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Sandwiched Surface Singularities And the Nash Resolution Problem

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§ 1. Introduction

Let X be an algebraic variety over $\boldsymbol{C}$. Consider the tower of morphisms

$$(*)\qquad \cdots \longrightarrow X^{(i+1)} \xrightarrow{\nu_i} X^{(i)} \longrightarrow \cdots \longrightarrow X'' \xrightarrow{\nu_1} X' \xrightarrow{\nu} X$$

where either all the ν_i are Nash modifications (abbreviated by N) or Nash modifications followed by normalizations (abbreviated by NN).

The Nash problem: Is $X^{(i)}$ nonsingular for $i \gg 0$?

It is known ([7], p. 300) that, in characteristic 0, N is an isomorphism if and only if X is nonsingular. In particular, if $\dim X=1$, a sequence of N desingularizes.

In this paper, we discuss the following.

Theorem 1.1. *Let* $\dim X=2$. *Then a sequence of NN desingularizes.*

For the rest of this paper all the varieties will be 2-dimensional algebraic varieties over $\boldsymbol{C}$ unless otherwise specified.

All the ν_i's in $(*)$ will be NN.

The following partial results were known previously:

Theorem 1.2 (González-Sprinberg, [2] pp. 176, 129–136). *If the singularities of X are rational double points or cyclic quotients, a sequence of NN desingularizes.*

Theorem 1.3 (Hironaka, [3], p. 110). *For any surface X, consider the sequence* $(*)$. *Then, for $i \gg 0$, $X^{(i)}$ birationally dominates a nonsingular surface* (*namely, the minimal resolution of X*).

Theorem 1.3 motivates the following definition:

Definition 1.1. *Let* $(\mathcal{O}, \mathcal{M})$ *be a normal local ring. We say that* $\mathcal{O}$ *has*

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a sandwiched singularitiy if $\mathcal{O}$ birationally dominates a regular local ring ($\mathcal{R}$, *m*), *that is,*

(i) $\mathcal{R}\subset\mathcal{O}$,

(ii) $m=\mathcal{R}\cap M$ *and*

(iii) $\mathcal{O}$ *and* $\mathcal{R}$ *have the same field or fractions.*

The strategy for proving Theorem 1.1 is:

(1) to classify sandwiched singularities,

(2) to generalize the technique used by González-Sprinberg for the cyclic quotients to the case of sandwiched singularities.

I am grateful to Kyoji Saito and Masataka Tomari of R.I.M.S., at Kyoto University, for very helpful discussions. Special thanks are due to my thesis adviser, Heisuke Hironaka, who suggested the problem to me and with whom I had many discussions on the subject.

Details and proofs of the theorems stated here, which form the contents of the author's Ph. D. thesis, will be published elsewhere.

§ 2. Classification of sandwiched singularities

Let S be a surface and ξ a point on S, such that $\mathcal{O}_{S,\xi}$ has a sandwiched singularity. Then there exists a nonsingular surface X_0, a point $\eta\in X_0$, and an ideal $I\subset\mathcal{O}_{X_0,\eta}$, primary to the maximal ideal $m_{X_0,\eta}$, such that the blowing-up of the ideal I in X_0 has exactly one singular point, ξ, such that we have an isomorphism of formal completions $\hat{\mathcal{O}}_{S,\eta}\cong\hat{\mathcal{O}}_{X_0,\eta}$. Since $\xi\in S$ is normal, we may take I to be integrally closed.

We fix some conventions for the rest of the paper. Since we are interested only in the analytic type of the singularity, and not in the global structure of the surfaces involved, we will assume that $X_0=\boldsymbol{C}^2=\operatorname{Spec}\boldsymbol{C}[u,v]$. π_0: $S\rightarrow\boldsymbol{C}^2$ will denote a fixed blowing-up of an integrally closed ideal I in $\boldsymbol{C}[u,v]$, having cosupport at 0. At some point we will replace S by its affine subset, containing ξ, but that will be explicitly stated. π: $X\rightarrow S$ will always denote the minimal resolution of S and Γ- the dual graph of ξ.

The map $\pi\circ\pi_0$: $X\rightarrow\boldsymbol{C}^2$ is a composition of point blowing-ups, and, in particular, is a blowing-up of some integrally closed ideal $\mathcal{J}$ having cosupport at 0. The full exceptional set $(\pi\circ\pi_0)^{-1}(0)$ contains some exceptional curves of the first kind (i.e. smooth rational curves with self-intersection -1). The remaining exceptional curves form a connected set with a negative definite intersection matrix. π is simply the blowing-down of this set to a point ξ.

