Advanced Studies in Pure Mathematics 9, 1986
Homotopy Theory and Related Topics
pp. 27–40

Stable Equivalence of G-Manifolds

Katsuo Kawakubo

Dedicated to Professor Minoru Nakaoka on his 60th birthday

§ 1. Introduction

Let G be a compact Lie group and M_1, M_2 be closed G-manifolds. Suppose that there exists a G-homotopy equivalence

$$f: M_1 \longrightarrow M_2.$$

Then a natural question is the following. What kind of consequences follow from it?

For example, the following theorem holds.

Theorem 1 [14], [16]. *We have the following equality in* $J_G(M_1)$:

$$J_G(T(M_1)) = J_G(f^*(T(M_2))).$$

Here $T(M_i)$ *denote the tangent G-vector bundles of* M_i $(i=1, 2)$ *and*

$$J_G: KO_G(M_1) \longrightarrow J_G(M_1)$$

denotes the equivariant J_G*-homomorphism.*

It is well-known that G-homotopy equivalent manifolds are not necessarily G-diffeomorphic in general [5], [13], [15].

Our first result of the present paper is the following theorem.

Theorem 2. *Let* M_1 *and* M_2 *be closed G-manifolds. If* f: $M_1 \to M_2$ *is a G-homotopy equivalence, then there exist G-vector bundles* π_i: $E_i \to M_i$ $(i=1, 2)$ *and a G-diffeomorphism*

$$\bar{f}: E_1 \longrightarrow E_2$$

such that the following diagram

Received February 25, 1985.
Research supported in part by Grant-in-Aid for Scientific Research.

$$\begin{array}{ccc} E_1 & \xrightarrow{\bar{f}} & E_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

is G-homotopy commutative.

Definition. A G-homotopy equivalence f: $M_1 \to M_2$ will be called a *tangential G-homotopy equivalence* if there exists a G-representation space V such that there is a G-vector bundle isomorphism:

$$T(M_1) \oplus V \cong f^* T(M_2) \oplus V$$

where V is the trivial G-vector bundle $M_1 \times V \to M_1$, $\oplus$ is the Whitney sum operation, and $f^*T(M_2)$ is the induced G-vector bundle of $T(M_2)$ *via* the map f.

Then we have the following theorem which we announced in [17].

Theorem 3. *Let M_1 and M_2 be closed G-manifolds and f: $M_1 \to M_2$ be a G-map. Then f is a tangential G-homotopy equivalence if and only if there exist a G-representation space V and a G-diffeomorphism*

$$\bar{f}: M_1 \times V \longrightarrow M_2 \times V$$

such that the following diagram

$$\begin{array}{ccc} M_1 \times V & \xrightarrow{\bar{f}} & M_2 \times V \\ \pi \downarrow & & \downarrow \pi \\ M_1 & \xrightarrow{\quad f \quad} & M_2 \end{array}$$

is G-homotopy commutative. Here G-actions on $M_i \times V$ are given by diagonal actions and π: $M_i \times V \to M_i$ denote the projection maps for $i=1, 2$.

Let: π: $E \to M$ be a differentiable G-vector bundle over a compact G-mainfold M. As is well-known, there is a G-invariant Riemannian metric $\langle\ ,\ \rangle$ on E. Concerning the metric $\langle\ ,\ \rangle$, we set

$$\|v\| = \sqrt{\langle v, v \rangle} \qquad \text{for } v \in E.$$

Then we put for $r > 0$,

$$E(r) = \{v \in E \mid \|v\| \leqq r\},$$
$$SE(r) = \{v \in E \mid \|v\| = r\},$$
$$\mathring{E}(r) = E(r) - SE(r) = \{v \in E \mid \|v\| < r\}.$$

It is obvious that $E(r)$ is a compact G-manifold and $\mathring{E}(r)=\text{Int}\,E(r)$ if M is a closed G-manifold and that $\mathring{E}(r)$ is G-diffeomorphic to E.

