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A Description of Discrete Series for Semisimple Symmetric Spaces

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§ 1. Introduction

Let G be a connected real semisimple Lie group, σ an involution of G, and H the connected component of the fixed-point group G^σ containing the identity. Then G/H is called a semisimple symmetric space ([1], [5]). We assume in this paper that G is a real form of a complex Lie group G_c. When G/H satisfies the condition

$$\operatorname{rank}(G/H)=\operatorname{rank}(K/K\cap H), \tag{1.1}$$

Flensted-Jensen [5] constructed countably many discrete series for G/H. Here K is a σ-stable maximal compact subgroup of G and "discrete series for G/H" are equivalence classes of the representations of G on minimal closed G-invariant subspaces in $L^2(G/H)$. In this paper we give a theorem that describes all the discrete series for G/H. Especially there is no discrete series when $\operatorname{rank}(G/H)\neq\operatorname{rank}(K/K\cap H)$.

The result of this paper can be described as follows.

Let $\mathfrak{g}$ be a semisimple Lie algebra and σ an involution ($\sigma^2=$identity) of $\mathfrak{g}$. Fix a Cartan involution θ such that $\sigma\theta=\theta\sigma$. Let $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ (resp. $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$) be the decomposition of $\mathfrak{g}$ into the $+1$ and -1 eigenspaces for σ (resp. θ). Let $\mathfrak{g}_c$ be the complexification of $\mathfrak{g}$ and let $\mathfrak{g}^d$, $\mathfrak{k}^d$ and $\mathfrak{h}^d$ be subalgebras in $\mathfrak{g}_c$ defined by

$$\mathfrak{g}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})+\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h})+\mathfrak{p}\cap\mathfrak{q},$$
$$\mathfrak{k}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h}),\qquad \mathfrak{h}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q}).$$

Extend σ and θ to complex linear involutions of $\mathfrak{g}_c$. The restrictions of σ and θ to $\mathfrak{g}^d$ are denoted by the same letters. Then $(\mathfrak{g}^d, \mathfrak{k}^d, \mathfrak{h}^d, \sigma, \theta)$ satisfies the same condition as $(\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \theta, \sigma)$.

Let G_c be a connected complex Lie group with Lie algebra $\mathfrak{g}_c$, and let G, K, H, G^d, K^d, H^d, H_c and K_c be the analytic subgroups of G_c corresponding to $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$, $\mathfrak{g}^d$, $\mathfrak{k}^d$, $\mathfrak{h}^d$, $\mathfrak{h}_c$ and $\mathfrak{k}_c$, respectively. Let $\hat{K}$ (resp. $\hat{H}^d$)

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denote the set of equivalence classes of finite-dimensional irreducible representations of K (resp. H^d) and let $\hat{H}^d(K)$ denote the subset of $\hat{H}^d$ formed by restrictions of holomorphic representations of K_c. Then $\hat{K}$ and $\hat{H}^d(K)$ are in one-to-one correspondence via holomorphic representations of K_c. Thus two corresponding elements of $\hat{K}$ and $\hat{H}^d(K)$ will be denoted by the same letter in the following argument.

Let $\boldsymbol{D}(G/H)$ and $\boldsymbol{D}(G^d/K^d)$ be the algebras of invariant differential operators on G/H and G^d/K^d, respectively. Then $\boldsymbol{D}(G/H)$ and $\boldsymbol{D}(G^d/K^d)$ are naturally isomorphic via holomorphic differential operators on G_c/H_c. Fix a maximal abelian subspace $\mathfrak{a}_\mathfrak{p}^d$ of $\mathfrak{p}^d=\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})+\mathfrak{p}\cap\mathfrak{q}$ and a positive system $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ of the root system $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$ of the pair $(\mathfrak{g}^d, \mathfrak{a}_\mathfrak{p}^d)$. Let λ be an element of $(\mathfrak{a}_\mathfrak{p}^d)_c^*$ (i.e. λ is a linear map of $\mathfrak{a}_\mathfrak{p}^d$ into $\boldsymbol{C}$). Then the algebra homomorphisms $\chi_\lambda^d\colon \boldsymbol{D}(G^d/K^d)\to\boldsymbol{C}$ and $\chi_\lambda\colon \boldsymbol{D}(G/H)\to\boldsymbol{C}$ are defined by the Harish-Chandra isomorphism $\boldsymbol{D}(G^d/K^d)\simeq S(\mathfrak{a}_\mathfrak{p}^d)^W$, where $S(\mathfrak{a}_\mathfrak{p}^d)$ is the complex symmetric algebra on $\mathfrak{a}_\mathfrak{p}^d$, W is the Weyl group of $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$ and $S(\mathfrak{a}_\mathfrak{p}^d)^W$ is the set of W-invariant elements in $S(\mathfrak{a}_\mathfrak{p}^d)$.

Now we define the following subspaces in $\mathscr{A}(G/H)$ and $\mathscr{A}(G^d/K^d)$ where $\mathscr{A}(X)$ denotes the space of analytic functions on a manifold X. For a $\delta\in\hat{K}\simeq\hat{H}^d(K)$ and $\lambda\in(\mathfrak{a}_\mathfrak{p}^d)_c^*$, we put

$$\mathscr{A}_\delta(G/H;\mathscr{M}_\lambda)=\{f\in\mathscr{A}(G/H)\mid f \text{ transforms according to } \delta \text{ under the action of } K \text{ and } Df=\chi_\lambda(D)f \text{ for all } D\in\boldsymbol{D}(G/H)\}$$

and

$$\mathscr{A}_\delta(G^d/K^d;\mathscr{M}_\lambda^d)=\{f\in\mathscr{A}(G^d/K^d)\mid f \text{ transforms according to } \delta \text{ under the action of } H^d \text{ and } Df=\chi_\lambda^d(D)f \text{ for all } D\in\boldsymbol{D}(G^d/K^d)\}.$$

Moreover we put

$$\mathscr{A}_K(G/H;\mathscr{M}_\lambda)=\bigoplus_{\delta\in\hat{K}}\mathscr{A}_\delta(G/H;\mathscr{M}_\lambda)$$

and

$$\mathscr{A}_{H^d}(G^d/K^d;\mathscr{M}_\lambda^d)=\bigoplus_{\delta\in\hat{H}^d(K)}\mathscr{A}_\delta(G^d/K^d;\mathscr{M}_\lambda^d).$$

Here the above sums are algebraic direct sums. Then the spaces $\mathscr{A}_K(G/H;\mathscr{M}_\lambda)$ and $\mathscr{A}_{H^d}(G^d/K^d;\mathscr{M}_\lambda^d)$ have the structure of $\mathfrak{g}_c$-modules. Flensted-Jensen has proved (Theorem 2.3 in [5]) that there is a $\mathfrak{g}_c$-isomorphism

$$\eta\colon \mathscr{A}_K(G/H;\mathscr{M}_\lambda)\xrightarrow{\sim}\mathscr{A}_{H^d}(G^d/K^d;\mathscr{M}_\lambda^d) \tag{1.2}$$

which is obtained by the analytic continuation in G_c/H_c.

Let $P^d = M^d A_\mathfrak{p}^d N^{+d}$ be the minimal parabolic subgroup of G^d determined by the pair $(\mathfrak{a}_\mathfrak{p}^d, \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+)$ and ρ be the element of $(\mathfrak{a}_\mathfrak{p}^d)^*$ defined by $\rho(Y) = \frac{1}{2}\operatorname{trace}(\operatorname{ad}(Y)|_{\mathfrak{n}^{+d}})$ for $Y \in \mathfrak{a}_\mathfrak{p}^d$. For $\delta \in \hat{H}^d(K)$ and $\lambda \in (\mathfrak{a}_\mathfrak{p}^d)_c^*$, we put

$$\mathscr{B}_\delta(G^d/P^d; L_\lambda) = \{f \text{ is a hyperfunction on } G^d \mid f \text{ transforms according to } \delta \text{ under the action of } H^d \text{ and } f(xman) = a^{\lambda-\rho} f(x) \text{ for } x \in G^d, m \in M^d, a \in A_\mathfrak{p}^d \text{ and } n \in N^{+d}\}$$

where $a^{\lambda-\rho} = e^{\langle \lambda-\rho, \log a\rangle}$. Moreover we put

$$\mathscr{B}_{H^d}(G^d/P^d; L_\lambda) = \bigoplus_{\delta \in \hat{H}^d(K)} \mathscr{B}_\delta(G^d/P^d; L_\lambda).$$

Then we define the Poisson transform

$$\mathscr{P}_\lambda : \mathscr{B}_{H^d}(G^d/P^d; L_\lambda) \longrightarrow \mathscr{A}_{H^d}(G^d/K^d; \mathscr{M}_\lambda^d) \tag{1.3}$$

by the formula

$$(\mathscr{P}_\lambda f)(x) = \int_{K^d} e^{\langle -\lambda-\rho, H(x^{-1}k)\rangle} f(k)\, dk$$

for $x \in G^d$ and $f \in \mathscr{B}_{H^d}(G^d/P^d; L_\lambda)$. Here $H(x) = Y_1$ if $x = k_1 \exp Y_1 n_1$, $k_1 \in K^d$, $Y_1 \in \mathfrak{a}_\mathfrak{p}^d$ and $n_1 \in N^{+d}$.

Remark 1. (i) Let (π, V) be a discrete series for G/H and V_K the subspace of K-finite elements in V. Then it is clear that there exists a λ in $(\mathfrak{a}_\mathfrak{p}^d)_c^*$ such that $V_K \subset \mathscr{A}_K(G/H; \mathscr{M}_\lambda) \cap L^2(G/H)$ and that $\operatorname{Re}\langle \lambda, \tilde{\alpha}\rangle \geq 0$ for all $\tilde{\alpha} \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$. (See Remark in § 4).

(ii) If $\operatorname{Re}\langle \lambda, \tilde{\alpha}\rangle \geq 0$ for all $\tilde{\alpha} \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$, then it follows from the result in [7] that $\mathscr{P}_\lambda$ is a $\mathfrak{g}_c$-isomorphism.

(iii) For every function f in $\mathscr{B}_{H^d}(G^d/P^d; L_\lambda)$, it is clear that the support of f is a union of H^d-orbits on G^d/P^d.

