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Introduction to the L^2-Cohomology of Arithmetic Quotients of Bounded Symmetric Domains

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Let

G = the group of real points on a semi-simple group $\boldsymbol{G}$ defined over $\boldsymbol{Q}$
Γ = a neat arithmetic subgroup of G (see beginning of § 3)
K = a maximal compact subgroup of G
$\mathfrak{X}$ = the symmetric space G/K, assigned a G-invariant Riemannian metric and assumed to possess a G-invariant complex structure.

It has been conjectured [29] that the L^2-cohomology of the quotient $V = \Gamma\backslash\mathfrak{X}$ is naturally isomorphic to the middle intersection cohomology [22] of the compactification V^* of V constructed by Satake [26] in certain cases and by Baily and Borel [2] in general. I recall that V^* is a projective, normal, but in general highly singular algebraic variety. This conjecture was verified by Zucker himself [29] for a few cases where $\boldsymbol{G}$ has rational rank one, by Borel for all remaining groups of rational rank one (an announcement appears in [4]), and by Zucker [30] for a few cases of rational rank two. Very recently Borel and I working together have concluded the proof for all rational rank two groups (an announcement [6] will appear soon).

On the one hand, the proof that Borel and I have concocted seems even to us extraordinarily complicated. On the other, large parts of it carry through for groups of arbitrary rational rank and it certainly looks plausible (to me, at least) that an extension of our techniques will eventually work in general. What I propose to do in this paper, therefore, is to explain bits and pieces of what we have done in a relatively informal way. Actually the paper may be divided into two relatively independent parts: the first two sections form a general introduction to L^2-cohomology, and the last three deal more particularly with Zucker's conjecture. There is much overlap in the first two sections with [15], but I have made more precise the connection between the obvious duality of L^2-cohomology and a corresponding, more technical, duality of associated complexes of sheaves.

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The point of the general treatment in these sections is that although much of what Borel and I have done looks at first very special to the particular case we are looking at, I think that when posed in the proper terminology many of the results—but of course few of the proofs—will hold for a larger class of manifolds with negligible boundary.

My talk in the conference at Tsukuba also included some (foolish) speculation on the analogous conjecture made in [15] for arbitrary singular projective varieties. If M^* is such a variety, embedded say in $\boldsymbol{P}^n(\boldsymbol{C})$, let M be the subset of its non-singular points, assigned the metric induced from a $\boldsymbol{U}(n)$-invariant metric on $\boldsymbol{P}^n(\boldsymbol{C})$. Unless $M=M^*$, this metric will be incomplete, and at first sight one might expect analysis on M to have an entirely different flavour from that on the complete Riemannian manifolds occurring in Zucker's conjecture. Nonetheless, one of the assertions implicit in the conjecture of [15]—that the L^2-cohomology of M and the middle intersection cohomology of M^*coincide—is that although questions involving Laplacian operators may be very difficult, at least M is likely to have a negligible boundary, and consequently much of the discussion in Sections 1 and 2 is relevant. I had hoped to include in this paper something more detailed along these lines—something less naive than what I said at the conference—including a conjecture relating D-modules and the domain of the Laplacian, but this proved too difficult for me.

I would like to take this opportunity to thank especially one of the organizers of the Tsukuba conference, Tatsuo Suwa, for his generous hospitality both at the conference and at his home university in Sapporo. I would also like to thank the National Science and Engineering Research Council of Canada, the Max Planck Institut für Mathematik, and the Institut des Hautes Etudes Scientifiques for financial support during the time I have been working on this material.

§ 1. Analysis and cohomology on Riemannian manifolds

Let M be any oriented Riemannian manifold. On the space of $\boldsymbol{C}$-valued differential forms one has the norm

$$\langle\varphi, \psi\rangle=\int_M \varphi\wedge *\bar{\psi}. \tag{1.1}$$

Let

$\Omega_2^{\cdot}(M, \boldsymbol{C}):=$the Hilbert space of square integrable differential forms on M.

Associate to the de Rham differential two unbounded operators on this Hilbert space. I shall write the first as d_M with domain

$$\mathrm{Dom}\,(d_M):=\{C^\infty \text{ forms } \omega \text{ on } M \mid \omega,\, d\omega \text{ are square-integrable}\}$$

and the second as $d_{c,M}$ with domain the space of all C^∞ forms on M with compact support. When confusion is unlikely, the subscript M will be left out. Because d possesses a formal adjoint δ, the graph closures of both of these are also well-defined unbounded operators $\bar{d}$ and $\bar{d}_c$ on $\Omega_2^\cdot(M, \boldsymbol{C})$. One can define similarly closed operators $\bar{\delta}$ and $\bar{\delta}_c$. It was apparently first proven by Gaffney (see [18] and [19]) that:

1.1. Proposition. *For any Riemannian manifold M, the pairs of operators $\bar{d}$ and $\bar{\delta}_c$ on the one hand and $\bar{d}_c$ and $\bar{\delta}$ on the other are adjoint pairs in the technical sense.*

The Appendix to [13] also contains a proof of this. I recommend here and elsewhere Chapter VII of [25] for a readable summary of the theory of unbounded operators on Hilbert spaces.

In general, the operators $\bar{d}$ and $\bar{d}_c$ will be quite different. This can be seen most simply when $M=(0, 1)$ with metric dx^2: the domain of $\bar{d}_c$ in $L^2(0, 1)$ consists of absolutely continuous functions f with $f(0)=f(1)=0$, while the domain of $\bar{d}$ consists of the restrictions to $(0, 1)$ of functions absolutely continuous on all of $\boldsymbol{R}$. If $\bar{d}$ and $\bar{d}_c$ are the same, then Gaffney says that M has *negligible boundary*. This is not quite accurate terminology, as will be seen in a moment.

Let Δ_c be the Laplacian differential operator with domain the space of all C^∞ forms of compact support. Of course this operator is formally self-adjoint, that is to say symmetric on its domain. Its closure will always be well-defined, but even when M has negligible boundary in Gaffney's sense this closure may not be self-adjoint. (Cheeger in [14] points out that the double covering of $\boldsymbol{C}$ ramified at 0 and assigned the metric induced from the standard metric on $\boldsymbol{C}$ offers an example of this as well as other interesting phenomena.)

Define an unbounded operator Δ_G by specifying its domain to be the space of all those forms ω in $\Omega_2^\cdot(M, \boldsymbol{C})$ such that $\delta\omega$, $d\omega$, $d\delta\omega$, $\delta d\omega$ are all square-integrable, and on this domain define $\Delta_G\omega$ to be just $(d\delta+\delta d)\omega$.

In general, this operator will not be symmetric. However:

1.2. Proposition. *When M has negligible boundary, Δ_G is self-adjoint.*

This is a result of [21] (see also [18]).

1.3. Proposition. *If M is complete, then*

(a) *M has negligible boundary;*

(b) *The closure of Δ_c is the same as Δ_G.*

Part (a) is due to Gaffney [20], and (b) seems to be at least implicit in his work, although it has been rediscovered many times independently. (See [16] for a more thorough result and references.)

The domain of $\bar{d}$ and $\bar{d}_c$ are both complexes. Define two kinds of L^2-cohomology:

$$H^{\bullet}_{(2)}(M, \boldsymbol{C}) := \text{cohomology of Dom}\,(\bar{d})$$

$$H^{\bullet}_{(2),c}(M, \boldsymbol{C}) := \text{cohomology of Dom}\,(\bar{d}_c).$$

Cheeger in the Appendix of [13] shows:

1.4. Proposition. *The inclusion of* Dom (d) *in* Dom $(\bar{d})$ *induces an isomorphism of cohomology.*

That is to say, this inclusion is a *quasi-isomorphism*. This result implies among other things that when M is compact the L^2-cohomology and the ordinary cohomology agree. Incidentally, the L^2-cohomology is often defined to be the cohomology of the complex Dom (d), but this is a definition almost impossible to work with directly.

1.5. Proposition. *If either $H^{\bullet}_{(2)}(M, \boldsymbol{C})$ or $H^{\bullet}_{(2),c}(M, \boldsymbol{C})$ is finite dimensional, then so is the other, and the two are canonically dual in complementary dimensions.*

Proof. If T is any closable operator on a Hilbert space H, then it is always true that

$$H = \overline{\operatorname{Ran}(T)} \oplus \operatorname{Ker}(T^*). \tag{1.2}$$

If T is a closed operator, then its domain is a Hilbert space with norm $\|x\|^2 + \|T(x)\|^2$. Furthermore, it is an exercise to see that if T has closed range then so does T^*. Therefore by the Closed Graph Theorem [27: Theorem 17.1], if the cohomology of the complex Dom $(\bar{d})$—for example—is finite-dimensional the ranges of (*a*) $\bar{d}$ and (*b*) $\bar{\delta}_c$ are closed. Then (1.2) with $T = \bar{d}$ together with 1.1 imply

$$H = \operatorname{Ran}(\bar{d}) \oplus \operatorname{Ker}(\bar{\delta}_c) \tag{1.3a}$$

$$H = \operatorname{Ker}(\bar{d}) \oplus \operatorname{Ran}(\bar{\delta}_c) \tag{1.3b}$$

One deduces Proposition 1.5 by applying the map $*$. □

Recall that if $K^{\bullet}$ is a complex, then the dual complex $L^{\bullet}$ is defined as the sum of graded modules

$$L^m := \operatorname{Hom}_{\boldsymbol{C}}(K^{-m}, \boldsymbol{C}),$$

with the transpose differential, and recall also that the shifted complex $K[l]^{\cdot}$ is defined by

$$K[l]^m := K^{l+m}$$

with the differential shifted in sign by $(-1)^l$. In accord with these conventions, another way to formulate 1.5 is this:

1.6. Corollary. *When $H^{\cdot}_{(2)}(M, \boldsymbol{C})$ or $H^{\cdot}_{(2),c}(M, \boldsymbol{C})$ is finite-dimensional, integration of the exterior product induces a quasi-isomorphism*

$$\mathrm{Dom}\,(\bar{d}) \to \mathrm{Hom}_{\boldsymbol{C}}\,(\mathrm{Dom}\,(\bar{d}_c), \boldsymbol{C})[-n]$$

where $n = \dim(M)$.

