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Sheaf Theoretic L^2-Cohomology

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If M is a compact manifold, then we have the famous de Rham isomorphism: $H_i(M)\cong\mathrm{Hom}\,(H^i_{DR}(M), \boldsymbol{R})$. In this paper, we show that this isomorphism can be generalized to the case of the so-called Thom-Mather's stratified spaces by replacing the simplicial homology by the intersection homology and the de Rham cohomology by the L^2-cohomology. This assertion has already been verified in [1] and [6]. Here we reconstruct the proof from the sheaf theoretic viewpoint developed by Goresky-MacPherson ([4]) and Cheeger ([1]).

§ 1. L^2-cohomology and intersection homology: Main Theorem

From now on, X is an n-dimensional compact stratified space without boundary. We will fix a stratification

$$X=X_n\supset X_{n-1}=X_{n-2}(=\Sigma)\supset X_{n-3}\supset\cdots\supset X_1\supset X_0,$$

and a tubular neighborhood system and, moreover, ~~the~~ PL-structure compatible with these structures.

Let g be a metric on $X-\Sigma$, and let d_i be the exterior derivative on $X-\Sigma$ with domain

$$\mathrm{dom}\, d_i=\{\omega\in A^i(X-\Sigma)\cap L^2A^i(X-\Sigma)\,|\,d\omega\in L^2A^{i+1}(X-\Sigma)\}.$$

The i-th cohomology group of the cochain complex $\{\mathrm{dom}\, d_i\}$ is called the i-th *L^2-cohomology group* and denoted by $H^i_{(2)}(X-\Sigma)$.

Next, taking account of the PL-structure of X, we define the intersection homology. Let $\bar{p}=(p_2, p_3, \cdots, p_n)$ be a *perversity*, i.e., a sequence of non-negative integers satisfying $p_2=0$ and $p_k\leq p_{k+1}\leq p_k+1$ for all k. The perversities which are of particular importance are as follows:

$\bar{0}=(0, \cdots, 0)$, the *zero perversity*,

$\bar{m}=(0, 0, 1, 1, 2, 2, \cdots)$, $m_k=[k/2]-1$, the (*lower*) *middle perversity*,

$\bar{t}=(0, 1, 2, 3, \cdots)$, the *top perversity*.

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We write $\bar{p}\leq\bar{q}$ when $p_k\leq q_k$ for all k. We also set

$$\bar{p}+\bar{q}=(p_2+q_2,\ p_3+q_3,\ \cdots).$$

The perversity $\bar{q}$ is said to be the *complementary perversity* of $\bar{p}$ if $\bar{p}+\bar{q}=\bar{t}$. Then, take an integer i. A subspace Y of X is called $(\bar{p}, i)$-*allowable* if $\dim Y\leq i$ and $\dim(Y\cap X_{n-k})\leq i-k+p_k$ for all k. For example, Y is $(\bar{0}, \dim Y)$-allowable means that Y and the strata are in general position. Now, let's set

$$IC_i^{\bar{p}}(X)=\left\{\xi\in C_i(X)\,\middle|\,\begin{matrix}|\xi| \text{ is } (\bar{p}, i)\text{-allowable and}\\ |\partial\xi| \text{ is } (\bar{p}, i-1)\text{-allowable.}\end{matrix}\right\}.$$

Then, the i-th homology group of the chain complex $\{IC_i^{\bar{p}}(X)\}$ is called the i-th *intersection homology group* with the perversity $\bar{p}$ and denoted by $IH_i^{\bar{p}}(X)$.

Now we may remark that the perversities which are interesting here or which we wish to treat here are the perversities which are smaller than the middle perversity, i.e., $\bar{p}\leq\bar{m}$. This restriction seems to be not essential (see the remark following Definition 1.1).

We now return to the L^2-cohomology and define the metric associated to a given perversity and then state the main theorem.

For a non-negative real number c and a Riemannian manifold Y with metric g, we set

$$C^c(Y)=\text{"the Riemannian manifold } (0, 1)\times Y \text{ with metric } dr\otimes dr+r^{2c}g\text{"}.$$

Now, fix a sequence of non-negative real numbers $\bar{c}=(c_2, c_3, \cdots, c_n)$. The metric g on $X-\Sigma$ is said to be *associated to* $\bar{c}$ if, for any point x of any non-empty stratum $X_{n-k}-X_{n-k-1}$, there exists a neighborhood $x\in U\subset X$ such that

$$U\cap(X-\Sigma)\underset{\text{quasi-isometry}}{\sim} C^{c_k}((\text{the link of } X_{n-k}-X_{n-k-1})\cap(X-\Sigma) \text{ with } g)$$
$$\underset{\text{product}}{\times}(U\cap(X_{n-k}-X_{n-k-1}) \text{ with Euclidean metric}).$$