We recall an old theorem of Zariski:

Theorem 2.1 ([8], Appendix 5, pp. 386, 389). *Let I be an integrally* *closed ideal in a regular 2-dimensional local ring $\mathcal{O}$. Then there exists a unique factorization*

$$I = P_1^{\alpha_1} \cdots P_r^{\alpha_r}$$

where each P_i is a simple integrally closed ideal. Moreover, every simple integrally closed ideal is a valuation ideal for some valuation ν of the field of fractions of $\mathcal{O}$.

Geometrically, if $\pi\colon X \to \operatorname{Spec} \mathcal{O}$ is the blowing-up of I, the P_i's correspond one to one to the irreducible components of the exceptional set. Thus, if $\pi\colon X \to \boldsymbol{C}^2$ is the blowing-up of an ideal $\mathcal{J} \subset \boldsymbol{C}[u, v]$, we can write

$$\mathcal{J} \cdot \boldsymbol{C}[u, v]_{(u,v)} = \prod_{i=1}^{r} P_i \cdot \prod_{j=1}^{s} Q_j$$

where the P_i's correspond to exceptional curves of the first kind, and the Q_j's to the remaining exceptional curves.

Then S is just the blowing-up of $\boldsymbol{C}^2$ by

$$I = \prod_{i=1}^{r} P_i.$$

Definition 2.1. *A sandwiched singularity is primitive if it can be obtained from a nonsingular surface by blowing-up one simple ideal.* (*In other words, the surface X contains exactly one exceptional curve of the first kind*).

Corollary 2.1. *Let $\xi \in S$ be a sandwiched singularity. There exists an* (*unordered*) *sequence $S_1, \cdots, S_r$ of surfaces, together with surjective birational proper maps $\alpha_i\colon S_i \to \boldsymbol{C}^2$ such that*:

(1) *each S_i has exactly one singularity ξ_i, which is primitive sandwiched, and each α_i is the blowing-up of a simple ideal in $\boldsymbol{C}^2$.*

(2) *S is the birational join of $S_1, \cdots, S_r$.*

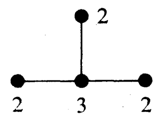
Moreover, if X is the minimal resolution of S, and X_i, the minimal resolutions of S_i, then X is the birational join of $X_1, \cdots, X_r$. In particular, the dual graph of ξ is the union of the dual graphs of the ξ_i (*as sets of vertices, with the obvious rules for taking unions*).

Example 2.1. A_n is a primitive sandwiched singularity. It is obtained by blowing-up the ideal $(u, v^{n+1}) \subset C[u, v]$.

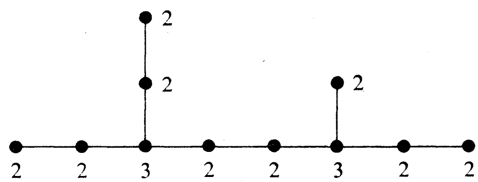
A few remarks about the place of sandwiched singularities among normal surface singularities.

Remark 2.1. A sandwiched singularity is rational. ([6] Proposition 1.2, p. 199).

Remark 2.2. A cyclic quotient singularity is always sandwiched. In fact, the cyclic quotients form a proper subclass of the minimal singularities, introduced by J. Kóllar in his thesis ([4], Definition 4.4.1). In our context, minimal singularities are the rational singularities with reduced fundamental cycle. Every minimal singularity is sandwiched, and a sandwiched singularity is minimal if and only if all its primitive components S_i of Corollary 2.1 are A_n, e.g.



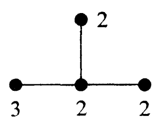
(a join of two A_3's)



(a join of three A_n's)

Here, and below, the number next to a vertex denotes minus the self-intersection number of the corresponding curve on X.

An example of a sandwiched singularity which is not minimal is



This is also primitive. It can be obtained by blowing-up $\boldsymbol{C}^2$ with the ideal

$$(u^2+v^3,\ u^3,\ v^4)\boldsymbol{C}[u, v].$$

Next, we classify the primitive sandwiched singularities.

Let $\xi \in S$ be primitive sandwiched. Let $P \subset \boldsymbol{C}[u, v]$ be the corresponding simple ideal. Let $D \subset S$ be the exceptional divisor of the blowing-up of the ideal P, in $\boldsymbol{C}^2$. The strict transform D' of D in X is an exceptional curve of the first kind. Let ν be the divisorial valuation of $\boldsymbol{C}(u, v)$ associated with the divisor D'. Then P is a ν-ideal. The image $\nu(\boldsymbol{C}(u, v)^*)$ is a discrete subgroup of $\boldsymbol{Q}$. Taking a suitable homogeneous linear transformation or the parameters (u, v) and rescaling ν, if necessary, we may assume:

