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Group Cohomology and Hecke Operators 2
Hilbert Modular Surface Case

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In the previous report [27], the authors developed the functorial behavior of Hecke operators operating on group cohomologies, and applied them to arithmetic Fuchsian groups to prove some congruence relations between eigenvalues of Hecke operators. There the authors promised to present similar congruence relations for Hilbert modular groups by the same principle. The author of the present report partly fulfills the promise. Namely, Theorem (3.4) in the last page of this report is a direct analogue for Hilbert-modular-surface case of Theorem (2.2.3) of the previous report [27].

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§ 1. Notations and known facts

1.1. Standard notations of $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}$ are used for the ring of integers, fields of rationals, reals, and complex numbers. In general, the notation K denotes a field of characteristic zero, usually $K=\boldsymbol{Q}$, $\boldsymbol{R}$ or $\boldsymbol{C}$. The finite field with q elements is denoted by $\boldsymbol{F}_q$. For an integer n, the cyclic group of order n is denoted by $\boldsymbol{Z}_n$ or by $\boldsymbol{Z}/n\boldsymbol{Z}$; if n is a prime number ℓ it is also denoted by $\boldsymbol{F}_\ell$. In general, for a module $\mathscr{M}$ and an integer n, the cokernel $\mathscr{M}/n\mathscr{M}$ of the n-multiplication $x \to nx$ is denoted by $\mathscr{M}_n$, and the kernel is denoted by ${}_n\mathscr{M}=\{x \in \mathscr{M}; nx=0\}$.

In this note, as a module $\mathscr{M}$ we consider $\mathscr{M}=\boldsymbol{C}, \boldsymbol{R}, \boldsymbol{Q}, \boldsymbol{Z}, \boldsymbol{Z}_n$, in particular $\boldsymbol{F}_\ell$, $\boldsymbol{C}/\boldsymbol{Z}$, $\boldsymbol{R}/\boldsymbol{Z}$, $\boldsymbol{Q}/\boldsymbol{Z}$, or a finite direct sum of them; so $\mathscr{M}_n$ and ${}_n\mathscr{M}$ are always finite modules.

For a group G, $G/[G, G]$ is denoted by G^{ab}; $[G, G]$ is denoted by $G^{(1)}$,

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$[G^{(1)}, G^{(1)}]$ by $G^{(2)}$, etc.

If a group Γ is operating on a set Ω, for a point $x \in \Omega$, the isotropy subgroup $\{\gamma \in \Gamma;\ \gamma(x)=x\}$ of x is denoted by Γ_x.

The direct product of two groups A, B is denoted by $A \times B$ here, but if A and B are both abelian groups, the notation $A \oplus B$ is also used. For an abelian group A, n-fold direct product (direct sum) is denoted either by A^n or by nA.

For a locally compact abelian group A, the Pontrjagin dual of A is denoted by $\hat{A}$.

For a real quadratic number filed $K=\boldsymbol{Q}(\sqrt{d})$, d denotes the discriminant, h the class number, ε_0 a fundamental unit.

The ring of integers in K is denoted by $\mathfrak{O}$, the group of units by $\mathfrak{O}^\times=\{\pm 1\}\times\{\varepsilon_0^n;\ n \in \boldsymbol{Z}\}$.

For a prime number p, we denote by e_p, f_p, g_p the ramification index, the degree of p in $\mathfrak{O}$, and the number of prime ideals of $\mathfrak{O}$ containing p respectively. Thus $e_p f_p g_p=2$; $e_p=2$ iff $p \mid d$; $(-1)^{g_p}=-(-1)^{f_p}=(d/p)=$ the quadratic residue symbol, for odd $p, p\nmid d$; and with $\nu=(d^2-1)/8, f_2=(1/2)(3-(-1)^\nu)$ and $g_2=(1/2)(3+(-1)^\nu)$ for odd d.

The two embeddings of K into $\boldsymbol{R}$ are denoted by φ_1 and φ_2; we rather consider K as already embedded in $\boldsymbol{R}$, and we denote by φ_1 the identity embedding, and by φ_2 the conjugation. Thus $\varphi_1(\sqrt{d})=\sqrt{d}>0$, and $\varphi_2(\sqrt{d})=-\sqrt{d}<0$.

$N(\alpha)$ denotes the norm $\varphi_1(\alpha)\cdot\varphi_2(\alpha)$ of an element $\alpha \in K$, $\operatorname{tr}(\alpha)=\varphi_1(\alpha)+\varphi_2(\alpha)$ denotes the trace.

For an ideal $\mathfrak{n}$ of $\mathfrak{O}$, $N(\mathfrak{n})$ denotes the norm $[\mathfrak{O}:\mathfrak{n}]$ of $\mathfrak{n}$.

The norm $N((1-\varepsilon_0^2))$ of the principal ideal $(1-\varepsilon_0^2)$ is particularly important in the later sections; we denote it by

$$m_0=N((1-\varepsilon_0^2))=|2-\operatorname{tr}(\varepsilon_0^2)|.$$

For an ideal $\mathfrak{n}$ of $\mathfrak{O}$, the reduction $\mathfrak{O}\to\mathfrak{O}/\mathfrak{n}$ mod $\mathfrak{n}$ is denoted by $\nu_\mathfrak{n}$. The same notation is used for the reduction: $M_2(\mathfrak{O})\to M_2(\mathfrak{O}/\mathfrak{n})$.

The image of the units-group $\mathfrak{O}^\times$ by the reduction $\nu_\mathfrak{n}$ mod $\mathfrak{n}$ is a subgroup of $(\mathfrak{O}/\mathfrak{n})^\times$; the subgroup will be denoted by $K=K(\mathfrak{n})$, the cokernel $(\mathfrak{O}/\mathfrak{n})^\times/K$ is denoted by $H=H(\mathfrak{n})$.

The order $h(\mathfrak{q})=|H(\mathfrak{q})|$ of the cokernel $H(\mathfrak{q})$, as a function of prime ideals $\mathfrak{q}$ is a very unpredictable function. But it is easy to see

Lemma (1.1.1). *If $\mathfrak{q}$ is of degree 2 over $\boldsymbol{Q}$, i.e. $N(\mathfrak{q})=q^2$ with a rational prime q, then $|K(\mathfrak{q})|$ must divide $2(q+1)$; thus, $(q-1)/2 \mid h(\mathfrak{q})$, if $f_q=2$.*

The upper half plane $\{z=x+\sqrt{-1}\,y \in \boldsymbol{C};\ y>0\}$ is denoted by $\mathfrak{H}$ or by $\mathfrak{H}^+$, on which $SL(2, \boldsymbol{R})$ operates by the fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}: z \longrightarrow (az+b)/(cz+d).$$

For a Fuchsian group $\Gamma_1 \subset SL(2, \boldsymbol{R})$ the quotient $\Gamma_1 \backslash \mathfrak{H}$ is denoted by $U = U(\Gamma_1)$. Γ_1 also operates on the lower half plane $\mathfrak{H}^- = \{z = x + \sqrt{-1}\, y;\ y < 0\}$. $\Gamma_1 \backslash \mathfrak{H}^-$ is denoted by $U^- = U^-(\Gamma_1)$.

A "Hilbert modular" group $\Gamma \subset SL(2, R)^m$ operates on $\mathfrak{H}^{\varepsilon_1} \times \mathfrak{H}^{\varepsilon_2} \times \cdots \times \mathfrak{H}^{\varepsilon_m}$ $(\varepsilon_i = \pm)$. The quotient $\Gamma \backslash \mathfrak{H}^{\varepsilon_1} \times \cdots \times \mathfrak{H}^{\varepsilon_m}$ is denoted by $U^{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m} = U^{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m}(\Gamma)$. $U^{+, \cdots, +}$ is denoted by U.

1.2. *Groups discussed here.* Let $K = \boldsymbol{Q}(\sqrt{d})$ be a real quadratic field with class number $h = 1$, and such that $N(\varepsilon_0) = -1$. The Hilbert modular group $SL(2; \mathfrak{O})$ is denoted by $\tilde{\Gamma}(1)$; $\tilde{\Gamma}(1)/\pm 1$ by $\Gamma(1)$. The ordered pair $\varphi = (\varphi_1, \varphi_2)$ of embeddings φ_1, and φ_2 of K into $\boldsymbol{R}$ embeds $\tilde{\Gamma}(1)$ into $SL(2, \boldsymbol{R}) \times SL(2, \boldsymbol{R}) = SL(2, \boldsymbol{R})^2$. Thus $\tilde{\Gamma}(1)$ and $\Gamma(1)$ act on the space $\mathfrak{H} \times \mathfrak{H} = \mathfrak{H}^2$.

For an ideal $\mathfrak{n}$ of $\mathfrak{O}$, we put

$$\tilde{\Gamma}_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}(1);\ c \equiv 0 \pmod{\mathfrak{n}} \right\}$$

and $\Gamma_0(\mathfrak{n}) = \tilde{\Gamma}_0(\mathfrak{n})/\pm 1$. Also we put

$${}^t\tilde{\Gamma}_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}(1);\ b \equiv 0 \pmod{\mathfrak{n}} \right\}$$

and ${}^t\Gamma_0(\mathfrak{n}) = {}^t\tilde{\Gamma}_0(\mathfrak{n})/\pm 1$. Also

$$\tilde{\Gamma}(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}(1);\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{n}} \right\}$$

and

$$\begin{aligned} \Gamma(\mathfrak{n}) &= \tilde{\Gamma}(\mathfrak{n}) && \text{if } \mathfrak{n} \not\ni 2, \\ &= \tilde{\Gamma}(\mathfrak{n})/\pm 1 && \text{if } \mathfrak{n} \ni 2. \end{aligned}$$

In this note we consider as $\mathfrak{n}$ a product:

$$\mathfrak{n} = \mathfrak{q}_1 \mathfrak{q}_2 \cdots \mathfrak{q}_n \cdot \mathfrak{q}_{n+1} \cdots \mathfrak{q}_m \tag{1.2.1}$$

of m distinct prime ideals $\mathfrak{q}_i$ $(i = 1, \cdots, m)$, among which $\mathfrak{q}_1, \cdots, \mathfrak{q}_n$ are of the degree 2 over $\boldsymbol{Q}$, i.e. $N(\mathfrak{q}_i) = q_i^2$, and $\mathfrak{q}_{n+1}, \cdots, \mathfrak{q}_m$ are of degree 1 over $\boldsymbol{Q}$, i.e. $N(\mathfrak{q}_i) = q_i$, where q_i $(i = 1, \cdots, m)$ are rational primes.

We assume furthermore that $(\mathfrak{n}, 6d)=1$, and that there exists a prime number ℓ such that

$$(1.2.2)\qquad \begin{array}{ll} (\ell, 6d\mathfrak{n})=1, & \\ q_i\equiv 1 \pmod{\ell} & \text{for } i=1, \cdots, n, \\ q_i\not\equiv 1 \pmod{\ell} & \text{for } i=n+1, \cdots, m, \\ (\ell, N(1-\varepsilon_0^2))=1. & \end{array}$$

Once such an ideal $\mathfrak{n}$ is chosen and fixed we denote $\tilde{\Gamma}_0(\mathfrak{n})$ by $\tilde{\Gamma}$, $\Gamma_0(\mathfrak{n})$ by Γ. We assume that Γ has no elliptic element. This is true if one of q_j $(j=n+1, \cdots, m)$ satisfies $(-1/q_j)=-1$, and one of q_j $(j=n+1, \cdots, m)$ satisfies $(-3/q_j)=-1$, and $d\neq 5$ (cf. [41], [42]).

In 1.4, we shall see that $H^1(\Gamma, C)=\mathrm{Hom}\,(\Gamma^{\mathrm{ab}}, C)=\{0\}$ (Th. (1.4.4)). This also comes from a result of Margulis [33].

Hence $\Gamma^{\mathrm{ab}}=\Gamma/[\Gamma, \Gamma]$ is a finite group, and by the congruence subgroup theorem of Bass-Milnor-Serre, there is an ideal $\mathfrak{m}$ of $\mathfrak{O}$ such that

$$[\tilde{\Gamma}, \tilde{\Gamma}]\supset\tilde{\Gamma}(\mathfrak{m}), \quad \text{and} \quad [\Gamma, \Gamma]\supset\Gamma(\mathfrak{m}).$$

Let us note here that $\tilde{\Gamma}\ni -1$, but $[\tilde{\Gamma}, \tilde{\Gamma}]\not\ni -1$ iff $\mathfrak{n}\not\ni 2$, so that

$$\Gamma^{\mathrm{ab}}=\tilde{\Gamma}^{\mathrm{ab}}/\pm 1 \qquad \text{for } \mathfrak{n}\not\ni 2.$$

In our case of $\Gamma=\Gamma(\mathfrak{n})$, $\mathfrak{n}=\mathfrak{q}_1, \cdots, \mathfrak{q}_m$, $\mathfrak{q}_i\neq\mathfrak{q}_j$ for $i\neq j$, and $(\mathfrak{n}, 6d)=1$, a standard calculation shows that we can take $\mathfrak{m}=6\mathfrak{n}$, and that

$$\begin{aligned} \tilde{\Gamma}^{\mathrm{ab}} &\cong(\tilde{\Gamma}/\tilde{\Gamma}(\mathfrak{m}))^{\mathrm{ab}} \\ &\cong(\tilde{\Gamma}(1)/\tilde{\Gamma}(6))^{\mathrm{ab}}\times\prod_{\mathfrak{q}|\mathfrak{n}}(\tilde{\Gamma}_0(\mathfrak{q})/\tilde{\Gamma}(\mathfrak{q}))^{\mathrm{ab}} \\ &\cong\prod_{\mathfrak{q}|6\mathfrak{n}}\Phi_{\mathfrak{q}}, \end{aligned}$$

where

$$\Phi_{\mathfrak{q}}=(\boldsymbol{F}_{N\mathfrak{q}})^{\times}\cong\boldsymbol{Z}_{(q^{f_q}-1)} \quad \text{with } N\mathfrak{q}=q^{f_q} \text{ for } \mathfrak{q}\,|\,\mathfrak{n},$$

$$\Phi_{\mathfrak{q}}=\begin{cases}\boldsymbol{Z}_2 & \text{iff } f_2=1 \\ \{1\} & \text{iff } f_2=2\end{cases} \quad \text{for } \mathfrak{q}\,|\,2,$$

and

$$\Phi_{\mathfrak{q}}=\begin{cases}\boldsymbol{Z}_3 & \text{iff } f_3=1 \\ \{1\} & \text{iff } f_3=2\end{cases} \quad \text{for } \mathfrak{q}\,|\,3.$$

We put

$$\begin{aligned} &\textstyle\prod_{\mathfrak{q}|2}\Phi_{\mathfrak{q}}=\Phi_2, \quad \text{which is } \cong\boldsymbol{Z}_2^{g_2} \text{ or } \cong\{1\}, \\ &\textstyle\prod_{\mathfrak{q}|3}\Phi_{\mathfrak{q}}=\Phi_3, \quad \text{which is } \cong\boldsymbol{Z}_3^{g_3} \text{ or } \cong\{1\}, \\ &\Phi_2\times\Phi_3=\Phi_6. \end{aligned}$$

The projection map $P_\mathfrak{q}\colon \tilde{\Gamma}\to\Phi_\mathfrak{q}$ of $\tilde{\Gamma}=\tilde{\Gamma}_0(\mathfrak{n})$ to a factor $\Phi_\mathfrak{q}$ of $\tilde{\Gamma}^{\mathrm{ab}}=\prod \Phi_\mathfrak{q}$ is given as follows:

For a prime ideal $\mathfrak{q}$ dividing $\mathfrak{n}$,

$$\begin{array}{ccc} P_\mathfrak{q}\colon \tilde{\Gamma}_0(\mathfrak{n}) & \longrightarrow & \Phi_\mathfrak{q}=\boldsymbol{F}_{N\mathfrak{q}}^{\times}=(\mathfrak{O}/\mathfrak{q})^{\times}, \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ \gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix} & \longmapsto & P_\mathfrak{q}(\gamma)=\nu_\mathfrak{q}(a)\end{array}$$

i.e. $P_\mathfrak{q}$ associates to a 2×2 matrix $\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ the reduction mod $\mathfrak{q}$ of its (1, 1)-entry a.

If $f_2=1$, thus $\Phi_2=\prod_{\mathfrak{q}|2}\Phi_\mathfrak{q}\cong \boldsymbol{Z}_2^{g_2}$, then for a prime ideal $\mathfrak{q}$ dividing 2, the projection map $P_\mathfrak{q}$ of $\tilde{\Gamma}$ to $\Phi_\mathfrak{q}\cong \boldsymbol{Z}_2$ is given as follows: for an element γ apply the reduction $\nu_\mathfrak{q}$, which sends $\tilde{\Gamma}$ onto $SL(2, \boldsymbol{Z}_2)$, which is isomorphic to the symmetric group $\sum_3$ of 3 letters, then consider the sign, sign $(\nu_\mathfrak{q}(\gamma))$ of the permutation $\nu_\mathfrak{q}(\gamma)$. The map $\mathrm{sign}\circ\nu_\mathfrak{q}\colon \tilde{\Gamma}\to\{\pm 1\}=\boldsymbol{Z}_2$ is the projection: $P_\mathfrak{q}=\mathrm{sign}\circ\nu_\mathfrak{q}$.