Definition 1.1. The metric g on $X-\Sigma$ is said to be *associated to the perversity* $\bar{p}$ $(\leq\bar{m})$ if g is associated to $\bar{c}=(c_2, c_3, \cdots, c_n)$:

$$\begin{cases}(k-1-2p_k)^{-1}\leq c_k<(k-3-2p_k)^{-1}; & 2p_k\leq k-3,\\ 1\leq c_k<\infty \quad ; & 2p_k=k-2.\end{cases}$$

If we want to treat the perversities which are larger than $\bar{m}$ or which are not comparable with $\bar{m}$, it will suffice to change (certain) c_k's to suitable negative numbers.

It is noteworthy here that, if $1 \leq c_k < \infty$ for all k and the metric g is associated to $\bar{c}$, then the metric g is associated to the middle perversity $\bar{m}$. This case was studied by J. Cheeger ([1]).

We will use the notation $(X-\Sigma)_{\bar{p}}$ in order to make it explicit that the metric under consideration is associated to $\bar{p}$. Then we can state the following.

Main Theorem. *If $\bar{p} \leq \bar{m}$, then*

$$IH_i^{\bar{p}}(X) \cong \mathrm{Hom}\,(H^i_{(2)}((X-\Sigma)_{\bar{p}}), \boldsymbol{R}).$$

The following two sections are preparations for the proof of Main Theorem from the sheaf theoretic viewpoint.

§ 2. Sheaf theoretic L^2-cohomology and intersection homology

As above, the non-singular part $X-\Sigma$ is endowed with metric g. Let $\Omega^{\cdot}$ be the complex of sheaves on X which is defined by

$$\Gamma(U, \Omega^i) = \left\{ \omega \in A^i(U \cap (X-\Sigma)) \;\middle|\; \begin{array}{l} \text{For any point } x \text{ of } U, \text{ there exists} \\ \text{a neighborhood } x \in V \subset U \text{ such that} \\ \displaystyle\int_{V \cap (X-\Sigma)} \omega \wedge *\omega < \infty \\ \displaystyle\int_{V \cap (X-\Sigma)} d\omega \wedge *d\omega < \infty \end{array} \right\}$$

with the sheaf maps $d\colon \Omega^i \to \Omega^{i+1}$ induced by the exterior derivative on $X-\Sigma$. In order to indicate that the metric g is associated to a given perversity $\bar{p}$, we will use the notation $\Omega^{\cdot}_{\bar{p}}$.

Next, paying attention to the PL-structure of X, we will define a complex of sheaves $\mathscr{I}\mathscr{C}^{\cdot}_{\bar{p}}$. We first define the sheaf $\mathscr{C}_i$ by

$\Gamma(U, \mathscr{C}_i) =$ "the group of locally finite i-dimensional simplicial chains with respect to the induced PL-structure of U".

For convenience, we set $\mathscr{C}^{\cdot} = \mathscr{C}_{-\cdot}$ and regard this as a complex of sheaves, being induced by the simplicial boundary operator. Then we define its subcomplex $\mathscr{I}\mathscr{C}^{\cdot}_{\bar{p}}$ by

$$\Gamma(U, \mathscr{I}\mathscr{C}^{-i}_{\bar{p}}) = \left\{ \xi \in \Gamma(U, \mathscr{C}^{-i}) \;\middle|\; \begin{array}{l} |\xi| \text{ is } (\bar{p}, i)\text{-allowable and } |\partial\xi| \text{ is} \\ (\bar{p}, i-1)\text{-allowable with respect} \\ \text{to the induced stratification of } U. \end{array} \right\}.$$

Now $\Omega^{\cdot}_{\bar{p}}$ and $\mathscr{I}\mathscr{C}^{\cdot}_{\bar{p}}$ are fine sheaves. Therefore we have

Lemma 2.1.

$$\mathcal{H}^i(X, \Omega^\bullet_{\bar{p}}) \cong H^i_{(2)}((X-\Sigma)_{\bar{p}}),$$
$$\mathcal{H}^{-i}(X, \mathcal{I}\mathcal{C}^\bullet_{\bar{p}}) \cong IH^{\bar{p}}_i(X).$$

Here $\mathcal{H}^*(X, \cdot)$ denotes the hypercohomology.

§ 3. Key Theorem due to Goresky and MacPherson

Let $\mathcal{S}^\bullet$ be a complex of sheaves on X which is constructible with respect to the given stratification $\{X_k\}$ (that is, for any j, $\mathcal{S}^\bullet|_{X_j - X_{j-1}}$ is cohomologically locally constant).