$$\nu(v)=1$$

$$\nu(u)=\frac{p_1}{q_1}>1.$$

Restrict ν to $\boldsymbol{C}[u, v]^*$ and let $\Lambda=\nu(\boldsymbol{C}[u, v]^*)$. Λ is a semigroup contained in $\boldsymbol{Q}$. Let $(1, (p_1/q_1), (p_2/q_2), \cdots, (p_n/q_n))$ be the minimal system of generators of Λ, where for each i, $p_i, q_i \in \boldsymbol{N}$ and g.c.d. $(p_i, q_i)=1$. For $1 \leq i \leq n$, define pairs of integers $p_i', q_i' \in \boldsymbol{N}$ satisfying g.c.d. $(p_i', q_i')=1$, as follows:

$$\frac{p_1'}{q_1'}=\frac{p_1}{q_1}$$

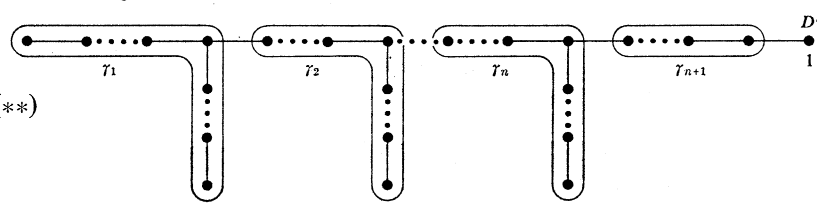
$$\frac{p_i'}{q_i'}=\text{l.c.m.}\,(q_1, \cdots, q_{i-1})\cdot\left(\frac{p_i}{q_i}-\frac{p_{i-1}}{q_{i-1}}\cdot\frac{\text{l.c.m.}\,(q_1, \cdots, q_{i-1})}{\text{l.c.m.}\,(q_1, \cdots, q_{i-2})}\right)$$

Theorem 2.2. *Let*

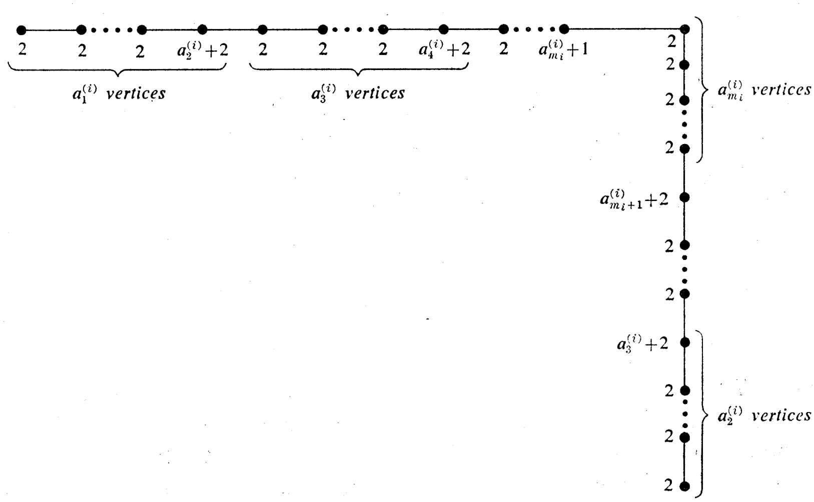
$$\frac{p_k'}{q_k'}=a_1^{(k)}+\cfrac{1}{a_2^{(k)}+\cfrac{1}{\cdots+\cfrac{1}{a_{m_k}^{(k)}}}}$$

be the expansion of p_k'/q_k' in continued fractions. Then the dual graph of ξ looks as follows. It is a tree with n stars. (By a star we shall mean a vertex which belongs to three or more arcs). We write $\Gamma=\bigcup_{i=1}^{n+1}\gamma_i$

(∗∗)



For $1 \leq i \leq n$, γ_i looks like



and Γ_{n+1} *looks like* A_t, $(\overset{2}{\bullet}\!\!-\!\!-\!\!-\overset{2}{\bullet}\cdots\overset{2}{\bullet})$ *where*

$$t=\left(\nu(P)-\frac{p_n}{q_n}\right)\cdot \text{l.c.m.}\,(q_1,\ \cdots,\ q_n)-1.$$

If, for some i, $a_1^{(i)}=0$, *then the weight of the star belonging to* Γ_{i-1}, *adjacent to* Γ_i, *becomes* $a_2^{(i)}+2$ *instead of* 2.

The -1 curve D' on the right of the diagram is not a part of the dual graph of ξ. The dual graph of ξ plus the -1 curve gives the total preimage $(\pi\circ\pi_0)^{-1}(0)\subset X$.

Next, we fix a dual graph as in (**), corresponding to a primitive sandwiched singularity ξ. We want to classify all the singularities with this dual graph up to an analytic isomorphism.