1.7. Corollary. *If M has negligible boundary and $H^{\cdot}_{(2)}(M, \boldsymbol{C})$ is finite-dimensional, then $H^{\cdot}_{(2)}(M, \boldsymbol{C})$ satisfies Poincaré duality.*

Assume for the rest of this section that M has negligible boundary and finite-dimensional L^2-cohomology. From (1.3a) we deduce the direct-sum decomposition

$$\Omega^{\cdot}_2(M, \boldsymbol{C}) = \mathrm{Ran}\,(\bar{d}) \oplus \mathfrak{H}^{\cdot}(M, \boldsymbol{C}) \oplus \mathrm{Ran}\,(\bar{\delta}), \tag{1.4}$$

where $\mathfrak{H}^{\cdot}(M, \boldsymbol{C})$ is $\mathrm{Ker}\,(\Delta_G)$, the space of *harmonic* forms. This implies that $H^{\cdot}_{(2)}(M, \boldsymbol{C})$ and $\mathfrak{H}^{\cdot}(M, \boldsymbol{C})$, are isomorphic. One consequence of this is that if M is a Kähler manifold then the decomposition of forms into (p, q)-type induces a corresponding decomposition of the L^2-cohomology groups. Another consequence is more technical. Define a *Sobolev form* to be a form ω such that all the forms $D\omega$, where D is any product of d and δ, are square-integrable. Define

$$\Omega^{\cdot}_{2,\infty}(M, \boldsymbol{C}) := \text{the space of all Sobolev forms on } M.$$

It is a reflexive Fréchet space with the semi-norms $\|D\omega\|^2$; When M has negligible boundary, this space can also be characterized as the intersection of the domains of all powers of Δ_G. By Sobolev's Lemma, any Sobolev form must be at least locally smooth. *Under the assumption that M has negligible boundary*, define also the space of *Sobolev currents*:

$$C^m_2(M, \boldsymbol{C}) := \text{strong dual of } \Omega^m_{2,\infty}(M, \boldsymbol{C}).$$

This becomes a complex according to the rule

$$\langle dF, f\rangle = \langle F, \delta f\rangle$$

for F a current, f a Sobolev form. Thus the inclusion of $\mathrm{Dom}(\bar{d})$ in $C_2^{\bullet}$ is compatible with differentials. In other words, we have inclusions of complexes

$$\Omega_{2,\infty}^{\bullet}(M, \boldsymbol{C}) \subseteq \mathrm{Dom}(\bar{d}_M) \subseteq C_2^{\bullet}(M, \boldsymbol{C}). \tag{1.5}$$

1.8. Proposition. *If M has negligible boundary and finite-dimensional L^2-cohomology, these inclusions are quasi-isomorphisms.*

This is because under the given hypothesis $\Omega_{2,\infty}^{\bullet}(M, \boldsymbol{C})$ possesses a decomposition analogous to (1.4). This result is suggestive rather than useful, because in practice one proves this result directly by a regularization procedure, and then applies it in order to deduce the finite-dimensionality of $H_{(2)}^{\bullet}(M, \boldsymbol{C})$.

It will appear in the next section that even if one's ultimate interest is a manifold M which is complete, one is forced to consider the L^2-cohomology of subsets of M—the intersections of M with neighborhoods of points in a compactification of M—which are not complete.

I have so far discussed only L^2-cohomology with trivial coefficients, but everything done so far applies with slight modification to L^2-cohomology with coefficients in any locally constant system E which may not be trivial and where one is given a Hermitian metric on the fibres. I shall write $d_{M,E}$ etc. for the corresponding objects. The most significant modification is that the duality of 1.5 is between L^2-cohomology with coefficients in E and "compactly supported" L^2-cohomology with coefficients in the dual system E^*.

§ 2. L^2-cohomology and sheaves

If M is embedded as a dense open subset of a compact Hausdorff space $\overline{M}$, I shall call $\overline{M}$ a *compactification* of M. The conditions of square-integrability on forms on M may then be interpreted as conditions local on $\overline{M}$. For example, we may define presheaves $\mathrm{Dom}(\bar{d})$ and $\Omega_{2,\infty}^{\bullet}$ on $\overline{M}$ by assigning $\mathrm{Dom}(\bar{d}_{U\cap M})$ or $\Omega_{2,\infty}^{\bullet}(U \cap M, \boldsymbol{C})$ to an open set U in $\overline{M}$. Let

$$\mathscr{D}^{\bullet} := \mathscr{D}om\,(\bar{d})$$

and $\Omega_{2,\infty,\mathrm{loc}}^{\bullet}$ be the corresponding sheaves. These are both complexes. The cohomology of the stalks of $\mathscr{D}^{\bullet}$ at a point x is called the *local L^2-cohomology* at x, while that of the stalks of $\Omega_{2,\infty,\mathrm{loc}}^{\bullet}$ I shall call the local $L^{2,\infty}$-cohomology. Proposition 1.4 implies that on M itself (a) the second sheaf is just the de Rham complex and (b) the inclusion of the second in the first is a quasi-isomorphism.

2.1. Lemma. *The sheaf $\mathscr{D}^{\cdot}$ is fine if for every x in $\bar{M}$ and open neighborhood U of x there exists a function f with support in U such that* (a) *f is identically* 1 *near x and* (b) *both f and df are bounded on $U \cap M$.*

In practice, when this is true it is simple to verify.

Assume from now on in this section that $\bar{M}$ is a compactification of M, that M has negligible boundary, and that $\mathscr{D}^{\cdot}$ is fine.

Recall that if $\mathscr{F}$ is any sheaf and U is an open set of $\bar{M}$, then $\Gamma_c(U, \mathscr{F})$ is the space of sections of $\mathscr{F}$ with support in a compact subset of U. If $\mathscr{F} = \mathscr{D}^{\cdot}$ (resp. $\Omega^{\cdot}_{2,\infty,\mathrm{loc}}$) then $\Gamma_c(U, \mathscr{F})$ may also be described as the set of ω in $\mathrm{Dom}(\bar{d}_{U \cap M})$ (resp. $\Omega^{\cdot}_{2,\infty}(U \cap M, \boldsymbol{C})$) with support in a compact subset of U.

Given U open in $\bar{M}$, define the restriction $d_{r, U \cap M}$ of $d_{U \cap M}$ by specifying its domain to be the subspace of ω in $\mathrm{Dom}(d_{U \cap M})$ such that the support of ω is compact in U.

2.2. Lemma. *For any open set U in $\bar{M}$, the operators $\bar{d}_{r, U \cap M}$ and $\bar{d}_{c, U \cap M}$ are the same.*

Proof. Because $d_c \subseteq d_r$ and $\bar{d}_c$ is the same as δ^*, it suffices to show that $d_r \subseteq d^*$, or that $\langle \varphi, \delta\psi \rangle = \langle d\varphi, \psi \rangle$ for any $\varphi \in \mathrm{Dom}(d_r)$ and $\psi \in \mathrm{Dom}(\delta)$ (I am dropping the subscript $U \cap M$ here). Since $\mathscr{D}^{\cdot}$ is fine, one can find Ψ in $\mathrm{Dom}(\bar{\delta}_M)$ agreeing with ψ on $\mathrm{supp}(\varphi)$. If one defines Φ to be φ inside U and 0 outside it, then Φ is also in $\mathrm{Dom}(\bar{d}_M)$. Since M has negligible boundary,

$$\begin{aligned}\langle \varphi, \delta\psi \rangle &= \langle \Phi, \delta\Psi \rangle \\ &= \langle d\Phi, \Psi \rangle. \\ &= \langle d\varphi, \psi \rangle.\end{aligned}$$

□

The dual of the sheaf complex $\mathscr{D}^{\cdot}$ is the presheaf complex

(2.1) $$\mathscr{L}^{\cdot} = \mathrm{Hom}_{\boldsymbol{C}}(\Gamma_c(\mathscr{D}^{\cdot}), \boldsymbol{C})$$

where the grading is the conventional one specified earlier. Since $\mathscr{D}^{\cdot}$ is fine this is in fact a sheaf. Integration of the exterior product gives a (graded) homomorphism of complexes

(2.2) $$\mathscr{D}^{\cdot} \to \mathscr{L}^{\cdot}[-n]$$

where $n = \dim_{\boldsymbol{R}}(M)$.