Here we prepare notation in the case of $\operatorname{rank}(G/H) = \operatorname{rank}(K/K \cap H)$. Let $\mathfrak{a}_\mathfrak{p}'$ be a maximal abelian subspace of $\mathfrak{p}^d \cap \mathfrak{h}^d$. Then $\mathfrak{a}_\mathfrak{p}'$ is a maximal abelian subspace in $\mathfrak{p}^d$, which is equivalent to $\operatorname{rank}(G/H) = \operatorname{rank}(K/K \cap H)$. By Section 3 Proposition 2 in [9] we can choose elements $x_1, \cdots, x_m$ of G^d such that $\operatorname{Ad}(x_j)\mathfrak{a}_\mathfrak{p}^d = \mathfrak{a}_\mathfrak{p}'$ and that $\{H^d x_j P^d \mid j = 1, \cdots, m\}$ is the set of all the closed H^d-orbits in G^d/P^d $(H^d x_i P^d \neq H^d x_j P^d$ if $i \neq j)$. For each j $(1 \leq j \leq m)$, we define $\Sigma(\mathfrak{a}_\mathfrak{p}')_j^+$, $\mathfrak{n}^{+j}$, $\lambda^j \in (\mathfrak{a}_\mathfrak{p}')_c^*$, $\rho^j \in (\mathfrak{a}_\mathfrak{p}')^*$ and $\rho_t^j \in (\mathfrak{a}_\mathfrak{p}')^*$ by

$$\Sigma(\mathfrak{a}_\mathfrak{p}')_j^+ = \{\tilde{\alpha} \circ \operatorname{Ad}(x_j)^{-1} \in \Sigma(\mathfrak{a}_\mathfrak{p}') \mid \tilde{\alpha} \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+\},$$

$$\mathfrak{n}^{+j} = \operatorname{Ad}(x_j)\mathfrak{n}^{+d}, \quad \lambda^j = \lambda \circ \operatorname{Ad}(x_j)^{-1}, \quad \rho^j = \rho \circ \operatorname{Ad}(x_j)^{-1}$$

$$\text{and} \quad \rho_t^j(Y) = \tfrac{1}{2}\operatorname{trace}(\operatorname{ad}(Y)|_{\mathfrak{n}^{+j} \cap \mathfrak{h}^d}) \quad \text{for } Y \in \mathfrak{a}_\mathfrak{p}',$$

respectively.

Now we can state the theorem of this paper as follows.

Theorem. *Let λ be an element of $(\mathfrak{a}_{\mathfrak{p}}^d)_c^*$ such that*

$$\mathrm{Re}\langle\lambda, \tilde{\alpha}\rangle \geq 0 \qquad \text{for all } \tilde{\alpha} \in \Sigma(\mathfrak{a}_{\mathfrak{p}}^d)^+. \tag{1.4}$$

(i) *If $\mathscr{A}_K(G/H; \mathscr{M}_\lambda) \cap L^2(G/H) \neq \{0\}$, then*

$$\mathrm{rank}(G/H) = \mathrm{rank}(K/K\cap H) \tag{1.5}$$

and

$$\mathrm{Re}\langle\lambda, \tilde{\alpha}\rangle > 0 \qquad \text{for any } \tilde{\alpha} \in \Sigma(\mathfrak{a}_{\mathfrak{p}}^d)^+. \tag{1.6}$$

In the following we assume the condition (1.5).

(ii) *Put*

$$\mathscr{B}_{H^d}^j(G^d/P^d; L_\lambda) = \{f \in \mathscr{B}_{H^d}(G^d/P^d; L_\lambda) \,|\, \mathrm{supp}\, f \subset H^d x_j P^d\}.$$

Then under the condition (1.6) *we have the surjective $\mathfrak{g}_c$-isomorphism*

$$\eta^{-1} \circ \mathscr{P}_\lambda \colon \bigoplus_{j=1}^m \mathscr{B}_{H^d}^j(G^d/P^d; L_\lambda) \xrightarrow{\sim} \mathscr{A}_K(G/H; \mathscr{M}_\lambda) \cap L^2(G/H)$$

by Flensted-Jensen's isomorphism and the Poisson transform.

(iii) *If the space $\mathscr{B}_{H^d}^j(G^d/P^d; L_\lambda)$ is non-trivial, then the following two conditions are satisfied.*

(a) *Let α be a compact simple root in $\Sigma(\mathfrak{a}_{\mathfrak{p}}')_j^+$* (i.e. $\mathfrak{g}^d(\mathfrak{a}_{\mathfrak{p}}'; \alpha) \subset \mathfrak{h}^d$). *Then*

$$\langle\lambda^j - \rho^j, \alpha\rangle \geq 0.$$

(b) *Put $\mu_\lambda^j = \lambda^j + \rho^j - 2\rho_t^j$. Then μ_λ^j belongs to the lattice in $(\mathfrak{a}_{\mathfrak{p}}')^*$ generated by the highest weights of all the finite-dimensional irreducible representations of K with $K\cap H$-fixed vectors.* (*Note that $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}}'$ is a maximal abelian subspace of $\mathfrak{k} \cap \mathfrak{q} = \sqrt{-1}\,(\mathfrak{p}^d \cap \mathfrak{h}^d)$.*)

(iv) *Suppose that $\mathscr{B}_{H^d}^j(G^d/P^d; L_\lambda) \neq \{0\}$. Then the $\mathfrak{g}_c$-module*

$$\mathscr{B}_{H^d}^j(G^d/P^d; L_\lambda)$$

is irreducible under the following condition (1.7).

Let $\mathfrak{a}_{\mathfrak{t}}^d$ be a maximal abelian subspace of $\mathfrak{m}^d$ and put $\mathfrak{a}_{\mathfrak{g}}^d = \mathfrak{a}_{\mathfrak{t}}^d + \mathfrak{a}_{\mathfrak{p}}^d$. Let $\Sigma(\mathfrak{a}_{\mathfrak{g}}^d)$ be the root system of the pair $(\mathfrak{g}_c, \mathfrak{a}_{\mathfrak{g}c}^d)$. For every $\alpha \in \Sigma(\mathfrak{a}_{\mathfrak{g}}^d)$ let $\bar{\alpha}$ denote the restriction of α to $\mathfrak{a}_{\mathfrak{p}}^d$. Choose a positive system $\Sigma(\mathfrak{a}_{\mathfrak{g}}^d)^+$ of $\Sigma(\mathfrak{a}_{\mathfrak{g}}^d)$

so that $\Sigma(\mathfrak{a}_{\mathfrak{g}}^{d})^{+}$ is compatible with $\Sigma(\mathfrak{a}_{\mathfrak{p}}^{d})^{+}$ (i.e. the condition $\alpha \in \Sigma(\mathfrak{a}_{\mathfrak{g}}^{d})^{+}$ and $\bar{\alpha}\neq 0$ implies $\bar{\alpha} \in \Sigma(\mathfrak{a}_{\mathfrak{p}}^{d})^{+}$). Put $\rho_{\mathfrak{m}}=\frac{1}{2}\sum \alpha$ where the sum is taken over all $\alpha \in \Sigma(\mathfrak{a}_{\mathfrak{g}}^{d})^{+}$ such that $\bar{\alpha}=0$. Then $-(\lambda+\rho_{\mathfrak{m}})$ parametrizes the infinitesimal character of the $\mathfrak{g}_c$-module $\mathscr{B}_{H^d}^{j}(G^d/P^d; L_\lambda)$.

(1.7) $\langle \lambda+\rho_{\mathfrak{m}}, \alpha\rangle \geq 0$ *for all* $\alpha \in \Sigma(\mathfrak{a}_{\mathfrak{g}}^{d})^{+}$.

This theorem is divided into three theorems. Theorem 1 is proved in Section 4–Section 7, Theorem 2 in Section 8 and Theorem 3 in Section 9 and Section 10.

Remark 2. (i) Suppose the condition

(1.8) μ_λ^j is equal to the highest weight of a finite dimensional representation τ of K with $K\cap H$-fixed vectors.

Let T^j be the distribution on K^d/M^d defined by

$$\langle T^j, \varphi\rangle=\int_{K\cap H} \varphi(kx_j)dk$$

for $\varphi \in C^\infty(K^d/M^d)$. Then T^j can be naturally identified with an element T_λ^j in $\mathscr{B}(G^d/P^d; L_\lambda)$ with support in $H^d x_j P^d$. When (1.8) is satisfied, it is proved in [5], Section 3 that T_λ^j transforms according to the representation contragredient to τ under the action of H^d. Thus we have

$$\mathscr{B}_{H^d}^{j}(G^d/P^d; L_\lambda)\neq\{0\}.$$

Put $\psi_\lambda^j=\eta^{-1}\circ \mathscr{P}_\lambda(T_\lambda^j)$. (This is the generating function of discrete series constructed by Flensted-Jensen [5].) If $\langle \lambda, \tilde{\alpha}\rangle>0$ for all $\tilde{\alpha} \in \Sigma(\mathfrak{a}_{\mathfrak{p}}^{d})^{+}$, then it follows from Theorem (ii) that $\psi_\lambda^j \in L^2(G/H)$. Hence we have proved the conjecture "$C=0$" in [5], p. 274. (This conjecture was already proved by the first author. C.f. [21].)

(ii) Suppose the condition (1.7). Then it is proved in Section 10, Lemma 11 that the pair of conditions (a) and (b) in Theorem (iii) is equivalent to the condition (1.8). Hence it follows from Theorem (iii) and the above remark in (i) that $\mathscr{B}_{H^d}^{j}(G^d/P^d; L_\lambda)\neq\{0\}$ (which is an irreducible $\mathfrak{g}_c$-module by Theorem (iv)) if and only if the conditions (a) and (b) in Theorem (iii) are satisfied.

(iii) If M^d is abelian, for instance when G/H is a group (i.e. $G=G_1\times G_1$ for some connected real semisimple Lie group G_1 and $H=\{(g, g) \in G \mid g \in G_1\}$) or when $\mathfrak{g}^d$ is a normal real form, then the condition (1.7) is equivalent to the condition $\langle \lambda, \tilde{\alpha}\rangle\geq 0$ for all $\tilde{\alpha} \in \Sigma(\mathfrak{a}_{\mathfrak{p}}^{d})^{+}$ which we always assume. (λ is real-valued on $\mathfrak{a}_{\mathfrak{p}}^{d}$ by Theorem (iii) (b).) Hence by the above remark in (ii), $\mathscr{B}^j_{H^d}(G^d/P^d; L_\lambda)\neq\{0\}$ if and only if the conditions (a) and (b) are satisfied. When G/H is a group, we have therefore given another proof of main results in [6].