The analytic type of the singularity at ξ is determined by the analytic type of the neighborhood of $\pi^{-1}(0)$ in X. X is obtained from $\boldsymbol{C}^2$ by a sequence of point blowing-ups. Every time we blow up a point on an exceptional curve of a previous blowing-up, which is not an intersection point of two such curves, we introduce a parameter in either $\boldsymbol{C}$ or $\boldsymbol{C}^*$. Let

$$N=\sum_{i=2}^{n} a_1^{(i)}+\left(\nu(P)-\frac{p_n}{q_n}\right)\cdot \text{l.c.m.}\,(q_1,\ \cdots,\ q_{n-1})-1.$$

Thus, a priori, we have a flat, equisingular family

$$\begin{array}{c}\mathfrak{X}\\ \Big\downarrow{\scriptstyle\alpha}\\ \boldsymbol{C}^N\times(\boldsymbol{C}^*)^n\end{array}$$

all of whose fibers have sandwiched singularities with dual graph Γ, and every singularity with dual graph Γ appears in this way. However, many of the fibers of α have analytically isomorphic singularities.

Let G be the solvable Lie group of automorphisms of $\boldsymbol{C}[u, v]_{(u,v)}$, modulo P^2, which preserve the property

$$\nu(v)=1.$$

$$\nu(u)=\frac{p_1}{q_1}.$$

Then G acts on $\boldsymbol{C}^N\times(\boldsymbol{C}^*)^n$ and $\mathfrak{X}$ in an obvious way, equivariantly with respect to α. Two points in $\boldsymbol{C}^N\times(\boldsymbol{C}^*)^n$, belonging to the same orbit of G, parametrize analytically isomorphic singularities. But this is not all. An isomorphism of singularities of fibers S_1 and S_2 of α induces an automorphism of $C[u, v]_{(u,v)}$ only if it extends to an isomorphism of neighborhoods of $(\pi \circ \pi_{0,1})^{-1}(0)$ and $(\pi \circ \pi_{0,2})^{-1}(0)$. This difficulty is taken care of by Theorem 2.3: Let

$$\begin{tikzcd}
X_1 \arrow[d, "\pi_1"] & & X_2 \arrow[d, "\pi_2"] \\
S_1 \arrow[dr, "\pi_{0,1}"] & & S_2 \arrow[dl, "\pi_{0,2}"'] \\
& C^2 &
\end{tikzcd}$$

be two sandwiched singularities corresponding to different fibers of α. Blow down the exceptional curve of the first kind in X_1 and X_2, respectively, to obtain new surfaces $\bar{X}_1$, $\bar{X}_2$.

$$\begin{tikzcd}
& X_1 \arrow[dl] \arrow[d, "\pi_1"] \arrow[ddr] & & & X_2 \arrow[d, "\pi_2"] \arrow[dr] \arrow[ddl] & \\
\bar{X}_1 \arrow[drr, "\kappa_1"'] & S_1 \arrow[dr, "\pi_{0,1}"] & & & S_2 \arrow[dll, "\pi_{0,2}"'] & \bar{X}_2 \arrow[dlll, "\kappa_2"] \\
& & C^2 & & &
\end{tikzcd}$$

Suppose there is an analytic isomorphism between classical neighborhoods of $\kappa_1^{-1}(0) \subset \bar{X}_1$ and $\kappa_2^{-1}(0) \subset \bar{X}_2$. Then the singularities of S_1 and S_2 are analytically isomorphic. In other words, the choice we make when blowing up the last curve does not matter: It gives rise to isomorphic singularities. Moreover, consider the action of C on $C^N \times (C^*)^n$, which is simply the translation of the N-th coordinate (corresponding to this last irrelevant parameter). Then the analytic types of the singularities with dual graphs are classified, in a one to one way, by the orbits of the induced action of $G \times C$ on $C^N \times (C^*)^n$.

The classification of analytic types is much more complicated for non-primitive sandwiched singularities, but there too, one can give a one to one correspondence with the orbits of a certain explicitly given action of a solvable Lie group on a certain rational variety.

An amusing application of this is the following: Many, but not all, of Laufer's taut and pseudo-taut singularities are sandwiched. The above considerations allow one to recover as a special case the sandwiched part of Laufer's list by a completely different method.

§ 3. The Nash problem for surfaces

To solve the Nash problem, we need only the first part of the preceding section: the classification of the dual graphs. We denote the singularity in question by $\xi \in S$. Consider the diagram

$$(***) \qquad \begin{tikzcd} X' \arrow[r, "b"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ S' \arrow[r, "\nu"] & S \end{tikzcd}$$

where ν is the NN of S, X' the minimal resolution of S'. We want to compare the dual graph Γ of ξ with the dual graphs of the singularities of S'. It is not true that Γ determines the dual graphs of the singularities of S', as we show by an example below.