For an element $\gamma=\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}$ the formula:

$$P_\mathfrak{q}(\gamma)=\nu_\mathfrak{q}(b)\in \mathfrak{O}/\mathfrak{q}=\boldsymbol{Z}_2$$

is observable easily.

If $f_3=1$, thus $\Phi_3=\boldsymbol{Z}_3^{g_3}$, then for a prime ideal $\mathfrak{q}$ dividing 3 the projection map $P_\mathfrak{q}\colon \tilde{\Gamma}\to\Phi=\boldsymbol{Z}_3$ is given as follows: $\nu_\mathfrak{q}(\tilde{\Gamma})$ is isomorphic to $SL(2, \boldsymbol{F}_3)$, which has a homomorphism onto A_4, the alternating group of 4 letters, which has a homomorphism onto $A_3\cong \boldsymbol{Z}_3$. $P_\mathfrak{q}$ is the combination of these 3 homomorphisms:

$$P_\mathfrak{q}\colon \tilde{\Gamma}\longrightarrow SL(2, \boldsymbol{F}_3)\longrightarrow A_4\longrightarrow A_3=\boldsymbol{Z}_3.$$

For an element γ of the shape $\gamma=\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}$, the formula

$$P_\mathfrak{q}(\gamma)=\nu_\mathfrak{q}(a^{-1}\cdot b)\in \mathfrak{O}/\mathfrak{q}\cong \boldsymbol{Z}_3$$

is observable easily.

We denote the order of cyclic groups $\Phi_\mathfrak{q}$ appearing in the decomposition $\tilde{\Gamma}^{\mathrm{ab}}=\prod \Phi_\mathfrak{q}$ by μ_i $(i=1, \cdots)$ in some ordering. Thus μ_i are one of $(q^{f_q}-1)$ or 2 or 3, and

$$\tilde{\Gamma}^{\mathrm{ab}}=\oplus \boldsymbol{Z}_{\mu_i}.$$

Thus

$$\Gamma^{\mathrm{ab}}=(\oplus \boldsymbol{Z}_{\mu_i})/\pm 1.$$

We denote $[\tilde{\Gamma}, \tilde{\Gamma}]$ by $\tilde{\Gamma}^{(1)}=\Gamma^{(1)}$. Note that $\tilde{\Gamma}^{(1)} \not\ni -1$. Also denote $[\Gamma^{(1)}, \Gamma^{(1)}]$ by $\Gamma^{(2)}$. $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are arithmetic subgroup, and we have

$$\begin{aligned}\Gamma^{(1)}&=\{\gamma \in \tilde{\Gamma};\ P_{\mathfrak{q}}(\gamma)=1,\ \mathfrak{q}\,|\,6\mathfrak{n}\}\\&=\left\{\gamma \in \tilde{\Gamma};\ P_6(\gamma)=1,\ \gamma\equiv\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix} \bmod \mathfrak{n}\right\},\\ \Gamma^{(1)}&\supset\Gamma^{(2)}\supset\Gamma((6\mathfrak{n})^m),\end{aligned}$$

for a sufficiently large integer m.

Thus $\Gamma^{(1)\mathrm{ab}}=\Gamma^{(1)}/\Gamma^{(2)}$ is a finite abelian group whose order divides some power of $6\mathfrak{n}$, and therefore, is coprime with ℓ. Namely,

Lemma (1.2.3). $H^1(\Gamma^{(1)}, \boldsymbol{F}_\ell)=\mathrm{Hom}\,(\Gamma^{(1)\mathrm{ab}}, \boldsymbol{F}_\ell)=\{0\}$.

$(\boldsymbol{R}\cup(\infty))\times(\boldsymbol{R}\cup(\infty))=(\boldsymbol{R}\cup(\infty))^2$ is a part of the boundary $\partial(\mathfrak{H}^2)$ of $\mathfrak{H}^2$ in $P^1(\boldsymbol{C})^2$. This part is denoted by $P^1(\boldsymbol{R})^2$. A map $\varphi=(\varphi_1, \varphi_2)$ of $K\cup(\infty)$ to $P^1(\boldsymbol{R})^2$ is defined by

$$\begin{cases}\varphi(\alpha)=(\varphi_1(\alpha), \varphi_2(\alpha)) \in \boldsymbol{R}^2 & \text{for } \alpha \in K,\\ \varphi(\infty)=(\infty, \infty).\end{cases}$$

A point in the image $\mathrm{Im}\,(\varphi)=\varphi(K\cup(\infty))\subset P^1(\boldsymbol{R})^2$ is called a cusp. $\mathrm{Im}\,(\varphi)$ is also denoted by $P^1(K)$, by identifying $\alpha \in K\cup(\infty)=P^1(K)$ with the image $\varphi(\alpha)=(\varphi_1(\alpha), \varphi_2(\alpha))$.

$SL(2, K)$ operates on $P^1(K)$, so does $\tilde{\Gamma}(1)$ and $\Gamma(1)$. In our case of $h=1$, the action of $\tilde{\Gamma}(1)$ on $P^1(K)$ is transitive. The isotropy group of $\infty=(\infty, \infty)$, denoted by $\tilde{\Gamma}_\infty(1)$, is

$$\begin{aligned}\tilde{\Gamma}_\infty(1)&=\{\gamma \in \tilde{\Gamma}(1);\ \gamma(\infty)=\infty\}=\left\{\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \tilde{\Gamma}(1)\right\}\\&=\left\{\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix};\ ad=1,\ a, b, d \in \mathfrak{O},\ a, d \in \mathfrak{O}^\times\right\}.\end{aligned}$$

Similarly, $\Gamma_\infty(1)=\{\gamma \in \Gamma(1):\ \gamma(\infty)=\infty\}=\tilde{\Gamma}_\infty(1)/\pm 1$ is defined.

Thus the space $P^1(K)$ of cusps is identified with $\tilde{\Gamma}(1)/\tilde{\Gamma}_\infty(1)=\Gamma(1)/\Gamma_\infty(1)$.

The isotropy subgroup $\tilde{\Gamma}_\alpha(1)=\{\gamma \in \Gamma(1);\ \gamma(\alpha)=\alpha\}$ of an arbitrary cusp $\alpha=g(\infty)$, $(g \in \tilde{\Gamma}(1))$, is $\tilde{\Gamma}_\alpha(1)=g\tilde{\Gamma}_\infty(1)g^{-1}$. Similarly $\Gamma_\alpha(1)$ is defined, and $\Gamma_\alpha(1)=g\Gamma_\infty(1)g^{-1}$ with the obvious implication.

For a subgroup $\Gamma\subset\Gamma(1)$, a Γ-orbit of cusps is called a Γ-cusp or simply a cusp. A Γ-cusp containing a cusp α is denoted by $[\alpha]$ or $[\alpha]_\Gamma$. For an index finite subgroup Γ in $\Gamma(1)$, the number $f=f_\Gamma$ of Γ-cusps is finite; and the set $\{c_1, c_2, \cdots, c_f\}$ of Γ-cusps is identified with the double-coset-space $\Gamma\backslash\Gamma(1)/\Gamma_\infty(1)$.

Choose a system $\{g_1, \cdots, g_f\}$ of representatives g_i $(i=1, \cdots, f)$ of double-cosets: $\Gamma(1)=\bigcup \Gamma g_i \Gamma_\infty(1)$, then $\alpha_i=g_i(\infty)$ $(i=1, \cdots, f)$ represent Γ-cusps $c_i=[\alpha_i]_\Gamma$ $(i=1, \cdots, f)$.

For a cusp $\alpha \in P^1(K)$, the isotropy subgroup $\Gamma_\alpha=\{\gamma \in \Gamma : \gamma(\alpha)=\alpha\}$ in Γ is obviously $\Gamma_\alpha=\Gamma \cap \Gamma_\alpha(1)$. If $\alpha, \beta \in P^1(K)$ are Γ-equivalent, then $\Gamma_\alpha, \Gamma_\beta$ are conjugate in Γ.

In our case of $\Gamma=\Gamma_0(\mathfrak{n})$, $\mathfrak{n}=\mathfrak{q}_1 \cdots \mathfrak{q}_m$, $\mathfrak{q}_i \neq \mathfrak{q}_j$ (for $i \neq j$), the number $f=f_\Gamma$ of Γ-cusps is $f_\Gamma=2^m$. The set $\{\mathfrak{q}_1, \cdots, \mathfrak{q}_m\}$ of prime ideals $\mathfrak{q}_i$ is abbreviated as $\{1, 2, \cdots, m\}$. A map D of $\{\mathfrak{q}_1, \cdots, \mathfrak{q}_m\}=\{1, 2, \cdots, m\}$ to the set $\{+1, -1\}$ is called a "sign-distribution" on $\{1, 2, \cdots, m\}$. There are altogether 2^m sign-distributions; and the set of them we denote by $\mathscr{D}$.

For each sign-distribution $D \in \mathscr{D}$, take an element $g_D \in \Gamma(1)$, such that

$$g_D \equiv \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \bmod \mathfrak{q}_i \qquad \text{if } D(\mathfrak{q}_i)=+1, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \bmod \mathfrak{q}_i \qquad \text{if } D(\mathfrak{q}_i)=-1, \end{cases}$$

for all $i=1, 2, \cdots, m$. Then, the double-coset space $\tilde{\Gamma}\backslash\tilde{\Gamma}(1)/\tilde{\Gamma}_\infty(1)$ is represented by $\{g_D ; D \in \mathscr{D}\}$, and thus Γ-cusps are represented by $\alpha_D= g_D(\infty)$ $(D \in \mathscr{D})$ (cf. [42]).

For a sign-distribution $D \in \mathscr{D}$, the product of prime ideals $\mathfrak{q}_i$ with $D(\mathfrak{q}_i)=-1$ is denoted by $\mathfrak{n}(D)=\prod_{D(i)=-1} \mathfrak{q}_i$, which is an ideal containing $\mathfrak{n}$. Also put $\mathfrak{m}(D)=\prod_{D(i)=+1} \mathfrak{q}_i$, so that $\mathfrak{n}=\mathfrak{n}(D) \cdot \mathfrak{m}(D)$.

The isotropy subgroup Γ_∞ of ∞ in Γ is obviously $\Gamma_\infty=\Gamma_\infty(1)$, since $\Gamma_\infty(1) \subset \Gamma_0(\mathfrak{n})=\Gamma$. The isotropy subgroup Γ_α of $\alpha=g(\infty)$, $(g \in \Gamma(1))$, is $\Gamma_\alpha=\Gamma_\alpha(1) \cap \Gamma=(g\Gamma_\infty(1)g^{-1}) \cap \Gamma=g(\Gamma_\infty(1) \cap g^{-1}\Gamma g)g^{-1}$. Thus Γ_α is isomorphic to $\Gamma_\infty(1) \cap g^{-1}\Gamma g$, which we will denote by Γ'_α. For $\alpha=\alpha_D= g_D(\infty)$, Γ_{α_D} is isomorphic to

$$\begin{aligned} \Gamma'_{\alpha_D} &= \Gamma_\infty(1) \cap g_{\alpha_D}^{-1}\Gamma_0(\mathfrak{n})g_{\alpha_D} = \left\{\gamma=\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma_\infty(1) ; b \in \mathfrak{n}(D)\right\} \\ &= \Gamma_\infty(1) \cap {}^t\Gamma_0(\mathfrak{n}(D)). \end{aligned}$$

By the same inner automorphism: $x \to g_D^{-1} x g_D$ of $\Gamma(1)$, which sends Γ_α to Γ'_α, $\Gamma=\Gamma_0(\mathfrak{n})$ is sent to the subgroup

$$g_D^{-1}\Gamma_0 g_D = g_D^{-1}\Gamma_0(\mathfrak{n})g_D = \Gamma_0(\mathfrak{m}(D)) \cap {}^t\Gamma_0(\mathfrak{n}(D)),$$

which we will denote by $\Gamma'=\Gamma'_D$.

As we can see easily,

$$\Gamma_\infty=\Gamma_\infty(1)=\left\{\begin{pmatrix} a & b \\ 0 & d\end{pmatrix};\ d=a^{-1},\ a\in\mathfrak{O}^\times,\ b\in\mathfrak{O}\right\}\Big/\pm 1,$$

is a semi-direct product of the "squares of units" $(\mathfrak{O}^\times)^2=\{\varepsilon^2;\ \varepsilon\in\mathfrak{O}^\times\}$ with the additive group $\mathfrak{O}$ of integers; i.e. $\Gamma_\infty=\Gamma_\infty(1)\cong(\mathfrak{O}^\times)^2\ltimes\mathfrak{O}$.

Also

$$\Gamma_{\alpha_D}\cong\Gamma'_{\alpha_D}=\Gamma_\infty(1)\cap{}^t\Gamma_0(\mathfrak{n}(D))=\left\{\begin{pmatrix} a & b \\ 0 & d\end{pmatrix};\ a\in\mathfrak{O}^\times,\ b\in\mathfrak{n}(D)\right\}\Big/\pm 1,$$

is isomorphic to the semi-direct product: $(\mathfrak{O}^\times)^2\ltimes\mathfrak{n}(D)$.

Also, it is easy to see that

$$[\Gamma_\infty,\Gamma_\infty]=\left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix};\ b\in(1-\varepsilon_0^2)\mathfrak{O}\right\},$$

$$[\Gamma'_{\alpha_D},\Gamma'_{\alpha_D}]=\left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix};\ b\in(1-\varepsilon_0^2)\mathfrak{n}(D)\right\}.$$

Thus

$$\text{(1.2.3}'\text{)}\qquad \begin{aligned}\Gamma_\infty^{\mathrm{ab}}&\cong(\mathfrak{O}^\times)^2\times(\mathfrak{O}/(1-\varepsilon_0^2)\mathfrak{O})=\boldsymbol{Z}\times\mathscr{G}_\infty,\\ \Gamma_{\alpha_D}^{\mathrm{ab}}&\cong(\mathfrak{O}^\times)^2\times(\mathfrak{n}(D)/(1-\varepsilon_0^2)\mathfrak{n}(D))=\boldsymbol{Z}\times\mathscr{G}_D,\end{aligned}$$

where

$$\text{(1.2.3}''\text{)}\qquad \begin{aligned}&\mathscr{G}_\infty=\mathscr{G}_\mathfrak{O}=\mathfrak{O}/(1-\varepsilon_0^2)\mathfrak{O},\\ &\mathscr{G}_D=\mathscr{G}_{\alpha_D}=\mathscr{G}_\alpha=\mathscr{G}_{\mathfrak{n}(D)}=\mathfrak{n}(D)/(1-\varepsilon_0^2)\mathfrak{n}(D),\end{aligned}$$

are finite abelian group of the order $|\mathscr{G}_\infty|=|\mathscr{G}_D|=|N(1-\varepsilon_0^2)|$.

We consider a direct-sum decomposition of the finite abelian groups $\mathscr{G}_\infty$ or $\mathscr{G}_\alpha$ into the direct sum:

$$\mathscr{G}_\infty=\oplus\boldsymbol{Z}_{m_{\infty,i}},\qquad \mathscr{G}_\alpha=\oplus\boldsymbol{Z}_{m_{\alpha,i}},$$

of cyclic group $\boldsymbol{Z}_m$'s. Orders m of cyclic groups are divisors of $N(1-\varepsilon_0^2)$.

The injection ι_α: $\Gamma_\alpha\hookrightarrow\Gamma$ induces the homomorphism

$$\Delta_\alpha:\ \Gamma_\alpha^{\mathrm{ab}}\longrightarrow\Gamma^{\mathrm{ab}}=\prod\Phi_\mathfrak{q}.$$

It is easy to see that for a prime ideal $\mathfrak{q}$ dividing $\mathfrak{n}$, the image of $P_\mathfrak{q}\circ\Delta_\alpha$ is the subgroup $K(\mathfrak{q})=\nu_\mathfrak{q}(\mathfrak{O}^\times)\subset(\mathfrak{O}/\mathfrak{q})^\times$ spanned by units of $\mathfrak{O}$ (see 1.1.1), which has the order $|K(\mathfrak{q})|$ coprime with ℓ.