(iv) Suppose that all the irreducible components of the root system $\Sigma(\mathfrak{a}^d_\mathfrak{p})$ are of type A_n, D_n or E_n $(n\geq 2)$. Then it is proved in Section 10, Lemma 10 that the pair of the conditions (a) and (b) is equivalent to the condition (1.8). Hence it follows from the remark in (ii) and Theorem (iii) that $\mathscr{B}^j_{H^d}(G^d/P^d; L_\lambda)\neq\{0\}$ if and only if the conditions (a) and (b) in Theorem (iii) are satisfied.

(v) In general, there are discrete series which cannot be obtained by the argument in (i) (c.f. [5], Section 8 when $\dim(\mathfrak{a}^d_\mathfrak{p})=1$).

(vi) When $\langle\lambda^j, \alpha\rangle=0$ for some noncompact (i.e. $\mathfrak{g}^d(\mathfrak{a}'_\mathfrak{p}; \alpha)\not\subset\mathfrak{h}^d$) simple root α in $\Sigma(\mathfrak{a}'_\mathfrak{p})^+_j$, $\eta^{-1}\circ\mathscr{P}_\lambda\mathscr{B}^j_{H^d}(G^d/P^d; L_\lambda)$ are the K-finite functions in a "limit of discrete series" for G/H.

(vii) The condition (1.4) is not necessary in the proof of Theorem (iii).

In a subsequent paper we will give a proof of the following.

Proposition. *Suppose the condition* (1.4). *Then* $\mathscr{B}^j_{H^d}(G^d/P^d; L_\lambda)\neq\{0\}$ *if and only if the condition* (b) *in Theorem* (iii) *and the following condition* (a′) *hold.*

(a′) *Let* $\{\beta_1, \cdots, \beta_k\}$ *be a sequence of roots in* $\Sigma(\mathfrak{a}'_\mathfrak{p})^+_j$ *satisfying the following* (i) *and* (ii).

(i) β_i *is a simple root in the set* $\{\alpha\in\Sigma(\mathfrak{a}'_\mathfrak{p})^+_j \mid \langle\alpha, \beta_1\rangle=\cdots=\langle\alpha, \beta_{i-1}\rangle=0\}$ *for* $i=1, \cdots, k$.

(ii) $\langle\beta_i, 2\rho^j_t-\rho^j\rangle<(\frac{1}{2}m_{\beta_i}+m_{2\beta_i})\langle\beta_i, \beta_i\rangle$ *for* $i=1, \cdots, k-1$ *and* $\langle\beta_k, 2\rho^j_t-\rho^j\rangle=(\frac{1}{2}m_{\beta_k}+m_{2\beta_k})\langle\beta_k, \beta_k\rangle$ *where* $m_\alpha=\dim\{X\in\mathfrak{g}^d \mid [Y, X]=\alpha(Y)X$ *for all* $Y\in\mathfrak{a}'_\mathfrak{p}\}$ *for* $\alpha\in\Sigma(\mathfrak{a}'_\mathfrak{p})$.

Then $\langle\mu^j_\lambda, \beta_k\rangle\geq 0$.

(*Note that the condition* (a) *in Theorem* (iii) *is equal to the condition for* $k=1$ *in* (a′).)

§ 2. Flensted-Jensen's isomorphism

We will use the standard notation $\boldsymbol{Z}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ for the ring of integers, the field of real numbers and the field of complex numbers, respectively. The set of nonnegative integers and nonnegative real numbers are denoted by $\boldsymbol{Z}_+$ and $\boldsymbol{R}_+$, respectively. For a real vector space E, let E^* denote the dual of E and E^*_c the complexification of E^*.

Let $\mathfrak{g}$ be a real semisimple Lie algebra and σ an involutive ($\sigma^2=$ identity) automorphism of $\mathfrak{g}$. Fix a Cartan involution θ of $\mathfrak{g}$ such that $\sigma\theta=\theta\sigma$. (See [1], [9] etc.) Let $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ (resp. $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$) be the decomposition of $\mathfrak{g}$ into the $+1$ and -1 eigenspaces for σ (resp. θ). Then we have a direct sum decomposition

$$\mathfrak{g}=\mathfrak{k}\cap\mathfrak{h}+\mathfrak{k}\cap\mathfrak{q}+\mathfrak{p}\cap\mathfrak{h}+\mathfrak{p}\cap\mathfrak{q}$$

of $\mathfrak{g}$.

Let $\mathfrak{g}_c$ be the complexification of $\mathfrak{g}$ and let $\mathfrak{g}^d$, $\mathfrak{k}^d$, $\mathfrak{p}^d$, $\mathfrak{h}^d$, $\mathfrak{q}^d$ and $\mathfrak{h}^a$ be subspaces of $\mathfrak{g}_c$ defined by

$$\begin{aligned}
&\mathfrak{g}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})+\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h})+\mathfrak{p}\cap\mathfrak{q},\\
&\mathfrak{k}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h}), \qquad \mathfrak{p}^d=\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})+\mathfrak{p}\cap\mathfrak{q},\\
&\mathfrak{h}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q}), \qquad \mathfrak{q}^d=\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h})+\mathfrak{p}\cap\mathfrak{q},\\
&\mathfrak{h}^a=\mathfrak{k}\cap\mathfrak{h}+\mathfrak{p}\cap\mathfrak{q}.
\end{aligned}$$

Then $\mathfrak{g}^d$, $\mathfrak{k}^d$, $\mathfrak{h}^d$ and $\mathfrak{h}^a$ are subalgebras in $\mathfrak{g}_c$. Extend involutions σ and θ to complex linear involutions of $\mathfrak{g}_c$. The restrictions of σ and θ to $\mathfrak{g}^d$ are denoted by the same letters. Then $(\mathfrak{g}^d, \mathfrak{k}^d, \mathfrak{h}^d, \sigma, \theta)$ satisfies the same condition as $(\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \theta, \sigma)$.

Let G_c be a connected complex Lie group with Lie algebra $\mathfrak{g}_c$, and let G, K, H, G^d, K^d, H^d, K_c, H_c and H^a be the analytic subgroups of G_c corresponding to $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$, $\mathfrak{g}^d$, $\mathfrak{k}^d$, $\mathfrak{h}^d$, $\mathfrak{k}_c$, $\mathfrak{h}_c$ and $\mathfrak{h}^a$, respectively. Let $\hat{K}$ (resp. $\hat{H}^d$) denote the set of equivalence classes of finite-dimensional irreducible representations of K (resp. H^d) and let $\hat{H}^d(K)$ denote the subset of $\hat{H}^d$ formed by restrictions of holomorphic representations of K_c. Then $\hat{K}$ and $\hat{H}^d(K)$ are in one-to-one correspondence via holomorphic representations of K_c. Thus two corresponding elements in $\hat{K}$ and $\hat{H}^d(K)$ will be denoted by the same letter in the following argument.

Let $\mathscr{A}(G/H)$ and $\mathscr{A}(G^d/K^d)$ be the spaces of analytic functions on G/H and G^d/K^d, respectively. For a $\delta\in\hat{K}(\simeq H^d(K))$ we put

$$\mathscr{A}_\delta(G/H)=\{f\in\mathscr{A}(G/H)\mid f \text{ transforms according to } \delta \text{ under the action of } K\}$$

and

$$\mathscr{A}_\delta(G^d/K^d)=\{f\in\mathscr{A}(G^d/K^d)\mid f \text{ transforms according to } \delta \text{ under the action of } H^d\}.$$

Moreover we put

$$\mathscr{A}_K(G/H)=\bigoplus_{\delta\in\hat{K}}\mathscr{A}_\delta(G/H)$$

and

$$\mathscr{A}_{H^d}(G^d/K^d)=\bigoplus_{\delta\in\hat{H}^d(K)}\mathscr{A}_\delta(G^d/K^d)$$

where the right hand sides are algebraic direct sums in $\mathscr{A}(G/H)$ and $\mathscr{A}(G^d/K^d)$, respectively.

Let $U(\mathfrak{g})=U(\mathfrak{g}^d)$ be the universal enveloping algebra of $\mathfrak{g}_c$ and $U(\mathfrak{g})^{\mathfrak{h}}=U(\mathfrak{g}^d)^{\mathfrak{k}^d}$ be the subalgebra of $U(\mathfrak{g})$ consisting of $\mathfrak{h}_c$-invariant elements. Then we have the following result by Flensted-Jensen.

Proposition 1 ([5], Theorem 2.3). *There exists a linear isomorphism*

$$\eta: \mathscr{A}_K(G/H) \xrightarrow{\sim} \mathscr{A}_{H^d}(G^d/K^d)$$

satisfying the following two conditions.

(i) $f^\eta(x)=f(x)$ *for* $f \in \mathscr{A}_K(G/H)$ *and* $x \in H^a$.