Let Ω_S^2 be the top order Kahler differentials on S, and

$$\tilde{\Omega}^2 := \pi^* \Omega_S^2/\text{torsion}.$$

According to [2], the problem reduces to computing $\tilde{\Omega}^2$, since b is the minimal nonsingular blowing-up such that $b^*\tilde{\Omega}^2/\text{torsion}$ is locally principal, and an irreducible curve $L \subset (b \circ \pi)^{-1}(\xi)$ contracts to a point in X' if and only if $\deg_L b^*\tilde{\Omega}^2 = 0$.

Represent S as a birational join of primitive sandwiched singularities:

$$\begin{tikzcd} & X \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_r"] \arrow[dd, "\pi"] & \\ X_1 \cdots \arrow[dd] & & \cdots X_r \arrow[dd] \\ & S \arrow[dl] \arrow[dr] \arrow[dd, "\pi_0"] & \\ S_1 \cdots \arrow[dr, "\beta_1"'] & & \cdots S_r \arrow[dl, "\beta_r"] \\ & \boldsymbol{C}^2 & \end{tikzcd}$$

Now, let $\bar{S}$ and $\bar{S}_i$ be affine neighborhoods of ξ and ξ_i. Let $e_i :=$ emb. dim. $(\bar{S}_i)$. Let $(f_1^{(i)}, \cdots, f_{ei}^{(i)})$: $X_i \to \boldsymbol{C}^{ei}$ be a morphism from X to $\boldsymbol{C}^{ei}$ whose image is $\bar{S}_i$. Let $e = \sum e_i$. Then $\alpha_i^* f_j^{(i)}$ give a morphism from X to $\boldsymbol{C}^e$ whose image is $\bar{S}$.

$\tilde{\Omega}^2$ is generated over O_X by things of form $d(\alpha_i^* f_j^{(i)}) \wedge d(\alpha_{i'}^* f_{j'}^{(i')})$. For $i, j \in \{1, \cdots, r\}$, let S_{ij} and X_{ij} be birational joins of S_i, S_j and X_i, X_j, respectively. Let $\tilde{\Omega}_{ij}^2 := \pi_{ij}^* \tilde{\Omega}_{S_{ij}}^2/\text{torsion}$. Then $\tilde{\Omega}^2$ is generated by all the $\alpha_{ij}^* \tilde{\Omega}_{ij}^2/\text{torsion}$ (as submodules of Ω_X^2). Here α_{ij}: $X \to X_{ij}$ denote the obvious maps. We compute the $\tilde{\Omega}^2_{ij}$ à la Gonazàlez-Sprinberg.

Computation of the fundamental cycle

Let Γ be the dual graph of a primitive sandwiched singularity $\xi \in S$. Let p_i/q_i and $a_j^{(i)}$ be as in **2**.

Notation: Let $x_1, \cdots, x_n$ be variables, and consider the continued fraction

$$x_1+\cfrac{1}{x_2+\cdots+\cfrac{1}{x_n}}.$$

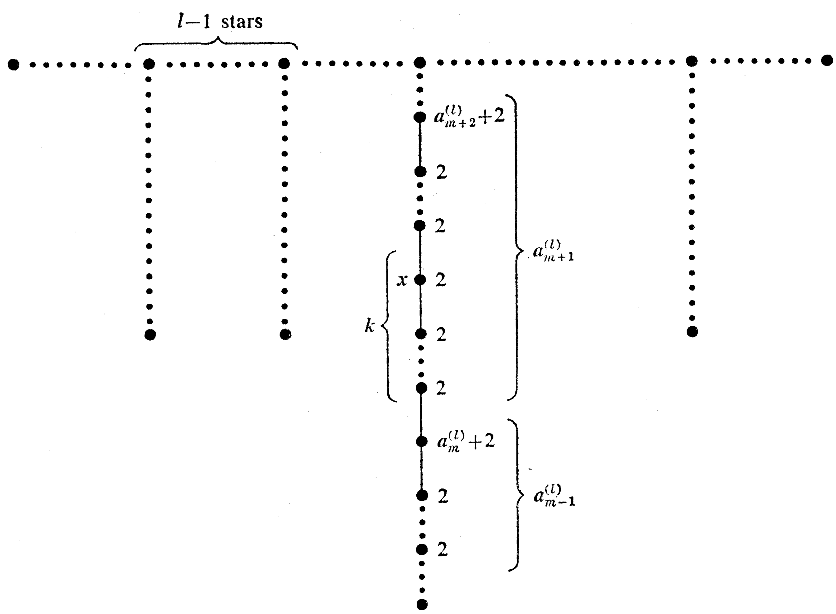
It can be written uniquely as a quotient of two polynomials having no common factors;

$$x_1+\cfrac{1}{x_2+\cdots+\cfrac{1}{x_n}}=\frac{P_n(x_1, \cdots, x_n)}{P_{n-1}(x_2, \cdots, x_n)}.$$

Let v denote, as usual, a coordinate on $\boldsymbol{C}^2$ such that $\nu(v)=1$. We think of $(\pi \circ \pi_0)^*(v)$ as a function: $X \to \boldsymbol{C}$.

