Advanced Studies in Pure Mathematics 4, 1984
Group Representations and Systems of Differential Equations
pp. 391–432

Boundary Value Problems for Systems of Linear Partial Differential Equations with Regular Singularities

Toshio Oshima

A concept of systems of linear partial differential equations with regular singularities and their boundary value problems were introduced by [K–O]. A typical example is the Laplacian $\Delta=(1-|z|^2)^2\partial^2/\partial z\partial\bar{z}$ on the unit disc in the complex plane $\boldsymbol{C}$, which has regular singularity along the boundary of the disc. In this case S. Helgason proved that any eigenfunction of Δ can be obtained by the Poisson integral of a hyperfunction on the boundary. The inverse correspondence is given by the map of taking the boundary value of the solution, which was defined in [K–O]. In general any simultaneous eigenfunction of the invariant differential operators on a Riemannian symmetric space of the non-compact type can be given by the Poisson integral of a hyperfunction on a boundary of the symmetric space. The main purpose of [K–O] was to prove this statement and in fact it was solved in [K–K–].

When we consider a realization of a Riemannian (or semisimple) symmetric space in a nice compact manifold (cf. [O 2] and [O–S]), the invariant differential operator has regular singularities along the boundaries. Hence for a deeper analysis on a symmetric space, we need a deeper study on systems of differential equations with regular singularities. This is a main motivation to write this paper and several applications of this paper to this subject will appear in subsequent papers. One of them will be found in [MaO].

We will mention some differences between [K–O] and this paper. In this paper we discuss a system of differential equations which has not necessarily one unknown function but finitely many. This enables us to study a system of differential equations defined in a vector bundle over a symmetric space. Moreover in [K–O] we only consider a system of differential equations whose number equals just the codimension of the boundary. But here we remove this restriction and we can consider more equations that the solution satisfies. Then the boundary value of the solution may satisfy some equations. These induced equations will be

Received August 10, 1983.

discussed in Section 6. As another result of the removal of this restriction, we will see that the definition of the boundary value of a function does not depend on equations that the function satisfies.

The most important difference is the following: In [K–O], to define boundary value of a solution of a system of differential equation with regular singularities, we needed a some restriction on the characteristic exponents of the system. Here we also remove the restriction but define the boundary value in a coordinate neighborhood. After that we will consider when the definition of the boundary value has nice properties with respect to coordinate transformations. Let us consider the equation

$$\left(t\frac{\partial}{\partial t}-\lambda\right)(1+Ct)^{-1}\left(t\frac{\partial}{\partial t}-\lambda-1\right)u(t, x)=0,$$

where λ is a complex number and C is a non-negative number. This equation has regular singularities along the hypersurface defined by $t=0$. The indicial equation equals $(s-\lambda)(s-\lambda-1)=0$ and the characteristic exponents are λ and $\lambda+1$. The solution $u(t, x)$ of the system on the domain defined by $t>0$ is of the form

$$u(t, x)=\phi_0(x)t^{\lambda}(1-Ct\log t)+\phi_1(x)t^{\lambda+1}.$$

In this paper we call $\phi_0(x)$ and $\phi_1(x)$ the boundary values of $u(t, x)$ with respect to the characteristic exponents λ and $\lambda+1$, respectively. If we use a coordinate system (t', x) with $t'=\alpha t(\alpha>0)$, then

$$u(t, x)=\phi_0(x)\alpha^{-\lambda}t'^{\lambda}(1-\alpha^{-1}Ct'\log t')+(\phi_1(x)-\phi_0(x)C\log\alpha)\alpha^{-\lambda-1}t'^{\lambda+1}.$$

Under the coordinate system (t', x), the pair of the boundary values $(\phi_0(x), \phi_1(x))$ changes into $(\phi_0(x)\alpha^{-\lambda}, (\phi_1(x)-\phi_0(x)C\log\alpha)\alpha^{-\lambda-1})$. In [K–O] we defined only the boundary value $\phi_0(x)\otimes(dt)^{\lambda}$ because its definition does not depend on the choice of local coordinate systems. We remark here that if $\phi_0(x)=0$, then $\phi_1(x)\otimes(dt)^{\lambda+1}$ has the same property. This important phenomenon will be discussed in Section 4.

In the first two sections we give preliminary results concerned with micro-differential operators. In Section 3 we define a system of differential equations with regular singularities and study its micro-local structure. In Section 4 we define boundary values of solutions of the system and give elementary properties of the boundary values. In Section 5 we study the solutions whose boundary values are all analytic. These solutions are called ideally analytic and were studied by [K–O] in a simple case. In the last section we discuss the induced equations that the boundary values satisfy. These equations are especially important to consider boundary value problems for several boundary components of a symmetric space.

§ 1. Micro-differential operators

In this section we define some notation used in this paper and give some properties of micro-differential operators.

We denote by N, N_+, Z, R, R_+ and C the set of non-negative integers, positive integers, integers, real numbers, positive real numbers and complex numbers, respectively. For a ring R and positive integers m_1, m_2 and m, $M(m_1, m_2; R)$ denotes the set of matrices of size $m_1 \times m_2$ with components in R, (r_{ij}) denotes an element of $M(m_1, m_2; R)$ whose (i, j)-component is $r_{ij}(1 \leqq i \leqq m_1, 1 \leqq j \leqq m_2)$ and R^m denotes $M(1, m; R)$. Then we can naturally define the map of $M(m_1, m_2; R) \times M(m_2, m_3; R)$ to $M(m_1, m_3; R)$. Moreover $R[X]$ denotes the ring of polynomials of l-variables $X = (X_1, X_2, \cdots, X_l)$ with coefficients in R ($l \in N$) and $R[X]^{(m)}$ denotes the set of polynomials in $R[X]$ of degree at most m.

For a sheaf $\mathscr{F}$ on a manifold we denote by $\mathscr{F}_p$ and $\mathscr{F}(U)$ the stalk of $\mathscr{F}$ at a point p and the set of sections of $\mathscr{F}$ over an open subset U, respectively.

Let X be a complex manifold of dimension $\tilde{n} = l + n$ with a local coordinate systems $z = (z_1, \cdots, z_{\tilde{n}})$, T^*X the cotangent vector bundle of X with a local coordinate system

$$(z, \eta) = (z_1, \cdots, z_{\tilde{n}}, \eta_1, \cdots, \eta_{\tilde{n}}) = \left(z, \sum_{i=1}^{\tilde{n}} \eta_i dz_i\right)$$

and $\mathcal{O}_X$ (resp. $\mathcal{O}_{T^*X}$) the sheaf of holomorphic functions on X (resp. T^*X). The projection of T^*X onto X will be denoted by π_X. Let $\mathscr{D}_X$ (resp. $\mathscr{E}_X$) be the sheaf of differential (resp. micro-differential) operators on X (resp. T^*X) of finite order (cf. Chap. II in [S–K–K]). Then $\mathscr{D}_X = \pi_X \mathscr{E}_X$. For any m in Z we denote by $\mathscr{E}_X^{(m)}$ the sheaf of micro-differential operators of order at most m and by $\mathcal{O}_{T^*X}^{(m)}$ the sheaf of homogeneous holomorphic functions of degree m with respect to η on T^*X. For an open subset U of T^*X every section $P(z, D_z)$ of $\mathscr{E}_X^{(m)}(U)$ is of the form

$$P(z, D_z) = \sum_{j \leqq m} P_j(z, D_z)$$

where $P_j(z, \eta) \in \mathcal{O}_{T^*X}^{(j)}(U)$ for all $j \in Z$ and satisfy

$$\lim_{i \to \infty} \sqrt[i]{|P_{-i}(z, \eta)|_K / i!} < \infty \tag{1.1}$$

for any compact subset K of U. In general for a subset B of U and an element $f = (f_{ij}) \in M(m_1, m_2; \mathcal{O}_{T^*X}(U))$, we put

$$|f|_B = \sup_{\substack{(z,\eta)\in B \\ 1\leq i\leq m_1, 1\leq j\leq m_2}} |f_{ij}(z,\eta)|.$$

The map σ_m of taking the principal symbols of $\mathscr{E}_X^{(m)}$ onto $\mathscr{O}_{T^*X}^{(m)}$ is defined by $\sigma_m(P)=P_m(z,\eta)$ and for any section (Q_{ij}) of $M(m_1, m_2; \mathscr{E}_X)$ we define ord $Q=\max\{\text{ord } Q_{ij}; 1\leqq i\leqq m_1, 1\leqq j\leqq m_2\}$ and $\sigma(Q)=(\sigma_{\text{ord}\,Q}(Q_{ij}))$. If $U=\pi_X^{-1}(V)$ with an open subset V of X, $P_j(z,\eta)$ are polynomials of η and $P_j=0$ for any $j<0$. In this case $P_j(z,\eta)$ is of the form

$$P_j(z,\eta)=\sum_{|\alpha|=j, \alpha\in N^{\tilde{n}}} p_\alpha(z)\eta^\alpha$$

with $p_\alpha\in\mathscr{O}_X(V)$ and P is the differential operator $\sum_{\alpha\in N^{\tilde{n}}} p_\alpha(z)D_z^\alpha$. Here $\eta^\alpha=\eta_1^{\alpha_1}\cdots\eta_{\tilde{n}}^{\alpha_{\tilde{n}}}$, $D_z=(\partial/\partial z_1,\cdots,\partial/\partial z_{\tilde{n}})$, $D_z^\alpha=\partial^{\alpha_1}/\partial z_1^{\alpha_1}\cdots\partial^{\alpha_{\tilde{n}}}/\partial z_{\tilde{n}}^{\alpha_{\tilde{n}}}$ and $|\alpha|=\alpha_1+\cdots+\alpha_{\tilde{n}}$ for any $\alpha=(\alpha_1,\cdots,\alpha_{\tilde{n}})\in Z^{\tilde{n}}$. Let $\hat{\mathscr{E}}_X^{(m)}$ be the sheaf of *formal* micro-differential operators of order at most m, i.e., $\hat{\mathscr{E}}_X^{(m)}=\varprojlim \mathscr{E}_X^{(m)}/\mathscr{E}_X^{(j)}$. Therefore the growth condition (1.1) is omitted for $\hat{\mathscr{E}}_X^{(m)}$. We put $\hat{\mathscr{E}}_X=\bigcup_{m\in Z}\hat{\mathscr{E}}_X^{(m)}$. Then we have the following algebraic properties of micro-differential operators:

Proposition 1.1 (§ 3 Chap. II in [S–K–K]).

(i) *$\mathscr{D}_X$, $\mathscr{E}_X$, $\mathscr{E}_X^{(0)}$, $\hat{\mathscr{E}}_X$ and $\hat{\mathscr{E}}_X^{(0)}$ are coherent Rings and their stalks are noetherian rings from the both sides.*

(ii) *$\hat{\mathscr{E}}_X$ is faithfully flat over $\mathscr{E}_X$ and $\mathscr{E}_X$ is flat over $\pi_X^{-1}\mathscr{D}_X$.*

(iii) *Let $\mathscr{M}_1\xrightarrow{\phi}\mathscr{M}_2\xrightarrow{\psi}\mathscr{M}_3$ be a complex of coherent $\mathscr{E}_X$-Modules and let $\mathscr{M}_j^{(0)}$ be a sub-$\mathscr{E}_X^{(0)}$-Modules of $\mathscr{M}_j$ such that $\phi(\mathscr{M}_1^{(0)})\subset\mathscr{M}_2^{(0)}$, $\psi(\mathscr{M}_2^{(0)})\subset\mathscr{M}_3^{(0)}$ and $\mathscr{E}_X\mathscr{M}_j^{(0)}=\mathscr{M}_j$ ($j=1,2,3$). If the induced complex $\mathscr{M}_1^{(0)}/\mathscr{E}_X^{(-1)}\mathscr{M}_1^{(0)}\to\mathscr{M}_2^{(0)}/\mathscr{E}_X^{(-1)}\mathscr{M}_2^{(0)}\to\mathscr{M}_3^{(0)}/\mathscr{E}_X^{(-1)}\mathscr{M}_3^{(0)}$ is an exact sequence, then $\mathscr{M}_1^{(0)}\to\mathscr{M}_2^{(0)}\to\mathscr{M}_3^{(0)}$ and $\mathscr{M}_1\to\mathscr{M}_2\to\mathscr{M}_3$ are also exact.*

A coherent left $\mathscr{E}_X$-Module $\mathscr{M}$ is called a *system of micro-differential equations* and its support is called its *characteristic variety*. Assume that $\mathscr{M}$ has N unknown functions $u_1,\cdots,u_N$, i.e., $\mathscr{M}$ is generated by $u_1,\cdots,u_N$. Let u be the column vector of length N whose i-th component is u_i and $\mathscr{J}$ a left sub-$\mathscr{E}_X$-Module of $(\mathscr{E}_X)^N$ which annihilates u. Then u is isomorphic to $(\mathscr{E}_X)^N/\mathscr{J}$. The *symbol Module* $\bar{\mathscr{J}}$ of u is a sub-$\mathscr{O}_{T^*X}$-Module of $(\mathscr{O}_{T^*X})^N$ generated by principal symbols $\sigma(P)$ of sections P of $\mathscr{J}$ and a system $\{P_1,\cdots,P_M\}$ of sections of $\mathscr{J}$ is called an *involutive base* if $\sigma(P_1),\cdots,\sigma(P_M)$ generate $\bar{\mathscr{J}}$.

Let U be an open subset of T^*X, m and m' integers and m_j positive integers ($j=1,2,3$). For a P in $M(m_1, m_2; \hat{\mathscr{E}}_{T^*X}^{(m)}(U))$ and a subset B of U we define

$$|P|_{B,k}=\sum_{j\geqq k}|P_j(z,\eta)|_B \quad (k\in Z) \tag{1.2}$$

by denoting $P=\sum_{j\in Z}P_j(z, D_z)$ with $P_j(z, \eta)\in M(m_1, m_2; \mathcal{O}_{T^*X}^{(j)}(U))$. Then $M(m_1, m_2; \hat{\mathscr{E}}_X^{(m)}(U))$ is a Fréchet space with the semi-norms $|\ \ |_{K,k}$ where K runs through the compact subsets of U and k runs through Z. Under the topology the map $(P, Q)\mapsto PQ$ of $M(m_1, m_2; \hat{\mathscr{E}}_X^{(m)}(U))\times M(m_2, m_3; \hat{\mathscr{E}}_X^{(m')}(U))$ to $M(m_1, m_3; \hat{\mathscr{E}}_X^{(m+m')}(U))$ is continuous.

Lemma 1.2. *Let $\mathscr{J}$ be a coherent left sub-$\hat{\mathscr{E}}_X$-Module of $(\hat{\mathscr{E}}_X)^N$. We put $\mathscr{J}(m)=\mathscr{J}\cap(\hat{\mathscr{E}}_X^{(m)})^N$ for every m in Z and $\bar{\mathscr{J}}(0)=\mathscr{J}(0)/\mathscr{J}(-1)$. Let $Q_1, \cdots, Q_M$ be sections of $\mathscr{J}(0)$ such that $\sigma_0(Q_1), \cdots, \sigma_0(Q_M)$ generate the $\mathcal{O}_{T^*X}^{(0)}$-Module $\bar{\mathscr{J}}(0)$ in a neighborhood of a point p of T^*X-X. Then there exist an open neighborhood U of p and C-linear maps ι_j $(j=1, \cdots, M)$ of $\mathscr{J}(U)$ to $\hat{\mathscr{E}}_X(U)$ such that $\sum_j \iota_j(P)Q_j=P$ for any P in $\mathscr{J}(U)$ and that the restrictions $\iota_j|_{\mathscr{J}(m)(U)}$ define continuous maps of $\mathscr{J}(m)(U)$ to $\hat{\mathscr{E}}_X^{(m)}(U)$.*

Proof. It is known (cf. p. 82 in [G–R]) that there exist a neighborhood U of p and continuous C-linear maps $\bar{\iota}_j$ of $\bar{\mathscr{J}}(0)(U)$ to $\mathcal{O}_{T^*X}^{(0)}(U)$ satisfying $\sum_j\bar{\iota}_j(f)\sigma_0(Q_j)=f$ $(f\in\bar{\mathscr{J}}(0)(U))$. Here $\mathcal{O}_{T^*X}^{(0)}(U)$ is a Fréchet space with the semi-norms $|\ \ |_K$ and $\bar{\mathscr{J}}(0)(U)$ is endowed with the topology induced by the Fréchet space $(\mathcal{O}_{T^*X}^{(0)}(U))^N$. We fix a non-vanishing section r of $\mathcal{O}_{T^*X}^{(1)}$ defined on a neighborhood of p. For a P in $\mathscr{J}(m)(U)$ we put $\iota_j(P)=\sum_{\nu\in N}R_{j\nu}(z, D_z)$ with $R_{j\nu}(z, \eta)\in\mathcal{O}_{T^*X}^{(m-\nu)}(U)$. Here $R_{j\nu}$ are inductively defined by

$$R_{j\nu}(z, \eta)=r^{m-\nu}\bar{\iota}_j\Big(r^{\nu-m}\sigma_{m-\nu}\Big(P-\sum_{j=1}^{M}\sum_{k=0}^{\nu-1}R_{jk}(z, D_z)Q_j\Big)\Big). \tag{1.3}$$

The lemma is clear by this definition. Q.E.D.

The following lemma will be used in Section 2. Here and in the sequel we identify X with the zero section of T^*X.

Lemma 1.3. *Let $\mathscr{J}$ be a coherent left sub-$\hat{\mathscr{E}}_X$-Module of $(\hat{\mathscr{E}}_X)^N$ (resp. sub-$\mathscr{E}_X$-Module of $(\mathscr{E}_X)^N$) and U an open subset of T^*X-X. Then $\mathscr{J}(m)(U)=(\mathscr{J}\cap(\hat{\mathscr{E}}_X^{(m)})^N)(U)$ is closed in $(\hat{\mathscr{E}}_X^{(m)}(U))^N$ (resp. $(\mathscr{E}_X^{(m)}(U))^N$ under the topology induced by $(\hat{\mathscr{E}}_X^{(m)}(U))^N$).*

Proof. Let $\{P_\nu\}_{\nu\in N}$ be a sequence in $\mathscr{J}(m)(U)$ which converges to an element P of $(\hat{\mathscr{E}}_X^{(m)}(U))^N$. To prove $P\in\mathscr{J}(m)(U)$ it is sufficient to show that at any point p of U the germ P_p defined by P is in $\mathscr{J}_p$. Under the notation in Lemma 1.2 we put $R_j=\lim\iota_j(P_\nu)$. Then $\sum_j R_jQ_j=\lim\sum_j\iota_j(P_\nu)Q_j=\lim P_\nu=P$, which proves $P_p\in(\hat{\mathscr{E}}_X\otimes\mathscr{J})_p$. Hence also in the case when $\mathscr{J}$ is a coherent $\mathscr{E}_X$-Module and $P_p\in(\mathscr{E}_X)_p^N$, we have $P_p\in\mathscr{J}_p$ because $\hat{\mathscr{E}}_X$ is faithfully flat over $\mathscr{E}_X$. Q.E.D.

Let V be an involutory submanifold of T^*X invariant under the action of $\boldsymbol{C}$ and $\mathscr{E}_V$ the sub-Ring of $\mathscr{E}_X$ generated by the sheaf $\{P \in \mathscr{E}_X^{(1)}; \sigma_1(P)|_V=0\}$. Then in [K–O] we prove that $\mathscr{E}_V$ is coherent and its stalks are noetherian and we say that a coherent left $\mathscr{E}_X$-Module $\mathscr{M}$ has *regular singularity* along V if $\mathscr{M}$ satisfies the following equivalent conditions:

(RS) *Every coherent sub-$\mathscr{E}_V$-Module $\mathscr{M}'$ of $\mathscr{M}$ defined on any open set U is coherent over $\mathscr{E}_X^{(0)}$.*

(RS)′ *For every point p there exist a neighborhood U of p and a coherent sub-$\mathscr{E}_V$-Module $\mathscr{M}^{(0)}$ of $\mathscr{M}|_U$ which is coherent over $\mathscr{E}_X^{(0)}$ and satisfy $\mathscr{E}_X\mathscr{M}^{(0)} =\mathscr{M}|_U$.*

Moreover if the degenerate locus $\Lambda=\{p \in V-X; (\iota_V^{-1} \sum_{i=1}^{n} \eta_i dz_i)(p)=0\}$ of V is holonomic (i.e. $\dim \Lambda = \dim X$ at every non-singular point of Λ), V is said to be *maximally degenerate.* Here ι_V is the inclusion map of V into T^*X. It is shown in [O 1] that in this case Λ is non-singular and V is locally equivalent to $\{(z, \eta); z_1=\eta_2=\cdots=\eta_d=0\}$ with $d=\operatorname{codim} V$ by a homogeneous canonical transformation. And we have the following proposition, which is a generalization of the fact that every formal solution of an ordinary differential equation with regular singularity converges.