Lemma (1.2.4). *$\Delta_\alpha(\Gamma_\alpha^{\mathrm{ab}})\subset\ell(\Gamma^{\mathrm{ab}})$, and $\Delta_\alpha(\Gamma_\alpha^{\mathrm{ab}})$ has the order coprime with ℓ.*

From this, it is also easy to see that

Lemma (1.2.5). *The subgroup* ${}_{\ell}(\Gamma^{\mathrm{ab}}/\Sigma_{\alpha}\Delta_{\alpha}(\Gamma_{\alpha}^{\mathrm{ab}}))$ *of* ℓ*-torsion elements in* $\Gamma^{\mathrm{ab}}/(\Sigma\Delta_{\alpha}(\Gamma_{\alpha}^{\mathrm{ab}}))$ *is isomorphic to* $\boldsymbol{F}_{\ell}^{n}$, *where n is the number of prime ideals* $\mathfrak{q}_i$ *in* $\mathfrak{n}$ *with* $N(\mathfrak{q}_i)=q_i^2$, $q_i\equiv 1 \pmod{\ell}$ (*see* 1.2.1, 2).

Comment. In this report, we restrict our attention to the Hilbert-modular group Γ, and keep off from quaternion-Hilbert-modular groups. Why? Because, for an arithmetic subgroup in the multiplicative group in a divison quaternion algebra, the congruence subgroup theorem (abbr. c.s.th) is not yet proven. Let B be a quaternion algebra over a totally real number field K. In B, we can construct arithmetic discontinuous groups $\Gamma=\Gamma_0(\mathfrak{n})$, similar to our Hilbert-modular cases. They operate on $\mathfrak{H}^2$ and produce algebraic surfaces $U=\Gamma\backslash\mathfrak{H}^2$ if B splits at exactly 2 ∞-places of K. While many of our results in this note are still valid for such Γ, the Lemma (1.2.3), which will play an essential role in a later section, is no longer valid because the lemma depends on the c.s.th. However if someday one could prove the c.s.th. for a Γ of this type, then total results of our note shall become valid for that Γ.

1.3. *Arrows discussed in this note.* The notation GR denotes one of the groups discussed in 1.2, i.e. GR is either Γ or Γ^{ab}, or Γ_{α} or $\Gamma_{\alpha}^{\mathrm{ab}}$, or a finite abelian group.

A short exact sequence:

$$\text{(SName)} \qquad 0 \dashrightarrow \mathcal{M}_1 \overset{\alpha}{\dashrightarrow} \mathcal{M}_2 \overset{\beta}{\dashrightarrow} \mathcal{M}_3 \dashrightarrow 0$$

of GR-trivial modules $\mathcal{M}_i$ is denoted by broken arrows $\dashrightarrow$; we call $\mathcal{M}_1$ the lst term, $\mathcal{M}_2$ the 2nd, $\mathcal{M}_3$ the 3rd.

The long exact sequence caused by the short exact sequence (SName) is denoted by (LName), i.e.,

$$\text{(LName)} \qquad \overset{\delta}{\dashrightarrow} H^r(GR, \mathcal{M}_1) \overset{\alpha}{\dashrightarrow} H^r(GR, \mathcal{M}_2) \overset{\beta}{\dashrightarrow} H^r(GR, \mathcal{M}_3) \cdots\cdots$$

In (LName) arrows are also denoted by broken arrows. Notations for induced maps of α (or of β) are also denoted by the same symbol α (or β). The connecting homomorphism $H^r(GR, \mathcal{M}_3) \dashrightarrow H^{r+1}(GR, \mathcal{M}_1)$ is denoted by δ.

The following is a table of short exact sequences used in this note.

$$(\mathrm{S}\boldsymbol{C}):\ 0 \dashrightarrow \boldsymbol{Z} \overset{i}{\dashrightarrow} \boldsymbol{C} \overset{\nu}{\dashrightarrow} \boldsymbol{C}/\boldsymbol{Z} \dashrightarrow 0$$

$$(\mathrm{S}\boldsymbol{R}):\ 0 \dashrightarrow \boldsymbol{Z} \overset{i}{\dashrightarrow} \boldsymbol{R} \overset{\nu}{\dashrightarrow} \boldsymbol{R}/\boldsymbol{Z} \dashrightarrow 0$$

$$(\mathrm{S}\boldsymbol{Q})\colon 0 \dashrightarrow \boldsymbol{Z} \overset{i}{\dashrightarrow} \boldsymbol{Q} \overset{\nu}{\dashrightarrow} \boldsymbol{Q}/\boldsymbol{Z} \dashrightarrow 0$$

$$(\mathrm{S}\nu_\ell)\colon 0 \dashrightarrow \boldsymbol{Z} \overset{\ell}{\dashrightarrow} \boldsymbol{Z} \overset{\nu_\ell}{\dashrightarrow} \boldsymbol{F}_\ell \dashrightarrow 0$$

$$(\mathrm{S}i_\ell)\colon 0 \dashrightarrow \boldsymbol{F}_\ell \overset{i_\ell}{\dashrightarrow} \boldsymbol{Q}/\boldsymbol{Z} \overset{\ell}{\dashrightarrow} \boldsymbol{Q}/\boldsymbol{Z} \dashrightarrow 0$$

Corresponding long exact sequences are denoted by (L$\boldsymbol{C}$), (L$\boldsymbol{R}$), $\cdots$, etc.

Combining long exact sequences (L$\boldsymbol{Q}$), (Lν_ℓ), (Li_ℓ), we have a commutative diagram:

(DNA. 1.3.1)

$$\begin{tikzcd}
 & {} \arrow[d, dashed] & {} \arrow[d, dashed, "\ell"] & {} \arrow[d, dashed, "\ell"] & \\
\cdots \arrow[r, dashed] & H^1(GR, \boldsymbol{Q}) \arrow[r, dashed] \arrow[d, dashed] & H^1(GR, \boldsymbol{Q}/\boldsymbol{Z}) \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\delta"] & H^2(GR, \boldsymbol{Z}) \arrow[r, dashed, dash] \arrow[d, dashed, "\nu_\ell"] & \cdots \\
 & 0 \arrow[r, dashed] \arrow[d, dashed] & H^2(GR, \boldsymbol{F}_\ell) \arrow[r, dashed, "\mathrm{id}"] \arrow[d, dashed, "i"] & H^2(GR, \boldsymbol{F}_\ell) \arrow[r, dashed] \arrow[d, dashed, "\delta"] & 0 \\
\cdots \arrow[r, dashed] & H^2(GR, \boldsymbol{Q}) \arrow[r, dashed] \arrow[d, dashed] & H^2(GR, \boldsymbol{Q}/\boldsymbol{Z}) \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\ell"] & H^3(GR, \boldsymbol{Z}) \arrow[r, dashed, dash] \arrow[d, dashed, "\ell"] & \cdots \\
\cdots \arrow[r, dashed] & H^2(GR, \boldsymbol{Q}) \arrow[r, dashed] \arrow[d, dashed, dash] & H^2(GR, \boldsymbol{Q}/\boldsymbol{Z}) \arrow[r, dashed] \arrow[d, dashed, dash] & H^3(GR, \boldsymbol{Z}) \arrow[r, dashed, dash] \arrow[d, dashed, dash] & \cdots \\
 & {} & {} & {} &
\end{tikzcd}$$

We call this diagram a DNA $(\boldsymbol{Q}-\nu-i_\ell)$. Similarly, DNA $(\boldsymbol{R}-\nu-i_\ell)$, DNA $(\boldsymbol{C}-\nu-i_\ell)$ are defined.

For a compact oriented manifold M with boundary ∂M the exact sequence of relative cohomologies with coefficient group $\mathscr{M}$:

$$(\mathrm{REL}1.3.2) \quad \xrightarrow{\delta} H^p(M, \partial M, \mathscr{M}) \longrightarrow H^p(M, \mathscr{M}) \xrightarrow{r} H^p(\partial M, \mathscr{M}) \longrightarrow$$

is denoted by unbroken arrows. The arrow $r\colon H^p(M, \mathscr{M}) \to H^p(\partial M, \mathscr{M})$ is called the restriction.

In our note, M is a manifold obtained from the Hilbert modular surface $U = \Gamma\backslash\mathfrak{H}^2$, by chopping off neighbourhoods of Γ-cusps; and

$$H^p(M, \mathscr{M}) \cong H^p(U, \mathscr{M}) \cong H^p(\Gamma, \mathscr{M})$$

and

$$H^p(\partial M, \mathscr{M}) \cong \bigoplus_D H^p(\Gamma_{\alpha_D}, \mathscr{M}).$$

Thus the restriction r combined with the projection $\oplus H^p(\Gamma_{\alpha_D}, \mathcal{M}) \to H^p(\Gamma_{\alpha_D}, \mathcal{M}) \to$ is denoted by r_{α_D}, which is the restriction of $H^p(\Gamma, \mathcal{M})$ to the subgroup Γ_{α_D}.

The natural mapping $\Gamma \xrightarrow{\Lambda} \Gamma^{\mathrm{ab}}$, (or $\Gamma_\alpha \xrightarrow{\Lambda_\alpha} \Gamma_\alpha^{\mathrm{ab}}$) induces homomorphisms $\lambda: H^p(\Gamma^{\mathrm{ab}}, \mathcal{M}) \to H^p(\Gamma, \mathcal{M})$, (or $\lambda_\alpha: H^p(\Gamma_\alpha^{\mathrm{ab}}, \mathcal{M}) \to H^p(\Gamma_\alpha, \mathcal{M})$). λ and λ_α are called lift of inflation; they are denoted by unbroken arrows.

The inflation $\lambda: H^2(\Gamma^{\mathrm{ab}}, \mathcal{M}) \to H^2(\Gamma, \mathcal{M})$ is a part of the Hochschild-Serre exact sequence:

(HS1.3.3) $$\longrightarrow H^1(\Gamma^{(1)}, \mathcal{M})\Gamma^{\mathrm{ab}} \longrightarrow H^2(\Gamma^{\mathrm{ab}}, \mathcal{M}) \xrightarrow{\lambda} H^2(\Gamma, \mathcal{M})$$

where $\Gamma^{(1)} = [\Gamma, \Gamma]$.

In particular if $H^1(\Gamma^{(1)}, \mathcal{M}) = 0$, the inflation $\lambda: H^2(\Gamma^{\mathrm{ab}}, \mathcal{M}) \to H^2(\Gamma, \mathcal{M})$ is injective. This, combined with the Lemma (1.2.3), gives

Lemma (1.3.4). *The inflation* λ: $H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell) \to H^2(\Gamma, \boldsymbol{F}_\ell)$ *is injective.*

Lemma (1.3.5). *The inflation* λ: $H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{C}/\boldsymbol{Z}) \to H^2(\Gamma, \boldsymbol{C}/\boldsymbol{Z})$ *is injective on the subgroup* ${}_\ell H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{C}/\boldsymbol{Z})$.

Proof. Since $H^1(\Gamma^{(1)}, \boldsymbol{C}/\boldsymbol{Z}) = \Gamma^{(1)\mathrm{ab}}$ has order coprime with ℓ.

Lemma (1.3.6). *The inflation* λ: $H^1(\Gamma^{\mathrm{ab}}, \mathcal{M}) \to H^1(\Gamma, \mathcal{M})$ *is always injective.*

Proof. Easy, or well known.

Broken arrows $\dashrightarrow$ are maps caused by operations on coefficient modules, and unbroken arrows $\rightarrow$ are maps caused by operations on manifolds or groups; and thus broken arrows and unbroken arrows are commutative.

The inclusion i_α: $\Gamma_\alpha \to \Gamma$ induces the homomorphism Δ_α: $\Gamma_\alpha^{\mathrm{ab}} \to \Gamma^{\mathrm{ab}}$, which makes the commutative diagram:

(1.3.7)
$$\begin{tikzcd} \Gamma_\alpha \arrow[r, "i_\alpha"] \arrow[d, "\Lambda_\alpha"'] & \Gamma \arrow[d, "\Lambda"] \\ \Gamma_\alpha^{\mathrm{ab}} \arrow[r, "\Delta_\alpha"] & \Gamma^{\mathrm{ab}} \end{tikzcd}$$

i_α induces the restriction r_α: $H^*(\Gamma, \mathcal{M}) \to H^*(\Gamma_\alpha, \mathcal{M})$; Λ induces the inflation λ: $H^*(\Gamma^{\mathrm{ab}}, \mathcal{M}) \to H^*(\Gamma, \mathcal{M})$. The homomorphism induced by Δ_α: $\Gamma_\alpha^{\mathrm{ab}} \to \Gamma^{\mathrm{ab}}$ we denote by ρ_α: $H^*(\Gamma^{\mathrm{ab}}, \mathcal{M}) \to H^*(\Gamma_\alpha^{\mathrm{ab}}, \mathcal{M})$: this is also called a "restriction". The homomorphism induced by Λ_α: $\Gamma_\alpha \to \Gamma_\alpha^{\mathrm{ab}}$ is denoted by λ_α: $H^*(\Gamma_\alpha^{\mathrm{ab}}, \mathcal{M}) \to H^*(\Gamma_\alpha, \mathcal{M})$, and is called inflation.

Inflations and restrictions λ, λ_α, r_α, and ρ_α form the commutative diagram:

$$
\begin{tikzcd}
H^*(\Gamma^{\mathrm{ab}}, \mathscr{M}) \arrow[r, "\rho_\alpha"] \arrow[d, "\lambda"'] & H^*(\Gamma_\alpha^{\mathrm{ab}}, \mathscr{M}) \arrow[d, "\lambda_\alpha"] \\
H^*(\Gamma, \mathscr{M}) \arrow[r, "r_\alpha"] & H^*(\Gamma_\alpha, \mathscr{M})
\end{tikzcd}
\tag{1.3.8}
$$

In later sections, the same symbol of arrows, (say r), appears in several different locations. In order to distinguish them, we put labels to them like $r[1]$, $r[2]$, $r[3]$, $\cdots$, $\lambda[1]$, $\lambda[2]$, $\lambda[3]$, $\cdots$ etc.

1.4. *From works of Hirzebruch/Harder.* Cohomologies $H^r(U, \boldsymbol{C})=H^r(\Gamma, \boldsymbol{C})$ of Hilbert modular varieties $U=\Gamma\backslash\mathfrak{H}^m$ are extensively investigated by Hirzebruch and Harder. Here Γ is an arithmetic subgroup of $SL(2, K)$, and K is a totally real number field with $[K: \boldsymbol{Q}]=m$. Here we quote some results from Harder [28], for $m=2$.

Let $(M, \partial M)$ be the 4-manifold M with boundary ∂M, obtained from the Hilbert modular surface $U=\Gamma\backslash\mathfrak{H}^2$ by chopping off neighborhoods of cusps. (for details see Harder [28]). Then

$$
\begin{aligned}
&H^r(M, -)=H^r(U, -)=H^r(\Gamma, -),\\
&H^r(\partial M, -)=\bigoplus_\alpha H^r(\Gamma_\alpha, -),
\end{aligned}
\tag{1.4.1}
$$

for a Γ-trivial module $-$.

Since M is a 4-fold with boundary and ∂M is a compact 3-fold without boundary and with f connected components, we have

$$
\begin{aligned}
&H^0(M, \boldsymbol{C})=\boldsymbol{C}, \quad H^0(M, \partial M, \boldsymbol{C})=0, \quad H^4(M, \boldsymbol{C})=0,\\
&H^4(M, \partial M, \boldsymbol{C})=\boldsymbol{C}, \quad H^0(\partial M, \boldsymbol{C})=\boldsymbol{C}^f, \quad H^3(\partial M, \boldsymbol{C})=\boldsymbol{C}^f.
\end{aligned}
\tag{1.4.2}
$$

Since $\Gamma_\alpha\cong\boldsymbol{Z}\ltimes\mathfrak{n}(D)$, $\Gamma_\alpha^{\mathrm{ab}}=\boldsymbol{Z}\oplus\mathscr{G}_\alpha$ for $\alpha=\alpha_D$, (see § 1.2), we have with the Poincaré duality:

$$
H^1(\partial M, \boldsymbol{C})=\bigoplus_\alpha H^1(\Gamma_\alpha, \boldsymbol{C})=\boldsymbol{C}^f, \quad H^2(\partial M, \boldsymbol{C})=\boldsymbol{C}^f.
$$

Consider the exact sequence of relative cohomologies:

$$
\begin{array}{l}
0\longrightarrow \underset{\{0\}}{\overset{\|}{H^0(M, \partial M, \boldsymbol{C})}} \longrightarrow \underset{\boldsymbol{C}}{\overset{\|}{H^0(M, \boldsymbol{C})}} \xrightarrow{r^0} \underset{\boldsymbol{C}^f}{\overset{\|}{H^0(\partial M, \boldsymbol{C})}}\\
\longrightarrow H^1(M, \partial M, \boldsymbol{C}) \longrightarrow H^1(M, \boldsymbol{C}) \xrightarrow{r^1} H^1(\partial M, \boldsymbol{C})
\end{array}
\tag{1.4.3}
$$

where $H^0(M,\partial M,\boldsymbol{C})=\{0\}$, $H^0(M,\boldsymbol{C})=\boldsymbol{C}$, $H^0(\partial M,\boldsymbol{C})=\boldsymbol{C}^f$, $H^1(M,\boldsymbol{C})=0$, $H^1(\partial M,\boldsymbol{C})=\boldsymbol{C}^f$.

$$
\begin{array}{ccccc}
 & & & & C^f \\
 & & & & \| \\
\longrightarrow H^2(M, \partial M, C) & \longrightarrow & H^2(M, C) & \xrightarrow{r^2} & H^2(\partial M, C) \\
 & & & & C^f \\
 & & & & \| \\
\longrightarrow H^3(M, \partial M, C) & \longrightarrow & H^3(M, C) & \xrightarrow{r^3} & H^3(\partial M, C) \\
C & & & & \\
\| & & & & \\
\longrightarrow H^4(M, \partial M, C) & \longrightarrow & 0. & &
\end{array}
$$

In this diagram r^i: $H^i(M, C)=H^i(\Gamma, C)\to H^i(\partial M, C)=\oplus H^i(\Gamma_\alpha, C)$ are the sums $r^i=\oplus r^i_\alpha$ of restrictions r^i_α: $H^i(\Gamma, C)\to H^i(\Gamma_\alpha, C)$. For these r^i, Harder showed that: r^0 is the diagonal map: $C\to C^f$; r^1 is the zero map; r^2 is surjective; and r^3 has codimension 1 image. He denoted the kernel of r^i by $H^i_f(U)=H^i_f(M)=H^i_f(\Gamma)$.