An equivariant simple homotopy theory has been developed by S. Illman [8], [10], [11], H. Hauschild [7], M. Rothenberg [22], D. R. Anderson [1] and S. Araki [2].

For a finite group G, any G-manifold M has a unique G-triangulation [9]. So an equivariant simple homotopy type is well-defined for a compact G-manifold.

Although a unique G-triangulation of a G-manifold is not known for a compact Lie group G, T. Matumoto and M. Shiota have shown that an equivariant simple homotopy type itself is well-defined for a compact G-manifold [18].

Theorem 4. *Let M_1 and M_2 be closed G-manifolds. If $f\colon M_1\to M_2$ is a G-simple homotopy equivalence, then there exist G-vector bundles $\pi_i\colon E_i\to M_i$ for $i=1, 2$, and a G-diffeomorphism*

$$\bar{f}\colon E_1(r)\longrightarrow E_2(r)$$

for any $r>0$ such that the following diagram

$$\begin{array}{ccc} E_1(r) & \xrightarrow{\bar{f}} & E_2(r) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

is G-homotopy commutative.

Corresponding to Theorem 3, we have the following theorem which we conjectured in [17].

Theorem 5 [3]. *Let M_1 and M_2 be closed G-manifolds and $f\colon M_1\to M_2$ be a G-map. Then f is a tangential G-simple homotopy equivalence if and only if there exist a G-representation space V and a G-diffeomorphism*

$$\bar{f}\colon M_1\times V(r)\longrightarrow M_2\times V(r)$$

for any $r>0$ such that the following diagram

$$\begin{array}{ccc} M_1\times V(r) & \xrightarrow{\bar{f}} & M_2\times V(r) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

is G-homotopy commutative. Here we regard V as a G-vector bundle over a point.

The techniques to prove Theorems 4 and 5 are valid for the proof of the following theorem.

Theorem 6 [3] (*Stable equivariant s-cobordism theorem*). *Let* $(W; X, Y)$ *be a* G *s-cobordism. Namely* $(W; X, Y)$ *is a* G *h-cobordism and the equivariant torsion* $\tau_G(W, X)$ *vanishes. Then there exists a G-representation space* V *such that for any* $r>0$, $W\times V(r)$ *is G-diffeomorphic to* $X\times I\times V(r)$ *where* I *denotes the interval* $[0, 1]$ *with trivial action.*

Remark. An equivariant s-cobordism theorem is stated in [22]. Unfortunately the assumption of the theorem is not stated in terms of the equivariant torsion $\tau_G(W, X)$ in the sense of Illman [8]. One of our tasks for the proofs of Theorems 4, 5 and 6 is to show that the torsions which will be defined successively vanish as well from the assumption $\tau_G(W, X) =0$.

Finally we show the following theorem.

Theorem 7. *An equivariant h-cobordism theorem and an equivariant s-cobordism theorem do not hold in general.*

§ 2. Equivariant infinite repetition

We modify the method of infinite repetition of Mazur [19], [20], [21] as follows.

Let G be a compact Lie group and M_i be compact G-manifolds with or without boundary $(i=1, 2)$.

Definition. Let $\mathscr{P}_1$ denote the set of G-maps $f\colon M_1\to M_2$ satisfying these properties:

(1) $f\colon M_1\to M_2$ is a G-embedding,
(2) $f(\operatorname{Int} M_1)$ is open in M_2,
(3) $f(M_1)\subset \operatorname{Int} M_2$.

For any sequence of G-manifolds and maps,

$$S\colon M_1\xrightarrow[f_1]{} M_2\xrightarrow[f_2]{} M_3\xrightarrow[f_3]{}\cdots \qquad f_i\in\mathscr{P}_1,$$

we denote by $\lim S$ the injective limit of the sequence S. A natural smooth structure and a G-action may be placed on $\lim S$ in an obvious manner. Clearly the G-action on $\lim S$ is smooth. Thus we get a G-manifold $\lim S$.