Proposition 1.4 (Theorem 3.13 in [K–O]). *Let $\mathscr{M}$ and $\mathscr{M}'$ be systems of micro-differential equations which have regular singularities along the maximally degenerate involutory manifold V with the degenerate locus Λ. Then*

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}_1, \mathscr{M}_2)|_\Lambda \xrightarrow{\sim} \boldsymbol{R}\mathscr{H}om_{\hat{\mathscr{E}}_X}(\hat{\mathscr{E}}_X\otimes\mathscr{M}_1, \hat{\mathscr{E}}_X\otimes\mathscr{M}_2)|_\Lambda.$$

Let Y be an n-dimensional submanifold of the $(l+n)$-dimensional complex manifold X, let $Y_1, \cdots, Y_l$ be non-singular hypersurfaces normally crossing at Y and let $(t, x)=(t_1, \cdots, t_l, x_1, \cdots, x_n)$ be a local coordinate system of X such that Y_i is defined by $t_i=0$ $(i=1, \cdots, l)$. We put

$$V=\pi_X^{-1}(Y)-\textstyle\bigcup_{i=1}^{l} T^*_{Y(i)}X-X,$$
$$\Lambda=T^*_YX\cap V,$$

where

$$Y(i)=Y_1\cap\cdots\cap Y_{i-1}\cap Y_{i+1}\cap\cdots\cap Y_l.$$

Using the local coordinate system $(t, x, \tau, \xi)=(t, x; \sum_{i=1}^{l} \tau_i dt_i + \sum_{j=1}^{n} \xi_i dx_j)$ of T^*X, $V=\{(t, x, \tau, \xi); t_1=\cdots=t_l=0, \tau_1\neq 0, \cdots, \tau_l\neq 0\}$ and

$$\Lambda=\{(t, x, \tau, \xi); t_1=\cdots=t_l=\xi_1=\cdots=\xi_n=0, \tau_1\neq 0, \cdots, \tau_l\neq 0\}.$$

We remark that V is a maximally degenerate involutory submanifold with the degenerate locus Λ. We denote by ϑ_i, ϑ, D_t, D_x and tD_x the (vectors of) operators $t_i\partial/\partial t_i$, $(\vartheta_1, \cdots, \vartheta_l)$, $(\partial/\partial t_1, \cdots, \partial/\partial t_l)$, $(\partial/\partial x_1, \cdots, \partial/\partial x_n)$ and $(t_1\partial/\partial x_1, t_1\partial/\partial x_2, \cdots, t_l\partial/\partial x_n)$. For any m in $\boldsymbol{Z}$ the sheaf $\mathscr{E}_*^{(m)}$ on Y is introduced in [K–O] whose sections consist of sections P of $\pi_X(\mathscr{E}_X^{(m)}|_\Lambda)$ satisfying the following equivalent conditions:

(1.4) *Each P is of the form*

$$P=\sum_{\alpha,\beta\in \boldsymbol{N}^l} P_{\alpha\beta}(x, D_x)t^\alpha D_t^{\alpha-\beta}.$$

(1.5) *Let v be the generator of the system of micro-differential equations:*

$$\begin{cases}(\vartheta_i-\lambda_i)v=0 \quad (i=1, \cdots, l),\\ \dfrac{\partial v}{\partial x_1}=\cdots=\dfrac{\partial v}{\partial x_n}=0.\end{cases}$$

Then if $\lambda_i\in \boldsymbol{C}-\{-1, -2, \cdots\}$ for $i=1, \cdots, l$ and $\phi(t, x)$ is a holomorphic function, there exists a holomorphic function $\psi(t, x)$ defined in a neighborhood of Y such that $P\phi(t, x)v=\psi(t, x)v$.

We put $\mathscr{E}_*^f=\bigcup_{m\in Z}\mathscr{E}_*^{(m)}$ and denote by $\mathscr{E}_*$ the sub-Ring of $\mathscr{E}_*^f$ generated by $\{P\in\mathscr{E}_*^{(1)}; \sigma_1(P)|_V=0\}$, i.e. $\mathscr{E}_*=\mathscr{E}_*^{(0)}[\vartheta]$. Moreover we put $\mathscr{D}_*^f=\mathscr{D}_X|_Y\cap\mathscr{E}_*^f$ and $\mathscr{D}_*=\mathscr{D}_X|_Y\cap\mathscr{E}_*$. Then any section P of $\mathscr{E}_*^{(0)}$ (resp. $\mathscr{E}_*$) satisfies ord $P_{\alpha\beta}\leqq|\beta|-|\alpha|$ (resp. $\leqq|\beta|$) in the expression (1.4) and therefore if a section P of $\mathscr{E}_*$ is of order 0, we can regard $\sigma_0(P)|_\Lambda$ as a section of $\mathcal{O}_Y$. Hence we can define a Ring homomorphism σ_* of $\mathscr{E}_*$ onto $\mathcal{O}_Y[s]=\mathcal{O}_Y[s_1, \cdots, s_l]$ so that

$$\begin{cases}\sigma_*(\vartheta_i)=s_i & \text{for } i=1, \cdots, l,\\ \sigma_*(P)=\sigma_0(P)|_\Lambda & \text{if ord } P=0.\end{cases}\tag{1.6}$$

In [K–O] we say that the system of differential equations

$$\mathscr{M}: P_1u=\cdots=P_lu=0$$

on X has regular singularities along the set of walls $\{Y_1, \cdots, Y_l\}$ if the following conditions are satisfied:

(RS–0) *There are differential operators G_{jk}^i of order less than $r_j+r_k-r_i$ such that*

$$[P_j, P_k]=\sum_{i=1}^{l} G^i_{jk}P_i$$

holds for any j, k, where $r_\nu=\operatorname{ord} P_\nu$.

(RS–1) $P_i \in \mathscr{D}_*(Y)$ *for* $i=1, \cdots, l$.

(RS–2) *Let* $a_i(x, s)$ *be homogeneous part of* $\sigma_*(P_i)$ *of degree* r_i $(=\operatorname{ord} P_i)$. *Then the solution of* $a_1(x, s)=\cdots=a_l(x, s)=0$ *is only the origin* $s=0$ *for any x in Y.*

To study a more general system of differential equations with regular singularities we will weaken these conditions in Section 3.

Finally in this section we will define micro-differential operators with *holomorphic parameters*. We assume that the manifolds X, Y and Y_i are of the forms $X'\times Z$, $Y'\times Z$ and $Y'_i\times Z$ $(i=1, \cdots, l)$, respectively. Here X' and Z are manifolds of dimensions $l+n'$ and $n-n'$, respectively, with an n' in N and therefore Y' and Y'_i are submanifolds of X' of codimension l and 1, respectively. We always choose the local coordinate system $(t, x)=(t, x', x'')$ such that $x'=(x_1, \cdots, x_{n'})$ and $x''=(x_{n'+1}, \cdots, x_n)$ are local coordinate systems of Y' and Z, respectively, and Y_i is defined by $t_i=0$. We put $D_{x'}=(\partial/\partial x_1, \cdots, \partial/\partial x_{n'})$ and $\xi'=(\xi_1, \cdots, \xi_{n'})$ and denote by $\pi_{X\to Z}$ the natural projection of X onto Z. Let $\mathscr{E}_{X/Z}$ be the subsheaf of $\mathscr{E}_X$ consisting of micro-differential operators P such that

$$[P, \phi\circ\pi_{X\to Z}]=0 \tag{1.7}$$

for every holomorphic function ϕ on Z. Since (1.7) equals

$$[P, x_j]=0 \quad (j=n'+1, \cdots, n),$$

P are of the forms $P=\sum_j P_j(t, x', x'', D_t, D_{x'})$ with $P_j(t, x', x'', \tau, \xi')\in \mathscr{O}^{(j)}_{T^*X}$ and called micro-differential operators on X' with a holomorphic parameter x'' in Z. Moreover we put

$$\begin{aligned}
&\mathscr{D}_{X/Z}=\pi_X(\mathscr{E}_{X/Z}) \\
&{}_Z\mathscr{E}^f_*=\mathscr{E}^f_*\cap\pi_X(\mathscr{E}_{X/Z}|_\Lambda), &&{}_Z\mathscr{D}^f_*=\mathscr{D}^f_*\cap\mathscr{D}_{X/Z}|_\Lambda,\\
&{}_Z\mathscr{E}^{(m)}_*={}_Z\mathscr{E}^f_*\cap\mathscr{E}^{(m)}_*, &&{}_Z\mathscr{D}^{(m)}_*={}_Z\mathscr{E}^{(m)}_*\cap\mathscr{D}_{X/Z}|_\Lambda,\\
&{}_Z\mathscr{E}_*={}_Z\mathscr{E}^f_*\cap\mathscr{E}_*, &&{}_Z\mathscr{D}_*=\mathscr{D}_*\cap\mathscr{D}_{X/Z}|_\Lambda,\\
&\hat{\mathscr{E}}_{X/Z}=\varprojlim\,(\mathscr{E}_{X/Z}\cap\mathscr{E}^f_X)/(\mathscr{E}_{X/Z}\cap\mathscr{E}^{(j)}_X),
\end{aligned}$$

and

$${}_Z\hat{\mathscr{E}}^{(m)}_*=\varprojlim\,{}_Z\mathscr{E}^{(m)}_*/{}_Z\mathscr{E}^{(j)}_*.$$

§ 2. Several Lemmas

In this section every situation is local. Therefore we fix a point in Y, denote it by 0 and put $\Lambda_0=\Lambda\cap\pi_X^{-1}(0)$. The local coordinate system (t, x) is chosen so that the point corresponds to $(0, 0)$. For an $m=(m_1, \cdots, m_l) \in \boldsymbol{R}^l$ and an $\alpha=(\alpha_1, \cdots, \alpha_l) \in \boldsymbol{Z}^l$ we put $m\vartheta=m_1\vartheta_1+\cdots+m_l\vartheta_l$ and $|m\alpha|=m_1\alpha_1+\cdots+m_l\alpha_l$. And for a $\sigma \in \boldsymbol{R}$ and a matrix S with components in $\pi_X(\hat{\mathscr{E}}_X|_\Lambda)$ we put

$$(2.1) \qquad S_\sigma^m=\sum_{\alpha\in N^l,|m\beta|=\sigma} S_{\alpha,\beta}(x, D_x)\vartheta^\alpha D_t^{-\beta}$$

by the expression

$$S=\sum_{\alpha\in N^l,\beta\in Z^l} S_{\alpha,\beta}(x, D_x)\vartheta^\alpha D_t^{-\beta}.$$

Lemma 2.1 ([K–O]). *Let $m \in \boldsymbol{R}_+^l$, $N \in \boldsymbol{N}_+$ and $A \in M(N; ({}_Z\hat{\mathscr{E}}_*^{(0)})_0)$. We put $\sigma_*(A)=(\lambda_{ij}(x))$. Assume that $[\partial/\partial t_\nu, A]=0$ for $\nu=1, \cdots, l$ and that if $i>j$, then $\lambda_{ij}=0$, the real part of $\lambda_{jj}(0)-\lambda_{ii}(0)$ is non-negative and moreover the number of the elements of the set $\{\alpha \in \boldsymbol{N}^l; |m\alpha|=\lambda_{jj}(0)-\lambda_{ii}(0)\}$ is 0 or 1. Then there exists an invertible matrix $U \in M(N; ({}_Z\hat{\mathscr{E}}_*^{(0)})_0)$ with $\sigma_*(U)=I_N$ such that $[\partial/\partial t_\nu, U]=0$ for $\nu=1, \cdots, l$ and the matrix*

$$(B_{ij})=m\vartheta-U(m\vartheta-A)U^{-1}$$

is of the form: $\sigma_*((B_{ij}))=\sigma_*(A)$ and if there exists an $\alpha(i,j) \in \boldsymbol{N}^l$ satisfying $\lambda_{ii}(0)-\lambda_{jj}(0)=|m\alpha(i,j)|$, then $B_{ij}=R_{ij}(x, D_{x'})D_t^{-\alpha(i,j)}$ with a differential operator $R_{ij}(x, D_{x'})$ of order $\leqq|\alpha(i,j)|$ and otherwise $B_{ij}=0$.

The above lemma is given in the proof of Theorem 5.3 in [K–O]. We assume there that $m_1, \cdots, m_l$ are linearly independent over the field of rational numbers. But concerning the statement in Lemma 2.1 we do not use this assumption.

Lemma 2.2. *If $m \in \boldsymbol{N}_+^l$, the U in Lemma 2.1 belongs to $M(N; ({}_Z\mathscr{E}_*^{(0)})_0)$.*

Proof. Let W be a $(1+l+n)$-dimensional complex manifold with the coordinate system $(t_0, \cdots, t_l, x_1, \cdots, x_n)$. Put $q=(0; dt_0) \in T^*W$ and $\vartheta_0=t_0\partial/\partial t_0$. For $P \in ({}_Z\hat{\mathscr{E}}_*^{(0)})_0$ with $[\partial/\partial t_\nu, P]=0$ $(\nu=1, \cdots, l)$, we put

$$\tilde{P}(\text{or } P^\sim)=\sum P_{\alpha,\beta}(x)D_{x'}^\alpha D_{t_1}^{(m_1-1)\beta_1}\cdots D_{t_l}^{(m_l-1)\beta_l}D_{t_0}^{-|m\beta|}$$

by the expression $P=\sum_{(\alpha,\beta)\in N^{n'+l}}P_{\alpha,\beta}(x)D_{x'}^\alpha D_t^{-\beta}$. Then $\tilde{P} \in (\hat{\mathscr{E}}_W^{(0)})_q$, $[m\vartheta, P]^\sim=[\vartheta_0, \tilde{P}]$ and $(PP')^\sim=\tilde{P}\tilde{P}'$. Therefore in Lemma 2.1, $\tilde{A}$ and $\tilde{B} \in M(N; (\mathscr{E}_W^{(0)})_q)$ and $\tilde{U}(\vartheta_0-\tilde{A})\tilde{U}^{-1}=\vartheta_0-\tilde{B}$, where $\tilde{A}=(\tilde{A}_{ij})$ etc. This means that the systems $\mathscr{M}: (\vartheta_0-\tilde{A})u=0$ and $\mathscr{N}: (\vartheta_0-\tilde{B})v=0$ with column vectors u and v of N-unknown functions are isomorphic by the correspondence $u=\tilde{U}v$. Since these systems have regular singularities along the maximally degenerate involutory submanifold defined by $t_0=0$ and since q belongs to the degenerate locus, Proposition 1.4 says $\tilde{U}v=Vv$ with a $V\in M(N;(\mathscr{E}_W^{(0)})_q)$. The Späth type theorem for micro-differential operators (cf. [S–K–K], Chap. II) shows that any section of $\mathscr{N}$ over a neighborhood of q has the unique expression Cv with a $C\in M(1,N;\mathscr{E}_W^{(0)})$ satisfying $[\partial/\partial t_0, C]=0$. This means $\tilde{U}\in M(N;(\mathscr{E}_W^{(0)})_q)$ because $[\partial/\partial t_0,\tilde{U}]=0$. Hence $U\in M(N;({}_Z\mathscr{E}_*^{(0)})_0)$ because in general $\tilde{P}\in(\mathscr{E}_W^{(0)})_q$ if and only if $P\in({}_Z\mathscr{E}_*^{(0)})_0$, which follows from (1.1). Q.E.D.

We will give other two lemmas for our purpose.

Lemma 2.3. *Let L, N and $R\in N_+$, $a_\mu\in C^l-Z^l$ and let $b_\nu\in C^l$ ($\mu=1,\cdots,L;\nu=1,\cdots,N$). Then there exists a $\beta\in N_+^l$ such that for any μ and ν*

$$(2.2)\qquad |a_\mu\beta|\notin Z \quad\text{and}\quad |b_\nu\beta|\neq|c\beta| \text{ if } c\in Z^l\cap(-R,\infty)^l-\{b_\nu\}.$$

Proof. Put $I_\mu=\{\alpha\in Z^l;|a_\mu\alpha|\in Z\}$ and $I_\mu^j=\{\alpha_j;$ there exists a $(0,\cdots,0,\alpha_j,\alpha_{j+1},\cdots,\alpha_l)\in I_\mu\}$. Then $I_\mu^j=Zk_\mu^j$ with some $k_\mu^j\in N$. Put $k=\prod_{k_\mu^j\neq0}k_\mu^j$ (1 if all k_μ^j are 0) and $V=\{t\in R_+^l;\sum_{j=1}^l(b_{\nu,j}-c_j)t_j\neq0$ for any $c\in Z^l\cap(-R,\infty)^l-\{b_\nu\}$ and $\nu=1,\cdots,N\}/R_+$. Since V is open dense in R_+^l/R_+, we can choose $\alpha\in N_+^l$ such that $R_+\alpha\in V$ and $\alpha_j\equiv1 \bmod k$ ($j=1,\cdots,l$). Then $\alpha\notin I_\mu$ because $(k_\mu^1,\cdots,k_\mu^l)\neq(1,\cdots,1)$. Q.E.D.

Lemma 2.4. *Let $m\in N^l$, N and $N'\in N_+$,*

$$A=(A_{ij})\in M(N;(\mathscr{D}_{X/Z})_0)\quad\text{and}\quad A'=(A'_{ij})\in M(N';(\mathscr{D}_{X/Z})_0).$$

We assume that A and A' are of the forms:

$$(2.3)\qquad\begin{cases}A_{ij}=A'_{ij}=0 & \text{if } i>j,\\ A_{ij}=A_{ij}(x,D_{x'}),\quad A'_{ij}=A'_{ij}(x,D_{x'}) & \text{if } i<j,\\ A_{ii}=\lambda_i(x''),\quad A'_{jj}=\lambda'_j(x''),\\ \lambda_1(0)=\lambda_i(0),\quad \lambda_1(0)-\lambda'_j(0)\notin Z,\\ A'_{ij}=0 \quad \text{if } \lambda'_i(0)-\lambda'_j(0)\notin Z.\end{cases}$$

Let $\mathscr{J}$ be a coherent left sub-$\mathscr{E}_X$-Module of

$$(\mathscr{E}_X)^{N+N'}=\sum_{i=1}^N\mathscr{E}_Xu_i+\sum_{j=1}^{N'}\mathscr{E}_Xu'_j$$

which is defined in a neighborhood of Λ_0 and satisfies

$$(2.4) \qquad m\vartheta u \equiv Au \quad \text{and} \quad m\vartheta u' \equiv A'u' \mod \mathscr{J}.$$

Here u (resp. u') *is the column vector of length* N (*resp.* N') *whose i-th component equals* u_i (*resp.* u_i').

(i) *Suppose* $\mathscr{J}$ *contains* $Pu+P'u'$ *with a* $P \in (\pi_X(\mathscr{E}_{X/Z}|_\Lambda))_0^N$ *and a* $P' \in (\pi_X(\mathscr{E}_{X/Z}|_\Lambda))_0^{N'}$. *Then for any* $\phi(x'', s) \in \mathscr{O}_Z[s]$ *and any* $\sigma \in \boldsymbol{Z}$, $P_\sigma^m \phi(x'', A)u$ *is a section of* $\mathscr{J}$ *over a neighborhood of* Λ_0.

(ii) *Let* $Q \in M(N; (\pi_X(\mathscr{E}_{x/z}|_\Lambda))_0)$ *and* $Q' \in M(N'; (\pi_X(\mathscr{E}_{X/Z}|_\Lambda))_0)$ *such that*

$$(2.5) \qquad (m\vartheta - A)(Qu+Q'u') \equiv 0 \mod \mathscr{J}.$$

Then $Qu+Q'u' \equiv Q_0^m u \mod \mathscr{J}$. *Moreover if* $\mathscr{J}$ *satisfies the following condition* (2.6) *and if* $(Q_0^m)_{ij}=0$ *in the case* $\lambda_i=\lambda_j$, *we have* $Q_0^m u \equiv 0 \mod \mathscr{J}$.

(2.6) *if* $\phi(x'')P \in \mathscr{J}$ *with a non-zero* $\phi \in \mathscr{O}_Z$ *and a* $P \in (\mathscr{E}_{X/Z})^{N+N'}$, *then* $P \in \mathscr{J}$.

(iii) *The statements* (i) *and* (ii) *hold even if we replace* $\mathscr{E}$ *by* $\hat{\mathscr{E}}$.