The global duality of Proposition 1.7 has the following local version:

2.3. Proposition. *Continue to assume that $\overline{M}$ is a compactification of M, that M has negligible boundary, and that the sheaf $\mathscr{D}om(\bar{d})$ is fine. If every point of $\overline{M}$ has a basis of neighborhoods U with the property that $H^{\cdot}_{(2)}(U \cap M, \boldsymbol{C})$ is finite-dimensional, then the sheaf complex $\mathscr{D}^{\cdot} = \mathscr{D}om(\bar{d})$ is self-dual, in the sense that the canonical map* (2.2) *is a quasi-isomorphism.*

Proof. For any U open in $\overline{M}$ the space $\Gamma_c(U, \mathscr{D}^{\cdot})$ is contained in $\mathrm{Dom}(\bar{d}_{r,U\cap M})$ and if W in turn is relatively compact in U then $\mathrm{Dom}(\bar{d}_{r,W\cap M})$ is contained in $\Gamma_c(U, \mathscr{D}^{\cdot})$. Hence the sheaf associated to the presheaf $L_0^{\cdot} = \mathrm{Hom}_{\boldsymbol{C}}(\mathrm{Dom}(\bar{d}_{r,U\cap M}), \boldsymbol{C})$ is identical with $\mathscr{L}^{\cdot} = \mathrm{Hom}_{\boldsymbol{C}}(\Gamma_c(\mathscr{D}^{\cdot}), \boldsymbol{C})$. The canonical map from $D_0^{\cdot} = \mathrm{Dom}(\bar{d})$ into $L_0^{\cdot}[-n]$ is a quasi-isomorphism by 2.2 and 1.5. Proposition 2.3 follows since cohomology commutes with direct limits. □

There does not seem to be any criterion as simple as 2.1 for deciding when $\Omega^{\cdot}_{2,\infty,\mathrm{loc}}$ is fine. *Assume for the rest of this section that it is.* For every open set U in $\overline{M}$ let $\Omega^{\cdot}_{2,\infty,r}(U, \boldsymbol{C})$ be the closure in $\Omega^{\cdot}_{2,\infty}(M, \boldsymbol{C})$ of the subspace of its elements with support relatively compact in U, and define the space of *tempered Sobolev currents* on U to be the strong dual $C_2^{\cdot}(U, \boldsymbol{C})$ of this closure. Define $\mathscr{C}_2^{\cdot}$ to be the sheaf of *Sobolev currents* associated to this presheaf. For any open U in $\overline{M}$, $\Gamma(U, \mathscr{C}_2^{\cdot})$ may also be characterized as the strong dual of the space $\Gamma_c(\Omega^{\cdot}_{2,\infty,\mathrm{loc}})$, which possesses naturally the structure of an *LF* space. The sections of $\mathscr{C}_2^{\cdot}$ over all of $\overline{M}$ are the same as the Sobolev currents on M defined before, and the tempered Sobolev currents on an open U are those Sobolev currents on U obtained by restriction from M.

2.4. Proposition. *Let U be open in $\overline{M}$. If the cohomology of either of the complexes $\Omega^{\cdot}_{2,\infty,r}(U, \boldsymbol{C})$ or $C_2^{\cdot}(U, \boldsymbol{C})$ is finite-dimensional, then so is that of the other, and the two are dual in complementary dimensions.*

This follows from

2.5. Lemma. *Let $A^{\cdot}$ be a complex of Fréchet spaces and $D^{\cdot}$ the topological dual complex.*

(a) *If $H^m(A^{\cdot})$ and $H^{m+1}(A^{\cdot})$ are both finite-dimensional, so is $H^{-m}(D^{\cdot})$;*

(b) *If $H^{-m}(D^{\cdot})$ and $H^{-m+1}(D^{\cdot})$ are both finite-dimensional, so is $H^m(A^{\cdot})$.*

In either case, $H^{-m}(D^{\cdot})$ is canonically isomorphic to the dual of $H^m(A^{\cdot})$.

Proof. It follows easily from the Closed Graph Theorem [27: Theorem 17.1] that if f is a continuous map of Fréchet spaces with finite cokernel, then its image is closed. Therefore if $H^{m+1}(A^{\cdot})$ is finite-dimensional, the subspace B^{m+1} of coboundaries in A^{m+1} is closed. The short exact sequence

$$0 \to Z^m \to A^m \to B^{m+1} \to 0$$

then gives by duality that the image of D^{-m-1} in D^{-m} is closed, and that the quotient of D^{-m} by this image may be identified with the dual of the cycles Z^m in A^m. If H^m is finite-dimensional as well, then B^m is closed in A^m, and the exact sequence

$$0 \to B^m \to Z^m \to H^m \to 0$$

gives by duality part (a) and half the last assertion.

Suppose now that $H^{-m+1}(D^\cdot)$ is finite-dimensional. Let C be the closure in A^m of the coboundaries B^m. Since B^m is dense in C, the map $C^* \to D^{-m+1}$ dual to the differential from A^{m-1} to C is an injection. Its image is the subspace of coboundaries in D^{-m+1}. Let F be a finite-dimensional complement in the subspace of cocyles, ι the injection of F into D^{-m+1}. Then the map

$$C^* \oplus F \xrightarrow{(d^*_{m-1}, \iota)} D^{-m+1}$$

is an injection onto the subspace of cocycles, hence in particular has closed image. Now this injection is the dual of (d_{m-1}, ι^*) from A^{m-1} to $C \oplus F^*$ The latter space is a Fréchet space, so by the criterion [27: Theorem 37.2] the latter map is surjective. In particular d_{m-1} maps A^{m-1} onto C, and $C = B^m$. In summary: $H^{m+1}(D^\cdot)$ finite-dimensional implies that the image of d_{m-1} is closed. But if $H^{-m}(D^\cdot)$ is also finite-dimensional, the image of d_m is also closed, and consequently so is that of d^*_m. The rest is elementary. □

In practice it often happens that in accord with the global result 1.8 the inclusions of the sheaf complexes $\Omega^\cdot_{2,\infty,\mathrm{loc}}$ in $\mathscr{D}^\cdot$ and of this in turn in $\mathscr{C}^\cdot_2$ are quasi-isomorphisms, but this seems to require some kind of regularization argument, and in particular does not seem to follow from the elementary arguments leading to 1.8 itself. Such regularization arguments are often preliminary to showing local finite-dimensionality.

These local notions also extend to an arbitrary locally constant Hermitian coefficient system E. In the rest of this paper I will be concerned with the problem of proving that in certain cases the sheaf $\mathscr{D}om(\bar{d}_E)$ on $\overline{M}$ is equivalent in a technical sense to the *middle perversity intersection complex* $\mathscr{I}\mathscr{C}^\cdot(E)$ on $\overline{M}$. The idea of the proof is, very roughly speaking, similar to that of the usual de Rham theorem, which asserts that the ordinary cohomology of any manifold is given in a natural way by its de

Rham complex. In particular, what is required here is to prove that the sheaf $\mathscr{D}om(\bar{d}_E)$ satisfies certain axioms which, as explained for example in [7: V.9] or [22:6.1], characterize $\mathscr{I}\mathscr{C}^{\cdot}(E)$.

Assume that $\bar{M}$ is a complex algebraic variety given a Whitney stratification $\bar{M} = \coprod M_i$ with M, a Zariski-open subvariety of the set of non-singular points on $\bar{M}$, equal to the complement of the proper strata. Let E be any locally constant coefficient system on M.

2.6. Proposition. *Let $\mathscr{L}^{\cdot}$ and $\mathscr{L}^{\cdot}_{*}$ be complexes of fine sheaves on $\bar{M}$. Then they are equivalent (in the derived category of $\mathbf{C}$-sheaves on $\bar{M}$) to $\mathscr{I}\mathscr{C}^{\cdot}(E)$ and $\mathscr{I}\mathscr{C}^{\cdot}(E^*)$ respectively if*:

(a) *They are non-trivial only in non-negative degrees*;

(b) *On M, $\mathscr{L}^{\cdot}$ is a resolution of E and $\mathscr{L}^{\cdot}_{*}$ of E^**;

(c) *The local cohomology groups of $\mathscr{L}^{\cdot}$ and $\mathscr{L}^{\cdot}_{*}$ are locally constant on each stratum M_i*;

(d) *$\mathscr{L}^{\cdot}$ and $\mathscr{L}^{\cdot}_{*}$ are dual to each other in the sense explained above*;

(e) *For $x \in M_i$, the local cohomology groups of each at x are trivial in degrees* $\geq \operatorname{codim}_C(M_i)$.