The vertices of Γ are ordered by the order in which the corresponding exceptional curves were created, starting from $\boldsymbol{C}^2$.

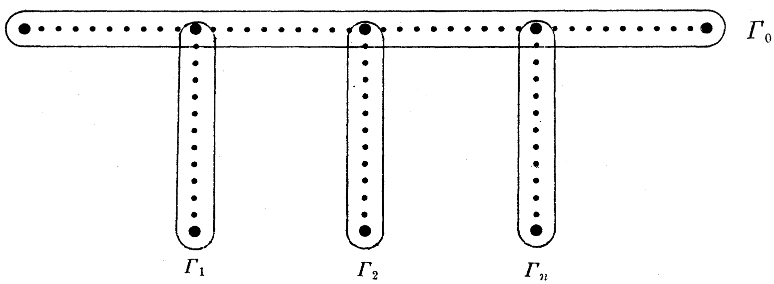
Let $x \in \Gamma$ be the vertex number $\sum_{i=1}^{l-1}(\sum_{j=1}^{m_i} a_j^{(i)})+\sum_{j=1}^{m} a_j^{(l)}+k$.



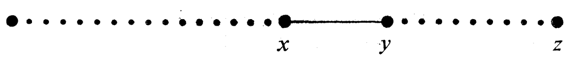
Let $L_x \subset X$ be the corresponding exceptional curve.

Lemma 3.1. $\mathrm{mult}_{L_x}(\pi \cdot \pi_0)^* v = \prod_{i=1}^{l-1} q_i' \cdot P_m(a_1^{(l)}, \cdots, a_m^{(l)}, k)$.

Next, let us write $\Gamma = \bigcup_{i=0}^{n} \Gamma_i$, where Γ_0 is the horizontal part of the tree, and $\Gamma_1, \cdots, \Gamma_n$-vertical branches:



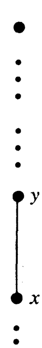
We define a cycle $\{m_x\}_{x \in \Gamma}$ recursively, starting from the right. For z the rightmost vertex, we set $m_z = 1$.



Next, let x and y be two adjacent vertices of Γ_0, as on the picture. Suppose m_y already defined. We define

$$m_x := \min_{m \in Z}\left\{ m \;\middle|\; \frac{m}{m_y} > \frac{\mathrm{mult}_{L_x}(\pi \circ \pi_0)^* v}{\mathrm{mult}_{L_y}(\pi \circ \pi_0)^* v} \right\}.$$

Finally, let x and y be two adjacent veritices of Γ_i, $1 \leq i \leq n$, as on the picture:



Suppose, m_y is already defined. Define

$$m_x := \min_{m \in Z}\left\{ m \,\middle|\, \frac{m}{m_y} \geq \frac{\mathrm{mult}_{L_x}(\pi \circ \pi_0)^* v}{\mathrm{mult}_{L_y}(\pi \circ \pi_0)^* v} \right\}.$$

For $x \in \Gamma$, let f_x denote the coefficient of L_x in the fundamental cycle.

Lemma 3.2. *For $x \in \Gamma$.*

$$f_x = \min\{m_x, \mathrm{mult}_{Lx}(\pi \circ \pi_0)^* v\}.$$

For a non-primitive sandwiched singulairty $\xi \in S$, let $X, S_1, \cdots, S_r, X_1, \cdots, X_r$ be as in Corollary 2.1. Let Z_i be the fundamental cycle of ξ_i. Then the fundamental cycle Z of ξ is given by $Z = \min_{1 \leq i \leq r} \alpha_i^* Z_i$.

Next, we compute $\tilde{\Omega}^2$ for a primitive singularity, and then $\tilde{\Omega}^2_{ij}$ for a joint of two primitive singularities. The philosophy is that, for a sandwiched singularity, among the functions f_i giving the morphism

$$X \xrightarrow{(f_1, \cdots, f_l) = \pi} S \hookrightarrow \boldsymbol{C}^l,$$

there are two God-given functions: $(\pi \circ \pi_0)^* u$ and $(\pi \circ \pi_0)^* v$, and $d((\pi \circ \pi_0)^* v) \wedge d((\pi \circ \pi_0)^* u)$ generates $\tilde{\Omega}^2$ at all the points of the exceptional set, sufficiently for to the left. (cf. the case of A_n, [2]).

Theorem 3.1. *Let (S, ξ) be a sandwiched singularity. Then $\tilde{\Omega}^2$ can be written*

$$\tilde{\Omega}^2 = I \cdot \mathscr{L}$$

where $\mathscr{L}$ is a line bundle and I an ideal sheaf having finite cosupport and satisfying the following conditions.