(ii) η *commutes with the left* $U(\mathfrak{g})$*-actions and with the right* $U(\mathfrak{g})^{\mathfrak{h}}$*-actions.*

Let $\boldsymbol{D}(G/H)$ and $\boldsymbol{D}(G^d/K^d)$ be the algebras of invariant differential operators on G/H and G^d/K^d, respectively. Clearly $\boldsymbol{D}(G/H)$ and $\boldsymbol{D}(G^d/K^d)$ are isomorphic via holomorphic differential operators on G_c/H_c.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}^d\cap\mathfrak{q}^d=\mathfrak{p}\cap\mathfrak{q}$ and $\mathfrak{a}_\mathfrak{p}^d$ a maximal abelian subspace of $\mathfrak{p}^d$ containing $\mathfrak{a}$. Let $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$ be the root system of the pair $(\mathfrak{g}^d, \mathfrak{a}_\mathfrak{p}^d)$. Namely for an $\tilde{\alpha}\in(\mathfrak{a}_\mathfrak{p}^d)^*$ we put $\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}^d;\tilde{\alpha})=\{X\in\mathfrak{g}^d \mid [Y,X]=\tilde{\alpha}(Y)X \text{ for all } Y\in\mathfrak{a}_\mathfrak{p}^d\}$ and we put

$$\Sigma(\mathfrak{a}_\mathfrak{p}^d)=\{\tilde{\alpha}\in(\mathfrak{a}_\mathfrak{p}^d)^*\backslash\{0\}\mid \mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}^d;\tilde{\alpha})\neq\{0\}\}.$$

Let $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ be a positive system of $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$ which is compatible with $\mathfrak{a}$. (i.e. If $\tilde{\alpha}\in\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ and $\tilde{\alpha}|_\mathfrak{a}\neq 0$, then $\sigma\theta\tilde{\alpha}\in\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$.) Let $\Sigma(\mathfrak{a})$ be the root system of the pair $(\mathfrak{g}^d,\mathfrak{a})$. (It can be easily proved that $\Sigma(\mathfrak{a})$ satisfies the axioms of root systems by the arguments in [17], p. 21 and p. 22. Another proof is given in [12]). Put $\mathfrak{n}^{+d}=\sum_{\tilde{\alpha}}\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}^d;\tilde{\alpha})$ where the sum is taken over all $\tilde{\alpha}\in\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ and put $\rho(Y)=\frac{1}{2}\operatorname{trace}(\operatorname{ad}(Y)|_{\mathfrak{n}^{+d}})$ for $Y\in\mathfrak{a}_\mathfrak{p}^d$.

Using the direct sum decomposition $U(\mathfrak{g}^d)=(\mathfrak{k}^dU(\mathfrak{g}^d)+U(\mathfrak{g}^d)\mathfrak{n}^{+d})\oplus U(\mathfrak{a}_\mathfrak{p}^d)$ of $U(\mathfrak{g}^d)$, we define a projection p of $U(\mathfrak{g}^d)$ onto $U(\mathfrak{a}_\mathfrak{p}^d)$. Let $W=W(\mathfrak{a}_\mathfrak{p}^d)$ be the Weyl group of $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$ and $U(\mathfrak{a}_\mathfrak{p}^d)^W$ be the subalgebra of $U(\mathfrak{a}_\mathfrak{p}^d)$ consisting of W-invariant elements in $U(\mathfrak{a}_\mathfrak{p}^d)$. Then it is known that the restriction of the map $D\to e^\rho\circ p(D)\circ e^{-\rho}$ to $U(\mathfrak{g}^d)^{\mathfrak{k}^d}$ defines an isomorphism

$$U(\mathfrak{g}^d)^{\mathfrak{k}^d}/U(\mathfrak{g}^d)^{\mathfrak{k}^d}\cap U(\mathfrak{g}^d)\mathfrak{k}^d \xrightarrow{\sim} U(\mathfrak{a}_\mathfrak{p}^d)^W.$$

It is clear that the left hand side is isomorphic to $\boldsymbol{D}(G/H)\simeq\boldsymbol{D}(G^d/K^d)$. For a $\lambda\in(\mathfrak{a}_\mathfrak{p}^d)_c^*$, we can define algebra homomorphisms $\chi_\lambda: \boldsymbol{D}(G/H)\to\boldsymbol{C}$ and $\chi_\lambda^d: \boldsymbol{D}(G^d/K^d)\to\boldsymbol{C}$ by the above isomorphism. Here we note that $\chi_\lambda=\chi_\mu$ (resp. $\chi_\lambda^d=\chi_\mu^d$) if and only if $\mu=w\lambda$ for some $w\in W$. Now we define following subspaces in $\mathscr{A}(G/H)$ and $\mathscr{A}(G^d/K^d)$.

$$\mathscr{A}_K(G/H;\mathscr{M}_\lambda)=\{f\in\mathscr{A}_K(G/H)\mid Df=\chi_\lambda(D)f \text{ for all } D\in \boldsymbol{D}(G/H)\},$$
$$\mathscr{A}_{H^d}(G^d/K^d;\mathscr{M}^d_\lambda)=\{f\in\mathscr{A}_{H^d}(G^d/K^d)\mid Df=\chi^d_\lambda(D)f \text{ for all } D\in \boldsymbol{D}(G^d/K^d)\}.$$

Then we have a $\mathfrak{g}_c$-isomorphism

$$\eta\colon \mathscr{A}_K(G/H;\mathscr{M}_\lambda)\xrightarrow{\sim}\mathscr{A}_{H^d}(G^d/K^d;\mathscr{M}^d_\lambda)$$

by Proposition 1.

§ 3. Boundary values and L^2-estimates

In this section, manifolds always mean real analytic manifolds and differential operators always mean linear partial differential operators of finite order whose coefficients are real analytic functions. A differential operator $P(x, D_x)$ defined on an n-dimensional manifold X is of the form

$$P(x, D_x)=\sum_{\alpha\in N^n} p_\alpha(x)D^\alpha_x,$$

where $x=(x_1, \cdots, x_n)$ is a local coordinate system and

$$D^\alpha_x=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

with $\alpha=(\alpha_1, \cdots, \alpha_n)$. The largest integer m which satisfies $p_\alpha\neq 0$ for at least one α with $m=\alpha_1+\cdots+\alpha_n$ is called the order of $P(x, D_x)$ and denoted by ord P. Then the principal symbol

$$\sigma(P)(x,\xi)=\sum_{\alpha_1+\cdots+\alpha_n=m} p_\alpha(x)\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$$

defines a function on the cotangent bundle T^*X of X, where $(x;\sum_i \xi_i dx_i)$ is a local coordinate system of T^*X. We denote by $\mathscr{A}(X)$ (resp. $\mathscr{B}(X)$) the vector space of all real analytic functions (resp. all hyperfunctions) defined on X.

In this section we will prove a proposition and two lemmas. The proposition reduces the question of the characterization of discrete series to a boundary value problem and secondly the two lemmas reduce the boundary value problem to a relation between the H^d-orbits structure on a boundary of the symmetric space G^d/K^d and a structure of the roots space for the symmetric pair.

For any function f in $\mathscr{A}_K(G/H;\mathscr{M}_\lambda)$, we can associate a function f^η in $\mathscr{A}_{H^d}(G^d/K^d;\mathscr{M}^d_\lambda)$ by Flensted-Jensen's isomorphism. Since $\chi^d_\lambda=\chi^d_{w\lambda}$ for any $w\in W$, we will fix $\lambda\in(\mathfrak{a}^d_\mathfrak{p})^*_c$ so that

(3.1) $$\operatorname{Re}\langle\lambda,\alpha\rangle\geq 0 \qquad \text{for any } \alpha\in\Sigma(\mathfrak{a}^d_\mathfrak{p})^+.$$

Let P^d be the minimal parabolic subgroup of G^d determined by the pair $(\mathfrak{a}_\mathfrak{p}^d, \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+)$ and let $P^d = M^d A_\mathfrak{p}^d N^{+d}$ be the corresponding Langlands decomposition. Then the Lie algebras of $A_\mathfrak{p}^d$ and N^{+d} equal $\mathfrak{a}_\mathfrak{p}^d$ and $\mathfrak{n}^{+d}$, respectively, and M^d is the centralizer of $\mathfrak{a}_\mathfrak{p}^d$ in K^d.

For a $\mu \in (\mathfrak{a}_\mathfrak{p}^d)_c^*$, we define the space of hyperfunction sections of class 1 principal series for G^d:

$$\mathscr{B}(G^d/P^d; L_\mu) = \{f \in \mathscr{B}(G^d) \mid f(xman) = a^{\mu-\rho} f(x) \text{ for } x \in G^d,\ m \in M^d,\ a \in A_\mathfrak{p}^d \text{ and } n \in N^{+d}\},$$

where $a^{\mu-\rho} = e^{\langle \mu-\rho, \log a \rangle}$. Then we have the Poisson transform

$$\mathscr{P}_\mu \colon \mathscr{B}(G^d/P^d; L_\mu) \longrightarrow \mathscr{B}(G^d/K^d)$$

by the formula

$$(\mathscr{P}_\mu f)(xK^d) = \int_{K^d} e^{\langle -\mu-\rho, H(x^{-1}k) \rangle} f(k)\, dk$$

for $x \in G^d$ and $f \in \mathscr{B}(G^d/P^d; L_\mu)$. Here $H(x) = Y_1$ if $x = k_1 \exp Y_1 n_1$, $k_1 \in K^d$, $Y_1 \in \mathfrak{a}_\mathfrak{p}^d$ and $n_1 \in N^{+d}$. Then $\mathscr{P}_\mu$ is a G^d-equivariant map and the image is contained in the following eigenspace of $\boldsymbol{D}(G^d/K^d)$:

$$\mathscr{A}(G^d/K^d; \mathscr{M}_\mu^d) = \{f \in \mathscr{A}(G^d/K^d) \mid Df = \chi_\mu^d(D) f \text{ for any } D \in \boldsymbol{D}(G^d/K^d)\}.$$

Now the main result in [7] says that the condition (3.1) for λ assures that the Poisson transform $\mathscr{P}_\lambda$ induces the G^d-isomorphism:

$$\mathscr{P}_\lambda \colon \mathscr{B}(G^d/P^d; L_\lambda) \xrightarrow{\sim} \mathscr{A}(G^d/K^d; \mathscr{M}_\lambda^d)$$

and the inverse of $\mathscr{P}_\lambda$ is given (up to a constant multiple) by the map β_λ of taking the boundary values. Hence for $\mu \in (\mathfrak{a}_\mathfrak{p}^d)_c^*$ and $\delta \in \hat{H}^d(K)$, denoting

$$\mathscr{B}_\delta(G^d/P^d; L_\mu) = \{f \in \mathscr{B}(G^d/P^d; L_\mu) \mid f \text{ transforms according to } \delta\}$$

and

$$\mathscr{B}_{H^d}(G^d/P^d; L_\mu) = \bigoplus_{\delta \in \hat{H}^d(K)} \mathscr{B}_\delta(G^d/P^d; L_\mu),$$

we have the $(U(\mathfrak{g}), H^d)$-isomorphisms

$$\mathscr{P}_\lambda \colon \mathscr{B}_{H^d}(G^d/P^d; L_\lambda) \xrightarrow{\sim} \mathscr{A}_{H^d}(G^d/K^d; \mathscr{M}_\lambda^d)$$

and

$$\beta_\lambda \colon \mathscr{A}_{H^d}(G^d/K^d; \mathscr{M}_\lambda^d) \xrightarrow{\sim} \mathscr{B}_{H^d}(G^d/P^d; L_\lambda)$$

and $\mathscr{P}_\lambda\beta_\lambda$ and $\beta_\lambda\mathscr{P}_\lambda$ are non-zero constant multiples of identity maps.