Denote the "variable" in $\mathfrak{H}^2$ by $z=(z_1, z_2)$, and write: $z_\alpha=x_\alpha+iy_\alpha$ ($\alpha=1, 2$). Put $w_\alpha=(1/4\pi i)(dz_\alpha\wedge d\bar{z}_\alpha/y_\alpha^2)$, ($\alpha=1, 2$). w_1 and w_2 are $SL(2, \boldsymbol{R})^2$-invariant closed two forms on $\mathfrak{H}^2$, so they determine de Rham cohomology classes on U; these cohomology classes we denote by the same symbols $w_1, w_2 \in H^2(U, C)$. w_1 and w_2 span a two dimensional subspace $W=Cw_1+Cw_2\subset H^2(U, C)$.

The upper half plane $\mathfrak{H}$ is also denoted by $\mathfrak{H}^+$, and the lower half plane $\{z=x+iy \in C; y<0\}$ is denoted by $\mathfrak{H}^-$. The arithmetic discontinuous group $\Gamma\subset SL(2, \boldsymbol{R})^2$ also operates on $\mathfrak{H}^+\times\mathfrak{H}^-$, $\mathfrak{H}^-\times\mathfrak{H}^+$, $\mathfrak{H}^-\times\mathfrak{H}^-$; and the quotients $\Gamma\backslash\mathfrak{H}^+\times\mathfrak{H}^-$, $\Gamma\backslash\mathfrak{H}^-\times\mathfrak{H}^+$, $\Gamma\backslash\mathfrak{H}^-\times\mathfrak{H}^-$ are denoted by U^{+-}, U^{-+}, U^{--} respectively. U is also denoted by U^{++}. The "partial-conjugation map"

$$\Theta^{+-}: \mathfrak{H}^+\times\mathfrak{H}^+ \ni (z_1, z_2)\longrightarrow(z_1, \bar{z}_2) \in \mathfrak{H}^+\times\mathfrak{H}^-$$

induces a diffeomorphism Θ^{+-} of U^{++} to U^{+-}. Similarly diffeomorphisms Θ^{-+}: $U^{++}\to U^{-+}$, Θ^{--}: $U^{++}\to U^{--}$ are defined.

The space of Γ-cusp forms of weight 2 on $\mathfrak{H}^2$ is denoted by $S_2(\Gamma)$.

For an automorphic form $\varphi(z)=\varphi(z_1, z_2) \in S_2(\Gamma)$, the holomorphic 2-form

$$w=w_\varphi=\varphi(z)dz_1\wedge dz_2$$

on $\mathfrak{H}^2$ is Γ-invariant; and w induces a holomorphic 2-form w on U. The de Rham cohomology class defined by $w=w_\varphi$ is also denoted by $w=w_\varphi$.

Then the map:

$$\Psi^{++}: S_2(\Gamma) \ni \varphi\longrightarrow w_\varphi \in H^2(U)$$

is injective. The image of the map Ψ^{++} is denoted by A^{++} or by S^{++}.

The space of Γ-cusp forms on $\mathfrak{H}^+\times\mathfrak{H}^-$ of weight 2 is denoted by $S_2(\Gamma, \mathfrak{H}^+\times\mathfrak{H}^-)$.

For an automorphic form $\varphi\in S_2(\Gamma, \mathfrak{H}^+\times\mathfrak{H}^-)$, the holomorphic 2-form $w=w_\varphi=\varphi(z_2, z_2)dz_1\wedge dz_2$ defines a de Rham cohomology class $w_\varphi\in H^2(U^{+-})$.

The image of

$$\begin{array}{ccccc} 0 \longrightarrow & S_2(\Gamma, \mathfrak{H}^+\times\mathfrak{H}^-) & \longrightarrow & H^2(U^{+-}) \\ & \cup & & \cup \\ & \varphi & \longmapsto & w_\varphi \end{array}$$

in $H^2(U^{+-})$ is denoted by S^{+-}.

Furthermore, we put $(\Theta^{+-})^*(S^{+-})=A^{+-}\subset H^2(U)$.

Similarly, we define

$$S_2(\Gamma, \mathfrak{H}^-\times\mathfrak{H}^+),\quad S^{-+}\subset H^2(U^{+-}),\quad A^{+-}=(\Theta^{-+})^*(S^{-+})\subset H^2(U);$$
$$S_2(\Gamma, \mathfrak{H}^-\times\mathfrak{H}^-),\quad S^{--}\subset H^2(U^{--}),\quad A^{--}=(\Theta^{--})^*(S^{--})\subset H^2(U).$$

We put

$$A^{++}+A^{+-}+A^{-+}+A^{--}=A=A(\Gamma).$$

Now, results of Harder [28] include

Theorem (1.4.4) (Harder).

$$H^1(\Gamma, \boldsymbol{C})=0,$$
$$H^2_f(\Gamma, \boldsymbol{C})=\ker(r^2;\ H^2(U, \boldsymbol{C})\longrightarrow H^2(\partial M, \boldsymbol{C}))=A\oplus W.$$

If the quadratic number field K has a unit ε_0 with $\varphi_1(\varepsilon_0)>0$, $\varphi_2(\varepsilon_0)<0$, then put $E_0=\begin{pmatrix}\varepsilon_0 & 0\\ 0 & 1\end{pmatrix}$. Then the map $E_0\colon z=(z_1, z_2)\longmapsto E(z)=(\varphi_1(\varepsilon_0)z_1, \varphi_2(\varepsilon_0)z_2)$ sends $\mathfrak{H}^+\times\mathfrak{H}^+$ to $\mathfrak{H}^+\times\mathfrak{H}^-$, and induces the bi-holomorphic isomorphism of U to U^{+-}, if

$$\text{(1.4.4}') \qquad E_0\Gamma=\Gamma E_0$$

is satisfied. For our group $\Gamma=\Gamma_0(\mathfrak{n})$, and for $\Gamma={}^t\Gamma_0(\mathfrak{n})$, $\Gamma=\Gamma(\mathfrak{n})$ the condition (1.4.4′) is satisfied.

The bi-holomorphic isomorphism

$$E_0\colon U\cong U^{+-}$$

induces the isomorphism:

$$E_0^*\colon S^{+-}\cong S^{++}$$

Similarly

$$E_0' = \begin{pmatrix} -\varepsilon_0^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

gives the isomorphisms: E_0': $U \cong U^{-+}$, $E_0'^*$: $S^{-+} \cong S^{++}$. Also

$$C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

induces isomorphisms: C: $U \cong U^{--}$, C^*: $S^{--} \cong S^{++}$. So one has

Lemma (1.4.5). *If* $N(\varepsilon_0) = -1$, *then* $A \cong 4S_2(\Gamma)$.

1.5. *Trivial information.* Since $(M, \partial M)$ in 1.2 is an oriented connected compact 4-manifold M with boundary $\partial M \neq \emptyset$, we have

$$(1.5.1') \qquad \begin{aligned} &H_0(M, \mathcal{M}) = \mathcal{M}, && H_0(M, \partial M, \mathcal{M}) = \{0\}, \\ &H^0(M, \mathcal{M}) = \mathcal{M}, && H^0(M, \partial M, \mathcal{M}) = \{0\}, \\ &H_4(M, \mathcal{M}) = \{0\}, && H_4(M, \partial M, \mathcal{M}) = \mathcal{M}, \\ &H^4(M, \mathcal{M}) = \{0\}, && H^4(M, \partial M, \mathcal{M}) = \mathcal{M}, \end{aligned}$$

for arbitrary constant coefficient-module $\mathcal{M}$.

Since $H_*^*(\Gamma, \mathcal{M}) = H_*^*(M, \mathcal{M})$, we have

$$(1.5.1) \qquad \begin{aligned} &H_0(\Gamma, \mathcal{M}) = \mathcal{M}, && H^0(\Gamma, \mathcal{M}) = \mathcal{M}, \\ &H_4(\Gamma, \mathcal{M}) = \{0\}, && H^4(\Gamma, \mathcal{M}) = \{0\}, \end{aligned}$$

for an arbitrary Γ-trivial module $\mathcal{M}$.

Since the cusp group Γ_α is the fundamental group of a compact orientable 3-manifold $Y_\alpha = \Gamma_\alpha \backslash \tilde{Y}_\alpha$, with contractible universal covering $\tilde{Y}_\alpha$ without boundary, (1.4 and Harder [28]), we have

$$(1.5.2) \qquad \begin{aligned} &H_0(\Gamma_\alpha, \mathcal{M}) = \mathcal{M}, && H^0(\Gamma_\alpha, \mathcal{M}) = \mathcal{M}, \\ &H_3(\Gamma_\alpha, \mathcal{M}) = \mathcal{M}, && H^3(\Gamma_\alpha, \mathcal{M}) = \mathcal{M}, \end{aligned}$$

for an arbitrary Γ_α-trivial module $\mathcal{M}$. Thus

$$(1.5.2') \qquad \begin{aligned} &H_0(\partial M, \mathcal{M}) = \mathcal{M}^f, && H^0(\partial M, \mathcal{M}) = \mathcal{M}^f, \\ &H^3(\partial M, \mathcal{M}) = \mathcal{M}^f, && H_3(\partial M, \mathcal{M}) = \mathcal{M}^f. \end{aligned}$$

Since $H_1(\Gamma_\alpha, \boldsymbol{Z}) = \Gamma_\alpha^{\mathrm{ab}}$, our knowledge of the structure of $\Gamma_\alpha^{\mathrm{ab}} = \boldsymbol{Z} \oplus \mathcal{G}_\alpha = \boldsymbol{Z} \oplus (\oplus \boldsymbol{Z}_{m_{\alpha,i}})$ and the universal-coefficients-theorem and Poincaré-duality imply

$$
\begin{aligned}
&H_1(\Gamma_\alpha, \boldsymbol{Z}) = \Gamma_\alpha^{\mathrm{ab}} = \boldsymbol{Z} \oplus \mathscr{G}_\alpha = \boldsymbol{Z} \oplus (\oplus \boldsymbol{Z}_m),\\
&H^1(\Gamma_\alpha, \boldsymbol{Z}) = \mathrm{Hom}\,(\Gamma_\alpha^{\mathrm{ab}}, \boldsymbol{Z}) = \boldsymbol{Z},\\
&H^2(\Gamma_\alpha, \boldsymbol{Z}) = H_1(\Gamma_\alpha, \boldsymbol{Z}) = \Gamma_\alpha^{\mathrm{ab}} = \boldsymbol{Z} \oplus (\oplus \boldsymbol{Z}_m),\\
&H_2(\Gamma_\alpha, \boldsymbol{Z}) = H^1(\Gamma_\alpha, \boldsymbol{Z}) = \boldsymbol{Z},\\
&H_1(\Gamma_\alpha, \mathscr{M}) = (H_1(\Gamma_\alpha, \boldsymbol{Z}) \otimes \mathscr{M}) \oplus \mathrm{Tor}(H_0(\Gamma_\alpha, \boldsymbol{Z}), \mathscr{M})\\
&\qquad = ((\boldsymbol{Z} \oplus \mathscr{G}_\alpha) \otimes \mathscr{M}) \oplus \mathrm{Tor}(\boldsymbol{Z}, \mathscr{M}) = \mathscr{M} \oplus (\mathscr{G}_\alpha \otimes \mathscr{M}) \oplus \{0\}\\
&\qquad = \mathscr{M} \oplus (\oplus \mathscr{M}_m),\\
&H^1(\Gamma_\alpha, \mathscr{M}) = \mathrm{Hom}\,(H_1(\Gamma_\alpha, \boldsymbol{Z}), \mathscr{M}) \oplus \mathrm{Ext}\,(H_0(\Gamma_\alpha, \boldsymbol{Z}), \mathscr{M})\\
&\qquad = \mathrm{Hom}\,(\boldsymbol{Z} \oplus \mathscr{G}_\alpha, \mathscr{M}) = \mathscr{M} \oplus \mathrm{Hom}\,(\mathscr{G}_\alpha, \mathscr{M}) = \mathscr{M} \oplus (\oplus_m \mathscr{M}),\\
&H_2(\Gamma_\alpha, \mathscr{M}) = H^1(\Gamma_\alpha, \mathscr{M}) = \mathscr{M} \oplus (\oplus_m \mathscr{M}),\\
&H^2(\Gamma_\alpha, \mathscr{M}) = H_1(\Gamma_\alpha, \mathscr{M}) = \mathscr{M} \oplus (\oplus \mathscr{M}_m).
\end{aligned}
\tag{1.5.3}
$$

for a Γ_α-trivial module $\mathscr{M}$.

In particular for a prime number ℓ, coprime to $N(1-\varepsilon_0^2)$

$$
H^1(\Gamma_\alpha, \boldsymbol{F}_\ell) = \boldsymbol{F}_\ell, \qquad H^2(\Gamma_\alpha, \boldsymbol{F}_\ell) = \boldsymbol{F}_\ell. \tag{1.5.4}
$$

Our knowledge of $\Gamma^{\mathrm{ab}} = \prod \Phi_{\mathrm{q}} = \oplus \boldsymbol{Z}_{\mu_i}$ implies

$$
\begin{aligned}
&H_1(\Gamma, \boldsymbol{Z}) = \Gamma^{\mathrm{ab}} = \prod \Phi_{\mathrm{q}} = \oplus \boldsymbol{Z}_{\mu_i},\\
&H^1(\Gamma, \boldsymbol{Z}) = \mathrm{Hom}\,(\Gamma^{\mathrm{ab}}, \boldsymbol{Z}) = \{0\},\\
&H^3(M, \partial M, \boldsymbol{Z}) = H_1(M, \boldsymbol{Z}) = H_1(\Gamma, \boldsymbol{Z}) = \Gamma^{\mathrm{ab}},\\
&H_3(M, \partial M, \boldsymbol{Z}) = H^1(M, \boldsymbol{Z}) = H^1(\Gamma, \boldsymbol{Z}) = \{0\}.
\end{aligned}
\tag{1.5.5}
$$

Thus the universal-coefficients-theorem gives

$$
\begin{aligned}
&H_1(\Gamma, \mathscr{M}) = (H_1(\Gamma, \boldsymbol{Z}) \otimes \mathscr{M}) \oplus \mathrm{Tor}\,(H_0(\Gamma, \boldsymbol{Z}), \mathscr{M}))\\
&\qquad = (\Gamma^{\mathrm{ab}} \otimes \mathscr{M}) \oplus \mathrm{Tor}\,(\boldsymbol{Z}, \mathscr{M}) = \Gamma^{\mathrm{ab}} \otimes \mathscr{M} = \oplus \mathscr{M}_{\mu_i},\\
&H^1(\Gamma, \mathscr{M}) = \mathrm{Hom}\,(\Gamma^{\mathrm{ab}}, \mathscr{M}) \oplus \mathrm{Ext}\,(H_0(\Gamma, \boldsymbol{Z}), \mathscr{M})\\
&\qquad = \mathrm{Hom}\,(\Gamma^{\mathrm{ab}}, \mathscr{M}) = \oplus ({}_{\mu_i}\mathscr{M}).
\end{aligned}
\tag{1.5.6}
$$

This information and Poincaré duality imply

$$
\begin{aligned}
&H_1(M, \mathscr{M}) = \Gamma^{\mathrm{ab}} \otimes \mathscr{M} = \mathscr{M}_{\mu_i},\\
&H^1(M, \mathscr{M}) = \mathrm{Hom}\,(\Gamma^{\mathrm{ab}}, \mathscr{M}) = \oplus {}_{\mu_i}\mathscr{M},\\
&H^3(M, \partial M, \mathscr{M}) = H_1(M, \mathscr{M}) = \oplus \mathscr{M}_{\mu_i},\\
&H_3(M, \partial M, \mathscr{M}) = H^1(M, \mathscr{M}) = \oplus {}_{\mu_i}\mathscr{M}.
\end{aligned}
\tag{1.5.6'}
$$

In order to estimate $H^3(\Gamma)\cong H^3(M)\cong H_1(M,\partial M)$, we need to use the exact sequence of relative cohomology:

$$(1.5.7)\qquad \begin{array}{ccccccc} \oplus\Gamma^{\mathrm{ab}}_{\alpha} & \xrightarrow{\quad\Delta\quad} & \Gamma^{\mathrm{ab}} & & H^3(\Gamma, Z) & & Z^f \\ \|\wr & & \|\wr & & \|\wr & & \|\wr \\ H^2(\partial M, Z) & \longrightarrow & H^3(M, \partial M, Z) & \longrightarrow & H^3(M, Z) & \longrightarrow & H^3(\partial M, Z) \end{array}$$

$$\begin{array}{ccc} & Z & \\ & \|\wr & \\ \longrightarrow & H^4(M, \partial M, Z) & \longrightarrow 0. \end{array}$$

Here the map $\Delta\colon \oplus\Gamma^{\mathrm{ab}}_{\alpha}\to\Gamma^{\mathrm{ab}}$ is the map $\Delta=\oplus(\Delta_\alpha\circ\mathrm{proj})$, described in 1.2. Thus

$$(1.5.8)\qquad H^3(\Gamma, Z)=Z^{f-1}\oplus(\Gamma^{\mathrm{ab}}/\oplus\Delta_\alpha\Gamma^{\mathrm{ab}}_\alpha))=Z^{f-1}\oplus(\text{torsion part}).$$

This, combined with Lemma (1.2.5), implies that the ℓ-torsions of $H^3(\Gamma, Z)$ is isomorphic to F^n_ℓ; thus:

Lemma (1.5.9). $\qquad {}_\ell H^3(\Gamma, Z)\cong F^n_\ell.$

By Lemma (1.2.4), we see also

Lemma (1.5.10). *The image of* $\delta\colon H^2(\partial M, Z)\to H^3(M, \partial M, Z)$ *is a subgroup of* $H^3(M, \partial M, Z)$ *with the order coprime to* ℓ.