These axioms are all to be verified locally on $\bar{M}$. Fineness gurantees that cohomology agrees with hypercohomology, and is usually straightforward. Properties (a) and (b) will be a matter of definition, (c) will require some geometrical knowledge which one would expect to come without too much trouble. The difficult point is to describe the cohomology of the complex of sections of $\mathscr{L}^{\cdot}$ over neighborhoods of points of $\bar{M}$: in particular, if a basis of neighborhoods can be found for which this is finite-dimensional, then (d) will follow by 2.3 (or, rather, the modification necessary to deal with non-trivial coefficient systems).

§ 3. The local geometry of Baily-Borel-Satake compactifications

I now use the notation of the introduction. The main reference for this section is [1: III.2–4, 6.1]. See also [2] for fine points.

I adopt the convention that if H is a rational subgroup of G then $\Gamma(H)$ is the intersection $\Gamma \cap H$, and that if R is a quotient of H then $\Gamma(R)$ is the image of $\Gamma(H)$ in R. Similarly for $K(H)$, $K(R)$. The assumption that Γ is neat means that if P is a $\boldsymbol{Q}$-rational parabolic subgroup of G with unipotent radical N then the image of $\Gamma(P)$ in P/N has no torsion.

For convenience I shall assume from now on that the algebraic group $\boldsymbol{G}$ is almost $\boldsymbol{Q}$-simple. The symmetric space $\mathfrak{X}$ may be embedded as a bounded domain in complex affine space. The *rational boundary components* $\mathfrak{X}_P$ of $\mathfrak{X}$ are subsets of the boundary of this domain, parametrized by the parabolic subgroups P of G which are maximal among the proper and $\boldsymbol{Q}$-rational ones. They are themselves bounded symmetric domains. The action of G on $\mathfrak{X}$ extends continuously to its boundary, and the parabolic subgroup P is the stabilizer of $\mathfrak{X}_P$. Conversely, $\mathfrak{X}_P$ is the set of points on the boundary fixed by all elements of the unipotent radical N_P of P.

There exists a canonical P-covariant fibration $\psi_P\colon \mathfrak{X}\to\mathfrak{X}_P$. For any $y\in\mathfrak{X}_P$ let $T(y)$ be $\psi_P^{-1}(y)$, which I call the *slice* of $\mathfrak{X}$ *transverse* to y. There exists in the center of N_P a rational, self-adjoint, homogeneous cone C_P stable under the adjoint action of P and another almost canonical P-covariant map $\varphi_P\colon \mathfrak{X}\to C_P$. The group N_P acts trivially on both $\mathfrak{X}_P$ and C_P, and the two projections ψ_P and φ_P allow one to identify the quotient $\mathfrak{X}/N_P$ with the quotient $\mathfrak{X}_P\times C_P$. Thus the slice $T(y)$ is N_P-stable, and its quotient by N_P is isomorphic to C_P.

Let $M_P=P/N_P$ be the Levi component of P, and let G_P be the kernel of the adjoint action of M_P on the centre of N_P, which is also $\mathrm{Ker}(M_P\to \mathrm{Aut}\,(C_P))$. It is a $\boldsymbol{Q}$-rational group, since the centre of N_P is $\boldsymbol{Q}$-rational. The group G_P is isogenous to the product of $\mathrm{Aut}\,\mathfrak{X}_P$ and a compact factor. It possesses in the kernel of the canonical action of M_P on $\mathfrak{X}_P$ a unique complement L_P which is also $\boldsymbol{Q}$-rational. The group L_P is isogenous to $\mathrm{Aut}(C_P)$ and hence C_P may be identified with the quotient $L_P/K(L_P)$. (Several slight contortions are necessary because it is not always true that the kernel of $P\to\mathrm{Aut}(\mathfrak{X}_P)$, which is called the centralizer of $\mathfrak{X}_P$, is rational.) If Z_P is the inverse image of L_P with respect to the canonical homomorphism $P\to M_P$, then it is also $\boldsymbol{Q}$-rational and for any y in $\mathfrak{X}_P$ the slice $T(y)$ may be identified with $Z_P/K(Z_P)$. Borel and I tentatively call Z_P the *rational centralizer* of $\mathfrak{X}_P$ and L_P, for reasons which will appear later on, its *link group*. Neither name is perfect, but the group Z_P does at least agree with the whole of the centralizer up to a compact factor.

Example. Let $G=Sp(2n,\boldsymbol{Q})$. Then $\mathfrak{X}$ is the Siegel space $\mathfrak{S}_n$ made up of $n\times n$ symmetric matrices $Z=X+iY$ with coefficients in $\boldsymbol{C}$ such that Y is positive definite. An element

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of G acts by taking Z to $(AZ+B)(CZ+D)^{-1}$. Let P be the parabolic subgroup of elements of G with $C=0$. The corresponding boundary component is a point, which may be described as the limit of itY, for any Y, as $t>0$ goes to infinity. The unipotent radical of P is the subgroup of elements with $A=D=\mathrm{I}$, hence isomorphic to the additive group of symmetric real $n\times n$ matrices. The Levi factor M_P is isomorphic to $GL_n(\boldsymbol{R})$. The cone C_P is the subset of all positive definite matrices and φ_P just takes Z

to Y. For other boundary components N_P will no longer be abelian, the group M_P will be the product $GL_m(\boldsymbol{R})\times Sp(2n-2m, \boldsymbol{R})\,(\sim L_P\times G_P)$ and C_P will be the cone of positive definite $m\times m$ matrices.

Let $\mathfrak{X}^*$ be the union of $\mathfrak{X}$ and all the $\mathfrak{X}_P$. The group Γ acts discretely on $\mathfrak{X}^*$, and V^* is the quotient $\Gamma\backslash\mathfrak{X}^*$. It is the disjoint union of varieties of the form $V_P=\Gamma(G_P)\backslash\mathfrak{X}_P$. These subvarieties are non-singular since Γ is neat and are the connected components of a Whitney stratification of the singular algebraic variety V^*. I define the *depth* (*niveau* in [6]) of a component V_P to be the maximal length m of a chain components $V=V_0$, $V_1, \cdots, V_m=V_P$ such that each V_{i+1} is contained in the closure of V_i. This is the same as the rational rank of L_P.

There exists in [1: p. 266] a relatively elegant description of the local geometry of V^*. Define a *rational core* of the cone C_P to be any open, $\Gamma(P)$-stable, subset of C_P which is commensurable with the convex hull of the intersection of C with a $\Gamma(P)$-stable lattice in the centre of N_P. (Around p. 120 of [1] several rather different-looking rational cores are described.) Now choose a point y in $\mathfrak{X}_P$. Let Y be a neighborhood of y in $\mathfrak{X}_P$ small enough to embed in $\Gamma(G_P)\backslash\mathfrak{X}_P$. Let $\mathfrak{C}$ be a rational core in C_P, and define

$$\mathfrak{X}(Y):=\Gamma(Z_P)\backslash\psi_P^{-1}(Y), \tag{3.1a}$$

$$\mathfrak{X}(Y, \mathfrak{C}):=\mathfrak{X}(Y)\cap\varphi_P^{-1}(\mathfrak{C}). \tag{3.1b}$$

3.1. Proposition. *For $\mathfrak{C}$ sufficiently small the quotient $\mathfrak{X}(Y, \mathfrak{C})$ embeds into V, and as $\mathfrak{C}$ and Y get smaller we obtain the intersections with V of a neighborhood basis of y in V^*.*

If x is a point of $\mathfrak{X}$ then according to [9: 1.1.9] there exists a unique subgroup $G_{P,x}$ of P mapping isomorphically onto G_P and stable under the Cartan involution of G corresponding to x. The $G_{P,x}$-orbit of x projects bijectively onto $\mathfrak{X}_P$, and I will call it a *horizontal section* of $\mathfrak{X}$ (over $\mathfrak{X}_P$). (This construction was suggested by the definition of the geodesic action in [9].) These horizontal sections form a P-invariant partition of $\mathfrak{X}$. They are everywhere transverse to what I called above the transverse slices of $\mathfrak{X}$ and hence give rise at each point of $\mathfrak{X}$ to a direct sum decomposition of the tangent space into horizontal and transverse components. One may thus, for example, define *transverse differential forms* at any point of $\mathfrak{X}$ to be those which have no horizontal components.

Using these horizontal sections, one can see that if $T(y, \mathfrak{C})$ is the intersection of $T(y)$ and $\varphi_P^{-1}(\mathfrak{C})$, then the set $\mathfrak{X}(Y, \mathfrak{C})$ is canonically isomorphic with a product:

$$\mathfrak{X}(Y, \mathfrak{C})\cong Y\times(\Gamma(Z_P)\backslash T(y, \mathfrak{C})). \tag{3.2}$$

The set $\varphi_P^{-1}(Y) \cap \psi_P^{-1}(\mathfrak{C})$ is stable under N_P as well as $\Gamma(Z_P)$. Consequently the set $\mathfrak{X}(Y, \mathfrak{C})$ can be represented as fibre bundle with fibre $\Gamma(N_P)\backslash N_P$ and base $Y \times (\Gamma(L_P)\backslash \mathfrak{C})$. The transverse component $\Gamma(Z_P)\backslash T(y, \mathfrak{C})$ is a fibre bundle with the same fibre and base $\Gamma(L_P)\backslash \mathfrak{C}$. The core $\mathfrak{C}$ may be chosen convex, and if this is done the intersection with V of the *topological* link of y in V^* transverse to V_P may also be expressed as a fibre bundle with the same fibre (which I therefore call the *unipotent link fibre*) but with base the quotient of the boundary of $\mathfrak{C}$ by $\Gamma(L_P)$.