In the present paper, we only deal with the following two types of sequences:

$$S(f_1, f_2): M_1 \xrightarrow[f_1]{} M_2 \xrightarrow[f_2]{} M_1 \xrightarrow[f_1]{} M_2 \xrightarrow[f_2]{} \cdots, \qquad f_i \in \mathscr{P}_1,$$

$$S(f): M \xrightarrow[f]{} M \xrightarrow[f]{} M \xrightarrow[f]{} \cdots, \qquad f \in \mathscr{P}_1.$$

We then put

$$X(f_1, f_2) = \lim S(f_1, f_2),$$
$$X(f) = \lim S(f).$$

Then the following lemma follows directly from the definition.

Lemma 8. *$X(f_2 \cdot f_1) \cong X(f_1, f_2) \cong X(f_2, f_1) \cong X(f_1 \cdot f_2)$. Here $\cong$ stands for a G-diffeomorphism.*

Definition. Let M be a compact G-manifold. We denote by $\mathscr{P}_2(M)$ the set of G-maps f: $M \to M$ satisfying these properties:

(1) $f \in \mathscr{P}_1$

(2) $f \simeq_G \mathrm{id}$

(3) for any f': $M \to M$ satisfying $f' \in \mathscr{P}_1$ and $f' \simeq_G \mathrm{id}$, there exists a G-diffeomorphism α: $M \to M$ such that the following diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ & {\scriptstyle f'} \searrow & \downarrow {\scriptstyle \alpha} \\ & & M \end{array}$$

is commutative. Here $f \simeq_G \mathrm{id}$ means that f is G-homotopic to the identity map.

Lemma 9. *If $\mathscr{P}_2(M)$ is non empty, then*

$$\mathscr{P}_2(M) = \{f\colon M \to M \mid f \in \mathscr{P}_1, f \underset{G}{\simeq} \mathrm{id}\}.$$

Proposition 10. *Let M be a compact G-manifold with $\mathscr{P}_2(M) \neq \phi$ and let f: $M \to M$ satisfy $f \in \mathscr{P}_1$ and $f \simeq_G$ id. Then we have*

$$X(f) \cong \mathrm{Int}\, M,$$

where $\cong$ stands for a G-diffeomorphism.

Proof. When the boundary ∂M of M is empty, then $f \in \mathscr{P}_1$ implies that f is a G-diffeomorphism. Hence Proposition 10 holds obviously.

In the following we assume that $\partial M \neq \phi$. Using the equivariant collar neighborhood,

$$c\colon \partial M \times I \longrightarrow M, \qquad c(\partial M \times 1) = \partial M,$$

we can construct a G-map

$$d\colon M \longrightarrow M$$

such that $d \in \mathscr{P}_1$, $d \simeq_G \mathrm{id}$ and that

$$\text{Image } d = M - c\left(\partial M \times \left(\frac{1}{2}, 1\right]\right).$$

According to Lemma 9, f belongs to $\mathscr{P}_2(M)$.

We now consider the following ladder:

$$\begin{array}{ccccccccc}
M & \xrightarrow{f} & M & \xrightarrow{f} & M & \xrightarrow{f} & M & \xrightarrow{f} & \cdots \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\
M & \xrightarrow[d]{} & M & \xrightarrow[d]{} & M & \xrightarrow[d]{} & M & \xrightarrow[d]{} & \cdots
\end{array}$$

which we explain in a moment. Since $f \in \mathscr{P}_2(M)$, there exists a G-diffeomorphism $\alpha_1\colon M \to M$ such that $\alpha_1 \cdot f = d$. It is easily seen that $\alpha_1 \simeq_G \mathrm{id}$. Hence we have $d \cdot \alpha_1 \simeq_G \mathrm{id}$. On the other hand, $d \in \mathscr{P}_1$ implies that $d \cdot \alpha_1 \in \mathscr{P}_1$. It follows from the fact $f \in \mathscr{P}_2(M)$ that there exists a G-diffeomorphism $\alpha_2\colon M \to M$ satisfying

$$\alpha_2 \cdot f = d \cdot \alpha_1.$$

Continuing these arguments, we obtain the ladder above.