Comment. For a quaternion-Hilbert-modular group $\Gamma=\Gamma_0(\mathfrak{n})\subset B^\times$, in a quaternion algebra B over a totally real number field, of which exactly two ∞-places split B, formulas in this section should be changed to

$$(1.5.1.1'\text{-B})\qquad \begin{aligned} &H^0(\Gamma, \mathcal{M})\cong H^0(U, \mathcal{M})\cong H_4(\Gamma, \mathcal{M})\cong H_4(U, \mathcal{M})\\ &\quad\cong H_0(\Gamma, \mathcal{M})=H_0(U, \mathcal{M})\cong H^4(\Gamma, \mathcal{M})\cong H^4(U, \mathcal{M})\cong\mathcal{M}. \end{aligned}$$

$$(1.5.5\text{-B})\qquad \begin{aligned} &H^1(\Gamma, Z)\cong H^1(U, Z)\cong H_3(\Gamma, Z)\cong H_3(U, Z)\cong\{0\},\\ &H_1(\Gamma, Z)\cong H_1(U, Z)\cong H^3(\Gamma, Z)\cong H^3(U, Z)\cong\Gamma^{\mathrm{ab}}\\ &\quad=\text{a finite abelian group.} \end{aligned}$$

$$(1.5.6\text{-B})\qquad \begin{aligned} &H_1(\Gamma, \mathcal{M})\cong H_1(U, \mathcal{M})\cong H^3(\Gamma, \mathcal{M})\cong H^3(U, \mathcal{M})\cong\Gamma^{\mathrm{ab}}\otimes\mathcal{M},\\ &H^1(\Gamma, \mathcal{M})=H^1(U, \mathcal{M})=H_3(\Gamma, \mathcal{M})=H_3(U, \mathcal{M})\\ &\quad=\mathrm{Hom}\,(\Gamma^{\mathrm{ab}}, \mathcal{M}). \end{aligned}$$

However the Lemma (1.5.9) is no longer valid. We can only say that

$$\dim_{F_\ell}({}_\ell H^3(\Gamma, Z))\geqq n.$$

1.6. *Finite abelian groups.* The multiplicative semi-group of $\boldsymbol{Z}$ is denoted by $(\boldsymbol{Z}, x)$. Let m be a space integer: $m \in \boldsymbol{Z}$. On an abelian group A, the m-multiplication $x \to mx$ of A into A is denoted by $(m)=(m)_A$.

For an integer $\nu \geqq 0$, the module A with the "modified" action $(m^\nu): x \to m^\nu x$ of the multiplicative semi-group $(\boldsymbol{Z}, x)$ on A is denoted by $A(\nu)$. In particular, $A(0)$ is a $(\boldsymbol{Z}, x)$-trivial module.

Let $\{\mathscr{G}_a\}_{a \in T}$ be a finite collection of finite cyclic groups $\mathscr{G}_a = \boldsymbol{Z}_{n(a)}$, $a \in T$, indexed by a finite set $T=\{1, 2, \cdots, t\}$. The function defined by: $T \ni a \to n(a) = |\mathscr{G}_a|$ is denoted by n: $T \to \boldsymbol{Z}_+$. We put $\mathscr{G}_T = \oplus_{a \in T} \mathscr{G}_a$. For a subset $S \subset T$, we put $\mathscr{G}_S = \oplus_{a \in S} \mathscr{G}_a$, and we consider it as a space subgroup of $\mathscr{G}_T$. Also we put $n(S)=$ the greatest common divisor of $n(a)$, $a \in S$. If $S=\phi$, we put $\mathscr{G}_\phi = \{0\}$, $n(\phi)=1$.

Let $\Omega = \{0, 1, 3, 5, 7, \cdots\}$ be the set of positive odd integers and zero. A function f of Ω to $\boldsymbol{Z}$ is defined by: $f(0)=0$, and $f(r)=(r+1)/2$, for odd r.

A function D: $T \to \Omega$ of T to Ω is called a "dimension-distribution". For a dimension-distribution D, we put $f(D) = \sum_{a \in T} f(D(a))$, $\deg(D) = \sum_{a \in T} D(a)$, $\operatorname{Supp}(D) = \{a \in T \colon D(a) > 0\}$, and $\boldsymbol{Z}[D] = \boldsymbol{Z}_{n(\operatorname{Supp} D)}(f(D))$. For a $\mathscr{G}_T$-trivial module $\mathscr{M}$ and for a dimension-distribution D, we put

$$\mathscr{M}[D] = \boldsymbol{Z}[D] \otimes \mathscr{M} = \mathscr{M}_{n(\operatorname{Supp} D)}(f(D)), \quad \text{and}$$
$$\mathscr{M}\langle D \rangle = \boldsymbol{Z}[D] * \mathscr{M} = {}_{n(\operatorname{Supp} D)}\mathscr{M}(f(D)),$$

where $A * B = \operatorname{Tor}(A, B)$.

For $\mathscr{M} = \boldsymbol{Z}$, $\boldsymbol{Z}\langle D \rangle = 0$, for $\mathscr{M} = \boldsymbol{Q}/\boldsymbol{Z}$, $\boldsymbol{Q}/\boldsymbol{Z}[D]=0$, and $\boldsymbol{Q}/\boldsymbol{Z}\langle D \rangle = \boldsymbol{Z}[D]$. For $\mathscr{M} = \boldsymbol{F}_\ell$, $\boldsymbol{F}_\ell\langle D \rangle = \boldsymbol{F}_\ell[D] = \boldsymbol{F}_\ell(f(D))$ if $\ell \mid n(D)$, $=0$ if $\ell \nmid n(D)$. Then we have an obvious exact sequence:

$$0 \longrightarrow \boldsymbol{F}_\ell[D] \longrightarrow \boldsymbol{Z}[D] \xrightarrow{\ \ell\ } \boldsymbol{Z}[D] \xrightarrow{\ \nu_\ell\ } \boldsymbol{F}_\ell[D] \longrightarrow 0. \tag{1.6.1}$$

If D_1, D_2 are two dimension-distributions with $\operatorname{Supp}(D_1) \cap \operatorname{Supp}(D_2) = \phi$, $D_1 \coprod D_2$ denotes the dimension distribution defined by $D_1 \coprod D_2 = \max(D_1, D_2)$. Then for such D_1 and D_2, we have

$$\boldsymbol{Z}[D_1] * \boldsymbol{Z}[D_2] = \boldsymbol{Z}[D_1 \textstyle\coprod D_2],$$
$$\boldsymbol{Z}[D_1] \otimes \boldsymbol{Z}[D_2] = \boldsymbol{Z}[D_1 \textstyle\coprod D_2].$$

For a positive integer r, and for a non-zero dimension-distribution D: $T \to \Omega$, with $s = |\operatorname{Supp}(D)|$, $d = \deg(D)$, we define an integer $m(r, D)$ by

$$m(r, D) = \binom{s-1}{r-d}$$

where

$$\binom{a}{b}=\begin{cases} a!/(b!(a-b)!) & \text{if } 0\leqq b\leqq a,\\ 1 & \text{if } 0=b=a,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$m(1, D)=\begin{cases} 1 & \text{iff } d=1,\ s=1,\ f(D)=1,\\ 0 & \text{otherwise,}\end{cases}$$

$$m(2, D)=\begin{cases} 1 & \text{iff } d=2,\ s=2,\ f(D)=2,\\ 0 & \text{otherwise,}\end{cases}$$

$$m(3, D)=\begin{cases} 1 & \text{if } d=3,\ s=3,\ f(D)=3,\\ 1 & \text{if } d=3,\ s=1,\ f(D)=2,\\ 1 & \text{if } d=2,\ s=2,\ f(D)=2,\\ 0 & \text{otherwise,}\end{cases}$$

$$m(4, D)=\begin{cases} 1 & \text{if } d=4,\ s=4,\ f(D)=4,\\ 1 & \text{if } d=4,\ s=2,\ f(D)=3,\\ 2 & \text{if } d=3,\ s=3,\ f(D)=3,\\ 0 & \text{otherwise.}\end{cases}$$

The effect of m-multiplication: (m): $\mathscr{G}_T \ni x \to mx \in \mathscr{G}_T$, $(m \in \boldsymbol{Z})$, on the homology group $H_*(\mathscr{G}_T, \mathscr{M})$, (or on the cohomology group $H^*(\mathscr{G}_T, \mathscr{M})$), is denoted by $(m)_*$ (or by $(m)^*$, respectively). With $(m)_*$ (or with $(m)^*$) $H_*(\mathscr{G}_T, \mathscr{M})$ (or $H^*(\mathscr{G}_T, \mathscr{M})$) is a $(\boldsymbol{Z}, x)$-module.

Theorem (1.6.2). *For a positive integer $r>0$*

$$H_r(\mathscr{G}_T, \boldsymbol{Z})=\bigoplus_D m(r, D)\boldsymbol{Z}[D],$$

where the summation extends over all the dimension-distributions D. Obviously, $H_0(\mathscr{G}_T, \boldsymbol{Z})=\boldsymbol{Z}$.

Theorem (1.6.3). *For a $\mathscr{G}_T$-trivial module $\mathscr{M}$, we have*

$$H_0(\mathscr{G}_T, \mathscr{M})=\mathscr{M},$$

$$H_1(\mathscr{G}_T, \mathscr{M})=\mathscr{G}_T\otimes\mathscr{M}=\bigoplus_{a\in T}\mathscr{M}_{n(a)}(1),$$

$$\begin{aligned} H_r(\mathscr{G}_T, \mathscr{M})&=\Sigma m(r, D)\boldsymbol{Z}[D]\otimes\mathscr{M}\oplus\Sigma m(r-1, D)\boldsymbol{Z}[D]\ {}_*\mathscr{M}\\ &=\Sigma m(r, D)\mathscr{M}[D]\oplus\Sigma m(r-1, D)\mathscr{M}\langle D\rangle. \end{aligned}$$

$$H^0(\mathscr{G}_T, \mathscr{M})=\{0\}.$$

$$H^1(\mathscr{G}_T, \mathscr{M}) = \mathrm{Hom}\,(\mathscr{G}_T, \mathscr{M}) = \oplus_{n(a)}\mathscr{M}(1),$$
$$H^r(\mathscr{G}_T, \mathscr{M}) = \Sigma m(r, D)\mathscr{M}\langle D\rangle \oplus \Sigma m(r-1, D)\mathscr{M}[D] \quad \text{for } r>0.$$

In particular,

Corollary (1.6.4). *For a field K of characteristic zero,*

$$H^r(\mathscr{G}_T, K) = 0 \qquad \textit{for } r>0,$$

$$H^r(\mathscr{G}_T, \boldsymbol{Q}/\boldsymbol{Z}) = \begin{cases} (\boldsymbol{Q}/\boldsymbol{Z})(0) & \textit{for } r=0 \\ \mathscr{G}_T(1) & \textit{for } r=1 \\ \Sigma m(r, D)(\boldsymbol{Q}/\boldsymbol{Z})\langle D\rangle & \textit{for } r>1, \end{cases}$$

$$H^r(\mathscr{G}_T, \boldsymbol{Z}) = \begin{cases} \boldsymbol{Z}(0) & \textit{for } r=0 \\ 0 & \textit{for } r=1 \\ \Sigma m(r-1, D)\boldsymbol{Z}[D] & \textit{for } r>1, \end{cases}$$

$$H^r(\mathscr{G}_T, \boldsymbol{F}_\ell) = \begin{cases} \boldsymbol{F}_\ell(0) & \textit{for } r=0 \\ t(\ell)\boldsymbol{F}_\ell(1) & \textit{for } r=1 \\ \Sigma(m(r, D)+m(r-1, D))\boldsymbol{F}_\ell[D] & \textit{for } r>1. \end{cases}$$

Here $t(\ell) = |T(\ell)|$ *is the number of elements of the set*

$$T(\ell) = \{a \in T \colon \ell \,|\, n(a)\},$$

and the sum Σ *extends over all the dimension-distributions D with* Supp(D) $\subset T(\ell)$.

In particular,

Corollary (1.6.5).

$$H^0(\mathscr{G}_T, \boldsymbol{Z}) = \boldsymbol{Z}(0),$$
$$H^1(\mathscr{G}_T, \boldsymbol{Z}) = \{0\},$$
$$H^2(\mathscr{G}_T, \boldsymbol{Z}) = \sum_{a\in T} \boldsymbol{Z}_{n(a)}(1) = \mathscr{G}_T(1),$$
$$H^3(\mathscr{G}_T, \boldsymbol{Z}) = \sum_{\{a,b\}\subset T} \boldsymbol{Z}_{n(a,b)}(2),$$
$$H^4(\mathscr{G}_T, \boldsymbol{Z}) = \sum_{\{a,b,c\}\subset T} \boldsymbol{Z}_{n(a,b,c)}(3) \oplus \sum_{a\in T} \boldsymbol{Z}_{n(a)}(2) \oplus \sum_{\{a,b\}\subset T} \boldsymbol{Z}_{n(a,b)}(2),$$

and

$$H^0(\mathscr{G}_T, \boldsymbol{F}_\ell) = \boldsymbol{F}_\ell(0),$$
$$H^1(\mathscr{G}_T, \boldsymbol{F}_\ell) = t\boldsymbol{F}_\ell(1),$$
$$H^2(\mathscr{G}_T, \boldsymbol{F}_\ell) = (t(t-1)/2)\boldsymbol{F}_\ell(2) \oplus t\boldsymbol{F}_\ell(1),$$

$$H^3(\mathscr{G}_T, \boldsymbol{F}_\ell)=(t(t-1)(t-2)/6)\boldsymbol{F}_\ell(3)\oplus t\boldsymbol{F}_\ell(2),$$
$$H^4(\mathscr{G}_T, \boldsymbol{F}_\ell)=(t(t-1)(t-2)(t-3)/24)\boldsymbol{F}_\ell(4)$$
$$\oplus(t(t-1)/2)\boldsymbol{F}_\ell(3)\oplus(t(t+1)/2)\boldsymbol{F}_\ell(2),$$

where $t=t(\ell)=|T(\ell)|$.