Proof. (i) We may assume $\sigma=0$ by considering $(D_t^\tau P, D_t^\tau P')$ in place of (P, P') with a suitable $\tau \in \boldsymbol{Z}^l$. We choose an open neighborhood V of 0 in Z such that λ_i and λ_j' are bounded and holomorphic in V, $|\lambda_i(x'')-\lambda_1(0)| \leq 1/3$ for any $x'' \in V$ and $\inf_{x'' \in V, k \in Z} |\lambda_i(x'')-\lambda_j'(x'')+k| > 0$. We put $C_1 = \max\{|\lambda_i|_V, |\lambda_j'|_V\}$ and $W=\{s \in \boldsymbol{C}; |s| \leq C_1+1\}$ and define a polynomial $B_\nu(s) \in (\mathscr{O}_Z(V))[s]$ by

$$\begin{aligned} B_\nu(s) &= \left\{\prod_{i=1}^{\nu}\prod_{j=1}^{N} \frac{(s-\lambda_j+i)(s-\lambda_j-i)}{-i^2}\right\}^N \cdot \left\{\prod_{i=0}^{\nu}\prod_{j=1}^{N'} \frac{(s-\lambda_j'+i)(s-\lambda_j'-i)}{-i^2-1}\right\}^{N'} \\ &= \left\{\prod_{i=1}^{\nu}\prod_{j=1}^{N}\left(1-\frac{(s-\lambda_j)^2}{i^2}\right)^N\right\} \cdot \left\{\prod_{i=0}^{\nu}\prod_{j=1}^{N'}\left(1-\frac{(s-\lambda_j')^2+1}{i^2+1}\right)\right\}^{N'} \end{aligned}$$

for every $\nu \in \boldsymbol{N}_+$. Then by the relations

$$\begin{aligned} m\vartheta P_k^* u^* &= [m\vartheta, P_k^*]u^* + P_k^* m\vartheta u^* \\ &\equiv P_k^*(A^*+k)u^* \mod \mathscr{J} \end{aligned}$$

and

$$\left(\prod_{j=1}^{N^*}(A^*-\lambda_j^*)\right)^{N^*} = 0,$$

we obtain

$$
\begin{aligned}
0 &\equiv B_\nu(m\vartheta)\{(\sum_{k\in \boldsymbol{Z}} P_k^m)u+(\sum_{k\in \boldsymbol{Z}} (P')_k^m)u'\} \mod \mathscr{J} \\
&\equiv \{\sum_{k\in \boldsymbol{Z}} P_k^m B_\nu(A+k)\}u+\{\sum_{k\in \boldsymbol{Z}} (P')_k^m B_\nu(A'+k)\}u' \mod \mathscr{J} \\
&\equiv P_0^m B_\nu(A)u+J_\nu \mod \mathscr{J}
\end{aligned}
$$

with

$$
J_\nu=\{\sum_{|k|>\nu} P_k^m B_\nu(A+k)\}u+\{\sum_{|k|>\nu}(P')_k^m B_\nu(A'+k)\}u'.
$$

Here $(P_k^*, u^*, N^*, \cdots)$ denotes $(P_k^m, u, N, \cdots)$ or $((P')_k^m, u', N', \cdots)$. Putting

$$
B_\nu(s+k)\equiv \sum_{i=1}^{(N^*)^2} b_{ik\nu}^*(x'')s^{i-1} \mod \mathcal{O}_Z(V)[s]\Big(\prod_{j=1}^{N^*}(s-\lambda_j^*)\Big)^{N^*}
$$

with $b_{ik\nu}^* \in \mathcal{O}_Z(V)$, we have

$$
B_\nu(A^*+k)=\sum_{i=1}^{(N^*)^2} b_{ik\nu}^*(x'')(A^*)^{i-1}
$$

and

$$
|b_{ik\nu}^*|_V \leqq C_2|B_\nu(s+k)|_{V\times W}
$$

with a suitable number $C_2 \in \boldsymbol{R}_+$ which does not depend on i, k and ν. Moreover if $(x'', s)\in V\times \boldsymbol{C}$, we have

$$
\begin{aligned}
|B_\nu(s)| &\leqq \Big\{\prod_{i=1}^{\nu}\Big(1+\frac{(|s|+C_1)^2}{i^2}\Big)\Big\}^{N^2}\cdot\Big\{\prod_{i=0}^{\nu}\Big(1+\frac{(|s|+C_1)^2+1}{i^2+1}\Big)\Big\}^{N'^2} \\
&\leqq (2+(|s|+C_1)^2)^{N'^2}\cdot\prod_{i=1}^{\infty}\Big\{1+\frac{(|s|+C_1+1)^2}{i^2}\Big\}^{N^2+N'^2} \\
&=(2+(|s|+C_1)^2)^{N'^2}\cdot\Big\{\frac{\sinh \pi(|s|+C_1+1)}{\pi(|s|+C_1+1)}\Big\}^{N^2+N'^2}.
\end{aligned}
$$

Hence there exists a $C_3 \in \boldsymbol{R}_+$ such that

$$
|b_{ik\nu}^*|_V \leqq C_3^{|k|+1} \qquad \text{for } i=1, \cdots, (N^*)^2,\ k\in \boldsymbol{Z} \text{ and } \nu \in \boldsymbol{N}_+.
$$

On the other hand since the sequence $\{B_\nu(s)\}_{\nu\in \boldsymbol{N}_+}$ converges to a holomorphic function $B(s)$ of (x'', s) on $V\times \boldsymbol{C}$ and the convergence is uniform on $V\times W$, the sequence $\{b_{i0\nu}\}_{\nu\in \boldsymbol{N}_+}$ uniformly converges to an element b_i of $\mathcal{O}_Z(V)$ for $i=1, \cdots, N^2$ and we have

$$
\inf_{x''\in V}\Big|\sum_{i=1}^{N^2} b_i(x'')\lambda_j(x'')^{i-1}\Big| = \inf_{x''\in V}|B(\lambda_j(x''))|>0
$$

for $j=1, \cdots, N$. Therefore we can choose an $F(s) \in \mathcal{O}_Z(V)[s]$ satisfying

$$F(s)\left(\sum_{i=1}^{N^2} b_i s^{i-1}\right) \equiv 1 \quad \bmod \mathcal{O}_Z(V)[s]\left(\prod_{j=1}^{N}(s-\lambda_j)\right)^N.$$

Hence $F(A)(\sum b_i A^{i-1}) = I_N$.

Let $p=(0; \sum_{i=1}^{l} \tau_i^0 dt_i)$ be an arbitrary point in Λ_0. We choose compact neighborhoods $K_\pm$ of the points $(0; \sum_{i=1}^{l}(2C_3)^{\pm m_i}\tau_i^0 dt_i)$ in T^*X, respectively, satisfying that $\pi_X(K_\pm) \subset X' \times V$ and that $\mathscr{J}$, P, P', A and A' are defined in a neighborhood of $K_+ \cup K_-$. Then there exist M_j and $M \in \boldsymbol{R}_+$ such that

$$|P_k^*|_{K_\pm, j} \leqq M_j \qquad \text{for } j \in \boldsymbol{Z}$$

and

$$|(A^*)^{i-1}|_{K_\pm, 0} \leqq M \qquad \text{for } i=1, \cdots, (N^*)^2.$$

Hence if we put $K'_\pm = \{(t, x, \tau, \xi) \in T^*X \mid ((2C_3)^{\mp m_1} t_1, \cdots, (2C_3)^{\mp m_l} t_l, x, (2C_3)^{\pm m_1}\tau_1, \cdots, (2C_3)^{\pm m_l}\tau_l, \xi) \in K_\pm\}$, respectively, and $K = K'_+ \cap K'_-$, then K is a neighborhood of p and

$$|P_k^*|_{K,j} \leqq (2C_3)^{\pm k} |P_k^*|_{K_\pm, j} \leqq (2C_3)^{\pm k} M_j \qquad \text{for any } (k, j) \in \boldsymbol{Z}^2.$$

Let U be an open neighborhood of p contained in K. Then we can choose $r \in \boldsymbol{N}$ and $L_j \in \boldsymbol{R}_+$ so that

$$|P_k^*(A^*)^{i-1}|_{U,j} \leqq L_j |P_k^*|_{K, j-r} \qquad \text{for } i=1, \cdots, (N^*)^2 \text{ and } k \in \boldsymbol{Z}.$$

Hence for each $\nu \in \boldsymbol{N}$ the above consideration proves

$$\begin{aligned}
&\left|(P_0^m B_\nu(A)u + J_\nu) - P_0^m\left(\sum_{i=1}^{N^2} b_i A^{i-1}\right)u\right|_{U,j} \\
&= \left|P_0^m \sum_{i=1}^{N^2}(b_{i0\nu} - b_i)A^{i-1}u + \left(\sum_{|k|>\nu} P_k^m \sum_{i=1}^{N^2} b_{ik\nu}A^{i-1}\right)u \right. \\
&\qquad \left. + \left(\sum_{|k|>\nu}(P')_k^m \sum_{i=1}^{N'^2} b'_{ik\nu}A'^{i-1}\right)u'\right|_{U,j} \\
&\leqq L_j\left(\sum_{i=1}^{N^2}|b_{i0\nu} - b_i|_V \cdot |P_0^m|_{K,j-r} + \sum_{|k|>\nu}\sum_{i=1}^{N^2}|b_{ik\nu}|_V \cdot |P_k^m|_{K,j-r}\right. \\
&\qquad \left. + \sum_{|k|>\nu}\sum_{i=1}^{N'^2}|b'_{ik\nu}| \cdot |(P')_k^m|_{K,j-r}\right) \\
&\leqq L_j M_{j-r}\left(\sum_{i=1}^{N^2}|b_{i0\nu} - b_i|_V + N^2 \sum_{|k|>\nu} C_3^{|k|+1}(2C_3)^{-|k|}\right. \\
&\qquad \left. + N'^2 \sum_{|k|>\nu} C_3^{|k|+1}(2C_3)^{-|k|}\right)
\end{aligned}$$

$$=L_j M_{j-r}\Big(\sum_{i=1}^{N^2} |b_{i0\nu}-b_i|_V+(N^2+N'^2)2^{1-\nu}C_3\Big),$$

which converges to 0 when ν becomes infinity. Since $P_0^m B_\nu(A)u+J_\nu$ is a section of $\mathscr{J}$, Lemma 1.3 shows that $P_0^m(\sum b_i A^{i-1})u$ is also a section of $\mathscr{J}$ over the neighborhood U of p. Moreover since p is an arbitrary point of Λ_0 and $0\equiv\phi(x'', m\vartheta)F(m\vartheta)P_0^m(\sum b_i A^{i-1})u\equiv P_0^m\phi(x'', A)F(A)(\sum b_i A^{i-1})u \equiv P_0^m\phi(x'', A)u \bmod \mathscr{J}$, $P_0^m\phi(x'', A)u$ defines a section of $\mathscr{J}$ over a neighborhood of Λ_0.

(ii) The statement (i) and the equation (2.5) entail $(m\vartheta-A)Q_\sigma^m u \equiv 0 \bmod \mathscr{J}$, which equals

$$Q_\sigma^m(A+\sigma)u\equiv AQ_\sigma^m u \bmod \mathscr{J} \quad (\sigma\in Z). \tag{2.7}$$

We put $Q_{\sigma,i}=((Q_\sigma^m)_{i1}, \cdots, (Q_\sigma^m)_{iN})$ and we shall prove $Q_{\sigma,i}u\equiv 0 \bmod \mathscr{J}$ for each $\sigma\neq 0$ by the induction whose hypothesis is $Q_{\sigma,\nu}u\equiv 0 \bmod \mathscr{J}$ if $\nu>i$. Then by the i-th components of (2.7) we have

$$\begin{aligned} Q_{\sigma,i}(A+\sigma)u &\equiv \lambda_i Q_{\sigma,i}u+\sum_{i<\nu\leqq N} A_{i\nu}Q_{\sigma,\nu}u \bmod \mathscr{J} \\ &\equiv \lambda_i Q_{\sigma,i}u \bmod \mathscr{J}. \end{aligned}$$

If $\sigma\neq 0$, we can choose a $G(s)\in \mathscr{O}_Z[s]$ satisfying $(A+\sigma-\lambda_i)G(A)=I_N$ as in the proof of (i) because $A+\sigma-\lambda_i$ is invertible. Hence we have $0\equiv Q_{\sigma,i}(A+\sigma-\lambda_i)u\equiv Q_{\sigma,i}(A+\sigma-\lambda_i)G(A)u\equiv Q_{\sigma,i}u \bmod \mathscr{J}$ from (i). Thus $Q_\sigma^m u\equiv 0 \bmod \mathscr{J}$ for any $\sigma\in Z-\{0\}$.

Putting $I(j)=\{\nu\,|\,1\leqq\nu\leqq N', \lambda'_\nu(0)=\lambda'_j(0)\}$ and $Q^{(j)}=(Q'_{i\nu})_{1\leqq i\leqq N, \nu\in I(j)}$ and $u^{(j)}=(u_\nu)_{\nu\in I(j)}$ and considering $Q^{(j)}u^{(j)}$ in place of Qu, we have $(Q^{(j)})_\sigma^m u^{(j)}\equiv 0 \bmod \mathscr{J}$ for any $\sigma\in Z$ and $j=1, \cdots, N'$ by the same argument as above because $\sigma+\lambda'_\nu(0)-\lambda_i(0)\neq 0$. Then $Q^{(j)}u^{(j)}\equiv 0$ and $Q'u'\equiv 0$. Thus we have $Qu+Q'u'\equiv Q_0^m u \bmod \mathscr{J}$.

We shall prove the last claim in (ii) by the same induction as above. The hypothesis of the induction implies $Q_{0,i}Au\equiv\lambda_i Q_{0,i}u \bmod \mathscr{J}$. Let E be the diagonal matrix whose ν-th diagonal component equals 1 if $\lambda_\nu=\lambda_i$ and 0 if $\lambda_\nu\neq\lambda_i$. Then $Q_{0,i}(A+E-\lambda_i)u\equiv 0 \bmod \mathscr{J}$ because $(Q_0^m)_{i\nu}=0$ if $\lambda_\nu=\lambda_i$. Since $A+E-\lambda_i$ is invertible for a generic x'', there exists a polynomial $H(s)$ with coefficients in the field of meromorphic functions of x'' such that $(A+E-\lambda_i)H(A)=I_N$. Since $H(s)=\phi^{-1}L(s)$ with a $\phi\in\mathscr{O}_Z$ and an $L(s)\in\mathscr{O}_Z[s]$, we have $0\equiv Q_{0,i}(A+E-\lambda_i)u\equiv Q_{0,i}(A+E-\lambda_i)L(A)u\equiv \phi Q_{0,i}u \bmod \mathscr{J}$. Hence we have $Q_{0,i}u\equiv 0$ from (2.6) and thus $Q_0^m u\equiv 0 \bmod \mathscr{J}$ by the induction.

(iii) This is clear by the above argument and Lemma 1.3. Q.E.D.

§ 3. Systems of differential equations with regular singularities

Let $\mathcal{M}$ be a system of differential equations on X' with a holomorphic parameter in Z, i.e. $\mathcal{M}$ is a coherent left $\mathcal{D}_{X/Z}$-Module on $X=X'\times Z$. We suppose $\mathcal{M}$ has N unknown functions $u_1, \cdots, u_N$. Let u denote the column vector formed by $u_1, \cdots, u_N$ and $\mathcal{J}$ the left sub-$\mathcal{D}_{X/Z}$-Module of $(\mathcal{D}_{X/Z})^N$ which annihilates u. Then $\mathcal{M}$ is isomorphic to $(\mathcal{D}_{X/Z})^N/\mathcal{J}$. Recall that Y'_j $(j=1, \cdots, l)$ are nonsingular hypersurfaces of X' normally crossing at Y'.

Definition 3.1. The system $\mathcal{M}$ is said to have regular singularities along the set of walls $\{Y'_1, \cdots, Y'_l\}$ with the edge Y' if there exists a positive integer m such that the following condition holds:

$$(3.1)\qquad \begin{aligned}(\mathcal{O}_Y[s]^{(m)})^N=\{\sigma_*(P);\ P\in({}_Z\mathcal{D}_*)^N\cap(\mathcal{J}|_Y) \text{ and } \operatorname{ord} P\leqq m\}\\ +(\mathcal{O}_Y[s]^{(m-1)})^N.\end{aligned}$$

In this sequel we study the system $\mathcal{M}$ which has regular singularities along the set of walls $\{Y'_1, \cdots, Y'_l\}$. Then there exists a positive integer M such that for any point in Y there exist a neighborhood V in Y of the point and sections $P_1, \cdots, P_M$ in $(({}_Z\mathcal{D}_*)^N\cap\mathcal{J})(V)$ (i.e. $P_iu=0$ and P_i are of the form $P_i(t, x, \vartheta, tD_{x'})$) and the following condition holds in V: Put $m_i=\operatorname{ord} P_i$, $p_i^*=\sigma_*(P_i)$ $(=P_i(0, x, s, 0)$ by definition), $\bar{\mathcal{J}}=\sum_{i=1}^M \mathcal{O}_Y[s]p_i^*$ and $\bar{\mathcal{M}}=(\mathcal{O}_Y[s])^N/\bar{\mathcal{J}}$. Then $\bar{\mathcal{M}}$ is a free $\mathcal{O}_Y$-Module of rank r with an $r\in N$. We call $\bar{\mathcal{M}}$ an *indicial equation* of $\mathcal{M}$.

The definition also implies that replacing M and $P_1, \cdots, P_M$ if necessary, we may assume

$$(3.2)\qquad \bar{\mathcal{J}}\cap(\mathcal{O}_Y[s]^{(k)})^N=\sum_{i=1}^{M}\mathcal{O}_Y[s]^{(k-m_i)}p_i^* \qquad \text{for } k\in N,$$

but in general we don't assume (3.2) for $P_1, \cdots, P_M$ in this sequel.

Let $\{\bar{v}_1, \cdots, \bar{v}_r\}$ be a basis of the $\mathcal{O}_Y$-Module $\bar{\mathcal{M}}$ and $\bar{v}$ the column vector formed by $\bar{v}_1, \cdots, \bar{v}_r$. Then there exist $\bar{Q}_j(x)\in M(r; \mathcal{O}_Y(V))$ such that $s_j\bar{v}=\bar{Q}_j(x)\bar{v}$ $(j=1, \cdots, l)$. The above condition implies that $[\bar{Q}_i, \bar{Q}_j]=0$ for $i, j=1, \cdots, r$ and that the system of the equations for $s=(s_1, \cdots, s_l)$

$$(3.3)\qquad \det\left(\sum_{j=1}^{l} C_j(s_j-\bar{Q}_j(x))\right)=0 \quad (C_j\in C)$$

has r roots including their multiplicity, which we denote by

$$\lambda_\nu(x)=(\lambda_{\nu,1}(x), \cdots, \lambda_{\nu,l}(x))\ (\nu=1, \cdots, r)$$

and call the *characteristic exponents* of $\mathcal{M}$ with respect to $\overline{\mathcal{M}}$. For simplicity we assume that λ_ν do not depend on $x' \in Y'$ but holomorphically depend on $x'' \in Z$, that is,

$$\lambda_\nu \in \mathcal{O}_Z(V \cap Z) \quad (\nu=1, \cdots, r). \tag{3.4}$$

Moreover we assume the existence of other M and $P_1, \cdots, P_M$ (which will be denoted by M' and $P'_1, \cdots, P'_{M'}$, respectively) such that they satisfy (3.2) and that the corresponding λ_ν also satisfy (3.4).

In this section we fix a point 0 in Y and study the system $\mathcal{M}$ in a neighborhood of 0. Putting $\lambda_\nu^\circ = \lambda_\nu(0)$, we may assume

$$\lambda_\mu^\circ - \lambda_\nu^\circ \notin N^l - \{0\} \qquad \text{if } \mu > \nu. \tag{3.5}$$

Moreover, choosing $\{\bar{v}_1, \cdots, \bar{v}_r\}$ suitably, we may assume

$$\begin{cases} \overline{Q}_j(x)_{\nu\nu} = \lambda_{\nu,j}(x'') & (j=1, \cdots, l; \nu=1, \cdots, r), \\ \overline{Q}_j(x)_{\mu\nu} = 0 & \text{if } \mu > \nu \text{ or } \lambda_\mu^\circ \neq \lambda_\nu^\circ. \end{cases} \tag{3.6}$$

We can choose $\overline{A}(x) \in M(N, r; \mathcal{O}_Z)$ and $\overline{B}(x, s) \in M(r, N; \mathcal{O}_Z[s])$ such that

$$\bar{u} = \overline{A}(x)\bar{v} \quad \text{and} \quad \bar{v} = \overline{B}(x, s)\bar{u}. \tag{3.7}$$

Here $\bar{u}$ denotes the column vector of length N whose i-th component is the residue class of $(\delta_{i1}, \cdots, \delta_{iN})$ in $\overline{\mathcal{M}}$ and δ_{ij} denotes Kronecker's delta. Hence (3.7) gives the isomorphim $\overline{\Phi}$ between the two $\mathcal{O}_Z[s]$-Modules

$$\overline{\mathcal{M}}: p_i^* \bar{u} = 0 \quad (i=1, \cdots, M)$$

and

$$\overline{\mathcal{N}}: (s_j - \overline{Q}_j(x))\bar{v} = 0 \quad (j=1, \cdots, l).$$

Theorem 3.2. *Retain the above notation and assume* (3.4), (3.5) *and* (3.6).