Obviously the ladder implies that

$$X(f) \cong X(d).$$

On the other hand, it is quite easy to prove that

$$X(d) \cong \mathrm{Int}\, M.$$

This completes the proof of Proposition 10.

Proposition 11. *Let $E \to M$ be a G-vector bundle over a closed G-manifold. Then there exists a G-representation space V such that for any $r > 0$,*

$$\mathscr{P}_2((E \oplus V)(r)) \neq \phi$$

where V denotes the trivial G-vector bundle $M \times V \to M$.

We now deduce Theorem 2 from Propositions 10 and 11.

As is well-known, there are G-representation spaces V_1 and V_2 such that M_1 and M_2 are G-embedded in V_1 and V_2 respectively. Denote by e_i $M_i \rightarrow V_i$ such G-embeddings and denote by ν the normal bundle of the G-embedding

$$f \times e_1 \colon M_1 \longrightarrow M_2 \times V_1.$$

According to [4], there are G-representation space V_3 and a G-vector bundle ξ over M_2 such that

$$T(M_2) \oplus \xi \cong V_3 = M_2 \times V_3.$$

We now define two G-vector bundles

$$E_1 = \nu \oplus M_1 \times (V_2 \oplus V_3),$$
$$E_2 = M_2 \times (V_1 \oplus V_2 \oplus V_3)$$

over M_1 and M_2 respectively.

Then one verifies the following

Proposition 12. *For any $r > 0$, there are G-maps*

$$f_1 \colon E_1(r) \longrightarrow E_2(r),$$
$$f_2 \colon E_2(r) \longrightarrow E_1(r)$$

satisfying $f_1, f_2 \in \mathscr{P}_1$ and

$$f_2 \cdot f_1 \underset{G}{\simeq} \mathrm{id} \quad \textit{and} \quad f_1 \cdot f_2 \underset{G}{\simeq} \mathrm{id}.$$

It follows from Proposition 12 that the composition

$$f_2 \cdot f_1 \colon E_1(r) \longrightarrow E_1(r)$$

belongs to $\mathscr{P}_1$ and $f_2 \cdot f_1 \simeq_G \mathrm{id}$.

On the other hand, there exists a G-representation space V such that

$$\mathscr{P}_2((E_i \oplus V)(r)) \neq \phi \qquad \text{for } i = 1, 2,$$

by Proposition 11. We now set

$$E_i' = E_i \oplus M_i \times V, \qquad i = 1, 2.$$

Let ε be a real number satisfying $0 < \varepsilon < 1$. Then define maps

$$f_1' = f_1 \oplus \bar{\varepsilon} \colon E_1'(r) = (E_1 \oplus M_1 \times V)(r) \longrightarrow E_2'(r) = (E_2 \oplus M_2 \times V)(r)$$
$$f_2' = f_2 \oplus \bar{\varepsilon} \colon E_2'(r) = (E_2 \oplus M_2 \times V)(r) \longrightarrow E_1'(r) = (E_1 \oplus M_1 \times V)(r)$$

where $\bar{\varepsilon}$ denotes the map defined by the scalar multiplication by $\varepsilon/\sqrt{2}$. Obviously $f'_2 \cdot f'_1$ also belongs to $\mathscr{P}_1$ and $f'_2 \cdot f'_1 \simeq_G \mathrm{id}$.

In view of Proposition 10, we have

$$X(f'_2 \cdot f'_1) \cong \operatorname{Int} E'_1(r) \cong E'_1.$$

Similarly we have

$$X(f'_1 \cdot f'_2) \cong \operatorname{Int} E'_2(r) \cong E'_2.$$

It follows from Lemma 8 that

$$E'_1 \cong X(f'_2 \cdot f'_1) \cong X(f'_1 \cdot f'_2) \cong E'_2.$$

It is easy to see that the following diagram

$$\begin{array}{ccc} E'_1 & \xrightarrow{\bar{f}} & E'_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

is G-homotopy commutative, where $\bar{f}$ denotes the G-diffeomorphism obtained above.