From these formulas, we have

Lemma (1.6.6). *If a homomorphism $\Delta\colon \mathscr{H}\to\mathscr{G}_T$ of a group $\mathscr{H}$ to the abelian group $\mathscr{G}_T$ has the image $\Delta(\mathscr{H})$ in $\ell\mathscr{G}_T$, then the "restriction" $\rho=\Delta^*\colon H^r(\mathscr{G}_T, \boldsymbol{F}_\ell)\to H^r(\mathscr{H}, \boldsymbol{F}_\ell)$ is the zero map for $r>0$.*

Let us apply this lemma to our $\mathscr{G}_T=\Gamma^{\mathrm{ab}}$, $\mathscr{H}=\Gamma^{\mathrm{ab}}_\alpha$, $\Delta=$the natural homomorphism $\Delta_\alpha\colon \Gamma^{\mathrm{ab}}_\alpha\to\Gamma^{\mathrm{ab}}$ (see § 1.2). Since $P_\mathfrak{q}\circ\Delta_\alpha(\Gamma^{\mathrm{ab}}_\alpha)\subset K(\mathfrak{q})$ for every $\mathfrak{q}\,|\,\mathfrak{n}$, we have $\Delta(\Gamma^{\mathrm{ab}}_\alpha)\subset\ell(\Gamma^{\mathrm{ab}})$, (see 1.2.4). So:

Lemma (1.6.7). *The restriction map $\rho_\alpha\colon H^r(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell)\to H^r(\Gamma^{\mathrm{ab}}_\alpha, \boldsymbol{F}_\ell)$ is the zero map for $r>0$.*

We put

$$(1.6.8)\qquad \begin{aligned} H^*(\mathscr{G}_T, \mathscr{M})^+&=\{x\in H^*(\mathscr{G}_T, \mathscr{M})\colon (-1)^*x=x\},\\ H^*(\mathscr{G}_T, \mathscr{M})^-&=\{x\in H^*(\mathscr{G}_T, \mathscr{M})\colon (-1)^*x=-x\},\end{aligned}$$

where $(-1)^*=$the effect of the (-1)-multiplication: $\mathscr{G}_T\ni x\to -x\in\mathscr{G}_T$ on the cohomology.

For $\mathscr{M}=\boldsymbol{F}_\ell$ with odd ℓ, we have the "eigenspace-decomposition"

$$H^*(\mathscr{G}_T, \boldsymbol{F}_\ell)=H^*(\mathscr{G}_T, \boldsymbol{F}_\ell)^+\oplus H^*(\mathscr{G}_T, \boldsymbol{F}_\ell)^-.$$

With the expression:

$$H^*(\mathscr{G}_T, \boldsymbol{F}_\ell)=\sum_D (m(r, D)+m(r-1, D))\boldsymbol{F}_\ell[D],$$

we have

$$H^*(\mathscr{G}_T, \boldsymbol{F}_\ell)^+=\sum_{D:f(D)\text{ even}} (m(r, D)+m(r-1, D))\boldsymbol{F}_\ell[D],$$
$$H^*(\mathscr{G}_T, \boldsymbol{F}_\ell)^-=\sum_{D:f(D)\text{ odd}} (m(r, D)+m(r-1, D))\boldsymbol{F}_\ell[D].$$

Since $H^1(\mathscr{G}_T, \boldsymbol{F}_\ell)=\mathrm{Hom}\,(\mathscr{G}_T, \boldsymbol{F}_\ell)=\boldsymbol{F}_\ell(1)$, so

$$H^1(\mathscr{G}_T, \boldsymbol{F}_\ell)=(\mathscr{G}_T, \boldsymbol{F}_\ell)^-.$$

Also

$$H^1(\mathscr{G}_T, \boldsymbol{C}/\boldsymbol{Z})=\hat{\mathscr{G}}_T(1)^{\vee}=H^1(\mathscr{G}_T, \boldsymbol{C}/\boldsymbol{Z})^-.$$

By the Kunneth theorem for $\mathscr{G}_T = \prod Z_{n(a)}$, we have

$$H^2(\mathscr{G}_T, F_\ell) = \{\bigoplus_{a,b} H^1(Z_{n(a)}, F_\ell) \otimes H^1(Z_{n(b)}, F_\ell)\} \oplus \{\oplus H^2(Z_{n(a)}, F_\ell)\};$$

and it is also observed easily:

Lemma (1.6.8).

$$H^2(\mathscr{G}_T, F_\ell)^+ = \bigoplus_{a,b} H^1(\mathscr{G}_a, F_\ell) \otimes H^1(\mathscr{G}_b, F_\ell),$$
$$H^2(\mathscr{G}_T, F_\ell)^- = \bigoplus_{a} H^2(\mathscr{G}_a, F_\ell).$$

Thus

$$\dim H^2(\mathscr{G}_T, F_\ell)^+ = t(t-1)/2; \ \dim H^2(\mathscr{G}_T, F_\ell)^- = t,$$

with $t = t(\ell)$. Also since $H^2(\mathscr{G}_T, Z) \cong \mathscr{G}_T(1)$, (1.6.5),

$$H^2(\mathscr{G}_T, Z)^- = H^2(\mathscr{G}_T, Z).$$

The short exact sequence: $0 \longrightarrow Z \xrightarrow{\ell} Z \xrightarrow{\nu_\ell} F_\ell \longrightarrow 0$ generates the long exact sequence:

(1.6.9)
$$\begin{array}{ccccccc}
0 \longrightarrow & H^1(\mathscr{G}_T, Z) & \longrightarrow & H^1(\mathscr{G}_T, Z) & \longrightarrow & H^1(\mathscr{G}_T, F_\ell) \\
& \| & & \| & & \| \\
& \{0\} & & \{0\} & & \oplus m(1, D) F_\ell[D] \\
\longrightarrow & H^2(\mathscr{G}_T, Z) & \longrightarrow & H^2(\mathscr{G}_T, Z) & \longrightarrow & H^2(\mathscr{G}_T, F_\ell) \\
& \| & & \| & & \| \\
& \oplus m(1, D) Z[D] & & \oplus m(1, D) Z[D] & & \{\oplus m(1, D) F_\ell[D]\} \\
& & & & & \oplus\{\oplus m(2, D) F_\ell[D]\} \\
\longrightarrow & H^3(\mathscr{G}_T, Z) & \longrightarrow \cdots & & & \\
& \| & & & & \\
& \oplus m(2, D) Z[D] & \text{, i.e.,} & & &
\end{array}$$

(1.6.10)
$$0 \longrightarrow \oplus m(1, D) F_\ell[D] \longrightarrow \oplus m(1, D) Z[D] \longrightarrow \oplus m(1, D) Z[D]$$
$$\longrightarrow \begin{cases} \oplus m(1, D) F_\ell[D] \\ \quad\oplus \\ \oplus m(2, D) F_\ell[D] \end{cases} \longrightarrow \oplus m(2, D) Z[D] \longrightarrow \cdots$$

This is just the "linking" of the exact sequences (1.6.1). In particular,

Lemma (1.6.11). *The* (LZ)*-sequence for* $GR=\mathscr{G}_T$ *is*

$$
\begin{array}{l}
0 \cdots\cdots\to H^1(\mathscr{G}_T, \boldsymbol{F}_\ell) \cdots\cdots\to H^2(\mathscr{G}_T, \boldsymbol{Z}) \cdots\cdots\to H^2(\mathscr{G}_T, \boldsymbol{Z}) \\
\overset{\nu_\ell}{\cdots\cdots\to} \left.\begin{array}{l} H^2(\mathscr{G}_T, \boldsymbol{F}_\ell)^- \cdots\cdots\to 0 \\ \oplus \\ \end{array}\right\} = H^2(\mathscr{G}_T, \boldsymbol{F}_\ell) \\
0 \cdots\cdots\to H^2(\mathscr{G}_T, \boldsymbol{F}_\ell)^+ \underset{\delta}{\cdots\cdots\to} H^3(\mathscr{G}_T, \boldsymbol{Z}) \cdots\cdots\to \quad , \text{ i.e.,}
\end{array}
$$

$$\nu_\ell(H^2(\mathscr{G}_T, \boldsymbol{Z})) = H^2(\mathscr{G}_T, \boldsymbol{F}_\ell)^-,$$

and

$$\delta \textit{ restricted on } H^2(\mathscr{G}_T, \boldsymbol{F}_\ell)^+ \textit{ is injective.}$$

We shall apply this lemma for $\mathscr{G}_T = \Gamma^{\mathrm{ab}}$ in Section 2.

The big diagram of DNA$(\boldsymbol{Q}-\nu-i_\ell)$ for $GR=\mathscr{G}_T$ (see 1.3.1) is, noting that $H^r(\mathscr{G}_T, \boldsymbol{Q})=0$ for $r>0$;

(1.6.12)
$$
\begin{tikzcd}
 & {} \arrow[d, dashed, "\ell"] & {} \arrow[d, dashed, "\ell"] & \\
0 \arrow[r, dotted] & H^1(\mathscr{G}_T, \boldsymbol{Q}/\boldsymbol{Z}) \arrow[r, dotted, "\cong"] \arrow[d, dashed, "\delta"] & H^2(\mathscr{G}_T, \boldsymbol{Z}) \arrow[r, dotted] \arrow[d, dashed, "\nu_\ell"] & 0 \\
 & H^2(\mathscr{G}_T, \boldsymbol{F}_\ell) \arrow[r, dotted, "=", "\mathrm{id}"'] \arrow[d, dashed, "i"] & H^2(\mathscr{G}_T, \boldsymbol{F}_\ell) \arrow[d, dashed, "\delta"] & \\
0 \arrow[r, dotted] & H^2(\mathscr{G}_T, \boldsymbol{Q}/\boldsymbol{Z}) \arrow[r, dotted, "\cong"] \arrow[d, dashed] & H^3(\mathscr{G}_T, \boldsymbol{Z}) \arrow[r, dotted] \arrow[d, dashed] & 0 \\
0 \arrow[r, dotted] & H^2(\mathscr{G}_T, \boldsymbol{Q}/\boldsymbol{Z}) \arrow[r, dotted] \arrow[d, dashed] & H^3(\mathscr{G}_T, \boldsymbol{Z}) \arrow[r, dotted] \arrow[d, dashed] & 0 \\
 & {} & {} &
\end{tikzcd}
$$

Hereafter, we identify $H^r(\mathscr{G}_T, \boldsymbol{Q}/\boldsymbol{Z}) = H^r(\mathscr{G}_T, \boldsymbol{R}/\boldsymbol{Z}) = H^r(\mathscr{G}_T, \boldsymbol{C}/\boldsymbol{Z})$ with $H^{r+1}(\mathscr{G}_T, \boldsymbol{Z})$ for $r>0$; this turns the ladder to a string.

(1.6.13)
$$
\begin{array}{l}
\cdots\cdots\to H^1(\mathscr{G}_T, \boldsymbol{Q}/\boldsymbol{Z}) = H^2(\mathscr{G}_T, \boldsymbol{Z}) \cdots\cdots\to H^2(\mathscr{G}_T, \boldsymbol{F}_\ell) \\
\cdots\cdots\to H^2(\mathscr{G}_T, \boldsymbol{Q}/\boldsymbol{Z}) = H^3(\mathscr{G}_T, \boldsymbol{Z}) \cdots\cdots\to
\end{array}
$$

(written horizontally).

§ 2. Less trivial information

2.1. Combining exact sequences (L$\boldsymbol{Q}$), (Lν_ℓ), and (Li_ℓ), we construct the diagram DNA ($\boldsymbol{Q}-\nu-i_\ell$) for $GR=\Gamma$. (See 1.3.1).

(DNA 2.1.1)

$$
\begin{tikzcd}
 & {} \arrow[d, dashed, "\ell"] & {} \arrow[d, dashed, "\ell"] & \\
H^1(\Gamma, \boldsymbol{C}) \arrow[r, dotted] \arrow[d, equal] & H^1(\Gamma, \boldsymbol{C}/\boldsymbol{Z}) \arrow[r, dotted, "\delta[1]"] \arrow[d, dashed, "\delta[2]"] & H^2(\Gamma, \boldsymbol{Z}) \arrow[r, dotted] \arrow[d, dashed, "\nu_\ell"] & H^2(\Gamma, \boldsymbol{C}) \\
\{0\} & H^2(\Gamma, \boldsymbol{F}_\ell) \arrow[r, phantom, "="] \arrow[d, dashed, "i_\ell"] & H^2(\Gamma, \boldsymbol{F}_\ell) \arrow[d, dashed, "\delta[3]"] & \\
{} \arrow[r, dotted] & H^2(\Gamma, \boldsymbol{C}/\boldsymbol{Z}) \arrow[r, dotted, "\delta[4]"] \arrow[d, dashed] & H^3(\Gamma, \boldsymbol{Z}) \arrow[r, dotted] \arrow[d, dashed] & {} \\
 & {} & {} &
\end{tikzcd}
$$

In this diagram:

Lemma (2.1.2). *Since $H^1(\Gamma, \boldsymbol{C})=0$, $\delta[1]$ is injective.*

The DNA for $GR=\mathscr{G}_T=\Gamma^{\mathrm{ab}}$, is (see 1.6.12):

(DNA 2.1.3)

$$
\begin{tikzcd}
 & {} \arrow[d, dashed] & {} \arrow[d, dashed] & \\
0 \arrow[r, dotted] & H^1(\Gamma^{\mathrm{ab}}, \boldsymbol{C}/\boldsymbol{Z}) \arrow[r, dotted] \arrow[d, dashed] & H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{Z}) \arrow[r, dotted] \arrow[d, dashed] & 0 \\
 & H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell) \arrow[r, equal] \arrow[d, dashed] & H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell) \arrow[d, dashed] & \\
0 \arrow[r, dotted] & H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{C}/\boldsymbol{Z}) \arrow[r, dotted] \arrow[d, dashed] & H^3(\Gamma^{\mathrm{ab}}, \boldsymbol{Z}) \arrow[r, dotted] \arrow[d, dashed] & 0 \\
 & {} & {} &
\end{tikzcd}
$$

Combining these two DNA's by inflations λ: $H^*(\Gamma^{\mathrm{ab}}, -)\to H^*(\Gamma, -)$, we have

(2.1.4)

$$\begin{tikzcd}[column sep=small]
 & & H^1(\Gamma, \boldsymbol{C}/\boldsymbol{Z}) \arrow[r, dotted] \arrow[dd, dashed] & H^2(\Gamma, \boldsymbol{Z}) \arrow[dd, dashed] \\
H^1(\Gamma^{\mathrm{ab}}, \boldsymbol{C}/\boldsymbol{Z}) \arrow[r, dotted, "\cong"] \arrow[urr, "\lambda[1]"] \arrow[dd, dashed] & H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{Z}) \arrow[urr, "\lambda[2]"'] \arrow[dd, dashed] & & \\
 & & H^2(\Gamma, \boldsymbol{F}_\ell) \arrow[r, equal] \arrow[dd, dashed] & H^2(\Gamma, \boldsymbol{F}_\ell) \arrow[dd, dashed] \\
H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell) \arrow[r, equal] \arrow[urr] \arrow[dd, dashed] & H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell) \arrow[urr, "\lambda[3]"'] \arrow[dd, dashed] & & \\
 & & H^2(\Gamma, \boldsymbol{C}/\boldsymbol{Z}) \arrow[r, dotted] & H^3(\Gamma, \boldsymbol{Z}) \\
H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{C}/\boldsymbol{Z}) \arrow[r, dotted, "\cong"] \arrow[urr, "\lambda[4]"] & H^3(\Gamma^{\mathrm{ab}}, \boldsymbol{Z}) \arrow[urr, "\lambda[5]"'] & &
\end{tikzcd}$$

Or, identifying $H^r(\Gamma^{\mathrm{ab}}, \boldsymbol{C}/\boldsymbol{Z})$ with $H^{r+1}(\Gamma^{\mathrm{ab}}, \boldsymbol{Z})$, we have

(2.1.5)

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[r, dotted] & H^1(\Gamma, \boldsymbol{C}/\boldsymbol{Z}) \arrow[rr, dotted, "\delta[1]"] \arrow[dr, dashed, "\delta[2]"'] & & H^2(\Gamma, \boldsymbol{Z}) \arrow[dl, dashed, "\nu_\ell[1]"] \\
H^1(\Gamma^{\mathrm{ab}}, \boldsymbol{C}/\boldsymbol{Z}) = H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{Z}) \arrow[urr, "\lambda[1]"] \arrow[urrrr, "\lambda[2]"'] \arrow[dd, dashed, "{\delta[5]=\nu_\ell[2]}"'] & & & & \\
 & & & H^2(\Gamma, \boldsymbol{F}_\ell) \arrow[ddl, dashed, "i[1]"'] \arrow[ddr, dashed, "\delta[3]"] & \\
H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell) \arrow[urrr, "\lambda[3]"] \arrow[dd, dashed, "{i[2]=\delta[6]}"'] & & & & \\
 & & H^2(\Gamma, \boldsymbol{C}/\boldsymbol{Z}) \arrow[rr, dotted, "\delta[4]"] & & H^3(\Gamma, \boldsymbol{Z}) \\
H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{C}/\boldsymbol{Z}) = H^3(\Gamma^{\mathrm{ab}}, \boldsymbol{Z}) \arrow[urr, "\lambda[4]"] \arrow[urrrr, "\lambda[5]"'] & & & &
\end{tikzcd}$$

In this diagram:

Lemma (2.1.6). *$\lambda[1]$ is an isomorphism:*

$$\lambda[1]\colon H^1(\Gamma^{\mathrm{ab}}, \boldsymbol{C}/\boldsymbol{Z}) \cong H^1(\Gamma, \boldsymbol{C}/\boldsymbol{Z}).$$

Proof. The inflation $\lambda[1]$ is injective, (1.3.6). Since $H^1(\Gamma, \boldsymbol{C}/\boldsymbol{Z}) = \mathrm{Hom}(\Gamma^{\mathrm{ab}}, \boldsymbol{C}/\boldsymbol{Z}) = H^1(\Gamma^{\mathrm{ab}}, \boldsymbol{C}/\boldsymbol{Z}) = \Gamma^{\mathrm{ab}}$ is a finite group, $\lambda[1]$ must be surjective.