Fix a G-invariant measure $d\mu$ on G/H and let $L^2(G/H)$ denote the Hilbert space formed by the square integrable functions on G/H with respect the measure. Our theorem characterizes the subspace

$$\beta_\lambda \circ \eta(\mathscr{A}_K(G/H; \mathscr{M}_\lambda) \cap L^2(G/H)) \quad \text{of } \mathscr{B}_{H^d}(G^d/P^d; L_\lambda).$$

Hence the first step toward the theorem is to characterize the image of $\mathscr{A}_K(G/H; \mathscr{M}_\lambda) \cap L^2(G/H)$ under the map η.

Let A denote the analytic subgroup of G_c with the Lie algebra $\mathfrak{a}$ and let $f \in \mathscr{A}_K(G/H; \mathscr{M}_\lambda)$. The condition that the function f belongs to $L^2(G/H)$ is determined by its behavior at infinity. Owing to the decomposition $G=KAH$, the restriction $f|_A$ controls the behavior because f is K-finite. More precisely, the growth condition of f at infinity is determined by the restrictions on A of the translations of f under the action of K. Here we remark that A is contained in both G and G^d, and therefore $f|_A=f^\eta|_A$.

To examine the asymptotic behavior of the function

$$f|_A \quad \text{for } f \in \mathscr{A}_{H^d}(G^d/K^d; \mathscr{M}_\lambda),$$

we use a realization of G^d/K^d in a compact manifold $\tilde{X}$ which is constructed in [11]. Then the asymptotic behavior of $f|_A$ at infinity is translated into the local behavior of $f|_A$ at some boundary points of G^d/K^d in $\tilde{X}$.

Let $\Psi=\Psi(\mathfrak{a}_\mathfrak{p}^d)=\{\tilde{\alpha}_1, \cdots, \tilde{\alpha}_{l'}\}$ be the set of simple roots in $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ and $\{\tilde{\omega}_1, \cdots, \tilde{\omega}_{l'}\}$ the dual basis of Ψ. We recall that we defined the order of $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$ so that the following condition holds:

$$\text{(3.2)} \qquad \text{If } \tilde{\alpha} \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+ \text{ and } \tilde{\alpha}|_\mathfrak{a} \neq 0, \text{ then } \sigma\theta\tilde{\alpha} \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+.$$

Hence we can define a compatible order on $\Sigma(\mathfrak{a})$. That is,

$$\Sigma(\mathfrak{a})^+ = \{\tilde{\alpha}|_\mathfrak{a};\, \tilde{\alpha} \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+ \text{ and } \tilde{\alpha}|_\mathfrak{a} \neq 0\}.$$

Then similarly, we denote by $\Psi(\mathfrak{a})=\{\alpha_1, \cdots, \alpha_l\}$ the set of simple roots in $\Sigma(\mathfrak{a})^+$ and $\{\omega_1, \cdots, \omega_l\}$ the dual basis of $\Psi(\mathfrak{a})$. We will identify $A_\mathfrak{p}^d$ and A with $(0, \infty)^{l'}$ and $(0, \infty)^l$ by the maps

$$\text{(3.3)} \qquad \begin{array}{ccc} (0, \infty)^{l'} & \longrightarrow & A_\mathfrak{p}^d \\ \cup\!\!\! & & \cup\!\!\! \\ t=(t_1, \cdots, t_{l'}) & \longmapsto & a_t=\exp\left(-\sum_i \log(t_i)\tilde{\omega}_i\right) \end{array}$$

and

$$\text{(3.4)} \qquad \begin{array}{ccc} (0, \infty)^{l} & \longrightarrow & A \\ \cup\!\!\! & & \cup\!\!\! \\ y=(y_1, \cdots, y_l) & \longmapsto & a(y)=\exp\left(-\sum_j \log(y_j)\omega_j\right), \end{array}$$

respectively.

Let Θ be a subset of $\Psi(\mathfrak{a}_{\mathfrak{p}}^d)$ and W_Θ the subgroup of W generated by the reflections $w_{\tilde{\alpha}}$ with respect to the roots $\tilde{\alpha}$ in Θ. Put $P_\Theta^d = P^d W_\Theta P^d$. Then P_Θ^d is a parabolic subgroup of G^d. Let $P_\Theta^d = M_\Theta^d A_\Theta^d N_\Theta^{+d}$ be the Langlands decomposition of P_Θ^d such that $A_\Theta^d \subset A_{\mathfrak{p}}^d$. Furthermore, put $M_\Theta^d(K) = M_\Theta^d \cap K^d$ and define a closed subgroup $P_\Theta^d(K) = M_\Theta^d(K) A_\Theta^d N_\Theta^{+d}$ of G^d.

The structure of the manifold $\tilde{X}$ plays a crucial role in our analysis of asymptotic behavior of functions in $\mathscr{A}_{H^d}(G^d/K^d; \mathscr{M}_\lambda^d)$. We will review the construction of $\tilde{X}$. For any $t \in \boldsymbol{R}^{l'}$, we put

$$\operatorname{sgn} t = (\operatorname{sgn} t_1, \cdots, \operatorname{sgn} t_{l'}) \in \{-1, 0, 1\}^{l'},$$

$$\Theta_t = \{\tilde{\alpha}_i \in \Psi \mid t_i \neq 0\}$$

and

$$a_t = \exp(-\sum_{\tilde{\alpha}_i \in \Theta_t} \log|t_i| \tilde{\omega}_i).$$

We note that if $t \in (0, \infty)^{l'}$, then a_t is the corresponding element of A^d under the map (3.3). Define the following equivalence relation on the product manifold $G^d \times \boldsymbol{R}^{l'}$:

Two elements (g, t) and (g', t') in $G^d \times \boldsymbol{R}^{l'}$ are equivalent if and only if $\operatorname{sgn} t = \operatorname{sgn} t'$ and $g a_t P_{\Theta_t}^d(K) = g' a_{t'} P_{\Theta_{t'}}^d(K)$.

Then the space $\tilde{X}$ is defined as the quotient space $(G^d \times \boldsymbol{R}^{l'})/\sim$ by the equivalence relation $\sim$. Let π be the natural projection of $G^d \times \boldsymbol{R}^{l'}$ onto $\tilde{X}$. The action of G^d by the left translation on the first factor of $G^d \times \boldsymbol{R}^{l'}$ defines an action of G^d on $\tilde{X}$ through the projection π. We can define a real analytic structure on $\tilde{X}$ so that the following properties hold (c.f. [11]):

The space $\tilde{X}$ is a simply connected compact real analytic manifold where G^d acts analytically. For any $g \in G^d$. the map

$$\begin{array}{ccc} N^{-d} \times \boldsymbol{R}^{l'} & \longrightarrow & \tilde{X} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (n, t) & \longmapsto & \pi((gn, t)) \end{array} \tag{3.5}$$

defines a diffeomorphism onto an open dense subset of $\tilde{X}$, where $N^{-d} = \sigma(N^{+d})$ with the Cartan involution σ of G^d. For elements (g, t) and (g', t') of $G^d \times \boldsymbol{R}^{l'}$, two points $\pi((g, t))$ and $\pi((g', t'))$ in $\tilde{X}$ belong to a same G^d-orbit if and only if $\operatorname{sgn} t = \operatorname{sgn} t'$. Moreover, the G^d-orbit containing $\pi((g, t))$ is naturally diffeomorphic to the homogeneous space $G^d/P_{\Theta_t}^d(K)$.

We identify G^d/K^d with the open orbit $G^d \pi(1, (1, \cdots, 1))$ of $\tilde{X}$. The G^d-orbits appeared in the boundary of G^d/K^d are called the boundary components of G^d/K^d. The compact boundary component of G^d/K^d, which is diffeomorphic to G^d/P^d and only one compact G^d-orbit in $\tilde{X}$, is called the distinguished boundary of G^d/K^d. Thus we identify G^d/P^d with this boundary component.

Another important feature of $\tilde{X}$ is concerned with the G^d-invariant differential operators (c.f. [11]): Any invariant differential operator in $\boldsymbol{D}(G^d/K^d)$ has an analytic extension on $\tilde{X}$. Since G^d/K^d is open in $\tilde{X}$, we can naturally identify $\boldsymbol{D}(G^d/K^d)$ with the ring of G^d-invariant differential operators on $\tilde{X}$. We fix homogeneous elements

$$p_1(\tilde{\omega}_1, \cdots, \tilde{\omega}_{l'}), \cdots, p_{l'}(\tilde{\omega}_1, \cdots, \tilde{\omega}_{l'}) \in U(\mathfrak{a}_{\mathfrak{p}}^d)^W$$

so that $\boldsymbol{C}[p_1, \cdots, p_{l'}] = U(\mathfrak{a}_{\mathfrak{p}}^d)^W$. Let $D_1, \cdots, D_{l'}$ be the elements of $\boldsymbol{D}(G^d/K^d)$ which correspond to $p_1, \cdots, p_{l'}$, respectively, by the Harish-Chandra isomorphism. For each $i \in \{1, \cdots, l'\}$, let Y_i be the hypersurface of $\tilde{X}$ defined by $t_i = 0$ through the map (3.5). Then the system of differential equations on $\tilde{X}$

$$\mathscr{M}_\lambda^d\colon (D_i - \chi_\lambda^d(D_i))u = 0 \qquad (i = 1, \cdots, l')$$

has regular singularities along the set of walls $\{Y_1, \cdots, Y_{l'}\}$ with the edge G^d/P^d.