This completes a sketch of the proof of Theorem 2.

The proof of Theorem 3 is essentially similar to that of Theorem 2.

§ 3. Decomposition of G-manifolds

We first introduce some basic notations. Let G be a compact Lie group. Whenever H is a closed subgroup of G, (H) denotes the conjugacy class of H in G and $N(H)$ denotes the normalizer of H in G. There is a partial ordering relation among the set of conjugacy classes of closed subgroups of G, i.e., $(H_1) \leqq (H_2)$ if and only if there exists $g \in G$ such that $gH_1g^{-1} \subset H_2$.

Let W be a compact G-manifold. We shall denote the isotropy group at $x \in W$ by G_x, namely

$$G_x = \{g \in G \mid g \cdot x = x\}.$$

For a subgroup H of G, we shall put

$$W^H = \{x \in W \mid G_x \supset H\},$$
$$W(H) = \{x \in W \mid (G_x) = (H)\}.$$

Since W is compact, there are only finite G-isotropy types, say

$$\{(G_x) \mid x \in W\} = (H_1) \cup \cdots \cup (H_k).$$

It is possible to arrange (H_i) in such order that $(H_i) \geqq (H_j)$ implies $i \leqq j$.

We shall get a filtration

$$W = W_1 \supset W_2 \supset \cdots \supset W_k$$

consisting of compact G-manifolds W_i with corners such that

$$\{(G_x) \mid x \in W_i\} = (H_i) \cup (H_{i+1}) \cup \cdots \cup (H_k)$$

as follows.

Since (H_1) is a maximal conjugacy class, $W(H_1)$ is a compact G-invariant submanifold. We identify the normal bundle ν_1 of $W(H_1)$ in W with an open tubular neighborhood of $W(H_1)$ in W and impose a Riemannian G-vector bundle structure on ν_1. Set

$$W_2 = W - \mathring{\nu}_1(1).$$

Then W_2 is a compact G-manifold with corner and satisfies

$$\{(G_x) \mid x \in W_2\} = (H_2) \cup (H_3) \cup \cdots \cup (H_k).$$

Suppose that we get a filtration

$$W = W_1 \supset W_2 \supset \cdots \supset W_i$$

such that

$$\{(G_x) \mid x \in W_j\} = (H_j) \cup (H_{j+1}) \cup \cdots \cup (H_k)$$

and W_j is a compact G-manifold with corner for every $j \leqq i$. Since (H_i) is a maximal conjugacy class among the set

$$\{(G_x) \mid x \in W_i\},$$

$W_i(H_i)$ is a compact G-invariant submanifold of W_i. We identify the normal bundle ν_i of $W_i(H_i)$ in W_i with an open tubular neighborhood of $W_i(H_i)$ in W_i and impose a Riemannian G-vector bundle structure on ν_i. Set

$$W_{i+1} = W_i - \mathring{\nu}_i(1).$$

Then W_{i+1} is a compact G-manifold with corner and satisfies

$$\{(G_x) \mid x \in W_{i+1}\} = (H_{i+1}) \cup \cdots \cup (H_k).$$

This completes the inductive construction.

Thus we have shown the following decomposition theorem.