For any $\eta \in$ cosupport (I), one of the following holds:

(1) *η belongs to only one irreducible component L of $\pi^{-1}(\xi)$. Choose local coordinates (u_0, v_0) on X such that $u_0 = 0$ is the local defining equation for L. Then there exists an integer $l \in N$ such that the stalk $I_{X,\eta}$ is given by*

$$I_{X,\eta} = (v_0, u_0^l)\mathcal{O}_{X,\eta}$$

(2) *$\eta = L_1 \cap L_2$, where $\{L_i\}_{i=1,2}$ are two distinct irreducible components of $\pi^{-1}(\xi)$. Then $I_{x,\eta}$ is generated by elements $f \in \mathcal{O}_{X,\eta}$ such that the zero set of f is contained in $L_1 \cup L_2$ near η.*

Remark 3.1. If $\tilde{\pi}\colon \tilde{X} \to X$ is any sequence of point blowing-ups then the sheaf $\tilde{\pi}^* \tilde{\Omega}^2/$torsion on $\tilde{X}$ still satisfies properties (1) and (2). If $\tilde{\Omega}^2$ is generated over $\mathcal{O}_X$ by subsheaves $\{\Omega_j\}_{1 \leq j \leq p}$ and (1) and (2) hold for each

Ω_j then (1) and (2) hold for $\tilde{\Omega}^2$. Hence, by the argument in the beginning of this section, it is sufficient to prove Theorem 3.1 in the case when (S, ξ) is the birational join of two primitive singularities. In this case, the proof is by a direct calculation, using the classification of primitive singularities.

Remark 3.2. For a vertex $x \in \Gamma$ let L_x denote the corresponding irreducible component of $\pi^{-1}(\xi)$ and $\gamma(x)$ the number of arcs of Γ coming out of x. It is known that for any rational singularity $-L_x^2+1 \geq \gamma(x)$ for any vertex $x \in \Gamma$. On the other hand, a singularity is minimal (cf. Remark 2.2) if and only if $-L_x^2 \geq \gamma(x)$ for any vertex $x \in \Gamma$.

Corollary 3.1. *Let (S, ξ) be a sandwiched singularity and Γ its dual graph. Recall the diagram* $(***)$. *Let $\tilde{\Gamma}$ denote the dual graph of the configuration $(b \circ \pi)^{-1}(\eta)$. Assume that ξ is the only singularity of S. Let $\xi'_1, \cdots, \xi'_r$ denote the singularities of S' and $\Gamma'_1, \cdots, \Gamma'_2, \cdots, \Gamma'_r$ their respective dual graphs. We have*

$$\coprod_{i=1}^{r} \Gamma'_i \underset{\substack{\text{as weighted}\\ \text{graphs}}}{\subset} \tilde{\Gamma} \underset{\substack{\text{as sets of}\\ \text{vertices}}}{\supset} \Gamma$$

Let $x \in \Gamma'_i$ be a vertex such that

$$-L_x^2+1=\gamma(x)$$

Then $x \in \Gamma$. In other words, NN creates no new vertices for which the above equality holds.

Corollary 3.2. *For $i \gg 0$ in* $(*)$, *$X^{(i)}$ has only minimal singularities.*

Indeed, for each i, let $\Gamma^{(i)}$ denote the disjoint union of the dual graphs of the singularities of $X^{(i)}$ and $\tilde{\Gamma}^{(i)}$ the dual graph of the total preimage of all the singular points of $X^{(i)}$ in the minimal resolution of $X^{(i+1)}$. Associated with $(*)$ is the sequence of dual graphs

$$(****) \qquad \cdots \supset \Gamma^{(i+1)} \subset \tilde{\Gamma}^{(i)} \supset \Gamma^{(i)} \subset \cdots \subset \tilde{\Gamma}' \supset \Gamma' \subset \tilde{\Gamma} \cap \Gamma$$

where $\Gamma^{(i+1)} \subset \tilde{\Gamma}^{(i)}$ as weighted graphs and $\Gamma^{(i)} \subset \tilde{\Gamma}^{(i)}$ as sets of vertices. Hironaka's theorem (Theorem 1.3) asserts that for any given vertex $x \in \Gamma$ and $i \gg 0$, $x \notin \Gamma^{(i)}$. Together with Corollary 3.1 this implies Corollary 3.2. Thus the Nash problem reduces to the case of minimal singularities.

