{n-k-1}\times I$. This completes the inducting step and gets a desired concordance T of the liftings s_0 and gs_1. q.e.d.

Proof of Theorem 4.1. Take a C^ω equivariant smoothing M_1 of M and a subanalytic triangulation τ: $|K|\to M_1/G$ compatible with orbit type decomposition. Let Δ^n be a simplex in a barycentric subdivision K' of K. Then, $q^{-1}(\tau(\Delta^n))$ satisfies the condition of Lemma 4.4 and we get a lifting s: $\Delta^n\to M$. Collecting G-cells $Gs(\tau(\Delta^n))$ for all simplexes of K' we get a G-CW complex X such that the underlying space $|X|$ is M.

If M has a non-empty boundary and $k\leqq\infty$, we consider the double DM with a $G\times Z_2$-action such that $M/G=DM/(G\times Z_2)$. (See [I3]). So, M/G has a "subanalytic" triangulation compatible with the orbit type decomposition and the above lifting argument implies the result. If M has a non-empty boundary and $k=\omega$, there is a C^ω manifold $\tilde M$ containing M as a G-invariant subanalytic set. So, M/G is also subanalytic and has a subanalytic triangulation compatible with the orbit type decomposition, which implies Theorem 4.1 by the above lifting argument. q.e.d.

Proof of Theorem 4.2. Let (X, ξ) and (Y, η) be two G-CW complex structures on M which induce "subanalytic" triangulations on M/G. Then, there are two C^ω equivariant smoothings f_i: $M_i \to M$ such that $\bar{\xi}'=\bar{f}_1^{-1} \circ \bar{\xi}$: $X/G \to M_1/G$ and $\bar{\eta}'=\bar{f}_2^{-1} \circ \bar{\eta}$: $Y/G \to M_2/G$ are subanalytic triangulations. By Theorem 1.2 (or 1.2′) there is a C^ω (or subanalytic C^1) equivariant diffeomorphism f: $M_1 \to M_2$ which is an approximation of $f_2^{-1} \circ f_1$. In any case the induced map $(\bar{\eta}')^{-1} \circ \bar{f} \circ \bar{\xi}'$: $X/G \to M_1/G \to M_2/G \to Y/G$ is a subanalytic homeomorphism and we can assume that f is equivariantly isotopic to $f_2^{-1} \circ f_1$. (See the proof of Theorems 1.2–1.2′.) Noticing that subanalytic triangulation isotopies of X/G and Y/G are covered by equivariant isotopies of X and Y, we may assume by (1) that $\bar{\xi}(X/G)$ and $\bar{f}_1^{-1} \circ \bar{f}^{-1} \circ \bar{f}_2 \circ \bar{\eta}(Y/G)$ have a common linear subdivision. It is easy to see that the linear subdivision of X/G naturally induces a subdivision of X in the sense of (2). So, since f is equivariantly isotopic to $f_2^{-1} \circ f_1$, it suffices to show Theorem in the case $\xi=\eta \circ f$ and $\bar{f}$: $X/G \to Y/G$ is a simplicial isomorphism. By using subdivisions of (2) again we may suppose also that $|K|=X/G$ gives the barycentric subdivision of a triangulation compatible with orbit type decomposition. For each simplex Δ^n in K there are two liftings s_0 and s_1 defined by the G-CW complex structures X and Y, which are concordant in the sense of Lemma 4.4. This is exactly what Theorem 4.2 asserts for a C^k G-manifold M without boundary.

If M has a non-empty boundary, the same argument can apply by using the "subanalytic" or subanalytic triangulation of the orbit space given in the last part of the proof of Theorem 4.1. q.e.d.

References

[B] G. E. Bredon, Introduction to compact transformation groups, Academic Press, New York-London (1972).

[Ha] R. M. Hardt, Triangulation of subanalytic sets and proper light subanalytic maps, Invent. Math., **38** (1977), 207–217.

[H1] H. Hironaka, Subanalytic sets, in Number theory, algebraic geometry and commutative algebra, in honor of Y. Akizuki, Kinokuniya, Tokyo (1973), 453–493.

[H2] ——, Triangulations of subanalytic sets, Proc. Symp. in Pure Math., Amer. Math. Soc., **29** (1975), 165–185.

[Hi] M. W. Hirsch, Differential topology, Graduate Texts in Math. Springer-Verlag, New York **33** (1976).

[I1] S. Illman, Whitehead torsion and group actions, Ann. Acad. Sci. Fenn., AI **588** (1974), 1–44.

[I2] ——, Smooth equivariant triangulations of G-manifolds for G a finite group, Math. Ann., **233** (1978), 199–220.

[I3] ——, The equivariant triangulation theorem for actions of compact Lie groups, Math. Ann., **262** (1983), 487–501.

[I4] ——, Actions of compact Lie groups and the equivariant Whitehead group, (to appear in Osaka J. Math.).

[J] F. E. A. Johnson, On the triangulation of stratified sets and singular varieties, Trans. Amer. Math. Soc., **275** (1983), 333–343.

[K] M. Kato, Elementary topology of analytic sets (in Japanese) Sugaku, **25** (1973), 38–51.

[M1] T. Matumoto, Equivariant *K*-theory and Fredholm operators J. Fac. Sci. Univ. Tokyo IA, **18** (1971), 109–125.

[M2] ——, On *G-CW* complexes and a theorem of J.H.C. Whitehead, Ibid IA, **18** (1971), 363–374.

[MZ] D. Montgomery and L. Zippin, Topological transformation groups, Wiley (interscience), New York (1955).

[Mo] G. D. Mostow, Equivariant embedding in euclidean space, Ann. of Math., **65** (1957), 432–446.

[P1] R. S. Palais, The classification of *G*-spaces, Mem. Amer. Math. Soc., **36** (1960).

[P2] ——, C^1 actions of compact Lie groups on compact manifolds are C^1-equivalent to C^∞ actions, Amer. J. Math., **92** (1970), 748–759.

[P3] ——, Equivariant real algebraic differential topology, Lecture Note, Brandeis Univ.

[Sh] K. Shiga, Differentiable manifolds and real analytic manifolds (in Japanese), Lecture Note, Tokyo Metropolitan Univ. (1963).

[S] M. Shiota, Piecewise linearlization of real analytic functions, Publ. Math. RIMS, Kyoto Univ., **20** (1984), 727–792.

[SY] M. Shiota and M. Yokoi, Triangulations of subanalytic sets and locally subanalytic manifolds, Trans. Amer. Math. Soc., **286** (1984), 727–750.

[V] A. Verona, Triangulation of stratified fibre bundles, Manuscripta Math., **30** (1980), 425–445. (Cf. Springer Lecture note **1102**)

[Wa] A. Wasserman, Equivariant differential topology, Topology, **8** (1969), 127–150.

[W] H. Weyl, The classical groups, Princeton Univ. Press, Princeton (1946).

[Wh] J. H. C. Whitehead, On C^1 complexes, Ann. of Math., (2) **41** (1940), 809–824.

[Y] C. T. Yang, The triangulability of the orbit space of a differentiable transformation group, Bull. Amer. Math. Soc., **69** (1963), 405–408.

T. Matumoto
Department of Mathematics
Faculty of Science
Hiroshima University
Hiroshima 730, Japan

M. Shiota
Department of Mathematics
College of General Education
Nagoya University
Nagoya 464, Japan