(i) *$\mathcal{E}_X \otimes \mathcal{M}$ is a system of micro-differential equations with regular singularity along V in a neighborhood of Λ.*

(ii) *There exists a surjective homomorphism Φ of a system*

$$\mathcal{N}: (\vartheta_j - Q_j(x, D_{x'}, D_t))v = 0 \quad (j=1, \cdots, l)$$

of micro-differential equations with the column vector v formed by r unknown functions $v_1, \cdots, v_r$ onto the system $\mathcal{E}_X \otimes \mathcal{M}$ in a neighborhood of $\Lambda_0 = \pi_X^{-1}(0) \cap \Lambda$ where

$$(3.8)\quad \begin{cases} Q_j \in M(r;\, {}_Z\mathscr{E}^{(0)}_*), \\ \sigma_*(Q_j)=\bar{Q}_j, \\ (Q_j)_{\mu\nu}=0 \quad \text{if } \lambda^\circ_\mu-\lambda^\circ_\nu \notin N^l \text{ or } \mu>\nu, \\ (Q_j)_{\mu\nu}=Q^\circ_{j,\mu,\nu}(x, D_{x'})D_t^{\lambda^\circ_\nu-\lambda^\circ_\mu} \text{ with some } Q^\circ_{j,\mu,\nu}\in \mathscr{D}_{Y/Z} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \lambda^\circ_\mu-\lambda^\circ_\nu \in N^l. \end{cases}$$

Moreover the homomorphism is given by the correspondence

$$(3.9)\qquad v=B(t, x, D_t, D'_x)u$$

with a $B\in M(r, N;\, {}_Z\mathscr{E}_*)$ *satisfying*

$$(3.10)\qquad \sigma_*(B)=\bar{B}(x, s)$$

and there exists an $A\in M(N, r;\, {}_Z\mathscr{E}^{(0)}_*)$ *such that*

$$(3.11)\qquad u=ABu \quad \text{and} \quad \sigma_*(A)=\bar{A}(x).$$

(iii) *Assume*

$$(3.12)\qquad \lambda^\circ_\mu-\lambda^\circ_\nu \notin N^l-\{0\} \qquad \text{for any } \mu \text{ and } \nu$$

or both

$$(3.13)\qquad \lambda_\mu-\lambda_\nu \notin N^l-\{0\} \qquad \text{for any } \mu \text{ and } \nu$$

and

(3.14) *the equation* $\phi Pu=0$ *with a non-zero* $\phi\in\mathcal{O}_Z$ *and a* $P\in(\mathscr{E}_{X/Z})^N$ *implies* $Pu=0$.

Then if we fix a system $\mathscr{N}$ *in* (ii), *the homomorphism* Φ *of* $\mathscr{N}$ *to* $\mathscr{M}$ *is uniquely determined by the condition* (3.10) *or* (3.11).

First we assume that P_i satisfy (3.2). Let $p^\circ_i(s)$ denote the homogeneous part of $p^*_i(0, s)$ of degree m_i and let $\{c_\nu(s);\, 1\leqq\nu\leqq r\}$ be a subset of $C[s]^N$ such that

$$(3.15)\qquad \sum_{i=1}^{M} C[s]^{(m-m_i)}p^\circ_i+\sum_{c_\nu\in(C[s]^{(m)})^N} Cc_\nu=(C[s]^{(m)})^N$$

for any $m\in N$. The existence of c_ν follows from (3.2).

Lemma 3.3. *Let* W *be a neighborhood of the origin of* C^L, *where* L *is a positive integer. Let* $p_i(w, s)$ *be vectors in* $\mathcal{O}_W[s]^N$ *satisfying* $p_i(0, s)=$

$p_i^\circ(s)$ $(i=1, \cdots, M)$. *Then in a neighborhood of the origin we have*

$$\sum_{i=1}^{M} \mathcal{O}_W[s]^{(m-m_i)} p_i + \sum_{c_\nu \in (C[s]^{(m)})^N} \mathcal{O}_W c_\nu = (\mathcal{O}_W[s]^{(m)})^N. \tag{3.16}$$

Proof. By the assumption we have linear maps T_i of $(C[s]^{(m)})^N$ to $C[s]^{(m-m_i)}$ and R_ν of $(C[s]^{(m)})^N$ to C such that $f=\sum T_i(f)p_i^\circ+\sum R_\nu(f)c_\nu$ for $f \in (C[s]^{(m)})^N$ and that $R_\nu=0$ if $\deg c_\nu > m$. Let $\tilde{T}_i$ and $\tilde{R}_\nu$ denote the $\mathcal{O}_W$-linear extensions of T_i and R_ν, respectively. For any $g \in (\mathcal{O}_W[s]^{(m)})^N$ we inductively define $g_j (j \in N)$ by $g_0=g$ and $g_{j+1}=\sum_i (p_i-p_i^\circ)\tilde{T}_i(g_j)$. In a small neighborhood of the origin, $\sum_{j=0}^\infty g_j$ converges into an $h \in (\mathcal{O}_W[s]^{(m)})^N$, so $g=\sum \tilde{T}_i(h)p_i+\sum \tilde{R}_\nu(h)c_\nu$. Q.E.D.

Proof of Theorem 3.2 *under the assumption* (3.2). Let $C(s)$ denote the matrix in $M(r, N; C[s])$ whose ν-th row equals $c_\nu(s)$. Put $w=C(\vartheta)u$. First we want to prove

$$({}_Z\mathcal{E}_*)^N u = ({}_Z\mathcal{E}_*^{(0)})^r w. \tag{3.17}$$

Put $p_i=\sigma(P_i)$, $s_j=t_j\tau_j$, $t_{jj'}=t_j\tau_j\tau_{j'}^{-1}$ and $\xi_{kj}=\xi_k\tau_j^{-1}$ $(i=1, \cdots, M; j, j'=1, \cdots, l; k=1, \cdots, n')$. Let W be a neighborhood of the origin of $C^{n+l^2+n'l}$ with the coordinate system $(x, t_{jj'}, \xi_{kj})$. Since $P_i \in ({}_Z\mathcal{D}_*)^N$, p_i are of the form $p_i=\sum_{|\alpha|=m_i} p_{\alpha,i}s^\alpha$ with some $p_{\alpha,i} \in (\mathcal{O}_W)^N$. Let $m \in N$. For any $Q \in ({}_Z\mathcal{E}_* \cap \mathcal{E}_*^{(m)})^N$ we use the similar expression $\sigma_m(Q)=\sum_{|\alpha|=m} q_\alpha s^\alpha$ with some $q_\alpha \in (\mathcal{O}_W)^N$. Since $\sum_{|\alpha|=m_i} p_{\alpha,i}s^\alpha=p_i^\circ(s)$, Lemma 3.3 assures the existence of $f_i \in \mathcal{O}_W[s]^{(m-m_i)}$ and $g_\nu \in \mathcal{O}_W$ such that $\sum f_i p_i+\sum g_\nu c_\nu=\sigma_m(Q)$. Let F_i and G_ν be sections of ${}_Z\mathcal{E}_*$ with the principal symbols f_i and g_ν, respectively. Then $Q-\sum F_i P_i-\sum G_\nu c_\nu(\vartheta) \in ({}_Z\mathcal{E}_* \cap \mathcal{E}_*^{(m-1)})^N$, so $Qu \in ({}_Z\mathcal{E}_* \cap \mathcal{E}_*^{(m-1)})^N u+({}_Z\mathcal{E}_*^{(0)})^r w$. Moreover, if $m=0$ and $Q \in (\mathcal{O}_X)^N$, we can choose f_i and g_ν in $\mathcal{O}_X$ such that $Q=\sum f_i P_i+\sum g_\nu c_\nu(\vartheta)$. This implies $(\mathcal{O}_X)^N u \subset (\mathcal{O}_X)^r w$ and therefore

$$({}_Z\mathcal{E}_*^{(0)})^N u={}_Z\mathcal{E}_*^{(0)}(\mathcal{O}_X)^N u \subset {}_Z\mathcal{E}_*^{(0)}(\mathcal{O}_X)^r w=({}_Z\mathcal{E}_*^{(0)})^r w.$$

Hence by the induction on m we have $Qu \in ({}_Z\mathcal{E}_*^{(0)})^r w$. Since $\bigcup_{m=0}^\infty ({}_Z\mathcal{E}_* \cap \mathcal{E}_*^{(m)})={}_Z\mathcal{E}_*$, we obtain (3.17).

Since $\mathcal{E}_X^{(0)}|_\Lambda \otimes (\pi_X|_\Lambda)^{-1}({}_Z\mathcal{E}_*)=\mathcal{E}_V|_\Lambda$ and $\mathcal{E}_X^{(0)}|_\Lambda \otimes (\pi_X|_\Lambda)^{-1}({}_Z\mathcal{E}_*^{(0)})=\mathcal{E}_X^{(0)}|_\Lambda$, we have $(\mathcal{E}_V)^N u=(\mathcal{E}_X^{(0)})^r w$ by (3.17) and therefore $(\mathcal{E}_V)^r w=(\mathcal{E}_V)^r C(\vartheta)u \subset (\mathcal{E}_V)^N u =(\mathcal{E}_X^{(0)})^r w$ in a neighborhood of Λ. This proves Theorem 3.2 (i) by definition.

By (3.17) we have $(\vartheta_j-R_j)w=0$ for $j=1, \cdots, l$ with some $R_j \in M(r; {}_Z\mathcal{E}_*^{(0)})$. Put $\bar{w}=C(s)\bar{u}$. The proof of (3.17) shows $(s_j-\sigma_*(R_j))\bar{w}=0$, $u=Fw$ and $\bar{u}=\sigma_*(F)\bar{w}$ with an $F \in M(N, r; \mathcal{O}_X)$. Then the assumption implies the existence of an invertible matrix $H \in M(r; \mathcal{O}_Y)$ such that $\bar{v}=$

$H\bar{w}$ and $H^{-1}\bar{Q}_j H = \sigma_*(R_j)$. Now we put $v' = Hw$ and $S'_j = HR_jH^{-1}$. Then $(\vartheta - S'_j)v' = 0$ and $\sigma_*(S'_j) = \bar{Q}_j$. By the Späth type theorem for micro-differential operators we can choose $S_j = M(r; {}_Z\mathscr{E}_*^{(0)})$ such that

$$(3.18) \qquad (\vartheta_j - S_j)v' = 0 \quad (j = 1, \cdots, l),$$

$$(3.19) \qquad \sigma_*(S_j) = \bar{Q}_j \text{ and } [\partial/\partial t_i, S_j] = 0 \quad (i, j = 1, \cdots, l).$$

Since $\bar{u} = \sigma_*(F)H^{-1}\bar{v}$ and $\bar{v} = HC(s)\bar{u}$, we have $\bar{A} = \sigma_*(F)H^{-1}$ and there exists $K_i(x, s) \in M(r, N; \mathscr{O}_Y[s])$ such that $\bar{B} = HC(s) + \sum K_i(x, s)\sigma_*(P_i)$. Put $A' = FH^{-1}$ and $B' = HC(\vartheta) + \sum K_i(x, \vartheta)P_i$. Then

$$(3.20) \qquad u = A'v', \qquad v' = B'u$$

and

$$(3.21) \qquad \sigma_*(A') = \bar{A}, \qquad \sigma_*(B') = \bar{B}.$$

By Lemma 2.3 we can choose an $m \in N_+^l$ such that

$$(3.22) \qquad \sum_{j=1}^{l} m_j(\lambda^\circ_{\mu,j} - \lambda^\circ_{\nu,j}) \begin{cases} \neq |m\alpha| & \text{for any } \alpha \in N^l - \{\lambda^\circ_\mu - \lambda^\circ_\nu\} \\ & \text{if } \lambda^\circ_\mu - \lambda^\circ_\nu \in Z^l, \\ \notin Z & \text{if } \lambda^\circ_\mu - \lambda^\circ_\nu \notin Z^l. \end{cases}$$

Put $S = \sum m_j S_j$. Then by Lemma 2.2 we have an invertible matrix $U \in M(r; {}_Z\mathscr{E}_*^{(0)})$ with $\sigma_*(U) = I_r$ such that $[\partial/\partial t_j, U] = 0$ for $j = 1, \cdots, l$ and that the matrix $T = m\vartheta - U(m\vartheta - S)U^{-1}$ is of the form

$$(3.23) \qquad \begin{cases} \sigma_*(T) = \sum m_j \bar{Q}_j, \\ T_{\mu\nu} = T^\circ_{\mu,\nu}(x, D_{x'})D_t^{\lambda^\circ_\nu - \lambda^\circ_\mu} \quad \text{with some } T^\circ_{\mu,\nu} \in \mathscr{D}_{Y/Z} \\ \qquad\qquad \text{if } \lambda^\circ_\mu - \lambda^\circ_\nu \in N^l, \\ T_{\mu\nu} = 0 \qquad \text{if } \lambda^\circ_\mu - \lambda^\circ_\nu \notin N^l. \end{cases}$$

Now we want to show that the matrix $B = UB'$ gives the desired correspondence (3.9). Let ι be a map of the set $\{\lambda^\circ_1, \cdots, \lambda^\circ_r\}$ to itself such that $\lambda^\circ_\nu - \iota(\lambda^\circ_\nu) \in Z^l$ for $\nu = 1, \cdots, r$ and that $\iota(\lambda^\circ_\mu) = \iota(\lambda^\circ_\nu)$ if $\lambda^\circ_\mu - \lambda^\circ_\nu \in Z^l$. Let D denote the diagonal matrix of size r whose ν-th diagonal component equals $D_t^{\lambda^\circ_\nu - \iota(\lambda^\circ_\nu)}$. We put $v = Uv'$, $T_j = \vartheta_j - U(\vartheta_j - S_j)U^{-1}$, $\tilde{v} = Dv$, $\tilde{T}_j = \vartheta_j - D(\vartheta_j - T_j)D^{-1}$ and $\tilde{T} = m\vartheta - D(m\vartheta - T)D^{-1}$. Then $\sigma_*(T_j) = \bar{Q}_j$ and T_j are of the form

$$(3.24) \qquad (T_j)_{\mu\nu} = \sum_{\alpha \in N^l} T^\alpha_{j,\mu,\nu}(x, D_{x'})D_t^{-\alpha}$$

with some $T^\alpha_{j,\mu,\nu} \in \mathscr{D}_{Y/X}$ and

$$(3.25)\qquad \begin{cases} \tilde{T}_{\nu\nu}=\sum m_j(\lambda_{\nu,j}(x'')-\lambda^\circ_{\nu,j}+\iota(\lambda^\circ_\nu)_j), \\ \tilde{T}_{\mu\nu}=T^\circ_{\mu,\nu}(x, D_{x'}) \quad \text{if } \lambda^\circ_\mu-\lambda^\circ_\nu \in Z^l, \\ \tilde{T}_{\mu\nu}=0 \quad\quad\quad\quad\quad \text{if } \lambda^\circ_\mu-\lambda^\circ_\nu \notin Z^l. \end{cases}$$

Here we remark

$$(3.26)\qquad \tilde{T}_{\mu\mu}(0)-\tilde{T}_{\nu\nu}(0)\begin{cases} =0 & \text{if } \lambda^\circ_\mu-\lambda^\circ_\nu \in Z^l, \\ \notin Z & \text{if } \lambda^\circ_\mu-\lambda^\circ_\nu \notin Z^l. \end{cases}$$

Since $(m\vartheta-\tilde{T})\tilde{v}=(\vartheta_j-\tilde{T}_j)\tilde{v}=0$, Lemma 2.4 (i), (3.25) and (3.26) show $(\vartheta_j-\tilde{Q}_j)\tilde{v}=0$, where $(\vartheta_j-\tilde{Q}_j)_{\mu\nu}$ equals 0 if $\lambda^\circ_\mu-\lambda^\circ_\nu \notin Z^l$ and $((\vartheta_j-T_j)^m_0)_{\mu\nu}$ if $\lambda^\circ_\mu-\lambda^\circ_\nu \in Z^l$. Put $Q_j=\vartheta_j-D^{-1}(\vartheta_j-\tilde{Q}_j)D$. Then $(\vartheta_j-Q_j)v=0$ and if $\lambda^\circ_\mu-\lambda^\circ_\nu \notin Z^l$, $(Q_j)_{\mu\nu}=0$. On the other hand, if $\lambda^\circ_\mu-\lambda^\circ_\nu \in Z^l$,

$$\begin{aligned}(\vartheta_j-Q_j)_{\mu\nu}&=D_t^{\iota(\lambda^\circ_\mu)-\lambda^\circ_\mu}(\vartheta_j-\tilde{Q}_j)_{\mu\nu}D_t^{\lambda^\circ_\nu-\iota(\lambda^\circ_\nu)} \\ &=D_t^{\iota(\lambda^\circ_\mu)-\lambda^\circ_\mu}((D_t^{\lambda^\circ_\mu-\iota(\lambda^\circ_\mu)}(\vartheta_j-T_j)_{\mu\nu}D_t^{\iota(\lambda^\circ_\nu)-\lambda^\circ_\nu})^m_0)D_t^{\lambda^\circ_\nu-\iota(\lambda^\circ_\nu)} \\ &=\vartheta_j\delta_{\mu,\nu}-((T_j)^m_{|m(\lambda^\circ_\mu-\lambda^\circ_\nu)|})_{\mu\nu}.\end{aligned}$$

Thus by (3.22) and (3.24) we have $(Q_j)_{\mu\nu}=0$ if $\lambda^\circ_\mu-\lambda^\circ_\nu \notin N^l$ and

$$(Q_j)_{\mu\nu}=T^{\lambda^\circ_\mu-\lambda^\circ_\nu}_{j,\mu,\nu}(x, D_{x'})D_t^{\lambda^\circ_\nu-\lambda^\circ_\mu} \qquad \text{if } \lambda^\circ_\mu-\lambda^\circ_\nu \in N^l,$$

so Q_j satisfy (3.8). Moreover, putting $B=UB'$ and $A=U^{-1}A'$, we have $v=Bu$, $u=Av=ABu$, $\sigma_*(A)=\sigma_*(U^{-1})\sigma_*(A')=\bar{A}$ and $\sigma_*(B)=\sigma_*(U)\sigma_*(B')=\bar{B}$ (cf. (3.20) and (3.21)). Hence we have Theorem 3.2 (ii) because Bu satisfies $\mathcal{N}$.

To prove (iii) we assume (3.12) or both (3.13) and (3.14). We moreover assume that the correspondence $v=B_1u$ with a $B_1 \in M(r, N; {}_Z\mathscr{E}_*)$ also given a homomorphism Φ_1 of $\mathcal{N}$ to $\mathcal{M}$. Put

$$\tilde{T}=m\vartheta-D\Big(m\vartheta-\sum_{j=1}^{l}\mathrm{m}_jQ_j\Big)D^{-1}.$$

This $\tilde{T}$ may be different from the $\tilde{T}$ defined before but has the form (3.25) with some $T^\circ_{\mu,\nu}(x, D_{x'}) \in \mathscr{D}_{Y/Z}$ and satisfies $(m\vartheta-\tilde{T})DBu=(m\vartheta-\tilde{T})DB_1u=0$. Hence Lemma 2.4 (ii) and (3.26) show

$$DBu-DB_1u=(D(B-B_1)AD^{-1})DBu=LDBu,$$

where $L \in M(r; {}_Z\mathscr{E}^f_*)$ and $L_{\mu\nu}=0$ if $\lambda^\circ_\mu-\lambda^\circ_\nu \notin Z^l$, and

$$L_{\mu\nu}=((D(B-B_1)AD^{-1})_{\mu\nu})^m_0=D_t^{\lambda^\circ_\mu-\iota(\lambda^\circ_\mu)}(((B-B_1)A)^m_{|m(\lambda^\circ_\mu-\lambda^\circ_\nu)|})_{\mu\nu}D_t^{\iota(\lambda^\circ_\nu)-\lambda^\circ_\nu}$$

if $\lambda^\circ_\mu-\lambda^\circ_\nu \in Z^l$. Therefore if $\lambda^\circ_\mu-\lambda^\circ_\nu \notin N^l$, we have $L_{\mu\nu}=0$ by (3.22) and if $\lambda_\mu^\circ=\lambda_\nu^\circ$ and $\sigma_*(B)=\sigma_*(B_1)$, we have also $L_{\mu\nu}=0$. Thus if $\sigma_*(B)=\sigma_*(B_1)=\bar{B}$, the assumption and Theorem 2.4 (ii) prove $LDBu=0$, so $Bu=B_1u$ and $\Phi=\Phi_1$. Next assume the existence of an $A_1\in M(N, r; {}_Z\mathscr{E}_*^{(0)})$ satisfying $A_1B_1u=u$ and $\sigma_*(A_1)=\bar{A}$. Since $\bar{B}\bar{A}\bar{v}=\bar{v}$, there exists $H_j(x, s)\in M(r; \mathscr{O}_Y[s])$ such that $\bar{B}\bar{A}+\sum_{j=1}^l H_j(x, s)(s_j-\bar{Q}_j)=I_r$. Put $J=BA_1+\sum_{j=1}^l H_j(x, \vartheta)(\vartheta_j-Q_j)-I_r$. Then

$$\sigma_*(J)=0 \quad \text{and} \quad DBu-DB_1u=D(BA_1-I_r)B_1u=(DJD^{-1})DB_1u.$$

Hence the same argument as above proves $\Phi=\Phi_1$. Thus we have proved Theorem 3.2 under the assumption (3.2). Q.E.D.