Let A_P be the topologically connected component of the (group of $\boldsymbol{R}$-rational points on the) maximal $\boldsymbol{Q}$-split torus in the centre of M_P. Since P is maximal, it is one-dimensional. If ρ_P is the square-root of the modulus character of P: $p \to |\det(\mathrm{Ad}_{\mathfrak{n}}(p))|^{1/2}$, then ρ_P is an isomorphism of A_P with $\boldsymbol{R}^{\mathrm{pos}}$. *Here and elsewhere small gothic letters denote complexified Lie algebras.* The group A_P is contained in L_P and acts on C_P by scalar multiplication. Let $A_P^+ = \rho_P^{-1}[1, \infty)$. I shall call a core *contractible* if it is stable under multiplication by scalars >1, or equivalently under A_P^+.

Since $G = PK$, the symmetric space $\mathfrak{X}$ can be identified with $P/K(P)$, and therefore the sets $\mathfrak{X}(Y)$ and $\mathfrak{X}(Y, \mathfrak{C})$ may be identified with open subsets $\psi_P^{-1}(Y)$ and $\psi_P^{-1}(Y) \cap \varphi_P^{-1}(\mathfrak{C})$ respectively in $\Gamma(Z_P)\backslash P/K(P)$. As above, there exists a unique copy of A_P in P stable under the Cartan involution corresponding to the choice of K. It commutes with $K(P)$, and therefore this copy of A_p acts on the right on $\Gamma(Z_P)\backslash P/K(P)$, giving rise to a one-parameter group of quasi-isometries in the neighborhood of a point y in $\mathfrak{X}_P$, preserving the transverse slices. (This is essentially the geodesic action of [9:3].)

Recall that the cone C_P can be identified with $L_P/K(L_P)$. Define a *core* in L_P to be the inverse image of a core in C_P, and define a core in Z_P to be the inverse image in Z_P of one in $L_P = Z_P/N_P$. More precisely, to the core $\mathfrak{C}$ in C_P associate the cores $\mathfrak{C}(L_P)$ and $\mathfrak{C}(Z_P)$. Then the transverse component of the set $\mathfrak{X}(Y, \mathfrak{C})$ may be identified with $\Gamma(Z_P)\backslash \mathfrak{C}(Z_P)/K(Z_P)$.

§ 4. Global and local L^2-cohomology of V^*

Let E be any smooth representation of G, and also (an abuse of language) the corresponding locally constant coefficient system on V. Explicitly, if $\mathrm{pr}_\Gamma\colon \mathfrak{X} \to V$ is the canonical projection, then to an open subset U of V is associated the space of locally constant, Γ-covariant functions from $\mathrm{pr}_\Gamma^{-1}(U)$ to E. The de Rham complex of this coefficient system similarly associates to U the space of Γ-invariant, C^∞ forms on $\mathrm{pr}_\Gamma^{-1}(U)$ with values in E—i.e.

$$(\Omega^\cdot(\mathrm{pr}_\Gamma^{-1}(U), \boldsymbol{C}) \otimes E)^\Gamma.$$

Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{s}$ be the orthogonal decomposition of $\mathfrak{g}$ with respect to the Killing form. Assume E to be finite-dimensional, and choose for it a Euclidean metric with respect to which the representation of K is unitary and that of $\exp(\mathfrak{s})$ is by self-adjoint operators. The conjecture of Zucker mentioned in the Introduction amounts to:

4.1. Conjecture. *The sheaf* $\mathscr{D}om(\bar{d}_E)$ *is equivalent in the derived category of sheaves on* V^* *to* $\mathscr{I}\mathscr{C}^{\cdot}(E)$.

This may be understood as a purely local statement. Borel and I have proven:

4.2. Theorem. *This conjecture is true on the union of the strata* V_P *of depth* $\leq$ *two.*

In effect, we show that the axioms 2.6 are satisfied. It is perhaps unfortunate that we use representation theory of Lie algebras in the proof, but as compensation we are able to describe in some detail what the local L^2-cohomology looks like. As I have mentioned in the Introduction, several special cases of this result have been proven also by Zucker, among them those where $G=\mathrm{Sp}(2n, Q)$ and several where $G=SU(p,q)$. His techniques are rather different from ours, and seem to exploit phenomena which do not occur in general.

I shall give first a rough idea of how representation theory comes in by telling what role it plays in describing the global L^2-cohomology.

Let $\mathrm{pr}_K\colon \Gamma\backslash G\to V=\Gamma\backslash\mathfrak{X}$ be the canonical projection (depending on the identification $G/K=\mathfrak{X}$). Let for the moment F be any $(\mathfrak{g}, K)$-module. In particular it is a representation of K, so associated to it is a fibre bundle over V with fibre F. If U is any open set in V, the space of smooth sections of this bundle over U may be identified with the space of all smooth, K-covariant functions from $\mathrm{pr}_K^{-1}(U)$ to F. More generally, the map $\omega\to \mathrm{pr}_K^*(\omega)$ induces an isomorphism between the space of smooth differential forms on U with values in F and that of all smooth forms η on $\mathrm{pr}_K^{-1}(U)$ such that (a) η is right K-invariant, and (b) $\kappa\,\lrcorner\,\eta=0$ for all $\kappa\in\mathfrak{k}$. An easy transformation identifies this in turn with the graded space

$$\mathrm{Hom}_K(\Lambda^{\cdot}(\mathfrak{g}/\mathfrak{k}),\ C^\infty(\mathrm{pr}_K^{-1}(U), F)). \tag{4.1}$$

The space $C^\infty(\mathrm{pr}_K^{-1}(U), F)$ is via the right regular representation a module over $(\mathfrak{g}, K)$, and the space (4.1) is that underlying its Koszul complex. The Koszul differential on this complex—that defining its relative Lie algebra cohomology—corresponds to a connection on the fibre bundle associated to F. If the representation of $(\mathfrak{g}, K)$ comes from a smooth representation of G itself, then this connection is the same as the one constructed at the beginning of this section. (The book [10] is a general reference for the cohomology of $(\mathfrak{g}, K)$-modules, and Chapter VII is particularly relevant here.)

Take F to be the finite-dimensional representation E. Square-integrable differential forms on U with values in E similarly lift to forms η on $\mathrm{pr}_K^{-1}(U)$ satisfying in addition to conditions (a) and (b) above the third condition (c) η is square-integrable. The space of all such forms may also be expressed as

$$\mathrm{Hom}_K(\Lambda^{\cdot}(\mathfrak{g}/\mathfrak{k}), L^2(\mathrm{pr}_K^{-1}(U))\otimes E).$$

For any open set W in $\Gamma\backslash G$, let $L^{2,\infty}(W)$ be the space of all C^∞ functions $f\colon W\to \boldsymbol{C}$ such that all derivatives $R_X f$ ($X \in U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$) are square-integrable. It is a $(\mathfrak{g}, K)$-module. Those forms on U which lift to elements of

$$\mathrm{Hom}_K(\Lambda^{\cdot}(\mathfrak{g}/\mathfrak{k}), L^{2,\infty}(\mathrm{pr}_K^{-1}(U))\otimes E) \tag{4.2}$$

are clearly Sobolev forms on U, although I do not see that the converse holds, except for the case $U=V$, when a result of Nelson [23] gives:

4.3. Proposition. *Lifting forms from V to $\Gamma\backslash G$ induces an isomorphism between the cohomology of the complex of Sobolev forms on V with values in E and the relative Lie algebra cohomology*

$$H^{\cdot}(\mathfrak{g}, K, L^{2,\infty}(\Gamma\backslash G)\otimes E).$$

Nelson's theorem also implies that at least the sheaves on V^* corresponding to Sobolev forms and the presheaf (4.2) are the same (this requires the result of [29] mentioned at the beginning of § 5).

Incidentally, the Laplacian Δ_E corresponds in this lifting to the Casimir element in the centre $Z(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$.

In view of these translations from space cohomology to $(\mathfrak{g}, K)$-cohomology, the regularization theorem of [3] implies directly:

4.4. Proposition. *The global L^2-cohomology of V is the same as that of the complex of Sobolev forms on V.*

In [5] we give a second proof, using Langlands' spectral decomposition for $L^2(\Gamma\backslash G)$. This spectral decomposition expresses $L^2(\Gamma\backslash G)$ as a sum of discrete and continuous components, where each continuous component is itself a summand of a representation induced from a parabolic subgroup of G. Because of 4.4, we were able in [5] to apply this decomposition and the techniques of relative Lie algebra cohomology, particularly Shapiro's Lemma, to deduce:

4.5. Theorem. *The L^2-cohomology of V with coefficients in E is finite-dimensional.*

Notice that these results are in accord with Proposition 1.8. It follows from the remarks just before this proposition that the L^2-cohomology of V may also be identified with the square-integrable forms annihilated by Δ_E.