Definition 3.1 ([4]). We say that $\mathcal{S}^\bullet$ satisfies the *axiom* $[\mathrm{AX1}]_{\bar{p}}$ provided:

(a) $\mathcal{S}^\bullet|_{X-\Sigma} \cong \boldsymbol{R}[n]^\bullet$ (the isomorphism in the derived category),
(b) $\mathcal{H}^i(\mathcal{S}^\bullet)=0$ for all $i<-n$,
(c) $\mathcal{H}^m(\mathcal{S}^\bullet|_{X-X_{n-k-1}})=0$ for all $m>p_k-n$,
(d) the attaching maps (in the derived category)

$$\mathcal{H}^m(j_k^*\mathcal{S}^\bullet|_{X-X_{n-k-1}}) \longrightarrow \mathcal{H}^m(j_k^*\mathcal{R}i_{k*}i_k^*\mathcal{S}^\bullet|_{X-X_{n-k-1}})$$

are isomorphisms for all $m \leq p_k - n$.

Here $\mathcal{H}^*(\cdot)$ denotes the cohomology sheaf. Also $i_k\colon X-X_{n-k} \to X-X_{n-k-1}$ and $j_k\colon X_{n-k}-X_{n-k-1} \to X-X_{n-k-1}$ are the inclusion maps.

Then, according to [4], we have

Key Theorem (Goresky and MacPherson).

(1) *The constructible complex of sheaves which satisfies the axiom* $[\mathrm{AX1}]_{\bar{p}}$ *is unique up to isomorphism in the derived category.*

(2) $\mathcal{I}\mathcal{C}^\bullet_{\bar{p}}$ *is constructible and satisfies the axiom* $[\mathrm{AX1}]_{\bar{p}}$.

(3) *If* $\bar{p}+\bar{q}=\bar{t}$, *then* $\mathcal{I}\mathcal{C}^\bullet_{\bar{p}} \cong \mathcal{R}\mathcal{H}om(\mathcal{I}\mathcal{C}^\bullet_{\bar{q}}, \mathcal{D}^\bullet_X)[n]^\bullet$, *where* $\mathcal{D}^\bullet_X$ *is the dualizing complex on* X, i.e., $\mathcal{D}^\bullet_X = f^!\boldsymbol{R}_{pt}$ *with* $f\colon X \to (point)$.

§ 4. Proof of Main Theorem

It suffices to prove

Assertion. $\Omega^\bullet_{\bar{p}}[n]$ *is constructible and satisfies the axiom* $[\mathrm{AX1}]_{\bar{q}}$, *where* $\bar{q}$ *is the complementary perversity*, $\bar{p}+\bar{q}=\bar{t}$.

In fact, if we assume the above, we have

Proof of Main Theorem. From Key Theorem (1) and Assertion, we have

$$\Omega_{\bar{p}}^{\cdot}[n] \cong \mathscr{I}\mathscr{C}_{\bar{p}}^{\cdot}.$$

Therefore, by substituting $\Omega_{\bar{p}}^{\cdot}[n]$ for $\mathscr{I}\mathscr{C}_{\bar{p}}^{\cdot}$ in Key Theorem (3), we get

$$\mathscr{I}\mathscr{C}_{\bar{p}}^{\cdot} \cong \mathscr{R}\mathscr{H}om(\Omega_{\bar{p}}^{\cdot}, \mathscr{D}_X^{\cdot})^{\cdot}.$$

Hence, by the Verdier duality theorem, we have

$$\mathscr{H}^{-i}(X, \mathscr{I}\mathscr{C}_{\bar{p}}^{\cdot}) \cong \mathrm{Hom}(\mathscr{H}^i(X, \Omega_{\bar{p}}^{\cdot}), \boldsymbol{R}).$$

Thus, combined with Lemma 2.1, the proof is complete.

Now we will prove Assertion. It suffices to examine (a)–(d) of $[\mathrm{AX1}]_{\bar{q}}$. The constructibility of $\Omega_{\bar{p}}^{\cdot}[n]$ will be shown on the way.

(a) Since $\Omega_{\bar{p}}^{\cdot}[n]|_{X-\Sigma}$ is the sheaf of C^∞-forms on $X-\Sigma$, we have $\Omega_{\bar{p}}^{\cdot}[n]|_{X-\Sigma} \cong \boldsymbol{R}_{X-\Sigma}[n]^{\cdot}$ because of the usual resolution.