Now we will examine the kernel of the map Φ in Theorem 3.2.

Theorem 3.4. *Retain the notation $\mathscr{M}$, u, A, Q, λ_ν° etc. in Theorem 3.2. Let*

$$P=\Big(\sum_{\alpha,\beta\in N^l} P_{\alpha,\beta,1}(x, D_{x'})t^\alpha D_t^{\alpha-\beta}, \cdots, \sum_{\alpha,\beta\in N^l} P_{\alpha,\beta,N}(x, D_{x'})t^\alpha D_t^{\alpha-\beta}\Big)$$

be a section of $({}_Z\mathscr{D}_*^f)^N$ defined in a neighborhood of 0 such that $Pu=0$ for the generator u of $\mathscr{M}$. Put $PA=R=(R_1, \cdots, R_r)$ and

$$R_\mu=\sum_{\alpha,\beta\in N^l} R_{\alpha,\beta,\mu}(x, D_{x'})\vartheta^\alpha D_t^{-\beta} \quad (\mu=1, \cdots, r). \tag{3.27}$$

Fix a $\nu\in\{1, \cdots, r\}$ and a $\gamma\in N^l$ and a polynomial

$$\phi(x'', s)=\sum_{\tau\in N^l}\phi_\tau(x'')s^\tau\in\mathscr{O}_Z[s]$$

defined in a neighborhood of 0. *Then for every $\mu\in\{1, \cdots, r\}$ put $\omega=\gamma-\lambda_\mu^\circ+\lambda_\nu^\circ$ and define a micro-differential operator $S_\mu(s)\in\pi(\mathscr{E}_{X/Z}|_\Lambda)[s]$ by*

$$S_\mu=\begin{cases} 0 & \text{if } \omega\notin N^l, \\ \sum_{\alpha,\tau\in N^l}\phi_\tau(x'')R_{\alpha,\omega,\mu}(x, D_{x'})D_t^{\lambda_\mu^\circ-\lambda_\nu^\circ}(s+\omega)^{\alpha+\tau} & \text{if } \omega\in N^l, \end{cases} \tag{3.28}$$

and moreover put $(S_1, \cdots, S_r)=\sum_{\beta\in N^l}G_\beta s^\beta$ ($G_\beta\in(\pi(\mathscr{E}_{X/Z}|_\Lambda))^N$ and

$$G(t, x, D_{x'}, D_t)=\sum_{\beta\in N^l}G_\beta\vartheta_1^{\beta_1}\cdots\vartheta_l^{\beta_l}. \tag{3.29}$$

Then $\Phi(Gv)=0$. Especially when $\lambda_\nu^\circ-\lambda_\mu^\circ\notin N^l$ for $\mu\in\{1, \cdots, r\}-\{\nu\}$, we have

$$\Phi\Big(\sum_{i=1}^N\sum_{\alpha\in N^l}P_{\alpha,0,i}(x, D_{x'})\sigma_*(A_{i\nu})\lambda_{\nu,1}(x'')^{\alpha_1}\cdots\lambda_{\nu,l}(x'')^{\alpha_l}v_\nu\Big)=0. \tag{3.30}$$

Proof under the assumption (3.2). Define the map ι, $m \in N^l$, $D \in M(r; \mathscr{E}_{X/Z})$ and $\tilde{T} \in M(r; \mathscr{D}_{X/Z})$ as in the proof of Theorem 3.2. Here we can assume $\iota(\lambda_\nu^\circ)=\lambda_\nu^\circ$. Since $PABu=Pu=0$, we have $\Phi(PAv)=0$ and $\Phi(PAD^{-1}\tilde{v})=0$ with $\tilde{v}=Dv$. Here we remark $(m\vartheta-\tilde{T})\tilde{v}=0$ and that D is a diagonal matrix whose μ-th diagonal component equals $D_t^{\lambda_\mu^\circ-\iota(\lambda_\mu^\circ)}$. Now applying Lemma 2.4 (i) with $\sigma=0$ to the kernel of Φ and its element $D_t^\gamma PAD^{-1}\tilde{v}$, we have $S'\tilde{v}=0$, where $S'=(S_1', \cdots, S_r')$ are defined by

$$(3.31)\qquad S_\mu'=\begin{cases} 0 & \text{if } \omega \notin N^l, \\ \sum_{\alpha\in N^l} R_{\alpha,\omega,\mu}(x, D_{x'})D_t^\gamma \vartheta^\alpha D_t^{-\gamma} & \text{if } \omega \in N^l, \end{cases}$$

for $\mu=1, \cdots, r$. Since $\phi(x'', \vartheta+\gamma)$ commutes with S', we have Ker $\Phi \ni S'\phi(x'', \vartheta+\gamma)Dv$. Here we note that $S'\phi(x'', \vartheta+\gamma)D=(S_1(\vartheta), \cdots, S_r(\vartheta))$. Thus we have $\Phi(Gv)=0$. Since $\vartheta_j v=Q_j v$, the last statement in the theorem is clear if we put $\gamma=0$ and $\phi=1$.

Proof of Theorem 3.2 *and Theorem* 3.4. Definition 3.1 implies the existence of sections $P_1', \cdots, P_{M'}'$ of $({}_Z\mathscr{D}_*)^N \cap \mathscr{J}$ on a neighborhood of 0 such that the following condition holds: Put $m_i'=\text{ord}\, P_i'$ and $\bar{\mathscr{J}}'=\sum_{i=1}^{M'} \mathscr{O}_Y[s]\sigma_*(P_i')$. Then $\bar{\mathscr{M}}'=(\mathscr{O}_Y[s])^N/\bar{\mathscr{J}}'$ is a free $\mathscr{O}_Y$-module of rank r' with an $r' \in N$ and

$$(3.32)\qquad \bar{\mathscr{J}}' \cap (\mathscr{O}_Y[s]^{(k)})^N=\sum_{i=1}^{M'} \mathscr{O}_Y[s]^{(k-m_i')}\sigma_*(P_i') \qquad \text{for } k \in N.$$

Moreover we may assume $\bar{\mathscr{M}}$ is a quotient $\mathscr{O}_Y[s]$-Module of $\bar{\mathscr{M}}'$. Let ρ be the natural projection of $\bar{\mathscr{M}}'$ onto $\bar{\mathscr{M}}$. Then we can choose a system of generators $\{\bar{v}_1', \cdots, \bar{v}_{r'}'\}$ of the $\mathscr{O}_Y$-Module $\bar{\mathscr{M}}'$ such that the following conditions hold: Let $\bar{v}'$ be the column vector of length r' formed by the generators and let $\bar{Q}_j'(x) \in M(r; \mathscr{O}_Y)$ with $(s_j-\bar{Q}_j'(x))\bar{v}'=0$ for $j=1, \cdots, l$. Then there exists a subset I of $\{1, \cdots, r'\}$ such that the number of the elements of I equals $r'-r$ and

$$(3.33)\qquad \rho(\bar{v}_\mu')=0 \qquad \text{for } \mu \in I,$$

$$(3.34)\qquad \bar{Q}_j'(x)_{\nu\nu}=\lambda_{\nu,j}'(x'') \qquad \text{with } \lambda_{\nu,j}'(x'') \in \mathscr{O}_Z \quad (j=1, \cdots, l;\ \nu=1, \cdots, r'),$$

$$(3.35)\qquad \bar{Q}_j'(x)_{\mu\nu}=0 \qquad \text{if } \mu>\nu \text{ or } \lambda_\mu'(0)\neq\lambda_\nu'(0) \text{ or } (\mu, \nu) \in I\times(\{1, \cdots, r\}-I),$$

and

$$(3.36)\qquad \lambda_\mu'(0)-\lambda_\nu'(0) \notin N^l-\{0\} \qquad \text{if } \mu>\nu,$$

where λ'_ν denote the column vector of length l formed by $\lambda'_{\nu,1}, \cdots, \lambda'_{\nu,l}$.

Then we can apply Theorem 3.2 to $\mathcal{M}$ and $\bar{\mathcal{M}}'$. To avoid confusion in this case, we denote by $\bar{A}'$, $\bar{B}'$, $\bar{\mathcal{N}}'$, $\mathcal{N}'$, Q'_j, v', A', B' etc. in place of $\bar{A}$, $\bar{B}$, $\bar{\mathcal{N}}$, $\mathcal{N}$, Q_j, v, A, B etc. in Theorem 3.2, respectively, and we will prove Theorem 3.2 for $\mathcal{M}$ and $\bar{\mathcal{M}}$.

Then condition (3.33) says the existence of $R_{\mu,j}(x, s) \in \mathcal{O}_Y[s]$ such that

$$\bar{v}'_\mu = \sum_{j=1}^{M} \bar{R}_{\mu,j}(x, s)\sigma_*(P_j)(x, s)\bar{A}'(x)\bar{v}' \qquad \text{for } \mu \in I. \tag{3.37}$$

We put $R_\mu = \sum_{j=1}^{M} \bar{R}_{\mu,j}(x, \vartheta)P_j$ and apply Theorem 3.4 to R_μ with $\gamma=0$ and $\phi=1$. Then we have

$$v'_\mu - \sum_{\nu=1}^{r'} S_{\mu,\nu}(x, D_{x'})D_t^{\lambda'_\nu(0)-\lambda'_\mu(0)}v'_\nu \in \operatorname{Ker} \Phi' \qquad \text{for } \mu \in I, \tag{3.38}$$

where $S_{\mu,\nu} \in \mathcal{D}_{Y/Z}$ and $S_{\mu,\nu}=0$ if $\lambda'_\mu(0)-\lambda'_\nu(0) \notin N^l-\{0\}$. Let v'' be the column vector of length r formed by $\{v'_\mu;\ \mu \notin I\}$ and arranged by the same order of the components of v'. Let $G \in M(r, r'; \boldsymbol{Z})$ with $v''=Gv'$. Then by using (3.38) we can define $F \in M(r', r;\ {}_Z\mathcal{E}_*^{(0)})$ so that

$$\Phi'(v'-Fv'')=0, \qquad G\sigma_*(F)v''=v'',$$

and then

$$F_{\mu\nu}=F^\circ_{\mu,\nu}(x, D_{x'})D^{\lambda'_\nu(0)-\lambda_\mu(0)} \qquad \text{with some } F^\circ_{\mu,\nu} \in \mathcal{D}_{Y/Z}$$
$$\text{if } \lambda_\mu(0)-\lambda'_\nu(0) \in N^l.$$

Since $\vartheta_j v''=GQ'_jv' \equiv GQ'_jFv''$ mod Ker Φ', we put $Q_j=GQ'_jF$. Then it is clear that Q_j satisfy (3.8). Moreover putting $B=GB'$ and $A=A'F$, we have Theorem 3.2 (ii).

Theorem 3.2 (i) is clear, and Theorem 3.2 (iii) and Theorem 3.4 are proved in the same way as with the assumption (3.2). Q.E.D.

§ 4. Definition of boundary values

We will define boundary values of hyperfunction solutions of systems of differential equations with regular singularities. We remark that Definition 3.1 contains a wider class of systems of differential equations than the definition in [K–O] (cf. § 1).

In this section and the next, we denote by X an $(l+n)$-dimensional real analytic manifold, by Y an n-dimensional real analytic submanifold of X. We fix l hypersurfaces $Y_1, \cdots, Y_l$ normally crossing at Y. We choose a local coordinate system $(t, x)=(t_1, \cdots, t_l, x_1, \cdots, x_n)$ of X such that Y_j is defined by $t_j=0$. We denote by X_+ the open subset of X defined by $t_1>0, \cdots, t_l>0$. We call Y_j the wall and Y the edge. We denote by Z an n'-dimensional complex manifold with a local coordinate system $z=(z_1, \cdots, z_{n'})$. We denote by ${}_Z\mathscr{D}_X$ the sheaf of differential operators on X with holomorphic parameter z in Z. If we denote a complex neighborhood of X by X_C, then ${}_Z\mathscr{D}_X=\mathscr{D}_{X_C\times Z/Z}|X\times Z$. Let $\sqrt{-1}\,S^*X$ denote the conormal spherical bundle over X in X_C. Let ${}_Z\mathscr{E}_X$ denote the sheaf of micro-differential operators which are defined on $\sqrt{-1}\,S^*X$ and have holomorphic parameter z in Z. So any section of ${}_Z\mathscr{E}_X$ is naturally identified with a section of $\mathscr{E}_{X_C\times Z/Z}\,|\,\pi^{-1}_{X_C\times Z}(X\times Z)$. To avoid using many symbols, hereafter we use the same symbol X for X_C if there exists no confusion.

Let ${}_Z\mathscr{B}_X$ (resp. ${}_Z\mathscr{D}'_X$ and ${}_Z\mathscr{A}_X$) be the sheaf of hyperfunctions (resp. distributions and real analytic functions) on X with the holomorphic parameter z in Z. If $\mathscr{F}$ is one of $\mathscr{B}$, $\mathscr{D}'$ or $\mathscr{A}$ and if U is an open subset of X, then we denote by ${}_Z\mathscr{F}(U)$ the space of sections of ${}_Z\mathscr{F}_X$ over $Z\times U$ and by $({}_Z\mathscr{F}(U))^N$ the space of all the column vectors of length N whose components are in ${}_Z\mathscr{F}(U)$. Also let ${}_Z\mathscr{C}_X$ denote the sheaf of microfunctions on $\sqrt{-1}\,S^*X$ with the holomorphic parameter. For a section f of ${}_Z\mathscr{B}_X$, the corresponding microfunction is denoted by $\mathrm{sp}(f)$ and then $\mathrm{sp}(f)$ is a section ${}_Z\mathscr{C}_X$. If Z is a point, which means that we don't consider the holomorphic parameter, then we usually don't write Z such as $\mathscr{B}_X$, $\mathscr{C}_X$, etc.

Now we consider a system of differential equations

$$(4.1)\qquad \mathscr{M}: \sum_{j=1}^{N} P'_{i,j}(t, x, z, D_t, D_x)u_j=0 \quad (i=1, \cdots, M')$$

with N unknown functions u_j which is defined on X and has the holomorphic parameter $z \in Z$. That is, $\mathscr{M}$ is a coherent ${}_Z\mathscr{D}_X$-Module $({}_Z\mathscr{D}_X)^N/\mathscr{J}$, where $\mathscr{J}=\sum_j {}_Z\mathscr{D}_X P'_j$ with $P'_j=(P'_{1,j}, \cdots, P'_{N,j})$. We assume that $\mathscr{M}$ has regular singularities in the sense of Definition 3.1. Moreover we assume the condition (3.4) and the assumption after (3.4). Hence the characteristic exponents $\lambda_1, \cdots, \lambda_r$ of $\mathscr{M}$ with respect to $\bar{\mathscr{M}}$ are holomorphic functions of z. In a typical case, $\sigma_*(P_i)$ which are used to define the indicial equation $\bar{\mathscr{M}}$ do not depend on x. In this case we usually choose $\bar{A}$ and $\bar{B}$ so that they also do not depend on x. As in Section 3 we fix a point 0 in $Y\times Z$ and discuss the system $\mathscr{M}$ in a neighborhood of 0.

To define the boundary value $\beta(u)$ of a solution $u \in (\mathscr{B}(X_+\times Z))^N$ of $\mathscr{M}$ as in the way in [K–O], we will discuss solutions of the system $\mathscr{N}$ introduced in Theorem 3.2. Put

$$(4.2)\qquad \begin{cases} I(\nu)=\{\mu \in N;\ \nu \leqq \mu \leqq r,\ \lambda_\nu^\circ-\lambda_\mu^\circ \in N^l\}, \\ \lambda_{\mu,j}^\nu(z)=\lambda_{\mu,j}(z)-\lambda_{\mu,j}^\circ+\lambda_{\nu,j}^\circ, \\ R_{\nu,j}(z,\vartheta_j)=\prod_{\mu\in I(\nu)}(\vartheta_j-\lambda_{\mu,j}^\nu(z)) \end{cases}$$

and define the system of equations

$$(4.3)\qquad \mathcal{N}_\nu'\colon\ R_{\nu,j}(z,\vartheta_j)w=0 \qquad (j=1,\cdots,l).$$

Then the ν-th component v_ν of every microfunction solution v of $\mathcal{N}$ satisfies $\mathcal{N}_\nu'$. We assume moreover that

$$(4.4)\qquad \operatorname{Re}\lambda_{\nu,j}(z)\in \boldsymbol{R}_+ \qquad \text{for any } z\in Z\ (\nu=1,\cdots,r;\ j=1,\cdots,l),$$

where for a $C \in \boldsymbol{C}$, Re C means the real part of C. We remark here that this assumption is always satisfied if we consider $u'=t_1^k\cdots t_l^k u$ in place of u with a sufficiently large number $k \in N$ because any component of characteristic exponents of the system satisfied by u' is larger by k than the corresponding one of $\mathcal{M}$. For any j and ν we define a set of hyperfunction solutions $\{w_{\mu,\nu,j}(t_j,z);\ \mu \in I(\nu)\}$ of the single equation

$$(4.5)\qquad \mathcal{N}_{\nu,j}'\colon\ R_{\nu,j}(z,\vartheta_j)w'=0$$

such that $w_{\mu,\nu,j}$ are hyperfunctions of t_j with the holomorphic parameter z and

$$(4.6)\qquad w_{\nu,\nu,j}(t_j,z)=t_j^{\lambda_{\nu,j}(z)} \qquad \text{if } t_j>0,$$

$$(4.7)\qquad w_{\mu,\nu,j}(t_j,z)=0 \qquad \text{if } t_j\leqq 0 \text{ and } \mu\in I(\nu)$$

and that for any $z(0)\in Z$, the solutions $w_{\mu,\nu,j}(t_j,z(0))$ $(\mu\in I(\nu))$ of $\mathcal{N}_{\nu,j}'$ with $z=z(0)$ are linearly independent over $\boldsymbol{C}$. For example we may define

$$(4.8)\qquad w_{\mu,\nu,j}(t_j,z)=t_j^{\lambda_{\mu,j}^\nu(z)}\left(1+\int_0^{t_j} s^{-\lambda_{\mu,j}^\nu(z)-1}w_{\mu',\nu,j}(s,z)ds\right)$$

if $\lambda_\nu^\circ-\lambda_\mu^\circ \in N^l$ and $\mu>\mu'$ and there exists no element ω in $I(\nu)$ with $\mu>\omega>\mu'$. Then for $\alpha\in I(\nu)^l$, we define

$$(4.9)\qquad w_{\alpha,\nu}(t,z)=w_{\alpha_1,\nu,1}(t_1,z)\cdots w_{\alpha_l,\nu,l}(t_l,z).$$

Then putting $(\nu)=(\nu,\cdots,\nu)\in I(\nu)^l$, we have

$$(4.10)\qquad w_{(\nu),\nu}(t,x)=t_+^{\lambda_\nu(z)} \quad \left(=\prod_{j=1}^l t_{j+}^{\lambda_{\nu,j}(z)}\right).$$

Let $\tilde{u}(t, x, z)$ be any hyperfunction solution of the system $\mathcal{M}$ with support contained in $\mathrm{Cl}_X(X_+)\times Z$, where $\mathrm{Cl}_X(X_+)$ denotes the closure of X_+ in X. Then using the $B \in {}_Z\mathscr{E}_*$ in Theorem 3.2, we uniquely define $\phi_{\alpha,\nu}(x, z) \in \mathscr{B}_{Y\times Z}$ $(\alpha \in I(\nu), \nu=1, \cdots, r)$ such that

$$B(t, x, z, D_t, D_x)\,\mathrm{sp}(\tilde{u})=\mathrm{sp}(v(t, x, z)) \tag{4.11}$$

with

$$v(t, x, z)_\nu=\sum_{\alpha\in I(\nu)^l}\phi_{\alpha,\nu}(x, z)w_{\alpha,\nu}(t, z) \tag{4.12}$$

in a neighborhood of $\Lambda=\{(0, x, z; (\sqrt{-1}\sum\tau_i dt_i)\infty) \in \sqrt{-1}\,S^*(X\times Z);$ $\tau_i\neq 0$ for $i=1, \cdots, l\}$ because the ν-th component of the left hand side of (4.11) is a solution of $\mathcal{N}'_\nu$. This is equivalent to say

$$\mathrm{sp}(\tilde{u})=A(t, x, z, D_t, D_x)\,\mathrm{sp}(v(t, x, z)) \tag{4.13}$$

with the $A \in M(N, r; {}_Z\mathscr{E}^{(0)}_*)$ in Theorem 3.5. Then we define the boundary value $\beta(\tilde{u}) \in (\mathscr{B}_{Y\times Z})^r$ of $\tilde{u}$ on the edge Y by

$$\beta(\tilde{u})_\nu=\phi_{(\nu),\nu} \qquad \text{for } \nu=1, \cdots, r. \tag{4.14}$$

We remark that the condition $\beta(\tilde{u})=0$ means $\tilde{u}=0$ in a neighborhood of $Y\times Z$. This is proved as follows: Suppose $\beta(\tilde{u})=0$ and $\mathrm{sp}(\tilde{u})\neq 0$. Let ν the largest number satisfying $\mathrm{sp}(v)_\nu\neq 0$. Since Q_j in Theorem 3.2 are upper triangular matrices, we have $(\vartheta_j-\lambda_{\nu,j}(z))\,\mathrm{sp}(v)_\nu=0$ for $j=1, \cdots, l$. Hence $\mathrm{sp}(v)_\nu=\mathrm{sp}(\psi(x, z)t_+^{\lambda_\nu(z)})$ with a $\psi \in \mathscr{B}_{Y\times Z}$ and therefore $\psi=\phi_{(\nu),\nu}=0$. This contradicts the assumption. Thus if $\beta(\tilde{u})=0$, then $\mathrm{sp}(\tilde{u})=0$ and hence $\tilde{u}=0$ in a neighborhood of $Y\times Z$ because of Holmgren's theorem for hyperfunctions.