Since, as we explain in [5], the L^2-cohomology of arithmetic quotients of symmetric spaces without invariant complex structures may not be finite-dimensional, this may be considered weak evidence in favor of Zucker's conjecture.

Fix for the rest of this section and the next as well (a) a maximal, proper, $\boldsymbol{Q}$-rational parabolic subgroup P, (b) a point y in a boundary component $\mathfrak{X}_P$, (c) a small contractible neighborhood Y of y in $\mathfrak{X}_P$ and (d) a small contractible rational core $\mathfrak{C}$. I shall suppress subscripts referring to P whenever possible.

Recall that the set $\mathfrak{X}(Y, \mathfrak{C})$ may be considered as a subset of $\Gamma(Z)\backslash P/K(P)$. Let

$$\mathfrak{P}(Y, \mathfrak{C}) := \text{inverse image of } \mathfrak{X}(Y, \mathfrak{C}) \text{ in } P.$$

Thus $\mathfrak{P}(Y, \mathfrak{C})$ is stable under left multiplication by $\Gamma(Z)$ and N and under right multiplication by $K(P)$. We want to relate forms on $\mathfrak{X}(Y, \mathfrak{C})$ to forms on $\Gamma(Z)\backslash\mathfrak{P}(Y, \mathfrak{C})$, but there is a technicality to deal with first. The forms on $\mathfrak{X}(Y, \mathfrak{C})$ are square-integrable with respect to the measure induced by a left-invariant measure on $P/K(P)$, but when we lift to $\Gamma(Z)\backslash\mathfrak{P}(Y, \mathfrak{C})$ we want, for important technical reasons, to refer to the right-invariant measure, which is not the same since the group P is not unimodular (as G is). Explicitly, we have the integral formula

$$\int f(p)d_l p = \int f(p)\rho^2(p)d_r p,$$

which implies that $f(p)$ is square-integrable with respect to $d_l p$ if and only if $f(p)\rho^{-1}(p)$ is square-integrable with respect to $d_r p$. Writing $f(p) = (f(p)\rho^{-1}(p))\rho(p)$ we see that the Z-representation L^2_{left} is isomorphic to the tensor product $L^2_{\text{right}} \otimes \boldsymbol{C}(\rho)$.

For any subset U of $\Gamma(Z)\backslash P$ let

$L^2(U) :=$ functions on U square-integrable with respect to $d_r p$

$L^{2,\infty}(U) := \{f \in C^\infty(U) \mid R_X f \in L^2(U) \quad \text{for all} \quad X \in U(\mathfrak{g})\}$.

Lifting forms to $\mathfrak{P}(Y, \mathfrak{S})$ we then see that the L^2-cohomology of $\mathfrak{X}(Y, \mathfrak{S})$ may be identified with the cohomology of the complex of elements F of the graded space

$$\mathrm{Hom}_{K(P)}(\Lambda^{\cdot}(\mathfrak{p}/\mathfrak{k}(P)), L^2(\Gamma(Z)\backslash\mathfrak{P}(Y, \mathfrak{S}))\otimes \boldsymbol{C}(\rho)\otimes E) \tag{4.3}$$

such that dF again lies in this space. The set $\mathfrak{P}(Y, \mathfrak{S})$ is not stable under right multiplication by elements of P, but if p is close to the identity then the right translate of $\mathfrak{P}(Y, \mathfrak{S})$ by p will be contained in some other $\mathfrak{P}(Y^*, \mathfrak{S}^*)$. In view of this, a homotopy argument and another application of the results of [3] prove:

4.6. Lemma. *The L^2-cohomology of $\mathfrak{X}(Y, \mathfrak{S})$ with coefficients in E may be identified with the relative Lie algebra cohomology*

$$H^{\cdot}(\mathfrak{p}, K(P), L^{2,\infty}(\Gamma(Z)\backslash\mathfrak{P}(Y, \mathfrak{S}))\otimes \boldsymbol{C}(\rho)\otimes E).$$

In other words, in the terminology of Section 2 the sheaves $\mathscr{D}om(\bar{d}_E)$ and $\Omega^{\cdot}_{2,\infty,\mathrm{loc}}(E)$ are quasi-isomorphic.

As I have mentioned already, the sets $\mathfrak{X}(Y, \mathfrak{S})$ possess a fibration with fibre $\Gamma(N)\backslash N$. A classical result of van Est [17] and Nomizu [24] implies that the cohomology of each fibre may be identified with the cohomology of the complex of N-constant forms on it. A similar argument implies here that the inclusion of the spaces of N-constants in the spaces $L^{2,\infty}(\Gamma(Z)\backslash \mathfrak{P}(Y, \mathfrak{S}))$ induces quasi-isomorphisms. If $\mathfrak{M}(Y, \mathfrak{S})$ is the quotient of $\mathfrak{P}(Y, \mathfrak{S})$ by the left action of N, then N-invariant functions of $\mathfrak{P}(Y, \mathfrak{S})$ may be identified with ones on $\mathfrak{M}(Y, \mathfrak{S})$. Hence:

4.7. Lemma. *The L^2-cohomology of $\mathfrak{X}(Y, \mathfrak{S})$ with coefficients in E may be naturally identified with the relative Lie algebra cohomology*

$$H^{\cdot}(\mathfrak{p}, K(P), L^{2,\infty}(\Gamma(L)\backslash\mathfrak{M}(Y, \mathfrak{S}))\otimes \boldsymbol{C}(\rho)\otimes E). \tag{4.5}$$

The set $\mathfrak{X}(Y, \mathfrak{S})$ is the quotient $\Gamma(Z)\backslash\varphi^{-1}(\mathfrak{S}) \cap \psi^{-1}(Y)$. Hence it makes sense to define the *N-constant forms* on $\mathfrak{X}(Y, \mathfrak{S})$ to be the space of $\Gamma(Z)$-invariant, N-invariant forms on $\varphi^{-1}(\mathfrak{S}) \cap \psi^{-1}(Y)$. Lemma 4.7 can be phrased without using the terminology of Lie algebra cohomology: roughly speaking, the L^2-cohomology of $\mathfrak{X}(Y, \mathfrak{S})$ is that of the complex of its N-constant Sobolev forms.

The group Z is normal in P with quotient isogenous to G_P. The Hochschild-Serre spectral sequence corresponding to (P, Z) has as E_2-term

$$H^p(\mathfrak{g}_P, K(G_P), H^q(\mathfrak{z}, K(Z), L^{2,\infty}(\Gamma(L)\backslash\mathfrak{M}(Y, \mathfrak{S}))\otimes \boldsymbol{C}(\rho)\otimes E).$$

Since Y is contractible and $\mathfrak{X}(Y, \mathfrak{S})$ has the product structure (3.2) this turns out to be null for $p>0$, and we obtain:

4.8. Lemma. *The L^2-cohomology of $\mathfrak{X}(Y, \mathfrak{C})$ with coefficients in E may be identified with the relative Lie algebra cohomology*

$$H^{\cdot}(\mathfrak{z}, K(Z), L^{2,\infty}(\Gamma(L)\backslash\mathfrak{C}(L))\otimes C(\rho)\otimes E). \tag{4.6a}$$

Define

$$L_r^{2,\infty}(\Gamma(L)\backslash\mathfrak{C}(L)) := \{f \in L^{2,\infty}(\Gamma(L)\backslash\mathfrak{C}(L)) \mid \mathrm{Supp}(f) \text{ is contained in some } t\mathfrak{C} \text{ with } t>1\}$$

$$\mathscr{L}(U) := \text{topological dual of } L_r^{2,\infty}(\Gamma(L)\backslash\mathfrak{C}(L)).$$

The space $L^{2,\infty}(U)$ embeds canonically into $\mathscr{L}(U)$. Another regularization argument shows:

4.9. Proposition. *The inclusion of $L^{2,\infty}(\Gamma(L)\backslash\mathfrak{C}(L))$ in $\mathscr{L}(\Gamma(L)\backslash\mathfrak{C}(L))$ induces an isomorphism of the cohomology* (4.6a) *with*

$$H^{\cdot}(\mathfrak{z}, K(Z), \mathscr{L}(\Gamma(L)\backslash\mathfrak{C}(L))\otimes C(\rho)\otimes E). \tag{4.6b}$$

If this is finite-dimensional, then so is

$$H^{\cdot}(\mathfrak{z}, K(Z), L_r^{2,\infty}(\Gamma(L)\backslash\mathfrak{C}(L))\otimes C(\rho)\otimes E), \tag{4.7}$$

and the two are canonically dual in complementary dimensions.

The space $L_r^{2,\infty}(\Gamma(L)\backslash\mathfrak{C}(L))$ is only an LF space, but nonetheless the last part turns out to be in essence a consequence of 2.4.