Theorem 13. *Let W be a compact G-manifold and $(H_1), \cdots, (H_k)$ be the isotropy types appearing in W. Arrange $\{(H_i)\}$ in such order that $(H_i) \geqq (H_j)$ implies $i \leqq j$. Then there exist compact G-manifolds M_i with corners and G-vector bundles $\nu_i \to M_i$ for $1 \leqq i \leqq k$ such that*

$$M_i(H_i) = M_i \underset{G}{\simeq} W(H_i)$$

and that we have a decomposition

$$W = \nu_1(1) \cup \nu_2(1) \cup \cdots \cup \nu_k(1).$$

Moreover if we set

$$W_i = \nu_i(1) \cup \nu_{i+1}(1) \cup \cdots \cup \nu_k(1),$$

we have

$$\{(G_x) \mid x \in W_i\} = (H_i) \cup (H_{i+1}) \cup \cdots \cup (H_k).$$

§ 4. Equivariant simple homotopy type

In this section, we give a sketch of the proof of Theorem 4.

Let $f: M_1 \to M_2$ be a G-homotopy equivalence where M_i are closed G-manifolds $(i=1, 2)$. According to Theorem 2, there exist G-vector bundles $\pi_i: E_i \to M_i$ $(i=1, 2)$ and a G-diffeomorphism

$$\bar{f}: E_1 \longrightarrow E_2$$

such that the following diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\bar{f}} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow[f]{} & M_2 \end{array}$$

is G-homotopy commutative.

In the manner of the proof of Lemma 3.2 in [16], we can prove the following lemma.

Lemma 14. *There is a G-representation space V satisfying the following conditions:*

(i) *for any non negative integer m, there is a G-diffeomorphism*

$$\bar{f}: E_1 \oplus V^m \longrightarrow E_2 \oplus V^m$$

such that

$$\bar{f}((E_1 \oplus V^m)(1)) \subset \operatorname{Int}(E_2 \oplus V^m)(1)$$

and that the following diagram

$$\begin{array}{ccc} (E_1 \oplus V^m)(1) & \xrightarrow{\bar{f}} & (E_2 \oplus V^m)(1) \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ M_1 & \xrightarrow[f]{} & M_2 \end{array}$$

is G-homotopy commutative, where V^m denotes the direct sum of m copies of V,

(ii) *if $m \geqq \dim G+3$, then $\bar{f}(S(E_1 \oplus V^m)(1))$ and $S(E_2 \oplus V^m)(1)$ are strong G-deformation retracts of*

$$(E_2 \oplus V^m)(1) - \bar{f}(\operatorname{Int}(E_1 \oplus V^m)(1)).$$

For notational convenience, we put

$$W = (E_2 \oplus V^m)(1) - \bar{f}(\operatorname{Int}(E_1 \oplus V^m)(1)),$$
$$X = \bar{f}(S(E_1 \oplus V^m)(1)),$$
$$Y = S(E_2 \oplus V^m)(1).$$

Since X and Y are strong G-deformation retracts of W, we can define equivariant torsions $\tau_G(W, X)$ and $\tau_G(W, Y)$ in the sense of Illman [8].

Then we have the following lemma.

Lemma 15. *Suppose that $f: M_1 \to M_2$ is a G-simple homotopy equivalence, $V^G \neq \{0\}$ and that $m \geqq \dim G+3$. Then the triad $(W; X, Y)$ is a G s-cobordism. Namely*

$$\tau_G(W, X) = \tau_G(W, Y) = 0.$$

Let $W = W_1 \supset W_2 \supset \cdots \supset W_k$ be the filtration in Theorem 13. Set

$$X_i = X \cap W_i.$$

Then by the observations of [2] and [16], we have the following lemma.

Lemma 16. *If $m \geqq \dim G+3$, then X_i are strong G-deformation retracts of W_i and $\tau_G(W, X)=0$ implies $\tau_G(W_i, X_i)=0$.*

Suppose now that the assumptions of Lemmas 14 and 15 are satisfied. Then by Lemma 16 we have $\tau_G(W_i, X_i)=0$. Hence we have easily that

$$\tau_G(W_i(H_i), X_i(H_i))=0.$$

Moreover we assume that

$$m \geqq \dim G+6.$$

Then we can prove that

$$\dim W_i^{H_i} \geqq \dim G+6$$

and that

$$\tau(W_i^{H_i}/N(H_i), X_i^{H_i}/N(H_i))=0.$$

Here $\tau(\ ,\ )$ denotes the non-equivariant Whitehead torsion. Hence by the classical s-cobordism theorem, we have that $W_i^{H_i}$ is diffeomorphic to $X_i^{H_i}\times I$. Furthermore one verifies that $W_i^{H_i}$ is $N(H_i)$-diffeomorphic to $X_i^{H_i}\times I$.