Let ${}_Z\mathscr{B}(X_+, \mathcal{M})$ denote the space of all solutions of $\mathcal{M}$ which are functions in $({}_Z\mathscr{B}_X(X_+))^N$. Let $\mathscr{C}'_*(X_+\times Z)$ denote the space of all distributions on $X_+\times Z$ which can be extended to distributions on $X\times Z$ and we put ${}_Z\mathscr{C}'_*(X_+, \mathcal{M})={}_Z\mathscr{B}(X_+, \mathcal{M})\cap(\mathscr{C}'_*(X_+\times Z))^N$. Let u be an element of ${}_Z\mathscr{B}(X_+, \mathcal{M})$. To define the boundary value $\beta(u)$ of u, we use an extension theorem for u to a function $\tilde{u}$ mentioned above. Then we require the following assumption

(4.15) *For any wall Y_j there exist an $m_j \in N$ and a matrix $R_j \in M(N; {}_Z\mathscr{D}_*)$ such that every component of R_j is of order $\leq m_j$ and R_j is of the form*

$$R_j=\bar{R}_j(t, x, z, \vartheta_j)+t_jS_j(t, x, z, \vartheta, D_x)$$

with $[\partial/\partial t_j, \bar{R}_j]=0$ and moreover $\det\,\bar{R}_j(t, x, z, s)$ *is a polynomial of just degree m_jN for any $(t, x, z) \in Y_j\times Z$.*

The condition (4.15) means that the equation $R_j u=0$ has regular singularity along the wall Y_j. Then we have

Lemma 4.1. *Assume* (4.15) *and moreover*

(4.16) det $\bar{R}_j(t, x, z, k)\neq 0$ *for any* $(t, x, z) \in Y_j\times Z$ *and any negative integer* k.

We put $X_j=\{(t, x) \in X;\ t_\nu>0$ *for* $\nu\neq j\}$. *Then for any* $u \in {}_Z\mathscr{B}(X_+, \mathscr{M})$ *there exists a unique* $\tilde{u} \in ({}_Z\mathscr{B}_X(X_j))^N$ *such that*

$$\text{(4.17)}\qquad \begin{cases} \tilde{u}\,|\,X_+\times Z=u, \\ \operatorname{supp}\tilde{u}\subset \text{the closure of } X_+\times Z \text{ in } X_j\times Z, \\ R_j\tilde{u}=0. \end{cases}$$

The extension $\tilde{u}$ *has the following property*:

(4.18) *If* $Pu=0$ *with a* $P \in M(1, N; {}_Z\mathscr{D}^f_*)$ (*i.e.* $P=P(t, x, z, \vartheta, D_x)$), *then* $P\tilde{u}=0$.

(4.19) *If* $u \in {}_Z\mathscr{C}'_*(X_+, \mathscr{M})$, *then* $\tilde{u} \in ({}_Z\mathscr{D}'(X_j))^N$.

Proof. If $N=1$, the theorem is proved in Corollary 4.7 and Corollary 5.11 in [K–O] and Lemma 2.18 in [O–S] and Theorem 3.1 in [O 2]. We put

$$\begin{aligned} R_j&=\sum_{k=0}^{m_j}\sum_{i=0}^{k} S_{i,k}(t, x, z, D_x)t^i\vartheta_j^{m_j-k} \\ &=\sum_{i+k\leqq m_j} S_{i,k}(t, x, z, D_x)D_t^{-i}\Big(\prod_{\nu=1}^{i}(\vartheta_j+\nu)\Big)\vartheta_j^{m_j-k} \end{aligned}$$

with matrices of differential operators $S_{i,k}$ of order $\leqq i$. The assumption says that $S_{0,0} \in M(N; {}_Z\mathscr{A}_X)$ is invertible. Hence there exists an invertible matrix $V \in M(N; {}_Z\mathscr{E}_X^{(0)})$ defined in a neighborhood of $(0; dt_j) \in T^*(X\times Z)$ and satisfies $R_j=V(\vartheta_j^{m_j}-\sum_{i=0}^{m_j-1}T_i\vartheta_j^i)$ with suitable $T_i \in M(N; {}_Z\mathscr{E}_X^{(0)})$. Hence by the same argument in the proof of Theorem 4.5 in [K–O], we can prove that the map

$$\text{(4.20)}\qquad R_j\colon (\mathscr{H}^0_{Y_j\times Z}(\mathscr{B}_{X\times Z}))^N\longrightarrow(\mathscr{H}^0_{Y_j\times Z}(\mathscr{B}_{X\times Z}))^N$$

is bijective, which corresponds to Corollary 4.6 in [K–O]. Then owing to the flabbyness of the sheaf of hyperfunction, we have a unique $\tilde{u} \in (\mathscr{B}(X_j\times Z))^N$ satisfying (4.17) (cf. Corollary 4.7 [K–O]). Since $R_j(\bar{\partial}_{z_k}\tilde{u})=\bar{\partial}_{z_k}R_j\tilde{u}=0$ and $\operatorname{supp}\bar{\partial}_{z_k}\tilde{u}\subset Y_j\times Z$ and since the map (4.20) is injective, we see that $\tilde{u} \in ({}_Z\mathscr{B}(X_j))^N$.

The above is proved also in an elementary way as in the proof of Theorem 3.1 in [O 2] by using Theorem 2.1 in [O 2], which mentions a property of the operation of the adjoint of R_j on the space of holomorphic functions.

The proof of (4.18) is the same as that of Proposition 5.10 and Corollary 5.11 in [K–O]. On the other hand (4.20) holds even if we replace $\mathscr{B}_{X\times Z}$ by $\mathscr{D}'_{X\times Z}$, which is easily obtained by the same way as in the proof of Lemma 2.18 in [O–S] (or Theorem 3.1 and Lemma 3.2 in [O 2] if we use the fact that $\bar{R}_j(t, x, z, k)$ is invertible for any negative integer k). This assures (4.19). Q.E.D.

This lemma immediately implies the following (cf. the proof of Theorem 5.12 in [K–O] and the argument before the Proposition 2.20 in [O–S]):

Theorem 4.2. *Assume* (4.15) *and* (4.16) *for* $j=1, \cdots, l$. *Then for any* $u \in {}_Z\mathscr{B}(X_+, \mathscr{M})$ *there exists a unique* $\tilde{u} \in {}_Z\mathscr{B}(X)$ *such that*

$$\begin{cases} \tilde{u}\,|\,X_+\times Z=u, \\ \operatorname{supp}\tilde{u}\subset \mathrm{Cl}_X(X_+)\times Z, \\ Q\tilde{u}=0 \text{ for any } Q \in ({}_Z\mathscr{D}^f_*)^N \text{ with } Qu=0. \end{cases} \tag{4.21}$$

Moreover if $\tilde{u} \in {}_Z\mathscr{C}'_*(X_+, \mathscr{M})$, *then* $\tilde{u} \in ({}_Z\mathscr{D}'(X))^N$.

Now we will define the boundary value $\beta(u)$ of $u \in {}_Z\mathscr{B}(X_+, \mathscr{M})$ when $\mathscr{M}$ satisfies the condition (4.15). Considering $t_1^k \cdots t_l^k u$ with a large positive integer k if necessary, we can assume the both conditions (4.4) and (4.16). Then we have a unique extension $\tilde{u}$ satisfying (4.21). Then by (4.11) and (4.12) and (4.14), we define $\beta(u)=\beta(\tilde{u})(\in ({}_Z\mathscr{B}(Y))^r)$. Moreover if $u \in {}_Z\mathscr{C}'_*(X_+, \mathscr{M})$, then $\beta(u) \in ({}_Z\mathscr{D}'(Y))^r$ because of Lemma 2.19 in [O–S] (cf. Proposition 2.20 in [O–S]).

Definition 4.3. *Assume that the system* $\mathscr{M}$ *with regular singularities along the set of walls* $\{Y_1, \cdots, Y_l\}$ *satisfies the condition* (4.15). *Then we can define the map of taking the boundary values*

$$\begin{array}{ccc} \beta\colon {}_Z\mathscr{B}(X_+, \mathscr{M}) & \longrightarrow & ({}_Z\mathscr{B}(Y))^r \\ \cup & & \cup \\ {}_Z\mathscr{C}'_*(X_+, \mathscr{M}) & \longrightarrow & ({}_Z\mathscr{D}'(Y))^r \end{array} \tag{4.22}$$

in the way mentioned above. The ν*-th boundary value* $\beta(u)_\nu$, *which will be denoted by* $\beta_\nu(u)$, *is called the boundary value of* u *with respect to the characteristic exponent* $\lambda_\nu(z)$.

If a function $u \in ({}_Z\mathscr{B}_X)^N$ satisfies $R_j u=0$ with the R_j in (4.15), then supp sp(u) is contained in the set $\{(t, x, z; \sqrt{-1}(\sum \xi_i dx_i + \sum \tau_j dt_j)\infty) \in \sqrt{-1}\, T^*X \times Z;\ \det(\sigma_{m_j}(R_j))(t, x, z, \tau, \xi)=0\}$. Hence by the argument in Proposition 2.15 in [O–S] proves

Theorem 4.4. *We fix a $z(0) \in Z$. Then there exists a neighborhood V of Y in X which has the following property: If a function $u \in {}_Z\mathscr{B}(X_+, \mathscr{M})$ satisfies $\beta(u(t, x, z(0)))=0$, then $u(t, x, z(0))$ is identically zero in $V \cap X_+$.*

Now we consider the coordinate dependence of the definition of boundary values. We only consider the coordinate transformation $(t', x') \mapsto (t, x)$ of the form $t'_j = c_j(t, x) t_j$ with $c_j(t, x) \in \mathscr{A}_X$ satisfying $c_j > 0$. Hence Y_j (resp. X_+) are also defined by $t'_j=0$ (resp. $t'_1>0, \cdots, t'_l>0$). Define the line bundle over $Y \times Z$

$$L_{\lambda_\nu} = (T^*_{Y_1 \times Z}(X \times Z))^{\otimes \lambda_{\nu,1}(z)} \underset{Y}{\otimes} \cdots \underset{Y}{\otimes} (T^*_{Y_l \times Z}(X \times Z))^{\otimes \lambda_{\nu,l}(z)}. \tag{4.23}$$

Then the following theorem gives a sufficient condition for $\beta_\nu(u)$ to be a hyperfunction section of $L_{\lambda_\nu(z)}$ (cf. Theorem 5.8 in [K–O]):

Theorem 4.5. *We fix a $\nu \in \{1, \cdots, r\}$ and define*

$$I(\nu)' = \{\mu \in N;\ \nu < \mu \leqq r \text{ and } \lambda^\circ_\nu - \lambda^\circ_\mu \in N^l - \{0\}\}$$
$$\cup \{\mu \in N;\ \nu < \mu \leqq r,\ \lambda^\circ_\nu = \lambda^\circ_\mu \text{ and } \bar{Q}_{j,\nu\mu}(z) \neq 0 \text{ with a suitable } j\}$$

and

$${}_V\mathscr{B}(X_+, \mathscr{M})_\nu = \{u \in {}_V\mathscr{B}(X_+, \mathscr{M});\ \beta_\mu(u)=0 \text{ for } \mu \in I(\nu)'\}$$

for a small neighborhood V of 0 in Z.

(i) *The definition ${}_V\mathscr{B}(X_+, \mathscr{M})_\nu$ does not depend on the choice of local coordinate systems.*

(ii) *For any $u \in {}_V\mathscr{B}(X_+, \mathscr{M})_\nu$, the hyperfunction valued section*

$$\beta_\nu(u)(dt_1)^{\lambda_{\nu,1}(z)} \cdots (dt_l)^{\lambda_{\nu,l}(z)} \tag{4.24}$$

of L_{λ_ν} depends neither on the choice of local coordinate system nor on generators $P_1, \cdots, P_M$ to define $\bar{\mathscr{M}}$. It depends only on the ${}_Z\mathscr{A}_Y[s]$-Module $\bar{\mathscr{M}}$ and its basis $\bar{v}_1, \cdots, \bar{v}_r$.

(iii) *Let $a(t, x, z)$ be a non-vanishing function in ${}_Z\mathscr{A}_X$. For any $P \in {}_Z\mathscr{D}_X$ and any $f \in {}_Z\mathscr{B}(X_+, \mathscr{M})_\nu$, we put $P^a = aPa^{-1}$ and denote by $\beta^a_\nu(af)$ the ν-th boundary value of af which is defined by replacing P_i and P'_j by P^a_i and P'^a_j, respectively. If we do not change $\bar{\mathscr{M}}$ and $\bar{v}_1, \cdots, \bar{v}_r$, then*

$$\beta^a_\nu(af) = a(0, x, z)\beta_\nu(f). \tag{4.25}$$

We remark here that $\overline{\mathscr{M}}$ is defined by $({}_Z\mathscr{A}_Y[s])^N/\sum_{i=1}^M {}_Z\mathscr{A}_Y[s]\sigma_*(P_i)$ with suitable $P_1, \cdots, P_M \in {}_Z\mathscr{D}_*$ satisfying $P_i u=0$ and that $\overline{\mathscr{M}}$ is a locally free ${}_Z\mathscr{A}_Y$-Module of rank r (*cf* § 3).

Define the ${}_Z\mathscr{A}_Y[s]$-Module

$$\overline{\mathscr{N}}_\nu\colon (s_j-\lambda_{\nu,j}(z))\bar{w}_\nu=0 \quad (j=1, \cdots, l) \tag{4.26}$$

with a free basis $\bar{w}_\nu$ over ${}_Z\mathscr{A}_Y$ and put

$$\mathscr{N}_\nu\colon (\vartheta_j-\lambda_{\nu,j}(z))w_\nu=0 \quad (j=1, \cdots, l) \tag{4.27}$$

with one unknown w_ν. Let $\overline{\mathscr{M}}_\nu$ be the quotient ${}_Z\mathscr{A}_Y[s]$-Module of $\overline{\mathscr{M}}$ defined by

$$\bar{v}_\mu=0 \qquad \text{for } \mu \in I(\nu)' \tag{4.28}$$

and let $\mathscr{M}_\nu$ be the quotient ${}_Z\mathscr{E}_*$-Module of ${}_Z\mathscr{E}_*\otimes\mathscr{M}=({}_Z\mathscr{E}_*)^N u$ defined by

$$\Phi(v_\mu)=0 \qquad \text{for } \mu \in I(\nu)' \tag{4.29}$$

with Φ and v_ν in Theorem 3.2. Let γ (resp. $\bar{\gamma}$) denote the natural projection of ${}_Z\mathscr{E}_*\otimes\mathscr{M}$ (resp. $\overline{\mathscr{M}}$) onto $\mathscr{M}_\nu$ (resp. $\overline{\mathscr{M}}_\nu$). We note that ${}_Y\mathscr{B}(X_+, \mathscr{M})_\nu =\{f \in {}_Y\mathscr{B}(X_+, \mathscr{M});\ \mathrm{sp}(\tilde{f}) \text{ is a solution of } \mathscr{M}_\nu\}$. To prove Theorem 4.5 we prepare the following

Lemma 4.6. (i) *The definition of $\mathscr{M}_\nu$ depends neither the choice of local coordinate system nor the choice of $\mathscr{N}$ in Theorem* 3.2

(ii) *There exists a homomorphism of $\mathscr{N}_\nu$ to $\mathscr{M}_\nu$ which is defined by $w_\nu=C_\nu(t, x, z, D_t, D_x)\gamma u$ with a $C_\nu \in M(1, N; {}_Z\mathscr{E}_*)$ and uniquely determined by the condition*

$$\bar{\gamma}(\bar{v}_\nu)=\sigma_*(C_\nu)\bar{\gamma}\bar{u}. \tag{4.30}$$

(iii) *Assume* (4.4) *and* (4.16). *For any $f \in {}_Y\mathscr{B}(X_+, \mathscr{M})_\nu$, let $\tilde{f}$ be the extension in Theorem* 4.2. *Then* $\mathrm{sp}(\tilde{f})$ *is a solution of $\mathscr{M}_\nu$ and*

$$\mathrm{sp}(\beta_\nu(f)(x, z)t_+^{\lambda_\nu(z)})=C_\nu(t, x, z, D_t, D_x)\ \mathrm{sp}(\tilde{f}). \tag{4.31}$$

Proof. Put $I(\nu)''=\{\nu+1, \nu+2, \cdots, r\}$. Only in the proof of (i) and (ii), we denote by $\overline{\mathscr{M}}_\nu$ and $\mathscr{M}_\nu$ the corresponding ones which are obtained when we replace $I(\nu)'$ by $I(\nu)''$ in the definition. Then we will prove (i) and (ii) by the induction on the number of the elements of $I(\nu)''$. We remark that without changing the condition (3.8) for Q_j, we can assume $I(\nu)'=I(\nu)''$ by changing the order of the basis $\{v_1, \cdots, v_r\}$.

Assume $\nu=r$. Then $\overline{\mathscr{M}}_\nu=\overline{\mathscr{M}}$ and $\mathscr{M}_\nu={}_Z\mathscr{E}_*\otimes\mathscr{M}$ and therefore (i) is trivial. Put C_ν the last row of B in Theorem 3.2. Then C_ν satisfies (4.30) and the correspondence $w_\nu \mapsto C_\nu u$ clearly defines a ${}_Z\mathcal{E}_*$-homomorphism of $\mathcal{N}_\nu$ to $\mathcal{M}_\nu$. Moreover the same argument as in the proof of Theorem 3.2 (iii), which is based on Theorem 2.4, proves the uniqueness assertion in the lemma.

Assume $\nu<r$. Then by the hypothesis of the induction, we see that the definition of $\mathcal{M}_{\nu+1}$ and the image of the homomorphism do not depend on the choice of coordinate systems. This proves (i) because $\mathcal{M}_\nu$ is naturally isomorphic to the quotient of $\mathcal{M}_{\nu+1}$ by the image. Define $Q_j^\nu \in M(\nu; {}_Z\mathcal{E}_*^{(0)})$ by $Q_{j,\mu\mu'}^\nu = Q_{j,\mu\mu'}$ for $\mu, \mu'=1, \cdots, \nu$. Let $v(\nu)$ denote the column vector formed by the residue classes of $\Phi(v_1), \cdots, \Phi(v_\nu)$ in $\mathcal{M}_\nu$. Then $\vartheta_j v(\nu) = Q_j^\nu v(\nu)$ for $j=1, \cdots, l$. Since Q_j^ν satisfy the same property as (3.8), we can prove (ii) by the same argument as in the case when $\nu=r$.