In some sense, everything up to this point is elementary if highly technical. The previous several results can be summarized without reference to Lie algebra cohomology: note first that the Lie algebra $\mathfrak{g}_P$ acts canonically from the left on the space of N-constant forms on $\mathfrak{X}(Y, \mathfrak{C})$. Writing a little loosely, what the transformations (4.3)–(4.6) accomplish is the identification of the L^2-cohomology of $\mathfrak{X}(Y, \mathfrak{C})$ with the cohomology of the complex of Sobolev forms (or currents) on $\mathfrak{X}(Y, \mathfrak{C})$ which are N-constant, $\mathfrak{g}_P$-constant, and transverse.

The group N is normal in Z. The Hochschild-Serre spectral sequence for this pair and the cohomology (4.6) has E_2-term

$$H^{\cdot}(\mathfrak{l}_P, K(L_P), \mathscr{L}(\Gamma(L)\backslash\mathfrak{C}(L))\otimes C(\rho)\otimes H^{\cdot}(\mathfrak{n}_P, E)). \tag{4.8}$$

Borel has proven:

4.10. Proposition. *The differential d_2 is null, and this spectral sequence collapses.*

One consequence of our formulae (or, more intuitively, the remarks just after 4.9) is that the local L^2-cohomology is locally constant on each stratum V_P. More precisely, it is clear that the local L^2-cohomology (4.6) is canonically a module over G_P, giving rise to a locally constant coefficient system on V_P just as E gave rise to one on V itself.

§ 5. Sketch of the argument

Recall that I have fixed P.

It is proven in [29] that the sheaves $\mathscr{D}om(\bar{d}_E)$ and $\Omega_{2,\infty,\mathrm{loc}}(E)$—and implicitly $\mathscr{C}_2(E)$ as well—are fine. In view of this together with 2.6 and 4.9, in order to prove Theorem 4.2 it will suffice to show that for any contractible core $\mathfrak{C}$ the relative Lie algebra cohomology groups

$$H^{\cdot}(\mathfrak{z}, K(Z), \mathscr{L}(\Gamma(L)\backslash\mathfrak{C}(L))\otimes \boldsymbol{C}(\rho)\otimes E) \tag{5.1}$$

are finite-dimensional and vanish in dimensions $\geq \mathrm{codim}_{\boldsymbol{C}}(V_P)$ when the depth of V_P is $\leq$two.

The foundation on which Borel and I construct the proof of this is a kind of Hodge theorem for local L^2-cohomology. Let

$\mathscr{A}(\Gamma(L)\backslash L) :=$ the space of automorphic forms on $\Gamma(L)\backslash L$

$\mathscr{A}_2(\Gamma(L)\backslash\mathfrak{C}(L)) :=$ the subspace of automorphic forms whose restrictions to the subset $\Gamma(L)\backslash\mathfrak{C}(L)$ are square-integrable.

I recall that an automorphic form on $\Gamma(L)\backslash L$ is a function F which is (a) of moderate growth and (b) contained in a finite-dimensional subspace stable under $K(L)$ and $Z(\mathfrak{l})$, where $Z(\mathfrak{l})$ is the centre of the universal enveloping algebra of $\mathfrak{l}$. (Refer to [8] for equivalent characterizations.) The space $\mathscr{A}_2(\Gamma(L)\backslash\mathfrak{C}(L))$ turns out to be independent of the choice of core $\mathfrak{C}$.

5.1. Conjecture. *When $\mathfrak{C}$ is contractible, then the inclusion of $\mathscr{A}_2(\Gamma(L)\backslash\mathfrak{C}(L))$ in $\mathscr{L}(\Gamma(L)\backslash\mathfrak{C}(L))$ induces an isomorphism of the cohomology* (4.6) *with*

$$H^{\cdot}(\mathfrak{z}, K(Z), \mathscr{A}_2(\Gamma(L)\backslash\mathfrak{C}(L))\otimes \boldsymbol{C}(\rho)\otimes E). \tag{5.2}$$

There are ways to make this look more like a Hodge theorem. Let

$$\mathfrak{m}_{\mathfrak{l},E} := \mathrm{Ann}_{Z(\mathfrak{l})}(H_{\cdot}(\mathfrak{n}, E^*)\otimes \boldsymbol{C}(\rho^{-1}))$$
$$\mathscr{A}_{2,E}(\Gamma(L)\backslash\mathfrak{C}(L)) := \{F \in \mathscr{A}_2(\Gamma(L)\backslash\mathfrak{C}(L)) \mid \mathfrak{m}_{\mathfrak{l},E}{}^k F = 0 \text{ for some } k\}.$$

Note that $H_{\bullet}(\mathfrak{n}, E^*)$ is the dual of $H^{\bullet}(\mathfrak{n}, E)$.

5.2. Lemma. *The inclusion of* $\mathscr{A}_{2,E}(\Gamma(L)\backslash\mathfrak{S}(L))$ *in* $\mathscr{A}_2(\Gamma(L)\backslash\mathfrak{S}(L))$ *induces an isomorphism of*

$$H^{\bullet}(\mathfrak{z}, K(Z), \mathscr{A}_{2,E}(\Gamma(L)\backslash\mathfrak{S}(L))\otimes \boldsymbol{C}(\rho)\otimes E) \tag{5.3}$$

with the cohomology (5.2).

Proof. Apply Hochschild-Serre to obtain a spectral sequence converging to (5.2) with E_2-term

$$H^{\bullet}(\mathfrak{l}, K(L), \mathscr{A}_2(\Gamma(L)\backslash\mathfrak{S}(L))\otimes \boldsymbol{C}(\rho)\otimes H^{\bullet}(\mathfrak{n}, E)). \tag{5.4}$$

On the one hand, the space $\mathscr{A}_2(\Gamma(L)\backslash\mathfrak{S}(L))$ is an algebraic sum of its primary constituents with respect to the maximal ideals of $Z(\mathfrak{l})$, and on the other Wigner's Lemma [10: 1.4.1] implies that all constituents other than those associated to the maximal ideals containing $\mathfrak{m}_{\mathfrak{l},E}$ contribute nothing to the cohomology (5.4). □

There is still another formulation, relatively free from the language of representation theory. Recall that $\mathfrak{X}(Y, \mathfrak{S})$ is an open subset of $\mathfrak{X}(Y)$. Let $\mathfrak{m}_E$ be the annihilator in $Z(\mathfrak{g})$ of E^*. Then 5.1 amounts to the assertion that L^2-cohomology of $\mathfrak{X}(Y, \mathfrak{S})$ may be identified with the cohomology of the complex of forms ω on $\mathfrak{X}(Y)$ satisfying (a) ω is square-integrable when restricted to $\mathfrak{X}(Y, \mathfrak{S})$; (b) ω has moderate growth on $\mathfrak{X}(Y)$; (c) ω is annihilated by some power of $\mathfrak{m}_E$; (*d*) ω is N-constant, $\mathfrak{g}_P$-constant, and transverse.

Note that the Laplacian Δ_E comes from the Casimir element in $Z(\mathfrak{g})$, so that (c) includes among other things the condition that ω be annihilated by a power of the Laplacian. As explained in [12], this is the best one can expect from a Hodge theory when the Laplacian has a continuous spectrum, as it does here on locally L^2 forms.

5.3. Theorem. *The conjecture above is true when* depth$(V_P)\leq 2$.

As far as I can see, proving Conjecture 5.1 is the main obstacle to proving Conjecture 4.1 unconditionally, although I do not know how to finish the proof even if 5.1 were known. Conjecture 5.1 in turn seems to depend mainly on some likely but unproven results in the theory of residual Eisenstein series for L. When the depth of V_P is two, the proof of 5.3 as well as the argument needed afterwards to deduce Zucker's conjecture are already both very complicated. What I propose to do in the rest of the paper is to discuss rather sketchily only a few aspects of the proof, exhibiting a little more detail only in the relatively simple case when V_P has depth one. Incidentally, it follows immediately from Conjecture 5.1, whenever proven, that the corresponding local L^2-cohomology is finite-dimensional. This is not too surprising, as in the proof of 5.3 we use at some point a Lemma we also used in [5] to prove global finite-dimensionality.