In general, under a local coordinate system $(x_1, \cdots, x_n, t_1, \cdots, t_r)$, the system of differential equations of the form

$$\mathscr{M}\colon P_i(x, t, tD_x, tD_t)u = 0 \qquad (i = 1, \cdots, r)$$

is said to have regular singularities along the set of walls $\{Y_1, \cdots, Y_r\}$ if the following conditions hold (c.f. [8]), where

$$tD_x = (t_1\partial/\partial x_1, t_1\partial/\partial x_2, \cdots, t_r\partial/\partial x_n), \qquad tD_t = (t_1\partial/\partial t_1, t_2\partial/\partial t_2, \cdots, t_r\partial/\partial t_r)$$

and each Y_i is defined by $t_i = 0$:

Put $m_i = \operatorname{ord} p_i$, $m = m_1 \times \cdots \times m_r$ and $a_i = P_i(x, 0, 0, s)$. Then there exist differential operators $R_{i,j}^k$ of order $< m_i + m_j - m_k$ so that $[P_i, P_j] = \sum_k R_{i,j}^k P_k$ $(i, j = 1, \cdots, r)$. Moreover, for each fixed x, the indicial equation

$$\overline{\mathscr{M}}\colon a_i(x, s) = 0 \qquad (i = 1, \cdots, r)$$

for $s \in \boldsymbol{C}^r$ has just m roots including their multiplicities. These roots are called characteristic exponents of $\mathscr{M}$.

In our case, the indicial equation is given by

$$\overline{\mathscr{M}}_\lambda^d\colon a_i(s) = 0 \qquad (i = 1, \cdots, l')$$

where $a_i(s)=p_i(\langle\rho, \tilde{\omega}_1\rangle-s_1, \cdots, \langle\rho, \tilde{\omega}_{l'}\rangle-s_{l'})-\chi_\lambda^d(D_i)$. Hence the indicial equation is constant on the edge G^d/P^d and there exist $|W|$ characteristic exponents

$$\lambda_w=(\langle\rho-w\lambda, \tilde{\omega}_1\rangle, \cdots, \langle\rho-w\lambda, \tilde{\omega}_{l'}\rangle) \tag{3.6}$$

parametrized by the elements $w \in W$. Moreover, the following statement holds:

For any point p of each wall Y_i, there exist differential operators S_i^j defined in a neighborhood of p such that the differential equation

$$\mathcal{M}_i\colon S_i u=0 \qquad \text{with } S_i=\sum_j S_i^j(D_j-\chi_\lambda^d(D_j))$$

has regular singularities along the hypersurface Y_i in the weak sense. Here "in the weak sense" means that by a coordinate transformation $t_i \mapsto t_i^k$ with a sufficiently large $k \in N$, $\mathcal{M}_i$ changes into a differential equation with regular singularities along Y_i in the original sense.

In fact, this is proved as follows: Fix an element g of G^d so that $gp \in \pi(\{1\}\times(-1, 1)^{l'})$. Then Proposition 11 in [11] assures that the map

$$\begin{array}{ccc} K/M\times(-1, 1)^{l'} & \longrightarrow & \tilde{X} \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ (kM, t) & \longmapsto & \pi((gk, t)) \end{array}$$

defines a local coordinate system in a neighborhood of p in $\tilde{X}$. Now it follows from Lemma 3.5 in [7] that there exist polynomials $S_i^1, \cdots, S_i^{l'}$ of $t_i\partial/\partial t_i$ such that the equation

$$\mathcal{M}_i\colon S_i u=0 \qquad \text{with } S_i=\sum_j S_i^j(D_j-\chi_\lambda^d(D_j))$$

has regular singularities in the weak sense along the hypersurface defined by $t_i=0$.

Under the following condition for a given $w \in W$ (c.f. (3.6))

$$\lambda_w-\lambda_{w'} \notin N^{l'}-\{0\} \qquad \text{for any } w' \in W, \tag{3.7}$$

we can define the map $\beta_{w\lambda}$ of taking the boundary value

$$\beta_{w\lambda}\colon \mathcal{A}(G^d/K^d; \mathcal{M}_\lambda^d)\longrightarrow\mathcal{B}(G^d/P^d; L(\lambda_w)) \tag{3.8}$$

by the method in [8]. Here for a $c=(c_1, \cdots, c_{l'}) \in C^{l'}$, $\mathcal{B}(G^d/P^d; L(c))$ is the space of all hyperfunction valued global sections of the line bundle

$$L(c)=(T^*_{Y_1}\tilde{X})^{\otimes c_1} \underset{G^d/P^d}{\otimes} \cdots \underset{G^d/P^d}{\otimes} (T^*_{Y_{l'}}\tilde{X})^{\otimes c_{l'}} \tag{3.9}$$

over G^d/P^d and $N=\{n \in Z \mid n \geq 0\}$.

Let $V=SP^d$ be an open subset of G^d/P^d and let $\mathscr{B}(V; L(c))$ be the space of all hyperfunction sections of $L(c)$ over the open set V. On the other hand, for any $\mu \in (\mathfrak{a}_{\mathfrak{p}}^d)_c^*$ we put

$$\mathscr{B}(V; L_\mu)=\{f \in \mathscr{B}(SP^d) \mid f(xman)=f(x)a^{\mu-\rho} \quad \text{for } x \in SP^d,\ m \in M^d,\ a \in A^d \text{ and } n \in N^{+d}\}. \tag{3.10}$$

Then the proof of Proposition 4.3 in [7] assures that $\mathscr{B}(V; L(\lambda_w))$ and $\mathscr{B}(V; L_{\lambda_w})$ are naturally isomorphic as local G^d-modules, which means the following. The isomorphism, say p, is given by their restrictions on $K^d \cap SP^d$ and if an element $g \in G^d$ and an open subset V_0 of G^d/P^d satisfy $gV_0 \subset V$, then $pg(f|_{V_0})=gp(f|_{V_0})$ for all $f \in \mathscr{B}(V; L(\lambda_w))$.

Hence under the assumption (3.7), we have a G^d-equivariant map

$$\beta_{w\lambda}\colon \mathscr{A}(G^d/K^d; \mathscr{M}_\lambda^d) \longrightarrow \mathscr{B}(G^d/P^d; L_{\lambda_w}). \tag{3.11}$$

When $w=1$, the condition (3.7) is always valid in view of (3.1) and the map β_λ mentioned before is obtained in this way. On the other hand, the condition (3.7) is too restrictive to define boundary values for our purpose and it is relaxed in [19] as in the following way.

Fix any point p in G^d/P^d and a coordinate neighborhood U of p in $\tilde{X}$. Put $V=U \cap G^d/P^d$. Then we can define $|W|$ maps

$$\beta_\lambda^w\colon \mathscr{A}(G^d/K^d; \mathscr{M}_\lambda^d) \longrightarrow \mathscr{B}(V) \qquad (w \in W) \tag{3.12}$$

under the fixed coordinate system (Definition 4.3 in [19]), where each β_λ^w corresponds to the characteristic exponent λ_w and if w satisfies (3.7), then $\beta_{w\lambda}|_V=\beta_\lambda^{w'}$ with a suitable $w' \in W$ satisfying $w\lambda=w'\lambda$. Fix a function u in $\mathscr{A}(G^d/K^d; \mathscr{M}_\lambda^d)$. If $\beta_\lambda^w(u)=0$ for all $w \in W$, then $u=0$ in a neighborhood of V (Theorem 4.3 in [19]) and therefore $u=0$ because u is real analytic. For a fixed $\lambda \in (\mathfrak{a}_{\mathfrak{p}}^d)_c^*$, there exists a semi-order $<_\lambda$ on W satisfying the following conditions (3.13), (3.14) and (3.15) (cf. Theorem 4.5 in [19]):

(3.13) If $\lambda_w-\lambda_{w'} \notin N^{l'}$ and $\lambda_{w'}-\lambda_w \notin N^{l'}$, then there exists no order between w and w'.

(3.14) If $\lambda_w-\lambda_{w'} \in N^{l'}-\{0\}$, then $w' <_\lambda w$.

(3.15) For any $w \in W$, putting

$$\mathscr{A}(V, G^d/K^d; \mathscr{M}_\lambda^d)_w=\{f \in \mathscr{A}(G^d/K^d; \mathscr{M}_\lambda^d) \mid \beta_\lambda^{w'}(u)=0 \text{ on } V \quad \text{for all } w' \in W \text{ with } w' <_\lambda w\},$$

the map β_λ^w induces the following map

$$\mathscr{A}(V, G^d/K^d; \mathscr{M}_\lambda^d)_w \longrightarrow \mathscr{B}(V; L(\lambda_w)) \tag{3.16}$$

whose definition does not depend on the choice of local coordinate systems.

Hence we have a linear map

$$\beta_w^V: \mathscr{A}(V, G^d/K^d; \mathscr{M}_\lambda^d)_w \longrightarrow \mathscr{B}(V; L_{w\lambda}) \tag{3.17}$$

which commutes with the local action of G^d.

The above consideration says the following: For any non-zero function $u \in \mathscr{A}(G^d/K^d; \mathscr{M}_\lambda^d)$ and for any open subset V of G^d/P^d, we can find at least one $w \in W$ so that u belongs to the domain of the above map β_w^V and moreover $\beta_w^V(u) \neq 0$. Furthermore assume $u \in \mathscr{A}_{H^d}(G^d/K^d; \mathscr{M}_\lambda^d)$ and assume u corresponds to a discrete series for G/H. Then it is an important problem to find such w by putting V an open H^d-orbit in G^d/P^d. This corresponds to an imbedding of the discrete series into a principal series for G/H, which will be discussed in a subsequent paper.

On the contrary, for any function $u \in \mathscr{A}(G^d/K^d; \mathscr{M}_\lambda^d)$ and for any $w \in W$, if we put

$$V = \{x \in G^d/P^d \mid \beta_\lambda^{w'}(u) = 0 \text{ in a neighborhood of } x \\ \text{for all } w' \in W \text{ with } w' <_\lambda w\},$$

then V is well-defined and $u \in V$ is in the domain of β_w^V.

Now for $u \in \mathscr{A}(G^d/K^d; \mathscr{M}_\lambda^d)$ and $w_0 \in W$, we define

(3.18) $\operatorname{supp} \beta_{w_0\lambda} u = \{x \in G^d/P^d \mid$ there exists $w \in W$ such that the function $\beta_\lambda^w(u)$ is not identically zero in any neighborhood of x and that $w\lambda = w_0\lambda$ or $w <_\lambda w_0\}$.