The statement in (iii) clearly follows from (ii) and the definition of the boundary value $\beta_\nu(f)$. Q.E.D.

Proof of Theorem 4.5. Lemma 4.6 (i) clearly implies Theorem 4.5 (i).

First assume the conditions (4.4) and (4.16). Then Lemma 5.9 in [K–O] and Lemma 4.6 prove Theorem 4.5 (ii) (cf. the proof of Theorem 5.8 in [K–O]). On the other hand it is clear that $\widetilde{af}=a\tilde{f}$ in (iii). Then if we put $C_\nu^a = a(0, x, z) C_\nu a^{-1}$ with the C_ν in Lemma 4.6 (ii), then

$$\operatorname{sp}(\beta_\nu^a(af) t_+^{\lambda_\nu(z)}) = C_\nu^a \operatorname{sp}(a\tilde{f}).$$

This follows from Lemma 4.6 (ii). Since

$$C_\nu^a \operatorname{sp}(a\tilde{f}) = a(0, x, z) C_\nu a^{-1} \operatorname{sp}(a\tilde{f}) = a(0, x, z) \operatorname{sp}(\beta_\nu(f) t_+^{\lambda_\nu(z)}),$$

we have (ii).

Now we consider the general case. For a $k \in Z$, we put $t^k = t_1^k \cdots t_l^k$. Let k be a sufficient large positive integer so that $u(k) = t^k u$ satisfies the conditions (4.4) and (4.16). Then P_i, $P'_{i'}$, R_j and s_j change into $P_i(k) = t^k P_i t^{-k}$, $P'_{i'}(k) = t^k P'_{i'} t^{-k}$, $R_j(k) = t^k R_j t^{-k}$ and $s_j(k) = s_j - k$, respectively. Let (t', x') be another coordinate system. Then $t'_j = c_j(t, x) t_j$ with $c_j>0$. Hence under the coordinate system (t', x'), $u(k)$ changes into $u(k)' = au(k)$. Here we put $a = c_1(t, x)^k \cdots c_l(t, x)^k$. Moreover P_i, $P'_{i'}$, R_j and s_j change into $P_i(k)' = aP_i(k)a^{-1}$, $P'_{i'}(k)' = aP'_{i'}(k)a^{-1}$ and $R_j(k)' = aR_j(k)a^{-1}$ and $s_j(k)' = s_j(k)$, respectively. Hence under the coordinate system (t, x) we have $\beta_\nu^a(t'^k u) = a(0, x)\beta_\nu(t^k u)$ owing to what we have just proved. Let β'_ν denote the map of taking the ν-th boundary value under the coordinate system (t', x'). Then we have

$$\beta_\nu^a(t'^k u) = \beta_\nu'^a(t'^k u) c_1(0, x)^{\lambda_{\nu,1}(z)+k} \cdots c_l(0, x)^{\lambda_{\nu,l}+k}.$$

Since $\beta'_\nu(u)=\beta'^a_\nu(t'^k u)$ and $\beta_\nu(u)=\beta_\nu(t^k u)$ by definition, we have the statement in (ii). Moreover the same argument as in Lemma 5.13 in [K–O] assures that the definition of $\beta_\nu(u)$ does not depend on the choice of k.

The statement in (iii) is clear from (ii) and the same statement under the conditions (4.4) and (4.16). Q.E.D.

Corollary 4.7. *Let $\mathcal{M}''$: $P''_i u''=0$ $(i=1, \cdots, M'')$ be another system of differential equations with regular singularities along $\{Y_1, \cdots, Y_l\}$, where P''_i are in $({}_Z\mathcal{D}_X)^N$ and u'' is a vector of N unknown functions. Suppose $\mathcal{M}''$ has the same indicial equation $\bar{\mathcal{M}}$ and also satisfies the conditions to be able to define the boundary values of the solutions. For any $f \in {}_Y\mathcal{B}(X_+, \mathcal{M})_\nu$, the ν-th boundary value of f defined by using $\mathcal{M}''$, $\bar{\mathcal{M}}$ and $\bar{v}_1, \cdots, \bar{v}_r$ coincides with $\beta_\nu(f)$ iff f is also a solution of $\mathcal{M}''$.*

In fact, to prove this we may assume $M''\geqq M$ and $P'_i=P''_i$ for $i=1, \cdots, M'$ (cf. (4.1)). Then the corollary is clear from Lemma 4.6.

§ 5. Ideally analytic solutions

The concept of ideally analytic solutions is introduced by [K–O]. We will generalize this in our situation. We will use the same notation and consider the same system $\mathcal{M}$ with regular singularities as in Section 4.

Definition 5.1. *A solution $u(t, x, z) \in {}_Z\mathcal{B}(X_+, \mathcal{M})$ is called ideally analytic if and only if $\beta(u)=({}_Z\mathcal{A}_Y)^r$.*

Let π denote the natural projection of $(\sqrt{-1}S^*X)\times Z$ (or $(T^*X)\times Z$) to $X\times Z$. We identify $(\sqrt{-1}S^*_YX)\times Z$ and $(T^*_YX)\times Z$ with $\sqrt{-1}S^*_{Y\times Z}(X\times Z)$ and $T^*_{Y\times Z}(X\times Z)$, respectively. For a coherent left ${}_Z\mathcal{D}^f_*$-Module $\mathcal{M}'$ we put

$$\text{SS}\,\mathcal{M}'=\{p \in (T^*X)\times Z;\ ({}_Z\mathcal{E}_X)_p\otimes\pi^{-1}\mathcal{M}'\neq 0\}, \tag{5.1}$$

where $({}_Z\mathcal{E}_X)_p$ is the stalk of ${}_Z\mathcal{E}_X$ at p. Then we have

Theorem 5.2. *For the system $\mathcal{M}=({}_Z\mathcal{D}_X)^N/\mathcal{J}$, we put*

$$\mathcal{M}'=({}_Z\mathcal{D}^f_*)^N/(\mathcal{J}\cap({}_Z\mathcal{D}^f_*)^N).$$

Then if

$$\text{SS}\,\mathcal{M}'\,|\,\pi^{-1}(Y\times Z)\subset(T^*_YX)\times Z, \tag{5.2}$$

all the solutions in ${}_Z\mathcal{B}(X_+, \mathcal{M})$ are ideally analytic.

Proof. It follows from the definition of the boundary value that

$u \in {}_Z\mathscr{B}(X_+, \mathscr{M})$ is ideally analytic if and only if

$$\text{supp}\,(\text{sp}\,(\widetilde{t^k u}))\,|\,\pi^{-1}(Y\times Z)\subset(\sqrt{-1}\,S_Y^*X)\times Z, \tag{5.3}$$

where k is a non-negative integer and $\widetilde{t^k u}$ is the extension which is used to define the boundary value. On the other hand it follows from Sato's fundamental theorem (Chapter II, Theorem 2.1.1 in [S–K–K]) that (5.2) implies (5.3). Q.E.D.

For simplicity we assume hereafter that Z is a point, which means that we do not consider holomorphic parameters. Hence the characteristic exponents are elements of $\boldsymbol{C}^l$. Put $\{\lambda_1\}\cup\cdots\cup\{\lambda_r\}=\{\lambda'_1, \cdots, \lambda'_{r'}\}$ so that the condition $i\neq j$ implies $\lambda'_i\neq\lambda'_j$.

Theorem 5.3. (i) *Assume a solution $u(t, x) \in \mathscr{B}(X_+, \mathscr{M})$ is locally of the form*

$$u(t, x)=\sum_j a_j(t, x)t^{c_j}q_j(\log t) \tag{5.4}$$

near the edge Y, where the sum is finite and $a_j \in M(N, 1; \mathscr{A}_X)$, $c_j \in \boldsymbol{C}^l$, $q_j \in \boldsymbol{C}[\log t]$ and $\log t=(\log t_1, \cdots, \log t_l)$. Then u is ideally analytic.

(ii) *There exist finite number of polynomials $q_{i,j}(\log t) \in \boldsymbol{C}[\log t]$ such that any ideally analytic solution $u \in \mathscr{B}(X_+, \mathscr{M})$ has the expression*

$$u(t, x)=\sum_{i,j} a_{i,j}(t, x)t^{\lambda'_i}q_{i,j}(\log t) \tag{5.5}$$

with suitable $a_{i,j} \in M(N, 1; \mathscr{A}_X)$ near the edge.

Let $u(t, x)$ be the ideally analytic solution of the form (5.5). *Fix a $\nu \in \{1, \cdots, r\}$ and also fix a $k \in \{1, \cdots, r'\}$ so that $\lambda'_k=\lambda_\nu$. If $\beta_\mu(u)=0$ for any μ with $\lambda_\nu-\lambda_\mu \in \boldsymbol{N}^l$, then $a_{i,j}=0$ for any i and j satisfying $\lambda'_k-\lambda'_i \in \boldsymbol{N}^l$. On the other hand, if $\beta_\nu(u)\neq 0$ and $\beta_\mu(u)=0$ for any $\mu \in \{1, \cdots, r\}$ with $\lambda'_k-\lambda_\mu \in \boldsymbol{N}^l-\{0\}$, then $\sum_j a_{k,j}(0, x)q_{k,j}(\log t)$ is not identically zero. More precisely, if $u \in \mathscr{B}(X_+, \mathscr{M})_\nu$ (cf. Theorem* 4.5), *we have*

$$\beta_\nu(u)(x)=\sum_j \bar{B}(x, \vartheta+\lambda'_k)_\nu a_{k,j}(0, x)q_{k,j}(\log t). \tag{5.6}$$

To prove the theorem we prepare

Lemma 5.4. *Let L be a positive integer and let C_j be upper triangular matrices in $M(L; \boldsymbol{C})$ $(j=1, \cdots, l)$. Suppose any diagonal component of C_j is not any negative integer. Put*

$$\mathscr{N}_0\colon \begin{cases} \dfrac{\partial}{\partial x_i}u_0=0 & \text{for } i=1, \cdots, n, \\ \vartheta_j u_0=C_j u_0 & \text{for } j=1, \cdots, l. \end{cases} \tag{5.7}$$

Then for any $P \in (\mathscr{E}_*^f)^L$, *we have*

$$Pu_0 = p(t, x)u_0 \tag{5.8}$$

with a suitable $p \in (\mathscr{A}_X)^L$. *Moreover if* $P \equiv P' \bmod \sum_i (\mathscr{E}_*^f)^L \partial/\partial x_i$ *with a* $P' \in \mathscr{E}_*$, *then we can choose* p *so that* $p(0, x) = \sum_{\alpha \in N^l} p_\alpha(x) C_1^{\alpha_1} \cdots C_l^{\alpha_l}$ *with* $\sigma_*(P) = \sum_{\alpha \in N^l} p_\alpha(x) s^\alpha$.

Proof. By Späth's theorem for micro-differential operators, we have

$$P = \sum_i S_i \partial/\partial x_i + \sum_j S_j'(\vartheta_j - C_j) + R(x, D_t)$$

with S_i, S_j' and $R \in (\mathscr{E}_*^f)^L$. Since $R \in (\mathscr{E}_*^f)^L$, R is of the form $R = \sum_{\alpha \in N^l} R_\alpha(x) D_t^{-\alpha}$ with the locally uniform estimate in a complex neighborhood of Y:

$$\lim_{|\alpha| \to \infty} \sqrt[|\alpha|]{|R_\alpha(x)|/|\alpha|!} < \infty.$$

This shows that $\sum_{\alpha \in N^l} t^\alpha R_\alpha(x) \prod_{j=1}^l \prod_{k=1}^{\alpha_j} (C_j + k)^{-1}$ converges to an element $p(t, x)$ of $(\mathscr{A}_X)^L$, which implies the lemma. In fact, by the equation

$$D_{t_j} t_j^k u_0 = k t_j^{k-1} u_0 + t_j^{k-1} \vartheta_j u_0 = (C_j + k) t_j^{k-1} u_0,$$

we have $D_{t_j}^{-1} t_j^{k-1} u_0 = (C_j + k)^{-1} t^k u_0$ and therefore $Ru_0 = p(t, x)u_0$. Q.E.D.

Proof of Theorem 5.3. Considering $t^k u$ if necessary, we may assume (4.4) and (4.16).

(i) We may moreover assume $\operatorname{Re} c_j \in \boldsymbol{R}_+^l$. Then $u(t, x)Y(t)$ $(= \sum_j a_j t_+^{c_j} q_j(\log t))$ is well-defined, where $Y(t)$ is the product of Heaviside's functions $Y(t_1), \cdots, Y(t_l)$. Since $\vartheta_j t_+^{c_j} = c_j t_+^{c_j}$, $Pu(t, x)$ has the similar expression $\sum_j a_j'(t, x) t_+^{c_j'} q_j'(\log t)$ for any $P \in \mathscr{D}_*^f$. Here $\{c_j'\} = \{c_j\}$. Hence if $Pu = 0$ for a $P \in M(1, N; \mathscr{D}_*^f)$, then $P(uY(t)) = 0$, which means $uY(t)$ is the extension $\tilde{u}$ used to define the boundary value. Since $\operatorname{supp} \operatorname{sp}(uY(t)) \subset \sqrt{-1}\, S_Y^* X$, we see that u is ideally analytic.

(ii) Let $P \in \mathscr{E}_*$. Then for any $a(t, x) \in \mathscr{A}_X$ and $\alpha \in N^l$, we can find $a_\beta(t, x) \in \mathscr{A}_X$ such that

$$P \operatorname{sp}(a(t, x)(\log t)^\alpha t_+^{\lambda_\nu}) = \sum_{i=1}^{l} \sum_{\beta_i=0}^{\alpha_i} a_\beta(t, x) \vartheta_1^{\beta_1} \cdots \vartheta_l^{\beta_l} \operatorname{sp}((\log t)^\beta t_+^{\lambda_\nu}) \tag{5.9}$$

and

$$\sum_\beta a_\beta(0, x) \vartheta^\beta (\log t)^\beta t_+^{\lambda_\nu} = \sigma_*(P)(x, \vartheta) a(0, x)(\log t)^\alpha t_+^{\lambda_\nu}. \tag{5.10}$$

In fact, $\operatorname{sp}((\log t)^\alpha t_+^{\lambda_\nu})$ satisfies the system

$$(5.11)\qquad \begin{cases} \dfrac{\partial}{\partial x_i} w(\alpha)=0 & \text{for } i=1,\cdots,n, \\ (\vartheta_j-\lambda_{\nu,j})^{\alpha_j+1} w(\alpha)=0 & \text{for } j=1,\cdots,l. \end{cases}$$

Using the matrix equation for $\{\vartheta_j^\beta w(\alpha);\ \beta\in N^l,\ 0\leqq\beta_j\leqq\alpha_j \text{ for } j=1,\cdots,l\}$, we have

$$(5.12)\qquad Pa(t,x)w(\alpha)=\sum_{i=1}^{l}\sum_{\beta_i=0}^{\alpha_i} a_\beta(t,x)\vartheta^\beta w(\alpha)$$

by Lemma 5.4 with suitable $a_\beta\in\mathscr{A}_X$ satisfying (5.10). This means (5.9).

In the definition of $\beta(u)$ we may assume $w_{\alpha,\nu}(t)$ (cf. (4.9)) are contained in $\{(\log t)^\beta t_+^{\lambda_\nu};\ \beta\in N^l\}$. Then

$$(5.13)\qquad (B(t,x,D_x,D_t)\,\mathrm{sp}(\tilde{u}))_\nu=\sum_\alpha \psi_{\alpha,\nu}(x)\,\mathrm{sp}((\log t)^\alpha t_+^{\lambda_\nu}).$$

Here the sum is finite and we remark that $\psi_{\alpha,\nu}\in\mathscr{A}_Y$ because u is ideally analytic. Since $\mathrm{sp}(\tilde{u})=AB\,\mathrm{sp}(\tilde{u})$ with the $A\in M(N,r;\mathscr{E}_*^{(0)})$ in Theorem 3.2 and since for an $L\in N$ the space spanned by $\{\mathrm{sp}((\log t)^\alpha t_+^{\lambda_\nu});\ 1\leqq\alpha_i\leqq L,\ 1\leqq i\leqq l\}$ over C is invariant by the maps ϑ_j, the above argument proves

$$(5.14)\qquad \mathrm{sp}(\tilde{u})=\sum_{\alpha,j} a_{\alpha,j}(t,x)\,\mathrm{sp}((\log t)^\alpha t_+^{\lambda'_j})$$

with suitable $a_{\alpha,j}\in M(N,1;\mathscr{A}_X)$. Then by Holmgren's theorem for hyperfunctions we have (5.5) in the intersection of X_+ and a neighborhood of the edge Y. Moreover the above argument also proves Theorem 5.3 (ii) except for (5.6).

We note that the systems of the equations $\mathscr{M}'$: $\sigma_*(P_i)(x,\vartheta)\bar{u}=0$ $(i=1,\cdots,M)$ and $\mathscr{N}$: $(\vartheta_j-\sigma_*(Q_j)(x,\vartheta))\bar{v}=0$ $(j=1,\cdots,l)$ are equivalent by the correspondence $\bar{u}=\sigma_*(A)(x,\vartheta)\bar{v}$ and $\bar{v}=\sigma_*(B)(x,\vartheta)\bar{u}$. We fix a $\nu\in\{1,\cdots,r\}$ and assume $u\in\mathscr{B}(X_+,\mathscr{M})_\nu$ and define $v(t,x)\in(\mathscr{C}_X)^r$ by

$$(5.15)\qquad v(t,x)_{\mu'}=\begin{cases} 0 & \text{if } \lambda_{\mu'}\neq\lambda_\nu, \\ B(\mathrm{sp}(\tilde{u}))_{\mu'} & \text{if } \lambda_{\mu'}=\lambda_\nu \end{cases}$$

for $\mu'\in\{1,\cdots,r\}$. Then the condition (3.8) assures that $v(t,x)_\nu=\beta_\nu(u)(x)\,\mathrm{sp}(t_+^{\lambda_\nu})$ and $v(t,x)$ is a solution of the system $\mathscr{N}$, which implies $\sigma_*(B)_\nu(x,\vartheta)\sigma_*(A)(x,\vartheta)v(t,x)=\beta_\nu(u)(x)\,\mathrm{sp}(t_+^{\lambda_\nu})$. Combining this with (5.9), (5.10), (5.13), (5.15) and the equation $\mathrm{sp}(\tilde{u})=A(t,x,D_x,D_t)(B\,\mathrm{sp}(\tilde{u}))$, we have (5.6) because $\sigma_*(A)(x,\vartheta)v(t,x)=\sum_j a_{k,j}(0,x)t_+^{\lambda'_k}q_{k,j}(\log t)$ under the notation in the last part of Theorem 5.3. Q.E.D.

Now we will review the definition of boundary values in a simple case. Hence suppose Z is a point and

$$\lambda_\mu - \lambda_\nu \notin N^l - \{0\} \qquad \text{for } \mu, \nu = 1, \cdots, r \tag{5.16}$$

and

$$\bar{Q}_j \text{ are diagonal matrices.} \tag{5.17}$$

Moreover suppose that $\sigma_*(P_i)$ do not depend on x. Here P_i are the operators used to define the indicial equation. Consider the system of differential equations

$$\sigma_*(P_i)(\vartheta)u = 0 \qquad (i = 1, \cdots, M). \tag{5.18}$$

We remark that (5.17) is equivalent to say that there exist r independent solutions of the form

$$u(\nu) = S_\nu^\circ t^{\lambda_\nu} \tag{5.19}$$

with suitable $S_\nu^\circ \in M(N, 1; C)$. We arbitrarily fix $u(\nu)$ and define $\bar{A} \in M(N, r; C)$ so that the ν-th column of $\bar{A}$ coincides with S_ν° $(\nu = 1, \cdots, r)$. We can find a $\bar{B} \in M(N, r; C[s])$ so that $\bar{A}\bar{B}(\vartheta) \sum c_\nu u(\nu) = \sum c_\nu u(\nu)$ for any $c = (c_1, \cdots, c_r) \in C^r$. By using these $\bar{A}$ and $\bar{B}$, we can define the map

$$\beta_\nu : \mathscr{B}(X_+, \mathscr{M}) \longrightarrow \{\text{hyperfunction sections of } L_{\lambda_\nu}\} \tag{5.20}$$

which has the following property: If $\beta_\nu(u) = 0$ for any $\nu = 1, \cdots, r$, then $u = 0$ in a neighborhood of the edge. Put $\{\lambda_1, \cdots, \lambda_r\} = \{\lambda'_1, \cdots, \lambda'_{r'}\}$ so that $\lambda'_i \neq \lambda'_j$ if $i \neq j$. If u is ideally analytic (i.e. all $\beta_\nu(u)$ are analytic), then

$$u = \sum_{i=1}^{r'} S_i(t, x) t^{\lambda'_i} \tag{5.21}$$

with suitable $S_i \in M(N, 1; \mathscr{A}_X)$ which satisfies

$$\sum_{i=1}^{r'} S_i(0, x) t^{\lambda'_i} = \sum_{\nu=1}^{r} \beta_\nu(u) S_\nu^\circ t^{\lambda_\nu}. \tag{5.22}$$

We remark last that for each i, the equation $S_i(0, x) \equiv 0$ implies that $S_i(t, x)$ is identically zero.