Conjecture 5.1 is analogous to a similar conjecture, due to Borel, that the ordinary cohomology of an arithmetic quotient $\Gamma\backslash\mathfrak{X}$ is the same as the cohomology of the complex of differential forms on $\Gamma\backslash\mathfrak{X}$ which are at the same time automorphic forms. This earlier conjecture has been proven in a number of special cases (see, for example, [12]) by means of a theorem analogous to the classical result of Paley-Wiener which characterizes functions on $\boldsymbol{R}$ with compact support by means of properties of their additive Fourier transforms. There is a general principle involved: just for one moment let G be an arbitrary Lie group with maximal compact subgroup K. The basic idea is that if U is a topological $(\mathfrak{g}, K)$-module and U^* its dual, then in order to show that the $(\mathfrak{g}, K)$-cohomology of U^* is the same as that of its subspace of $Z(\mathfrak{g})$-finite elements, one should try to prove some sort of analogue of Paley-Wiener for U itself. That idea works here, but the process is not so direct. In view of 4.9, the space for which one would at first try to prove a Paley-Wiener theorem is $L^{2,\infty}_r(\Gamma(L)\backslash\mathfrak{C}(L))$. If the depth of V_P is one, it turns out that such a theorem is not only simple, but even another classical result of Paley-Wiener, who also characterized functions in $L^2(0, \infty)$ by their (additive) Fourier transforms. But when the depth of V_P is more than one, this space is in some sense not amenable to such treatment: to be more precise, if a function in the space $L^{2,\infty}_r(\Gamma(L)\backslash\mathfrak{C}(L))$ is decomposed into its cuspidal and Eisenstein components, following Langlands' prescription, then these components themselves may not have support on $\mathfrak{C}(L)$. Therefore this space must be replaced by the larger space of functions in $L^{2,\infty}(\Gamma(L)\backslash L)$ which are required to vanish rapidly, along with all their $U(\mathfrak{l})$-derivatives, in the direction away from the component V_P. This amounts to considering on $\mathfrak{X}(Y)$ the Sobolev forms which are similarly rapidly decreasing. This replacement of functions concentrated on cores by those satisfying global conditions on $\Gamma(L)\backslash L$ seems natural when one reflects that, after all, automorphic forms are themselves rather global in nature.

When depth$(V_P)\geq 2$, making this definition precise involves introducing compactifications of the reductive symmetric space C and of its quotient $\Gamma(L)\backslash C$. (The closure of the open cone C and the union of C and its rational boundary cones [1: II.3] in this closure make up part of this compactification.) When depth$(V_P)=1$, this step is very simple, amounting to the compactification of the multiplicative group $\boldsymbol{R}^{\mathrm{pos}}$ by adding points at 0 and ∞, but in general these compactifications have apparently not been constructed explicitly in the literature. In the depth one case, the space occurring is modelled on

$$L^{2,\infty}_{\mathscr{S}}(\boldsymbol{R}^{\mathrm{pos}}):=\{F \in L^{2,\infty}(\boldsymbol{R}^{\mathrm{pos}})\,|\,F \text{ is the restriction to } (0,\infty) \text{ of a } C^{\infty} \text{ function on } (0,\infty) \text{ with all derivatives vanishing at } 0\}$$

If one specializes the arguments for depth $\leq$ two to depth one, then this is the space which must be characterized by Fourier (i.e. Mellin) transforms. This characterization is an interesting exercise. The rest of the proof of 5.1 in the depth one case is presented, in essential details, in [11].

In the rest of the paper I want to give a very rough idea of the transition from Theorem 5.3 to Theorem 4.2. I begin by looking at the simplest case, that of $G=SL_2(\boldsymbol{Q})$, $\Gamma=SL_2(\boldsymbol{Z})$. Then $N\cong\boldsymbol{R}$, L is (up to ± 1) the same as A, and $\Gamma(L)$ is $\{\pm 1\}$. Hence the automorphic forms on $\Gamma(L)\backslash L=A$ are the linear combinations of functions $x^s\log^k|x|$, with $s\in\boldsymbol{C}$, k in $\boldsymbol{N}$:

$$\mathscr{A}(A)=\sum \boldsymbol{C}(s)\otimes\mathfrak{Log}$$

where $\mathfrak{Log}$ is the module of all polynomials in $\log|x|$ and the sum is over $s\in\boldsymbol{C}$. The ones in $\mathscr{A}_2(A^+)$ are those for which $\mathrm{Re}(s)<0$. Since xd/dx is surjective on the A-module $\mathfrak{Log}$, it is $\mathfrak{a}$-acyclic, and it is simple to deduce that the cohomology $H^p(\mathfrak{a},\mathscr{A}_2(A^+)\otimes\boldsymbol{C}(\rho)\otimes H^q(\mathfrak{n},E))$ is null unless $p=q=0$, which is the simplest version of Zucker's conjecture.

In general the space $\mathscr{A}_2(\Gamma(L)\backslash\mathfrak{S}(L))$ will contain the $(\mathfrak{l},K(L))$-stable subspace

$$\mathscr{A}_{2,\mathrm{disc}}(\Gamma(L)\backslash\mathfrak{S}(L)):=\text{automorphic forms on } \Gamma(L)\backslash L \text{ which are square-integrable on } \Gamma(L)\backslash\rho^{-1}(1,\infty).$$

Note that $\rho^{-1}(1,\infty)$ will include $\Gamma(L)\backslash\mathfrak{S}(L)$ if $\mathfrak{S}$ is small enough. (Recall that $\mathscr{A}_2(\Gamma(L)\backslash\mathfrak{S}(L))$ does not depend on $\mathfrak{S}$.) This subrepresentation has a relatively simple structure. The space of square-integrable automorphic forms on $\Gamma(L)A\backslash L$ (essentially the discrete spectrum of a Casimir element of $Z(\mathfrak{l})$) will be a direct sum $\sum\sigma$ of irreducible unitary representations σ, each occurring with finite multiplicity. Furthermore $L\cong(L/A)\times A$. Hence the space $\mathscr{A}_{2,\mathrm{disc}}(\Gamma(L)\backslash\mathfrak{S}(L))$ can be expressed as the direct sum

$$\sum\sigma\otimes\boldsymbol{C}(s)\otimes\mathfrak{Log}$$

where the sum is over all these σ and $s\in\boldsymbol{C}$ with $\mathrm{Re}(s)<0$.

5.4. Theorem. *The $(\mathfrak{z},K(Z))$-cohomology of the space $\mathscr{A}_{2,\mathrm{disc}}(\Gamma(L)\backslash\mathfrak{S}(L))$ is null in dimensions $\geq\mathrm{codim}_{\boldsymbol{C}}(V_P)$.*

This is true without any assumption on the depth of V_P. The proof begins with the spectral sequence (4.8), combined with the remarks just made: the cohomology is the direct sum over all the representations σ, $s \in \boldsymbol{C}$ with $\mathrm{Re}(s)<0$, and integers p, q of

$$H^p(\mathfrak{l}, K(L), \sigma\otimes\mathfrak{Log}\otimes\boldsymbol{C}(s+\rho)\otimes H^q(\mathfrak{n}, E)).$$

Let τ be any irreducible L-constituent of $\boldsymbol{C}(\rho)\otimes H^q(\mathfrak{n}, E)$; applying the Hochschild-Serre spectral sequence for the pair (L, A) and the acyclicity of $\mathfrak{Log}$, we see that

$$H^p(\mathfrak{l}, K(L), \sigma\otimes\mathfrak{Log}\otimes\boldsymbol{C}(s)\otimes\tau)\cong H^p(\mathfrak{l}/\mathfrak{a}, K(L), \sigma\otimes H^0(\mathfrak{a}, \boldsymbol{C}(s)\otimes\tau)).$$

Since $\mathrm{Re}(s)< 0$ the τ for which this expression is not trivial must have the property that their restrictions to A^+ are >1. Since A is split over $\boldsymbol{R}$ and τ is essentially algebraic, it must also be true that the s which contribute are real. The representation σ must be unitary, in particular equivalent to its own conjugate dual, and this implies [10: II.6.12] the same must be true of τ. (This is an elementary consequence of Wigner's Lemma [10: I.4.1] but absolutely basic to much of what Borel and I do.) A relatively elementary geometrical argument [11:2.6] implies that q must be $<1/2\dim(\mathfrak{n})$. When G has real rank one the group L/A is compact and the component V_P is just a point, and this fact alone proves not only 5.4 but also Zucker's conjecture. But in general L/A will not be compact, and in order to finish the proof one needs a rather delicate argument involving the theorems of [28] which describe completely those unitary representations σ which might possibly contribute to cohomology. Until recently this was done in a case-by-case analysis which in fact excluded some exceptional groups. But now Borel has found a proof which, although complicated, deals with all cases more or less at once.

When L has rational rank one, the space $\mathscr{A}_{2,\mathrm{disc}}(\Gamma(L)\backslash\mathfrak{S}(L))$ is all of $\mathscr{A}_2(\Gamma(L)\backslash\mathfrak{S}(L))$, so that 5.4 implies Zucker's conjecture immediately in this case. When L has rational rank two, the cohomology of the quotient of the second by the first is part of the local L^2-cohomology of the given strata of depth two as embedded in the closures of the strata of depth one meeting it, with coefficients in the local L^2-cohomology of these strata as embedded in V^*. The situation here is similar to what happens in describing the ordinary cohomology of arithmetic groups of semi-simple $\boldsymbol{Q}$-rank one, as partially explained in [12]. In particular, the extra cohomology is contributed by Eisenstein series. In this case, therefore, Zucker's conjecture is proven by Theorem 5.4 and induction on depth. One interesting feature is that in order to prove the conjecture even for trivial coefficients it is necessary to know it for non-trivial coefficients on the strata of lower depth.

The last step in the proof was suggested by a formula due to J. Arthur (unpublished) for the trace of Hecke operators on the L^2-cohomology of $\Gamma\backslash G$. The structure of the local L^2-cohomology one obtains seems to fit in nicely with what one expects to be useful for applying Zucker's conjecture to find the Hasse-Weil zeta function of V^* associated to its intersection cohomology, when V is a Shimura variety.

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