Theorem 3.2. *Let (S, ξ) be a minimal singularity. Keep the notation of Corollary* 3.1. *Then all the ξ'_i are minimal singularities and*

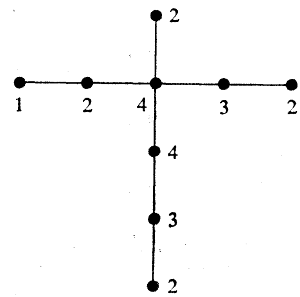
$$\max_{1 \leq i \leq r} (\#\{\textit{vertices of } \Gamma'_i\}) \leq \tfrac{1}{2}(\#\{\textit{vertices of } \Gamma\}).$$

In particular, for $j \gg 0$ in (****), $\Gamma^{(j)} = \emptyset$ *and hence* $X^{(j)}$ *in* (*) *is non-singular.*

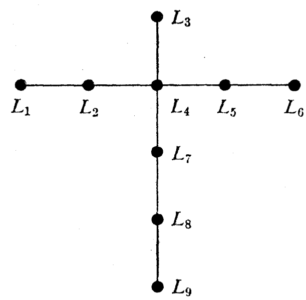
Theorem 1.1 follows.

Example 3.1.

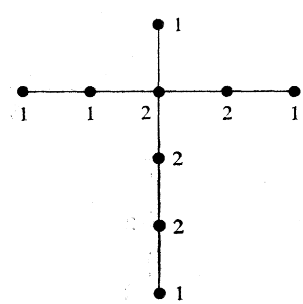
$\Gamma =$



Number the exceptional curves:



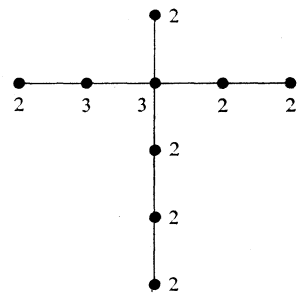
The fundamental cycle Z is:



Case I. The cross-ratio of the points $L_2 \cap L_4$, $L_3 \cap L_4$, $L_5 \cap L_4$ and $L_7 \cap L_4$ on L_4 is different from $1/2$.

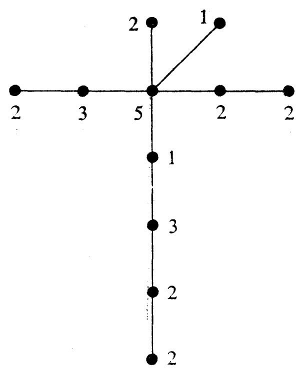
$$\tilde{\Omega}^2 = J\mathcal{O}_X(-M + K_X)$$

where K_X is the canonical divisor of X, M is the divisor given by

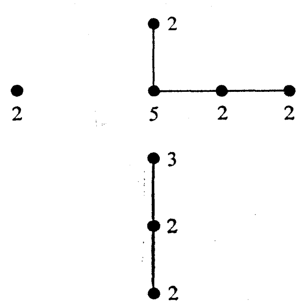


cosupport $(J)=\{\eta_1, \eta_2\}$, where $\eta_1=L_4\cap L_7$, $J_{\eta_1}=m_{X,\eta_1}$, $\eta_2=$the point on L_4 such that the cross-ration $(L_2\cap L_4, L_3\cap L_4, L_5\cap L_4, \eta_2)=1/2$, $J_{\eta_2}=m_{X,\eta_2}$.

In this case $\tilde{\Gamma}=$



and $\Gamma'=$



Case II. The cross-ratio $(L_2\cap L_4, L_3\cap L_4, L_5\cap L_4, L_7\cap L_4)=1/2$. Then,

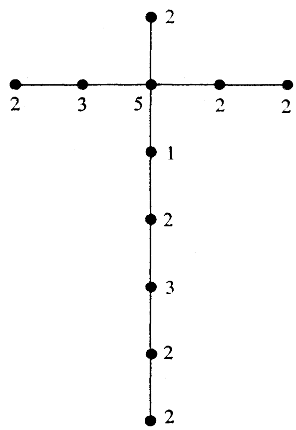
$$\tilde{\Omega}^2=J\mathcal{O}_X(-M+K_X)$$

where M is the same as before, and cosupport $(J)=\{\eta_1\}$, where $\eta_1=L_4\cap L_7$. If (u_1, v_1) are local coordinates at η_1 such that L_4 is given by $u_1=0$ and L_7 by $v_1=0$,

$$J_{\eta_1}=(u_1, v_1^2)\mathcal{O}_{X,\eta_1}$$

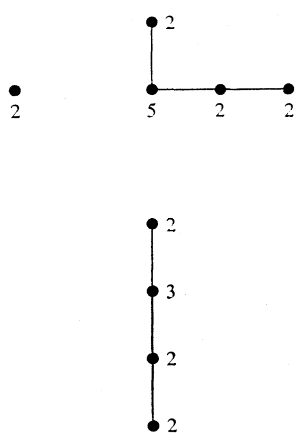
Thus,

$\tilde{\Gamma}=$



and

$\Gamma=$



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