Then $\operatorname{supp} \beta_\mu(gu) = g(\operatorname{supp} \beta_\mu u)$ for any $g \in G^d$ and any $\mu \in W\lambda$. We remark that if there exists a non-trivial $w \in W$ with $w\lambda = \lambda$, then the support of $\beta_\lambda(u)$ is contained in $\operatorname{supp} \beta_\lambda u$ defined above, but may differ $\operatorname{supp} \beta_\lambda u$. In this paper, we always use the notation $\operatorname{supp} \beta_\lambda u$ in the above meaning.

Another important feature concerning boundary values is the concept of ideally analytic solutions. For an open subset V of G^d/P^d and a function $u \in \mathscr{A}(G^d/K^d; \mathscr{M}_\lambda^d)$, we say u is ideally analytic in a neighborhood of V if $\beta_\lambda^w(u)|_V$ is real analytic for any $w \in W$. Then in a neighborhood of V, u is of the following form (Theorem 5.3 in [19]):

$$u(x, t) = \sum_{\mu \in W\lambda} \sum_{i=1}^{m} a_{\mu,i}(x, t) t^{\rho-\mu} q_{\mu,i}(\log t). \tag{3.19}$$

Here m is a certain positive integer and (x, t) is a local coordinate system such that each Y_i is defined by $t_i=0$ and G^d/K^d is defined by $t_1>0$, $\cdots, t_{l'}>0$. The functions $a_{\mu,i}$ are real analytic in a neighborhood of V and

$$t^{\rho-\mu}=t_1^{\langle\rho-\mu,\tilde{\omega}_1\rangle}\cdots t_{l'}^{\langle\rho-\mu,\tilde{\omega}_{l'}\rangle} \tag{3.20}$$

and $q_{\mu,i}(\log t)$ are polynomials of $(\log t_1, \cdots, \log t_{l'})$. We can show that $m=|W_\lambda|$, where $W_\lambda=\{w \in W | \frac{1}{2}(\lambda_e-\lambda_w) \in N^{l'}\}$, and that $q_{\mu,i}$ are harmonic polynomials corresponding to W_λ, but we will not use this.

We will use the following fact for the above ideally analytic solution u, which is the result in Theorem 5.3 in [19]: Fix a $\nu \in W\lambda$ and assume

$$\begin{aligned} a_{\mu,i}=0 \quad &\text{for } i=1, \cdots, m \text{ and all } \mu \in W\lambda \text{ satisfying} \\ &(\langle\mu-\nu, \tilde{\omega}_1\rangle, \cdots, \langle\mu-\nu, \tilde{\omega}_{l'}\rangle) \in N^{l'}-\{0\}. \end{aligned} \tag{3.21}$$

Then the three conditions (3.22), (3.23) and (3.24) are equivalent:

$$\sum_{i=1}^{m} a_{\nu,i}(x, 0)t^{\rho-\nu}q_{\nu,i}(\log t)=0. \tag{3.22}$$

$$\sum_{i=1}^{m} a_{\nu,i}(x, t)t^{\rho-\nu}q_{\nu,i}(\log t)=0. \tag{3.23}$$

$$\beta_\lambda^w(u)=0 \qquad \text{for any } w \in W \text{ with } w\lambda=\nu. \tag{3.24}$$

Especially in the case when $\langle\lambda-w\lambda, \tilde{\omega}_i\rangle \notin Z$ for all $w \in W-\{1\}$ and $i=1, \cdots, l'$, we have $m=1$, $q_{\mu,i}=1$ and $\beta_\lambda^w(u)=a_{w\lambda,1}(x, 0)$ with the expression (3.19) and the condition $a_{\mu,1}|_{t_1=\cdots=t_{l'}=0}=0$ implies $a_{\mu,1}=0$.

Now we return to our problem to characterize $\eta(\mathscr{A}_K(G/H; \mathscr{M}_\lambda)\cap L^2(G/H))$. For $\alpha \in \Sigma(\mathfrak{a})$, we put $\mathfrak{g}(\mathfrak{a}; \alpha)=\{X \in \mathfrak{g} | [Y, X]=\alpha(Y)X$ for all $Y \in \mathfrak{a}\}$, $p_\alpha=\dim(\mathfrak{g}(\mathfrak{a}; \alpha)\cap\mathfrak{h}^a)$ and $q_\alpha=\dim\mathfrak{g}(\mathfrak{a}; \alpha)-p_\alpha$, and define a function $D(y)$ on $(0, \infty)^l$ by

$$D(y)=\prod_{\alpha\in\Sigma(\mathfrak{a})+}|y^\alpha-y^{-\alpha}|^{p_\alpha}(y^\alpha+y^{-\alpha})^{q_\alpha}.$$

Here we use the notatoin

$$y^\nu=y_1^{\langle\nu,\omega_1\rangle}\cdots y_l^{\langle\nu,\omega_l\rangle} \tag{3.25}$$

for any $\nu \in \mathfrak{a}_c^*$ (or $\in (\mathfrak{a}_\mathfrak{p}^d)_c^*$). Then the invariant measure $d\mu$ on G/H satisfies

$$\int_{G/H}\varphi d\mu=C\int_{K\times(0,\infty)^l}\varphi(ka(y)H)D(y)dk\frac{dy_1}{y_1}\cdots\frac{dy_l}{y_l}$$

for all continuous functions φ on G/H with compact support (c.f. p. 263 in [5]), where C is a positive constant, dk is the normalized Haar measure on K and $a(y) \in A$ is the one given by (3.4). Let $W(\mathfrak{a})$ denote the Weyl group of the root system $\Sigma(\mathfrak{a})$ and fix a representative $\bar{w} \in K$ for every $w \in W(\mathfrak{a})$ (c.f. [18], Lemma 7.2 in [20]). Then the above integral can be written in the following form

$$\int_{G/H} \varphi d\mu = C \sum_{w \in W(\mathfrak{a})} \int_{K \times (0,1)^l} \varphi(ka(y)\bar{w}H)D(y)dk \frac{dy_1}{y_1} \cdots \frac{dy_l}{y_l}. \tag{3.26}$$

We remark that there exist positive constants C_1 and C_2 so that

$$C_1 y^{-2\rho} \leq 1 + D(y) \leq C_2 y^{-2\rho} \qquad \text{for all } y \in (0, 2)^l. \tag{3.27}$$

By the map

$$\begin{array}{ccc} A_{\mathfrak{p}}^d \times A & \longrightarrow & A_{\mathfrak{p}}^d \\ \cup\!\!\!\!\! & & \cup\!\!\!\!\! \\ (a_t, a(y)) & \longmapsto & a_t a(y), \end{array} \tag{3.28}$$

any function f on $A_{\mathfrak{p}}^d$ can be lifted to a function $\tilde{f}$ on $A_{\mathfrak{p}}^d \times A$. We will express it by using the identifications (3.3) and (3.4). If $a_t = a(y)$, then $t_i = \exp(-\langle \tilde{\alpha}_j, \log a(y) \rangle) = \exp(\langle \tilde{\alpha}_i, \sum_j (\log y_j)\omega_j \rangle) = \Pi_j y_j^{\langle \tilde{\alpha}_i, \omega_j \rangle}$. Therefore we have

$$\tilde{f}(t, y) = f(t_1 \Pi_j y_j^{\langle \tilde{\alpha}_1, \omega_j \rangle}, \cdots, t_l \Pi_j y_j^{\langle \tilde{\alpha}_l, \omega_j \rangle}). \tag{3.29}$$

We remark that

$$\langle \tilde{\alpha}_i, \omega_j \rangle = \begin{cases} 1 & \text{if } \tilde{\alpha}_i|_{\mathfrak{a}} = \alpha_j, \\ 0 & \text{otherwise,} \end{cases}$$

and that

$$\widetilde{t^\nu} = t^\nu y^\nu \qquad \text{for all } \nu \in (\mathfrak{a}_{\mathfrak{p}}^d)_c^*.$$

Let p be any boundary point of the subset AK^d of G^d/K^d in $\tilde{X}$. Then any $u \in \mathscr{A}_{H^d}(G^d/K^d; \mathscr{M}_\lambda^d)$ is ideally analytic in a neighborhood of p, which will be proved later, and thus we have an expression for u as is given in (3.19). Especially, the point $a_t K^d$ in G^d/K^d converges to a point in G^d/P^d when $t \to 0$, which belongs to an open H^d-orbit in G^d/P^d. Hence it is expected that if $\eta^{-1}u \in L^2(G/H)$, then some terms in the expression (3.19) should vanish. This means the vanishing of the corresponding boundary values on the open H^d-orbit. In fact, we have

Proposition 2. *Let f be an element of $\mathscr{A}_K(G/H; \mathscr{M}_\lambda)$. Then f belongs* *to* $L^2(G/H)$ *if and only if* supp $\beta_\mu f^\eta$ *contains no inner points in* G^d/P^d *for any* $\mu \in W\lambda$ *which satisfies the condition*

$$(\mathrm{Re}\langle\mu, \omega_1\rangle, \cdots, \mathrm{Re}\langle\mu, \omega_l\rangle) \notin (-\infty, 0)^l. \tag{3.30}$$

Proof. Let $\{f_1, \cdots, f_r\}$ be a basis of the linear span of $\{\pi_k(f) \mid k \in K\}$. In general, we will denote by π_g an action of an element $g \in G_c$ on a function space. The induced action by the Lie algebra is also denoted by $\pi_X (X \in \mathfrak{g}_c)$. Here $(\pi_k f)(x) = f(k^{-1}x)$. Let u be the column vector formed by $f_1^\eta, \cdots, f_r^\eta$ and put supp $\beta_\mu u = \bigcup_i$ supp $\beta_\mu f_i^\eta$. Since f^η is H^d-finite, supp $\beta_\mu f^\eta$ is a union of H^d-orbits in G^d/P^d and supp $\beta_\mu u =$ supp $\beta_\mu f^\eta$. Moreover we remark that if $f \in L^2(G/H)$, then

$$f_i \in L^2(G/H) \qquad \text{for } i = 1, \cdots, r.$$

First suppose there exist an open H^d-orbit V in G^d/P^d, an element $\mu \in W\lambda$ and an index $i_0 \in \{1, \cdots, l\}$ such that $\mathrm{Re}\langle\mu, \omega_{i_0}\rangle \geq 0$ and supp $\beta_\mu u \supset V$. We want to prove that $f \notin L^2(G/H)$, which means the condition in Proposition 2 is necessary for f to be in $L^2(G/H)$. We may assume without loss of generality that $i_0 = 1$ and that if $\nu \in W\lambda$ satisfies $\mathrm{Re}\langle\nu - \mu, \omega_1\rangle > 0$, then supp $\beta_\nu u \cap V = \phi$.