Corollary (2.1.7). *$\lambda[2]$ is injective.*

Lemma (2.1.8). *$\lambda[3]$ is injective.*

Proof. This is Lemma (1.3.4) restated.

Comment. The Lemma (2.1.8), which is essential in a later section, is valid only for our $\Gamma = \Gamma_0(\mathfrak{n})$ of the Hilbert-modular case, and not valid for the similar $\Gamma = \Gamma_0(\mathfrak{n})$ of the quaternion case, since Lemma (2.1.8), which is same as Lemma (1.3.4), which depends on Lemma (2.1.3), whose proof used the congruence subgroup theorem. However "If we replace $H(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell)$ by adelically continuous cohomology group, then $\lambda[3]$ become injective?" might be an approachable conjecture, which shall be also helpful for our purpose if it is true.

Lemma (2.1.9). *In $H^1(\Gamma, \boldsymbol{F}_\ell)$,*

$$\nu_\ell[1](H^2(\Gamma, \boldsymbol{Z})_{\mathrm{tor}}) \cap \lambda[3](H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell)^+) = \{0\},$$

where $H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell)^+$ is the $(+1)$-eigenspace of $(-1)^*$. (*See* 1.6.8).

Proof. Chasing the diagram (2.1.5),

$$\begin{aligned}\nu_\ell(H^2(\Gamma, \boldsymbol{Z})_{\mathrm{tor}}) &= \mathrm{Im}\,(\nu_\ell[1] \circ \delta[1]) = \mathrm{Im}\,(\nu_\ell[1] \circ \delta[1] \circ \lambda[1]) \\ &= \mathrm{Im}\,(\lambda[3] \circ \delta[5]) = \lambda[3](H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell)^-),\end{aligned}$$

by the Lemma (1.6.11). Since $\lambda[3]$ is injective (Lemma (2.1.8)), and since

$$H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell)^+ \cap H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell)^- = \{0\},$$

we have (2.1.9). Q.E.D.

Lemma (2.1.10). *In $H^2(\Gamma, \boldsymbol{F}_\ell)$, put*

$$\lambda[3](H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell)^+) \cap \mathrm{Ker}\,(\delta[3]) = E.$$

Then dim $(E) \geqq n(n-3)/2$.

In particular if $n \geqq 4$, *then* $E \neq \{0\}$. *Here*, $n =$ *the number of prime ideals of order* 2, $\mathfrak{q} \mid \mathfrak{n}$, *with the properties described in* (1.2.1, 2).

Proof. Since $\lambda[3]$ is injective,

$$\dim \lambda[3](H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell)^+) = \dim H^2(\Gamma^{\mathrm{ab}}, \boldsymbol{F}_\ell)^+ = n(n-1)/2,$$

(apply 1.6.8′ for $\mathscr{G}_T = \Gamma^{\mathrm{ab}}$, $t = n$). Also $\mathrm{Im}\,(\delta[3]) = {}_\ell H^3(\Gamma, \boldsymbol{Z}) = \boldsymbol{F}_\ell^n$, (see 1.5.9). So, $\dim(E) \geqq (n(n-1)/2) - n = n(n-3)/2$. Q.E.D.

We put

$$\tilde{H}^2(\Gamma, \boldsymbol{F}_\ell) = H^2(\Gamma, \boldsymbol{F}_\ell)/\nu_\ell(H^2(\Gamma, \boldsymbol{F}_\ell)_{\mathrm{tor}}), \tag{2.1.11}$$

and the projection map of $H^2(\Gamma, \boldsymbol{F}_\ell)$ to $\tilde{H}^2(\Gamma, \boldsymbol{F}_\ell)$ is denoted by μ.

Also we put

$$\tilde{\nu}_\ell = \mu \circ \nu_\ell. \tag{2.1.12}$$

Put

$$\tilde{E} = \mu(E). \tag{2.1.12′}$$

By Lemma (2.1.9).

$$\tilde{E} \cong E. \tag{2.1.12″}$$

Put

$$E_0 = \lambda[3]^{-1}(E). \tag{2.1.12‴}$$

Since $\lambda[3]$ is injective,

$$E_0 \cong E \cong \tilde{E}. \tag{2.1.13}$$

Furthermore, we combine restrictions

$$r_\alpha\colon H^*(\Gamma, -) \longrightarrow H^*(\Gamma_\alpha, -),$$
$$\rho_\alpha\colon H^*(\Gamma^{\mathrm{ab}}, -) \longrightarrow H^*(\Gamma_\alpha^{\mathrm{ab}}, -),$$

and inflations

$$\lambda_\alpha\colon H^*(\Gamma_\alpha^{\mathrm{ab}}, -) \longrightarrow H^*(\Gamma_\alpha, -)$$

to the diagram (2.1.4), and have a commutative diagram:

(2.1.14)

$$
\begin{tikzcd}
 & & H^2(\Gamma, Z) \arrow[r] \arrow[d, dashed] & H^2(\Gamma_\alpha, Z) \arrow[d, dashed] \\
 & & H^2(\Gamma, Z) \arrow[r, "r_\alpha[1]"] \arrow[d, dashed, "\nu_\ell"] & H^2(\Gamma_\alpha, Z) \arrow[d, dashed, "\nu_\ell[2]"] \\
 & & E \subset H^2(\Gamma, F_\ell) \arrow[r, "r_\alpha[2]"] \arrow[dd, "\delta"] & H^2(\Gamma_\alpha, F_\ell) \arrow[dd, dashed] \\
H^2(\Gamma_\alpha^{\mathrm{ab}}, F_\ell) \arrow[r, "\substack{\rho_\alpha \\ \| \\ 0}"'] \arrow[urr, "\lambda[3]"] & H^2(\Gamma_\alpha^{\mathrm{ab}}, F_\ell) \arrow[urr, "\lambda_\alpha"] & & \\
 & & H^3(\Gamma, Z) \arrow[r, "r_\alpha"] & H^3(\Gamma_\alpha, Z)
\end{tikzcd}
$$

In this diagram:

Lemma (2.1.15). $r_\alpha[2](E)=\{0\}$, $r_\alpha[2] \circ \lambda[3]=0$.

Proof.
$$
\begin{aligned}
r[2](E) &\subset r[2] \circ \lambda[3](H^2(\Gamma^{\mathrm{ab}}, F_\ell)) \\
&= \lambda_\alpha \circ \rho_\alpha(H^2(\Gamma^{\mathrm{ab}}, F_\ell))=0 \text{ since } \rho_\alpha=0,
\end{aligned}
$$
(see Lemma 1.6.7). Q.E.D.

Put

$$
r=r[1]=\sum_{\alpha=1}^{f} r_\alpha[1]:
$$
$$
H^2(\Gamma, Z)=H^2(M, Z) \longrightarrow \oplus H^2(\Gamma_\alpha, Z)=H^2(\partial M, Z),
$$
$$
\rho=\rho[1]=\sum_{\alpha} \rho_\alpha[1]:
$$
$$
H^2(\Gamma, F_\ell)=H^2(M, F_\ell) \longrightarrow \oplus H^2(\Gamma_\alpha, F_\ell)=H^2(\partial M, F_\ell).
$$

Lemma (2.1.16). $E \subset \nu_\ell(\ker(r[1]))$.

Proof. Since $\delta(E)=0$, $E \subset \nu_\ell(H^2(\Gamma, Z))$. For an arbitrary element $x \in E$, take a representative $y \in H^2(\Gamma, Z)$: $x=\nu_\ell(y)$. Put $z_\alpha=r_\alpha[1](y) \in H^2(\Gamma_\alpha, Z)$, $z=(z_\alpha)=r[1](y) \in \oplus H^2(\Gamma_\alpha, Z)=H^2(\partial M, Z)$. Since $\nu_\ell[2](z)=\nu_\ell[2] \circ r[1](y)=r[2] \circ \nu_\ell(y)=r[2](x)=0$, (see Lemma 2.1.15), hence $z=r[1](y) \in \ell(H^2(\partial M, Z))$. Take an element $u=(u_\alpha) \in \oplus H^2(\Gamma_\alpha, Z)=$

$H^2(\partial M, \boldsymbol{Z})$ such that $z=r[1](y)=\ell u$. The image $\delta(u) \in H^3(M, \partial M, \boldsymbol{Z})$ of u under δ: $H^2(\partial M, \boldsymbol{Z}) \to H^3(M, \partial M, \boldsymbol{Z})$ is denoted by $\bar{u}$.

By Lemma (1.5.10), the order k of $\bar{u}$ is coprime with ℓ: so take integers $a, b \in \boldsymbol{Z}$ such that $ak+\ell b=1$. Since $\delta(ku)=k\delta(u)=k\bar{u}=0$, $ku \in \ker(\delta)=\operatorname{Im}(r[1])$. Take an element $v \in H^2(\Gamma, \boldsymbol{Z})$ such that $r[1](v)=ku$. Put $w=ky-\ell v$. Then $r[1](w)=k(r[1](y))-\ell(r[1](v))=kz-\ell ku=k\ell u-\ell ku=0$.

Hence: $w \in \ker(r[1])$. Moreover $\nu_\ell(w)=k\nu_\ell(y)-\ell\nu_\ell(v)=kx$. Hence $kx \in \nu_\ell(\ker(r[1]))$.

So, $x=1x=(ak+b\ell)x=akx+b\ell x=a(kx) \in \nu_\ell(\ker(r[1])$. Q.E.D.

In the following diagram (2.1.17), we put

$$\mathscr{E}=\ker(r[1]),$$
$$\mathscr{E}'=\ker(r[0]\circ i)=i^{-1}(\ker(r[0])).$$

(2.1.17)
$$\begin{array}{ccc}
H^2(\Gamma, \boldsymbol{C})=H^2(M, \boldsymbol{C}) & \xrightarrow{r[0]} & H^2(\partial M, \boldsymbol{C}) \\
\uparrow i[1] & & \uparrow i[2] \\
\mathscr{E}\subset\mathscr{E}'\subset H^2(\Gamma, \boldsymbol{Z})=H^2(M, \boldsymbol{Z}) & \xrightarrow{r[1]} & H^2(\partial M, \boldsymbol{Z}) \\
 & & \uparrow \delta \\
 & & H^1(\partial M, \boldsymbol{C}/\boldsymbol{Z})
\end{array}$$

Then,

Lemma (2.1.18). $m_0\mathscr{E}'\subset\mathscr{E}\subset\mathscr{E}'\subset H^2(\Gamma, \boldsymbol{Z})$,

where $m_0=N(1-\varepsilon_0^2)$.

Proof. Take an element $x \in \mathscr{E}'$. Then,

$$0=r[0]\circ i[1](x)=i[2]\circ r[1](x),$$

so $r[1](x) \in \delta(H^1(\partial M, \boldsymbol{C}/\boldsymbol{Z}))=H^2(\partial M, \boldsymbol{Z})_{\mathrm{tor}}=\oplus\mathscr{G}_\alpha$ (see 1.5.3). Since $|\mathscr{G}_\alpha|=m_0$, $0=m_0r[1](x)=r[1](m_0x)$. Hence $m_0x \in \mathscr{E}$. Q.E.D.

Corollary (2.1.19). *In* $\nu_\ell H^2(\Gamma, \boldsymbol{Z})$, $\nu_\ell(\mathscr{E}')=\nu_\ell(\mathscr{E})$.

Proof. $\nu_\ell(\mathscr{E}')\supset\nu_\ell(\mathscr{E})\supset\nu_\ell(m_0\mathscr{E}')=m_0\nu_\ell(\mathscr{E}')=\nu_\ell(\mathscr{E}')$ because $(\ell, m_0)=1$.

Comments. For a quaternion Hilbert modular group $\Gamma=\Gamma_0(\mathfrak{n})$ in a quaternion algebra B, with $B\otimes\boldsymbol{R}=M_2(\boldsymbol{R})^2\oplus\boldsymbol{H}^{m-2}$, only statements up to 2.1.7, are known to be true. Statements after 2.1.7, depending on Lemma (2.1.8), the injectivity of $\lambda[3]$, are not yet authorized. In order to prove the injectivity, we need not have the whole c.s.th., but it is sufficient to know that $[[\Gamma,\Gamma],[\Gamma,\Gamma]]$ is a congruence subgroup.

Or, for our purpose, a weaker conjecture:

$$\dim_{F_\ell}(H^1(\Gamma^{(1)},F_\ell)\Gamma^{\mathrm{ab}})<c_1\dim_{F_\ell}({}_\ell\Gamma^{\mathrm{ab}})+c_2,$$

shall support similar results.

§ 3. Hecke operators

3.1. For a definition of Hecke operators see [16], [24], [27]. For the action of Hecke operators on group-cohomologies see [16], [27]. Here, we use the same notations as in [27].

Hecke operators are also considered as algebraic correspondences of U, sending cusps to cusps. More precisely, for $T=\Gamma\xi\Gamma$, put $U_\xi=(\Gamma\cap\xi^{-1}\Gamma\xi)\backslash\mathfrak{H}^2$, $f_1=$the natural covering map of U_ξ to U, $f_\xi=$the morphism of U_ξ onto U induced from $z\rightarrow\xi(z)\in\mathfrak{H}^2$; then $U_\xi\underset{f_\xi}{\overset{f_1}{\rightrightarrows}}U$ is the algebraic correspondence. Maps f_1 and f_ξ send cusps of $U_\xi((\Gamma\cap\xi^{-1}\Gamma\xi)$-cusps) to cusps of $U(\Gamma$-cusps). Let $(M,\partial M)$ be the manifold M with boundary $\partial M=\coprod_\alpha Y_\alpha$ (Y_α are connected components) obtained from U by chopping off neighborhoods of cusps, and let $(M_\xi,\partial M_\xi)$ be a manifold M_ξ with boundary $\partial M_\xi=\coprod Y_{\xi,\beta}$ ($Y_{\xi,\beta}$ are connected components) obtained from U_ξ in the same way.

Then $(f_1(M_\xi),f_1(\partial M_\xi))$ is homotopic to $(M,\partial M)$ in U in a space canonical way. Also $(f_\xi(M_\xi),f_\xi(\partial M_\xi))$ is homotopic to $(M,\partial M)$ in U in a similar way; and thus we have homomorphisms:

$$f_1^*\colon H^*(M,-)\longrightarrow H^*(M_\xi,-),$$
$$f_\xi^*\colon H^*(M,-)\longrightarrow H^*(M_\xi,-).$$

Since f_1, f_ξ are finite coverings, we can define "the trace map", or "the transfer":

$$f_1^\sharp\colon H^*(M_\xi,-)\longrightarrow H^*(M,-),$$
$$f_\xi^\sharp\colon H^*(M_\xi,-)\longrightarrow H^*(M,-),$$

which are defined as follows: a cochain c_ξ on M_ξ is given. For a chain z on M, put $c(z)=c_\xi(f_1^{-}(z))$. c defines a cocycle on M if c_ξ is a cocycle; and $c_\xi\rightarrow c$ induces a homomorphism $f_1^\sharp\colon H^*(M_\xi,-)\rightarrow H^*(M,-)$. The homomorphism $f_\xi^\sharp: H^*(M_\xi, -)\to H^*(M, -)$ is defined similarly.

The action T^* of a Hecke operator $T=(\Gamma\xi\Gamma)$ on $H^*(M, -)$ is defined by

$$T^*=f_\xi^\sharp \circ f_1^*: H^*(M, -)\longrightarrow H^*(M, -).$$

The action coincides with the action of $T=(\Gamma\xi\Gamma)$ on $H^*(\Gamma, -)$ defined in [16], [24], [27]. Also, since $f_1, f_\xi: \partial M_\xi\to\partial M$ (with adjustment by the canonical homotopy: $f(\partial M_\xi)\to\partial M$), are finite coverings, we can define the action:

$$T^*=f_\xi^\sharp \circ f_1^*: H^*(\partial M, -)\longrightarrow H^*(\partial M, -)$$

of Hecke operator $T=(\Gamma\xi\Gamma)$ on the cohomology of the boundary.

Also actions T^* of Hecke operator T on the relative cohomology $H^*(M, \partial M, -)$ are defined similarly, and we can observe that the exact sequence:

$$\text{(Rel)} \qquad \to H^*(M, \partial M, -)\longrightarrow H^*(M, -)\longrightarrow H^*(\partial M, -)$$

of relative cohomology is compatible with the actions T^* of T.