Using the fact that (H_i) is a maximal isotropy type of W_i, we can easily prove that there are natural G-diffeomorphisms:

$$W_i(H_i) \cong G \underset{N(H_i)}{\times} W_i^{H_i},$$
$$X_i(H_i) \cong G \underset{N(H_i)}{\times} X_i^{H_i}.$$

It follows that $W_i(H_i)$ is G-diffeomorphic to $X_i(H_i)\times I$.

Denote by $\nu_i(W_i(H_i))$ (resp. $\nu_i(X_i(H_i))$) the ν_i of W (resp. X) of Theorem 13. By the equivariant homotopy property of G-vector bundles, we get an isomorphism

$$\nu_i(W_i(H_i)) \cong \nu_i(X_i(H_i))\times I$$

of G-vector bundles.

Paying attention to the attaching maps, we can prove that W is G-diffeomorphic to $X\times I$ by Theorem 13. Hence we finish a sketch of the proof of Theorem 4.

The idea of the proofs of Theorems 5 and 6 are similar.

§ 5. Equivariant h-cobordism and s-cobordism theorems do not hold in general

In Section 4, we mentioned that an equivariant stable s-cobordism theorem holds. In this section, we shall show Theorem 7. Namely equivariant h-cobordism and s-cobordism theorems do not hold in general.

According to Giffen [6] and Sumners [23], for each pair of integers

(n, p) with $n \geqq 2$ and $p \geqq 2$, there are infinitely many knots (S^{n+2}, kS^n) which admit smooth semi-free Z_p-actions such that the fixed point set is kS^n.

Choose arbitrary two points x and y from kS^n. Let $D(x)$ and $D(y)$ be Z_p-invariant closed tubular neighborhoods of x and y respectively in S^{n+2} satisfying

$$D(x) \cap D(y) = \phi.$$

Then we put

$$W = S^{n+2} - \text{Int } D(x) - \text{Int } D(y),$$
$$X = SD(x), \qquad Y = SD(y).$$

Note that W is diffeomorphic to $S^{n+1} \times I$ and W^{Z_p} is diffeomorphic to $X^{Z_p} \times I = S^{n-1} \times I$.

It follows from [12] that X and Y are Z_p-deformation retracts of W. Namely $(W; X, Y)$ is a Z_p h-cobordism. Since W has a Z_p-triangulation [9], X is a strong G-deformation retract of W [8]. Hence we can define the equivariant torsion $\tau_{Z_p}(W, X)$.

If $n \geqq 3$, each component of any X^H, $H \subset Z_p$, is simply connected. It follows from [8] that the equivariant Whitehead group $Wh_{Z_p}(X)$ vanishes for $p = 2, 3, 4$ or 6 and for $n \geqq 3$. Namely $(W; X, Y)$ is a Z_p s-cobordism in this case.

On the other hand, one verifies the following lemma.

Lemma 17. *If a knot (S^{n+2}, kS^n) is non trivial, then the pair $(W, W \cap kS^n)$ is not diffeomorphic to the pair $(X, X \cap kS^n) \times I$.*

Remark. Lemma 17 does not hold in general for knots of codimension greater than two.

It follows from Lemma 17 that W is not Z_p-diffeomorphic to $X \times I$.

This completes the proof of Theorem 7.

Added in November 1985. Professor K. H. Dovermann kindly informed the author that some results related with our Theorem 3 are obtained by S. Kwasik [24].

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Department of Mathematics
Osaka University
Toyonaka 560, Japan