§ 6. Induced equations

In this last section, we will discuss differential equations which are satisfied by the boundary values defined in Section 4. We use the same notation and the system $\overline{\mathscr{M}}$ with regular singularities as in Section 4. Especially we use the same notation $\overline{\mathscr{J}}$, $\overline{\mathscr{M}}$, $\overline{\mathscr{N}}$, $\overline{A}$, $\overline{B}$, $\overline{Q}_j$ and $\mathscr{N}$, A, B, Q_j and the characteristic exponents $\lambda_1, \cdots, \lambda_r$.

For example, for any solution of $\mathscr{M}$, we can define its boundary value on every wall. Then the boundary value satisfies many differential equations and the system of the equations also has regular singularities. It is very important to consider such induced equations. One of the application of the result here will be found in [MaO].

Theorem 6.1. *Fix a solution* $u(t, x, z) \in {}_Z\mathscr{B}(X_+, \mathscr{M})$ *and an index* $\nu \in \{1, \cdots, r\}$. *Denoting by* $G \in M(r; Z)$ *the diagonal matrix whose i-th component equals* 1 *if* $\lambda_i^\circ = \lambda_\nu^\circ$ *and* 0 *otherwise, put* $\beta_*(u) = G\beta(u)$. *Suppose*

(6.1) $\quad \beta_i(u) = 0$ *for any* $i \in \{1, \cdots, r\}$ *with* $\lambda_\nu^\circ - \lambda_i^\circ \in N^l - \{0\}$.

(i) *Let* $P(t, x, z, \vartheta, D_x) \in ({}_Z\mathscr{D}_*^f)^N$ *with* $Pu = 0$. *Then* $\beta_*(u)$ *is a solution of the system*

$$\text{(6.2)} \qquad \begin{cases} P(0, x, z, s, D_x)\overline{A}(x, z)s^\alpha w = 0 & (\alpha \in N^l), \\ s_j w = \overline{Q}_j(x, z)w & (j = 1, \cdots, r). \end{cases}$$

Especially when $\lambda_i^\circ \neq \lambda_\nu^\circ$ *for any* $i \in \{1, \cdots, r\} - \{\nu\}$, *we have*

$$\text{(6.3)} \qquad P(0, x, z, \lambda_\nu(z), D_x)\overline{A}(x, z)\beta_*(u) = 0.$$

(ii) *Let* $P(t, x, z, \vartheta, D_x) \in M(N; {}_Z\mathscr{D}_*^f)$. *Suppose* $Pu = 0$ *and*

$$P(0, x, z, s, D_x) = Q(x, z, s, D_x)I_N$$

with a scalar $Q \in {}_Z\mathscr{D}_Y[s]$ *and moreover suppose* $\overline{B}$ *does not depend on* x. *Then*

$$\text{(6.4)} \qquad Q(x, z, \lambda_\nu(z), D_x)\beta_\nu(u) = 0.$$

Proof. We may assume (4.4) and (4.16). Then the statement in (i) immediately follows from the definition of the boundary value and Theorem 3.4 with $\gamma = 0$ and $\phi = s^\alpha$.

Since $\overline{B}(z, s)\overline{A}(x, z) \equiv I_r \bmod \sum_j {}_Z\mathscr{A}_Y[s](s_j - \overline{Q}_j)$, we have (6.4) by considering $\overline{B}(z, \vartheta)_i\, P$ in place of P satisfying the condition $\lambda_i^\circ = \lambda_\nu^\circ$. Here $\overline{B}(z, s)_i$ denotes the i-th row of $\overline{B}(z, s)$. Q.E.D.

To define boundary values we assumed (4.15), which assures that the system $\overline{\mathscr{M}}$ has regular singularities along any subset of the walls (at least after the coordinate transformation $t_j \mapsto t_j^k$ with a suitable $k \in N_+$). Hence it is natural to discuss the boundary value problem corresponding to this subset of the original walls and the induced equations for the boundary values.

Fix an $l' \in N$ so that $0<l'<l$ and put $t=(t', t'')$ with $t'=(t_1, \cdots, t_{l'})$ and $t''=(t_{l'+1}, \cdots, t_l)$. In the same way, we put $\vartheta=(\vartheta', \vartheta'')$ and $s=(s', s'')$. We assume that there exist $P_i'' \in (\mathscr{J} \cap ({}_Z\mathscr{D}_*)^N)$ such that the system of differential equations

$$\mathscr{M}': P_i''u=0 \qquad (i=1, \cdots, M'') \tag{6.5}$$

has regular singularities along the set of walls $\{Y_1, \cdots, Y_{l'}\}$ with the edge $Y'=Y_1 \cap \cdots \cap Y_{l'}$. Let $\bar{\mathscr{M}}'$ be the indicial equation defined by P_i'' and let $\lambda_1', \cdots, \lambda_{r'}'$ be the characteristic exponents. Then we moreover assume the following: $\sigma_*(P_i'') \in \bar{\mathscr{J}}$, any characteristic exponent does not depend on (t'', x) but holomorphically depends on z and there exists a similar system which satisfies (3.2) together with the above conditions.

For example, the system $R_ju=0$ $(j=1, \cdots, l')$ satisfies all the above conditions if the following hold: Any R_j belongs to $M(N; {}_Z\mathscr{D}_*)$, $\sigma_*(P)$ belongs to $\bar{\mathscr{J}}$ for any row P of R_j, and any root of $\det \bar{R}_j(t, x, z, s)|_{Y'\times Z}=0$ does not depend on $(t'', x) \in Y'$ but holomorphically depends on z for $j=1, \cdots, l'$.

Using the system $\mathscr{M}'$, we can define the map of taking the boundary values

$$\beta': {}_Z\mathscr{B}(X_+, \mathscr{M}) \longrightarrow ({}_Z\mathscr{B}(Y_+'))^{r'}, \tag{6.6}$$

where Y_+' is an open subset of Y' defined by $t_{l'+1}>0, \cdots, t_l>0$. Fix a characteristic exponent λ_μ' and put

(6.7) ${}_Z\mathscr{B}(X_+, \mathscr{M})_\mu'=\{u \in {}_Z\mathscr{B}(X_+, \mathscr{M}); \beta_i'(u)=0$ in the intersection of a neighborhood of Y and Y_+' for any $i \in \{1, \cdots, r'\}$ satisfying $\lambda_\mu'(0)-\lambda_i'(0) \in N^{l'}-\{0\}\}$.

Let $G' \in M(r'; Z)$ be a diagonal matrix whose i-th diagonal component equals 1 if $\lambda_i'(0)=\lambda_\mu'(0)$ and 0 otherwise, and put $\beta_*'(u)=G'\beta'(u)$. Then Theorem 6.1 proves that

$$\begin{cases} P(0, t'', x, z, s', \vartheta'', 0, t''D_x)s'^{\alpha}\bar{A}'(t'', x, z)\beta_*'(u)=0 & (\alpha \in N^{l'}), \\ (I_{r'}-G')\beta_*'(u)=0 \end{cases} \tag{6.8}$$

with

$$s_j'\beta_*'(u)=\bar{Q}_j'(t'', x, z)\beta_*'(u) \quad (j=1, \cdots, l') \tag{6.9}$$

for any $P(t', t'', x, z, \vartheta', \vartheta'', t'D_x, t''D_x) \in (\mathscr{J} \cap ({}_Z\mathscr{D}_*)^N)$. The equations (6.8) and (6.9) assure that $\beta'_*(u)$ satisfies a system of differential equations with regular singularities along the set of walls $\{Y_{l+1} \cap Y', \cdots, Y_r \cap Y'\}$ with the edge Y. We will determine its indicial equation.

We remark that in a neighborhood of 0 we have the following direct sum decomposition as ${}_Z\mathscr{A}_Y[s]$-Modules

$$\bar{\mathscr{M}} = \bar{\mathscr{M}}_* \oplus \bar{\mathscr{M}}^* \tag{6.10}$$

where

$$\bar{\mathscr{M}}_* = \{w \in \bar{\mathscr{M}};\ \prod_{\lambda'_\nu(0)=\lambda'_\mu(0)} (s_i - \lambda'_{\nu,i})^{r'} w = 0 \text{ for } i=1, \cdots, l\}$$

and

$$\bar{\mathscr{M}}^* = \{w \in \bar{\mathscr{M}};\ w \in \prod_{i=1}^{l'} {}_Z\mathscr{A}_Y[s](s_i - \lambda'_\mu(0)_i)w\ \}.$$

Then it follows from the definition of β' that the map induced by $\bar{u} \mapsto \bar{A}'G'\bar{B}'\bar{u}$ defines the projection map of $\bar{\mathscr{M}}$ onto $\bar{\mathscr{M}}_*$.

Now consider the system of differential equations (6.8) for P_i $(i=1, \cdots, M)$ which are used to define $\bar{\mathscr{M}}$. Then its indicial equation, which is a coherent ${}_Z\mathscr{A}_Y[s'']$-Module and will be denoted by $\bar{\mathscr{M}}''$, also has a structure of a $C[s']$-Module. So we consider it as a ${}_Z\mathscr{A}_Y[s]$-Module. Then $\bar{\mathscr{M}}''$ is naturally isomorphic to $\bar{\mathscr{M}}_*$ as ${}_Z\mathscr{A}_Y[s]$-Modules.

Put

$$\begin{aligned} {}_Z\mathscr{B}(Y'_+, \mathscr{M}_*) = \{u_*(t'', x, z) \in ({}_Z\mathscr{B}(Y'_+))^{r'};\ & u_* \text{ is a solution of} \\ & \text{(6.8) and (6.9) for any } P \in (\mathscr{J} \cap ({}_Z\mathscr{D}_*)^N)\} \end{aligned}$$

and $I_* = \{\nu \in \{1, \cdots, r\};\ (\lambda^\circ_{\nu,1}, \cdots, \lambda^\circ_{\nu,l'}) = \lambda'_\mu(0)\}$ and let r'' be the number of the elements of I_*. Then we can define the map of taking the boundary values

$$\beta'' : {}_Z\mathscr{B}(Y'_+, \mathscr{M}_*) \longrightarrow ({}_Z\mathscr{B}(Y))^{r''}$$

because the desired extension of the solution is assured by the induced equations corresponding to $R_j (j=l'+1, \cdots, l)$. On the other hand, we can also define the map

$$\begin{array}{ccc} \beta_* : {}_Z\mathscr{B}(X_+, \mathscr{M}) & \longrightarrow & ({}_Z\mathscr{B}(Y))^{r''} \\ \cup & & \cup \\ u & \longmapsto & (\beta_\nu(u))_{\nu \in I_*} \end{array}$$

Here we denote by $\bar{A}'$, $\bar{B}'$, A' and B' (resp. $\bar{A}''$, $\bar{B}''$, A'' and B'') the matrices which are used to define β' (resp. β''), corresponding to the matrices $\bar{A}$, $\bar{B}$, A and B used to define β. Fix a matrix $G_* \in M(r'', r; Z)$ so that $\beta_* = G_*\beta$. Then we have

Theorem 6.2. *For given β and β', we can define β'' so that*

$$\beta_* = \beta'' G' \beta' \tag{6.11}$$

on ${}_Z\mathscr{B}(X_+, \mathscr{M})'_\mu$. More precisely, we have the following:

Given B and B' corresponding to β and β', respectively, we can define B'' so that

$$G_*\sigma_*(B)\bar{u} = \sigma_*(B'')G'\sigma_*(B')\bar{u} \quad \text{and} \quad G_*Bu = B''G'B'u, \tag{6.12}$$

for all microfunction solutions u of ${}_Z\mathscr{E}_*\otimes\mathscr{M}$ satisfying $B'_i u = 0$ for any $i \in \{1, \cdots, r'\}$ with $\lambda'_\mu(0) - \lambda'_i(0) \in N^{l'} - \{0\}$.

Proof. We remark that A' and B' are originally defined in a neighborhood of

$$\Lambda' = \{(t, x, z; \textstyle\sum \tau_i dt_i + \sum \xi_j dx_j) \in (T^*X)\times Z;\ t_1 = 0, \cdots, t_{l'} = 0,\ \tau_1 \neq 0, \cdots, \tau_{l'} \neq 0,\ \tau_{l'+1} = 0, \cdots, \tau_l = 0,\ \xi_1 = 0, \cdots, \xi_n = 0,\ (t, x, z) \in V\},$$

where V is a small neighborhood of 0 in $X\times Z$. Since P''_i are of the forms

$$P''_i(t', t'', x, z, \vartheta', \vartheta'', t'D_x, t''D_x) \in ({}_Z\mathscr{D}_*)^N,$$

our construction of A' and B' in Theorem 3.2 shows that A' is of the form

$$A'(t, x, z, D_{t'}, \vartheta'', D_x) = \sum_{\alpha\in N^{l'}} A'_\alpha(t, x, z, \vartheta'', D_x) D_{t'}^{-\alpha}$$

and B' is of the form

$$B'(t, x, z, D_{t'}, \vartheta, D_x) = \sum_{|\beta|\leq \operatorname{ord} B', \beta\in N^{l'}} \sum_{\alpha\in N^{l'}} B'_{\alpha,\beta}(t, x, z, \vartheta'', D_x) D_{t'}^{-\alpha} \vartheta'^\beta$$

with matrices A'_α and $B'_{\alpha,\beta}$ of differential operators of order $\leqq|\alpha|$ which are of the above forms. Let

$$(\vartheta_i - Q'_i(t'', x, z, \vartheta'', t''D_x, D_{t'}))v' = 0 \qquad \text{for } i = 1, \cdots, l' \tag{6.13}$$

be the system which is obtained by applying Theorem 3.2 to define β'. Then the estimate (1.1) assures that A', B' and Q'_i are defined in a neighborhood of

$$\Lambda'_0 = \{(t, x, z; \textstyle\sum \tau_i dt_i + \sum \xi_j dx_j) \in (T^*X)\times Z;\ t_1 = 0, \cdots, t_l = 0,\ \tau_1 \neq 0, \cdots, \tau_{l'} \neq 0,\ \xi_1 = 0, \cdots, \xi_n = 0,\ (t, x, z) \in V\}.$$

Applying Späth's theorem of micro-differential operators to BA', we define $C \in M(r, r'; {}_Z\mathscr{E}_*)$ by

$$C(t'', x, z, D_x, D_t) = BA' + \sum_{i=1}^{l'} R_i(\vartheta_i - Q_i')$$

with suitable matrices R_i of micro-differential operators. Put

$$C(t'', x, z, D_x, D_t) = \sum_{\alpha \in N^{l'}} C_\alpha(t'', x, z, D_x, D_{t''}) D_{t'}^{-\alpha}.$$

Let $\tilde{\mathscr{M}} = {}_Z\mathscr{E}_* \tilde{u}$ be the direct sum of the system satisfied by Bu and that by $B'u$ with the vector u of the generators of ${}_Z\mathscr{E}_* \otimes \mathscr{M}$, and put

$$\tilde{C} = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} \in M(r+r', {}_Z\mathscr{E}_*).$$

Then the correspondence $\tilde{u} \mapsto \tilde{C}\tilde{u}$ gives a ${}_Z\mathscr{E}_*$-endomorphism of $\tilde{\mathscr{M}}$. Hence by the same argument as in the proof of Theorem 3.2 (iii), we have $\tilde{C}\tilde{u} = \begin{pmatrix} 0 & C' \\ 0 & 0 \end{pmatrix} \tilde{u}$ by putting

$$C'_{\nu i} = \begin{cases} C_\alpha D_{t'}^{-\alpha} & \text{if } \alpha = (\lambda^\circ_{\nu,1}, \cdots, \lambda^\circ_{\nu,l'}) - \lambda_i'(0) \in N^{l'}, \\ 0 & \text{otherwise.} \end{cases} \tag{6.14}$$

This means $Bu = BA'B'u = CB'u = C'B'u$. On the other hand, if $u(t, x, z) \in {}_Z\mathscr{B}(X_+, \mathscr{M})'_\mu$, then $B' \operatorname{sp}(\tilde{u}(t, x, z))_i = 0$ for $i \in \{1, \cdots, r'\}$ satisfying $\lambda'_\mu(0) - \lambda'_i(0) \in N^{l'} - \{0\}$. Hence by putting $B'' = G_* C' G'$, we have

$$B''G'B' \operatorname{sp}(\tilde{u}) = G_* C' G' B' \operatorname{sp}(\tilde{u}) = G_* B \operatorname{sp}(\tilde{u}) \qquad \text{if } u \in {}_Z\mathscr{B}(X_+, \mathscr{M})'_\mu.$$

Moreover (6.14) proves B'' is of the form $B''(t'', x, z, D_x, D_t'')$. Thus we obtain the desired B'' and by using B'' we can define β'' so that (6.11) holds. Q.E.D.

Theorem 6.2 immediately implies

Corollary 6.3. *Under the notation in Theorem 6.2, we have*

$$ {}_Z\mathscr{B}(X_+, \mathscr{M})'_\mu = \{u \in {}_Z\mathscr{B}(X_+, \mathscr{M});\ \beta_\nu(u) = 0 \text{ for any } \nu \in \{1, \cdots, r\} \text{ which satisfies } \lambda'_\mu(0) - (\lambda^\circ_{\nu,1}, \cdots, \lambda^\circ_{\nu,l'}) \in N^{l'} - \{0\}\}.$$

Especially, if $\lambda'_\mu(0) \neq (\lambda^\circ_{\nu,1}, \cdots, \lambda^\circ_{\nu,l'})$ *for* $\nu = 1, \cdots, r$, *then* $\beta'_\mu(u) = 0$ *in a neighborhood of the edge* Y *for any* $u \in {}_Z\mathscr{B}(X_+, \mathscr{M})'_\mu$. *On the other hand, for a given* $\nu \in \{1, \cdots, r\}$, *if* $(\lambda^\circ_{\nu,1}, \cdots, \lambda^\circ_{\nu,l'}) \neq \lambda'_i(0)$ *for* $i = 1, \cdots, r'$, *then* $\beta_\nu(u) = 0$ *for any* $u \in {}_Z\mathscr{B}(X_+, \mathscr{M})_\nu$.

In fact, the first statement is clear from Theorem 6.2 with Theorem 4.5, and the second and last follow from Theorem 6.2 with Theorem 4.4.

References

[G–R] R. C. Gunning and H. Rossi, Analytic functions of several complex variables, Prentice-Hall, 1965.

[K–K–] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima and M. Tanaka, Eigenfunctions of invariant differential operators on a symmetric space, Ann. of Math., **107** (1978), 1–39.

[K–O] M. Kashiwara and T. Oshima, Systems of differential equations with regular singularities and their boundary value problems, Ann. of Math., **106** (1977), 145–200.

[MaO] T. Matsuki and T. Oshima, A description of discrete series for semisimple symmetric spaces, Advanced Studies in Pure Math., **4** (1984), 331–390.

[MiO] K. Minemura and T. Oshima, Boundary value problem with regular singularity and Helgason-Okamoto conjecture, Publ. RIMS Kyoto Univ., **12** Suppl. (1977), 257–265.

[O 1] T. Oshima, Singularities in contact geometry and degenerate pseudo-differential equations, J. Fac. Sci. Univ. of Tokyo Sec. IA, **21** (1974), 43–87.

[O 2] ——, A realization of Riemannian symmetric spaces, J. Math. Soc. Japan, **30** (1978), 117–132.

[O 3] ——, A definition of boundary values of solutions of partial differential equations with regular singularities, Publ. RIMS Kyoto Univ., **19** (1983), 1203–1230.

[O–S] T. Oshima and J. Sekiguchi, Eigenspaces of invariant differential operators on an affine symmetric space, Invent. Math., **57** (1980), 1–81.

[S–K–K] M. Sato, T. Kawai and M. Kashiwara, Micro-functions and pseudo-differential equations, Proc. Conf. at Katata, 1971, Lecture Notes in Math. No. 287, Springer-Verlag, Berlin-Heidelberg-New York, 1973, pp. 265–529.

Department of Mathematics
Faculty of Science
University of Tokyo
Tokyo 113, Japan