3.2. For our $\Gamma=\Gamma_0(\mathfrak{n})$, we take as Δ,

$$\Delta=\Delta_0(\mathfrak{n})=\left\{\xi=\begin{pmatrix} a & b \\ c & d\end{pmatrix}; \begin{array}{l} \det\xi\gg 0,\ c\equiv 0 \bmod \mathfrak{n} \\ a\equiv d \bmod 3,\ (ad, 6\mathfrak{n})=1\end{array}\right\}.$$

The coset-space $\Delta/\Gamma^{(1)}$ is a finite abelian group, which we denote by Δ^{ab}. Δ^{ab} is isomorphic to $\Phi_6\times\prod_{\mathfrak{q}|\mathfrak{n}}\Phi_{\mathfrak{q}}^2$, and $\Delta^{\mathrm{ab}}/\Gamma^{\mathrm{ab}}\cong\prod_{\mathfrak{q}|\mathfrak{n}}\Phi_{\mathfrak{q}}$. Thus,

Lemma (3.2.1). *The action T^* of $T\in\mathscr{R}(\Gamma^{\mathrm{ab}}, \Delta^{\mathrm{ab}})$ on $H^*(\Gamma^{\mathrm{ab}}, -)$ is the scalar-multiplication of* $\deg(T)$, *where $-$ is a Δ^{ab}-trivial module.* (*See* [27], *Theorem* 1.5.2).

In the diagram (2.1.17), we put

$$\mathscr{E}'=i[1]^{-1}(\ker(r[0]))=\ker(r[0]\circ i[1]),$$
$$\mathscr{E}=\ker(r[1]).$$

Then

$$H^2(\Gamma, \boldsymbol{Z})_{\mathrm{tor}}\subset\mathscr{E}\subset\mathscr{E}'\subset H^2(\Gamma, \boldsymbol{Z}),$$

and $H^2(\Gamma, \boldsymbol{Z})/\mathscr{E}'$ has no torsion element. We put

$$H^2(\Gamma, \boldsymbol{Z})_{\text{free}} = H^2(\Gamma, \boldsymbol{Z})/H^2(\Gamma, \boldsymbol{Z})_{\text{tor}},$$
$$\mathscr{E}'_{\text{free}} = \mathscr{E}'/H^2(\Gamma, \boldsymbol{Z})_{\text{tor}},$$
$$\mathscr{E}_{\text{free}} = \mathscr{E}/H^2(\Gamma, \boldsymbol{Z})_{\text{tor}}.$$

Then $\mathscr{E}_{\text{free}}$, $\mathscr{E}'_{\text{free}}$, $H^2(\Gamma, \boldsymbol{Z})_{\text{free}}$ are torsion free and

$$\mathscr{E}_{\text{free}} \subset \mathscr{E}'_{\text{free}} \subset H^2(\Gamma, \boldsymbol{Z})_{\text{free}}.$$

Take a $\boldsymbol{Z}$-basis $\langle e_1, e_2, \cdots, e_k\rangle$ of $\mathscr{E}'_{\text{free}}$, then $\langle e_1, e_2, \cdots, e_k\rangle$ is extendable to a $\boldsymbol{Z}$-basis $\langle e_1, e_2, \cdots, e_k, e_{k+1}, \cdots, e_h\rangle$ of $H^2(\Gamma, \boldsymbol{Z})_{\text{free}}$, and $\langle i(e_1), i(e_2), \cdots, i(e_k)\rangle$ is a $\boldsymbol{C}$-basis of $\ker(r[0])$, and $\langle i(e_1), i(e_2), \cdots, i(e_h)\rangle$ is a $\boldsymbol{C}$-basis of $H^2(\Gamma, \boldsymbol{C})$. Since $\langle e_1, e_2, \cdots, e_k\rangle$ is a $\boldsymbol{Z}$-basis of the free $\boldsymbol{Z}$-module $\mathscr{E}'_{\text{free}}$, applying the reduction $\tilde{\nu}_\ell = \mu \circ \nu_\ell$, $\langle \tilde{e}_1 = \tilde{\nu}_\ell(e_1), \cdots, \tilde{e}_k = \tilde{\nu}_\ell(e_k)\rangle$ is an $\boldsymbol{F}_\ell$-basis of

$$\tilde{\nu}_\ell(\mathscr{E}'_{\text{free}}) = \nu_\ell(\mathscr{E}')/\nu_\ell(H^2(\Gamma, \boldsymbol{Z})_{\text{tor}}) = \nu_\ell(\mathscr{E})/(H^2(\Gamma, \boldsymbol{Z})_{\text{tor}}),$$

which we denoted by $\tilde{\mathscr{E}}$; and we know that

$$\tilde{\mathscr{E}} \supset \tilde{E} \cong E_0 \qquad \text{(see 2.1.13).}$$

Take a Hecke operator $T \in \mathscr{R}(\Gamma, \Delta)$. The characteristic polynomial of T^* on $H^2(\Gamma, \boldsymbol{C}) = H^2(U, \boldsymbol{C})$, (or on $\ker(\lambda[0]) = H^2_f(\Gamma, \boldsymbol{C})$) is denoted by $P(T, u)$ (or by $P_f(T, u)$). Let (t_{ij}) be the matrix representing the action T^* of T on $\mathscr{E}'_{\text{free}}$ with respect to the $\boldsymbol{Z}$-basis $\langle e_1, e_2, \cdots, e_k\rangle$. (t_{ij}) is a matrix with entries in $\boldsymbol{Z}$. Since $\langle i(e_1), i(e_2), \cdots, i(e_k)\rangle$ is a $\boldsymbol{C}$-basis of $\ker(r[0])$,

$$P_f(T, u) = \det(uI - (t_{ij})) \in \boldsymbol{Z}[u].$$

On the other hand, since $\langle \tilde{\nu}_\ell(e_1), \cdots, \tilde{\nu}_\ell(e_k)\rangle$ is an $\boldsymbol{F}_\ell$-basis of $\tilde{\mathscr{E}}$, we have

$$\tilde{P}^\ell_f(T, u) = (\text{the characteristic polynomial of } T^* \text{ on } \tilde{\mathscr{E}}) \in \boldsymbol{F}_\ell[u],$$

where $\tilde{P}^\ell =$ the reduction modulo ℓ of P.

Now, E_0, E, $\tilde{E}$ are isomorphic as T-modules, and since the action of T on $E_0 \subset H^2(\Gamma^{\text{ab}}, \boldsymbol{F}_\ell)$ is scalar-multiplication by $\deg(T)$, (see Lemma (3.2.1)), we have:

$$T^*|_{\tilde{E}} = \deg(T) I_{\tilde{E}}.$$

Since $\tilde{E} \subset \tilde{\mathscr{E}}$, as $\boldsymbol{F}_\ell$-linear subspace, we have

$$(u - \deg(T))^h \mid \tilde{P}^\ell_f(u),$$

where $h=\dim(E)\geqq n(n-3)/2$. (see Lemma (2.1.10)). Thus,

Lemma (3.2.2). *The characteristic polynomial $P_f(T, U)$ of Hecke-operator $T \in \mathscr{R}(\Gamma, \Delta)$ on $H_f^2(\Gamma, C)$ is in $Z[u]$, and divisible by $(u-\deg(t))^h$ in modulo ℓ, where $h=n(n-3)/2$.*

Comments. For $T=T_\mathfrak{p}=\Gamma\begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix}\Gamma$ of a prime ideal $\mathfrak{p}=(\pi)$, $\pi\gg 0$, of $\mathfrak{O}$ such that $(\mathfrak{p}, 6\mathfrak{n})=1$, $\deg(T_\mathfrak{p})=N(\mathfrak{p})+1$.

Lemma (3.2.2) implies that the eigenvalues λ of T on $H_f^2(\Gamma, C)$ are algebraic integers, and there are at least h eigenvalues λ_i such that $\lambda_i\equiv\deg(T) \bmod \mathfrak{l}$ for a prime divisor $\mathfrak{l}$ of ℓ in the field $Q(\lambda, \cdots)$ generated by eigenvalues of T.

Also the lemma implies: $\dim_C H_f^2(\Gamma, C)\geqq n(n-3)/2$. Is this a trivial inequality?

3.3. *Hecke operators on $S_2(\Gamma)$.* Since

$$H_f^2(U, C)=A(\Gamma)\oplus W, \qquad W=Cw_1\oplus Cw_2,$$

and since

$$T^*(w_\alpha)=\deg(T)w_\alpha, \qquad (\alpha=1, 2)$$

is easily seen, we have:

$$P_f(T, u)=P_A(T, u)\cdot(u-\deg(T))^2,$$

where $P_A(T, u)$ is the characteristic polynomial of T^* on the subspace A. So,

$$P_A(T, u) \in Z[u],$$

and if $n(n-3)/2\geqq 2$, then with $h'=n(n-3)/2-2=(n-4)(n+1)/2$,

$$(u-\deg(t))^{h'}\,|\,\tilde{P}_A^\ell(T, u).$$

The Hecke operator $T=\Gamma\xi\Gamma=\coprod_{i=1}^d \Gamma\xi_i$ operates on $S_2(\Gamma, \mathfrak{H}^\alpha\times\mathfrak{H}^\beta)$, $(\alpha, \beta=+, -)$ by

$$T(\varphi)=\sum_i \varphi(\xi_i(z))\cdot n(\det\xi_i)((c_i^{(1)}z_1+d_i^{(1)})\cdot(c_i^{(2)}z_2+d_i^{(2)}))^{-2},$$

for $\varphi \in S_2(\Gamma, \mathfrak{H}^\alpha\times\mathfrak{H}^\beta)$, where

$$\xi_i=\left(\begin{pmatrix}a_i^{(1)} & b_i^{(1)}\\ c_i^{(1)} & d_i^{(1)}\end{pmatrix}, \begin{pmatrix}a_i^{(2)} & b_i^{(2)}\\ c_i^{(2)} & d_i^{(2)}\end{pmatrix}\right).$$

The map $\psi^{\alpha\beta}$: $S_2(\Gamma, \mathfrak{H}^\alpha \times \mathfrak{H}^\beta) \rightarrow H^2(U^{\alpha\beta})$ is commutative with actions of T^*; so $S^{\alpha,\beta}$ is a Hecke-ring-stable subspace of $H^2(U^{\alpha,\beta})$. (For notations ψ, θ etc. see 1.4).

Since the (partial) conjugation map: $\theta^{\alpha,\beta}$: $\mathfrak{H}^+ \times \mathfrak{H}^+ \rightarrow \mathfrak{H}^\alpha \times \mathfrak{H}^\beta$ commutes with the action of $g \in GL^+(2, R)^2$, $\theta^{\alpha,\beta}$: $H^2(U^{\alpha,\beta}) \rightarrow H^2(U)$ also commutes with Hecke operator actions T^*. Thus $A^{\alpha,\beta}$ $(\alpha, \beta = +, -)$ are Hecke-ring-stable.

Denote the characteristic polynomial of T^* on $S_2(\Gamma, \mathfrak{H}^\alpha \times \mathfrak{H}^\beta)$ by $P_{S^{\alpha,\beta}}(T, u)$. Then it is also the characteristic polynomial of T^* on $S^{\alpha,\beta}$ and on $A^{\alpha,\beta}$, and

$$\begin{aligned} P_A(T, u) &= P_{S^{++}}(T, u) P_{S^{+-}}(T, u) P_{S^{-+}}(T, u) P_{S^{--}}(T, u) \\ &= |P_{S^{++}}(T, u) P_{S^{+-}}(T, u)|^2, \end{aligned}$$

since $P_{S^{++}} = \overline{P_{S^{--}}}$, and $P_{S^{+-}} = \overline{P_{S^{-+}}}$.

We assume that $N(\varepsilon_0) = -1$; let us assume that $\varphi_1(\varepsilon_0) > 0$, $\varphi_2(\varepsilon_0) < 0$ without loss of generality. Put

$$E_0 = \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & 1 \end{pmatrix},$$

then the map E_0: $\mathfrak{H}^+ \times \mathfrak{H}^+ \rightarrow \mathfrak{H}^+ \times \mathfrak{H}^-$ etc defined in Section 1.4 gives an isomorphism E_0^*: $S^{+-} \cong S^{++}$, etc. For $T = \Gamma \xi \Gamma \in \mathscr{R}(\Gamma, \Delta)$, define T^{E_0} by

$$T^{E_0} = E_0 \Gamma \xi \Gamma E_0^{-1} = \Gamma (E_0 \xi E_0^{-1}) \Gamma,$$

and extend $T \rightarrow T^{E_0}$ to the automorphism $T \rightarrow T^{E_0}$ of the ring $\mathscr{R}(\Gamma, \Delta)$.

Then for $w \in S^{+-}$, we have

$$E_0^*(T^*(w)) = T^{E_0^*}(E_0^*(w)).$$

Thus

$$P_{S^{+-}}(T, u) = P_{S^{++}}(T^{E_0}, u).$$

Similarly define

$$E_0' = \begin{pmatrix} \varepsilon_0' & 0 \\ 0 & 1 \end{pmatrix},$$

with $\varepsilon_0' = -\varepsilon_0^{-1}$, and $T^{E_0'} = E_0' T E_0'^{-1}$, $E_0'^*$: $S^{-+} \cong S^{++}$; then we have

$$E_0'^*(T^*(w)) = T^{E_0'^*}(E_0'^*(w)) \qquad \text{for } w \in S^{-+}$$

and

$$P_{S^{-+}}(T, u) = P_{S^{++}}(T^{E_0'}, u).$$

Also, define

$$C=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},$$

and $T^C=CTC^{-1}$, C^*: $S^{--}\cong S^{++}$; then we have $C^*(T^*(w))=T^{C*}(C^*(w))$ for $w\in S^{--}$; and

$$P_{S^{--}}(T, u)=P_{S^{++}}(T^C, u).$$

Thus,

$$P_A(T, u)=P_S(T, u)P_S(T^{E_0}, u)P_S(T^{E_0'}, u)P_S(T^C, u),$$

where $P_S(T, u)$ is the abbreviation of $P_{S^{++}}(T, u)$. In particular:

Lemma (3.1.1). *If* $T=T^{E_0}=T^{E_0'}=T^C$, *then*

$$P_A(T, u)^4=P_S(T, U)^4.$$

The assumption $T=T^{E_0}=T^{E_0'}=T^C$ is true if $T=\Gamma\xi\Gamma$ with $\xi=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$. In particular, it is true for $T=T_\mathfrak{p}$ with a prime ideal $\mathfrak{p}$ of $\mathfrak{O}$ such that $(\mathfrak{p}, 6\mathfrak{n})=1$, such $T_\mathfrak{p}$ is defined as

$$T_\mathfrak{p}=\Gamma\begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix}\Gamma$$

with

$$\mathfrak{p}=(\pi), \qquad \pi\gg 0.$$

These automorphisms E_0, E_0', $C=E_0\circ E_0'$ of the Hecke-ring $\mathscr{R}(\Gamma, \Delta)$ form an abelian group $\mathscr{D}=\{E_0, E_0', C, 1\}\cong Z_2\times Z_2$ with $1=$id. So $T=T^{E_0}=T^{E_0'}=T^C$ is true for $T=T_1+(T_1)^{E_0}+(T_1)^{E_0'}+(T_1)^C$ or for $T=(T_1)^2$ with some $T_1\in\mathscr{R}(\Gamma, \Delta)$.

3.4. *The final result.* Summarizing all of the above, we have

Theorem (3.4). *Under assumptions described below, the characteristic polynomial* $P_S(T, u)$ *of the Hecke operator action* T *on* $S_2(\Gamma)$ *is divisible by* $(u-\deg(T))^h$ *modulo* $\mathfrak{l}$ *where* $h=[(n-4)(n+1)/8]+1$.

Assumptions.

(1) $K=Q(\sqrt{d})$, $d>0$, with $h=1$, $N(\varepsilon_0)=-1$, $d\neq 5$.

(2) $\Gamma=\Gamma_0(\mathfrak{n})$ in the Hilbert modular group with $\mathfrak{n}=\mathfrak{q}_1\cdots\mathfrak{q}_m$; $\mathfrak{q}_i\neq\mathfrak{q}_j$ $(i\neq j)$; $N\mathfrak{q}_i=q_i^2$ for $i=1,\cdots,n$; $N\mathfrak{q}_j=q_j$ for $j=n+1,\cdots,m$, where q_j are rational primes; $(-1/q_j)=-1$ for some q_j $(j=n+1,\cdots,m)$ also $(-3/q_j)=-1$ for one of $j=n+1,\cdots,m$; $(\mathfrak{n}, 6)=1$.

(3) There is a rational prime number ℓ such that: $(\ell, 6\mathfrak{n})=1$, $q_i\equiv 1$ mod ℓ for $i=1,\cdots,n$: $q_j\not\equiv 1$ mod ℓ for $j=n+1,\cdots,m$; $(\ell, N(1-\varepsilon_0^2))=1$.

(4) $T=\Gamma\xi\Gamma\in\mathscr{R}(\Gamma,\Delta)$ with $\xi=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$.

In particular, $T=T_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p}$ of $\mathfrak{O}$ with $(\mathfrak{p}, 6\mathfrak{n})=1$.

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