(b) If $i<-n$, then $(\Omega_{\bar{p}}^{\cdot}[n])^i = \Omega_{\bar{p}}^{i+n} = 0$. Therefore $\mathscr{H}^i(\Omega_{\bar{p}}^{\cdot}[n]) = 0$ for all $i<-n$.

(*Preparation for* (c) *and* (d)) For a point x of $X_{n-k} - X_{n-k-1}$, take a suitable neighborhood U and the link L of the stratum at x. Then we have

$$\mathscr{H}^j(\Omega_{\bar{p}}^{\cdot})_x \cong H_{(2)}^j(U \cap (X-\Sigma)). \tag{4.1}$$

Strictly writing, the right hand side of (4.1) should be the inductive limit $\varinjlim_U H_{(2)}^j(U \cap (X-\Sigma))$. But, for sufficiently small U, it is naturally isomorphic to $H_{(2)}^j(U \cap (X-\Sigma))$ because the L^2-cohomology is invariant under the quasi-isometric transformation. Hence, also, $\Omega_{\bar{p}}^{\cdot}$ can be regarded as constructible. Moreover, (4.1) is isomorphic to

$$\begin{aligned} &H_{(2)}^j(C^{c_k}(L \cap (X-\Sigma)) \times (U \cap (X_{n-k} - X_{n-k-1}))) \\ &\cong H_{(2)}^j(C^{c_k}(L \cap (X-\Sigma))) \\ &\cong \begin{cases} H_{(2)}^j(L \cap (X-\Sigma)); & j < \dfrac{1}{2}\left(k-1+\dfrac{1}{c_k}\right), \\ 0 & ; \quad j \geq \dfrac{1}{2}\left(k-1+\dfrac{1}{c_k}\right), \end{cases} \end{aligned}$$

through the natural extension maps ([6, Lemma 3.12]). Hence

$$\mathscr{H}^j(\Omega_{\bar{p}}^{\cdot})_x \cong \begin{cases} H_{(2)}^j(L \cap (X-\Sigma)); & j \leq q_k, \\ 0 & ; \quad j > q_k, \end{cases} \tag{4.2}$$

because

$$q_k < \frac{1}{2}\left(k-1+\frac{1}{c_k}\right) \leq q_k+1.$$

(c) This is equivalent to the assertion that, if $j>q_k$, then $\mathscr{H}^j(\Omega_{\bar{p}}^{\cdot})_x=0$ for any point x of $X_{n-k}-X_{n-k-1}$. Hence, by (4.2), this is true.

(d) This is equivalent to the assertion that, if $j\leq q_k$, then the attaching maps

$$\mathscr{H}^j(\Omega_{\bar{p}}^{\cdot}|_{X-X_{n-k-1}})_x \longrightarrow \mathscr{H}^j(i_{k*}i_k^*\Omega_{\bar{p}}^{\cdot}|_{X-X_{n-k-1}})_x \tag{4.3}$$

are isomorphisms for any point x of $X_{n-k}-X_{n-k-1}$.

In order to prove this assertion, first remark that a cross section of $\Omega_{\bar{p}}^{\cdot}|_{X_{n-k-1}}$ resp. $i_{k*}i_k^*\Omega_{\bar{p}}^{\cdot}|_{X-X_{n-k-1}}$ is a smooth form which and whose image by the exterior derivative are square-integrable near any point of $X-X_{n-k-1}$ resp. $X-X_{n-k}$. (For a cross section ω of $i_{k*}i_k^*\Omega_{\bar{p}}^{\cdot}|_{X-X_{n-k-1}}$, it is not necessary to claim that ω and $d\omega$ are square-integrable near any point of $X_{n-k}-X_{n-k-1}$.) Therefore we have the natural sheaf map

$$\Omega_{\bar{p}}^{\cdot}|_{X-X_{n-k-1}} \longrightarrow i_{k*}i_k^*\Omega_{\bar{p}}^{\cdot}|_{X-X_{n-k-1}}.$$

And this induces the attaching map (4.3). Now, from the property of $i_{k*}i_k^*\Omega_{\bar{p}}^{\cdot}|_{X-X_{n-k-1}}$ mentioned above, we have

$$\mathscr{H}^j(i_{k*}i_k^*\Omega_{\bar{p}}^{\cdot}|_{X-X_{n-k-1}})_x \cong \begin{cases} H_{(2)}^j(L\cap(X-\Sigma)); & j<k, \\ 0 & ; \; j\geq k, \end{cases} \tag{4.4}$$

for any point x of $X_{n-k}-X_{n-k-1}$. Hence, for $j\leq q_k$, the identity map from the right hand side of (4.2) to the right hand side of (4.4) is just the attaching map (4.3). Thus the proof of (d) is complete.

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