The H^d-orbits in G^d/P^d are completely parametrized by [9] (c.f. § 4). It follows from the condition (3.2) that there exists a representative $\bar{w} \in G^d$ of an element w of W such that $V = H^d \bar{w} P^d$ and $\bar{w} A \bar{w}^{-1} = A$. We put $p = \bar{w} P^d$.

We note that u satisfies

$$\mathscr{M}: \begin{cases} \pi_X(u) = A(X)u & \text{for any } X \in \mathfrak{h}^d, \\ D_i u = \chi_\lambda^d(D_i)u & \text{for } i = 1, \cdots, l', \end{cases} \tag{3.31}$$

where $A(X)$ is an $r \times r$-matrix and π_X is the differential operator corresponding to the vector field v_X by the action of $\exp(-tX)$ on $\tilde{X} (t \in \boldsymbol{R})$. Since p belongs to an open H^d-orbit in G^d/P^d, $T_p(G^d/P^d) = \{(v_X)_p \mid X \in \mathfrak{h}^d\}$ and hence the equation (3.31) satisfies the condition $\mathrm{SS}\,\tilde{\mathscr{M}} \mid \pi^{-1}(G^d/P^d) \subset T^*_{G^d/P^d}\tilde{X}$ in Theorem 5.2 in [19]. Therefore u is ideally analytic in a neighborhood of p.

Let $X_1, \cdots, X_n$ be elements of $\mathfrak{h}^d$ so that $n = \dim G^d/P^d$ and

$$T_p(G^d/P^d) = \sum_i \boldsymbol{R}(v_{X_i})_p$$

and moreover the map $(-1, 1)^n \ni x \mapsto \exp(\sum x_i X_i)\bar{w}P^d \in G^d/P^d$ defines an into diffeomorphism. Then for an $\varepsilon > 0$ the map

$$(3.32)\qquad \begin{array}{ccc} (-\varepsilon, \varepsilon)^{n+l'} & \longrightarrow & \tilde{X} \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ (x, t) & \longmapsto & \pi((\exp(\sum x_i X_i)\bar{w}, t)) \end{array}$$

defines a local coordinate system and we have the expression

$$(3.33)\qquad u(x, t)=\sum_{\nu\in W\lambda}\sum_{i=1}^{m} a_{\nu,i}(x, t)t^{\rho-\nu}q_{\nu,i}(\log t)$$

for $(x, t)\in(0, \varepsilon)^{n+l'}$. By the assumption we have $a_{\nu,i}=0$ if $\mathrm{Re}\langle\nu-\mu, \omega_1\rangle>0$ and

$$\sum_{\mathrm{Re}\langle\nu-\mu,\omega_1\rangle=0}\sum_{i=1}^{m} a_{\nu,i}(x, 0)t^{\rho-\nu}q_{\nu,i}(\log t)\neq 0.$$

Here the vectors $a_{\nu,i}$ are analytic in a neighborhood of $\bar{w}P^d$ and $q_{\nu,i}$ are certain polynomials. Put $c=\mathrm{Re}\langle\rho-\mu, \omega_1\rangle$, which is not larger than $\langle\rho, \omega_1\rangle$, and $I=\{\langle\rho-\nu, \omega_1\rangle\,|\,\nu\in W\lambda$ and $\mathrm{Re}\langle\rho-\nu, \omega_1\rangle\geq c\}$ and $I_0=\{\xi\in I\,|\,\mathrm{Re}\,\xi=c\}$. Then it follows from (3.29) that

$$u(\exp(\sum x_i X_i)\bar{w}a_t\exp(y_1\omega_1))=\sum_{\xi\in I}\sum_{j=0}^{m'} b_{\xi,j}(x, t, y_1)y_1^{\xi}(\log y_1)^j,$$

where m' is a suitable non-negative integer and $b_{\xi,j}(x, t, y_1)$ are vectors of functions which are analytic in $(0, 3\varepsilon_1)^{n+l'}\times(-\varepsilon_1, 2\varepsilon_1)$ with a small positive number ε_1 and the function

$$\sum_{\xi\in I_0}\sum_{j=0}^{m'} b_{\xi,j}(x, t, 0)y_1^{\xi}(\log y_1)^j$$

is not identically zero. Let h be the smallest integer so that $b_{\xi,j}|_{y_1=0}=0$ for all $\xi\in I_0$ and $j=h+1, \cdots, m'$. Then for a suitable positive number C, we have the following uniform estimate

$$\begin{aligned}&|u(\exp(\sum x_i X_i)\bar{w}a_t\exp(y_1\omega_1))-\sum_{\xi\in I_0} b_{\xi,h}(x, t, 0)y_1^{\xi}(\log y_1)^h|\\ &\quad<Cy_1^{c}|\log y_1|^{h-1}\end{aligned}$$

for all $(x, t, y_1)\in(\varepsilon_1, 2\varepsilon_1)^{n+l'}\times(0, \varepsilon_1)$. Here for a vector u, $|u|$ means the maximum of the absolute values of the components of u. Choose $(x(0), t(0))\in(\varepsilon_1, 2\varepsilon_1)^{n+l'}$ so that $b_{\xi,h}(x(0), t(0), 0)\neq 0$ for a suitable $\xi\in I_0$. Put $a_{t(0)}=a'a(y(0))$ with an $a'\in A^d\cap H^d$ and moreover put $\varepsilon_2=y(0)_1\varepsilon_1$ and $h_0=(\exp(\sum x(0)_i X_i))\bar{w}a'\bar{w}^{-1}$. We remark that $h_0\in H^d$ because $\mathrm{Ad}(\bar{w})(\mathfrak{h}^d\cap\mathfrak{a}_{\mathfrak{p}}^d)=\mathfrak{h}^d\cap\mathfrak{a}_{\mathfrak{p}}^d$. Then we can choose a positive number C', an open neighborhood V_0 of $(y(0)_2, \cdots, y(0)_l)$ in $(0, \infty)^{l-1}$ and vectors of analytic functions $b'_{\xi}(y')$ on V_0 such that

(3.34) $$|u(h_0\bar{w}a(y)) - \sum_{\xi \in I_0} b'_\xi(y')y_1^\xi(\log y_1)^h| < C'y_1^c|\log y_1|^{h-1}$$

for all $y=(y_1, y') \in (0, \varepsilon_2)\times V_0$ and moreover $\sum_{\xi \in I_0} b'_\xi(y')y_1^\xi$ is not identically zero.

Since $\pi_{h_0^{-1}}(u)=Tu$ with an invertible matrix T, we have the same estimate for u with $h_0=1$ if we replace b'_ξ and C' by other analytic functions and a positive number, respectively. Hence we may assume $h_0=1$ in the estimate (3.34). Moreover we remark that $\sum_{\xi \in I_0} b'_\xi(y')y_1^\xi$ is still not identically zero.

Let $\tilde{f}$ be the column vector formed by $f_1, \cdots, f_r$. Since $\bar{w} \in K^d$ and $\bar{w}A\bar{w}^{-1}=A$, we have the estimate

$$|\tilde{f}(k\bar{w}a(y)\bar{w}^{-1}H) - T(k^{-1})\sum_{\xi \in I_0} b'_\xi(y')y_1^\xi(\log y_1)^h| < C'y_1^c|\log y_1|^{h-1}$$

for all $k \in K$, $(y_1, y') \in (0, \varepsilon_2)\times V_0$. Here $T(k^{-1})$ are the invertible matrices determined by $\pi_k(\tilde{f})=T(k)\tilde{f}$. Moreover we remark the following: There exists a $w' \in W(\mathfrak{a})$ so that $\bar{w}'a\bar{w}'^{-1}=\bar{w}a\bar{w}^{-1}$ for all $a \in A$. There exists a point $y'(0) \in V_0$ so that $\sum_{\xi \in I_0} b'_\xi(y'(0))y_1^\xi(\log y_1)^h \neq 0$. Moreover, $T(k^{-1})$ and $b'_\xi(y')$ are real analytic functions, and $\mathrm{Re}\,\xi = c \leq \langle \rho, \omega_1 \rangle$.

Combining the above estimate with (3.26) and (3.27), we can conclude that at least one of f_i does not belong to $L^2(G/H)$. Since f_i belongs to the linear span of $\{\pi_k f | k \in K\}$, this means $f \notin L^2(G/H)$. Thus we have proved that if $f \in L^2(G/H) \cap \mathscr{A}_K(G/H; \mathscr{M}_\lambda)$, then supp $\beta_\mu f^\eta$ contains no inner points in G^d/P^d for all $\mu \in W\lambda$ which satisfy (3.30).

Next we will prove the inverse part of Proposition 2. Fix $w_0 \in W(\mathfrak{a})$ arbitrarily and fix a representative $\bar{w} \in K^d$ of an element w of $W(\mathfrak{a}_\mathfrak{p}^d)$ so that $\bar{w}a\bar{w}^{-1}=\bar{w}_0^{-1}a\bar{w}_0$ for all $a \in A$ (c.f. [10] or Lemma 7.2 in [21]). Choose $\mu_0 \in \mathfrak{a}^*$ so that supp $\beta_\mu f^\eta \not\supset H^d\bar{w}P^d$ for all $\mu \in W\lambda$ which satisfy

(3.35) $$(\mathrm{Re}\langle \mu-\mu_0, \omega_1\rangle, \cdots, \mathrm{Re}\langle \mu-\mu_0, \omega_l\rangle) \notin (-\infty, 0]^l.$$

In fact the best possible μ_0 is given by $\mu_0=\sum C_i\alpha_i$ with

$$C_i = \max\{\mathrm{Re}\langle \mu, \omega_i\rangle | \mu \in W\lambda \text{ and supp } \beta_\mu f^\eta \supset H^d\bar{w}P^d\}.$$

Then we will prove that there exist $C>0$ and $m \in N$ such that

(3.36) $$|f(ka(y)\bar{w}_0H)| \leq Cy^{\rho-\mu_0}(1-\log y)^m$$

for all $y \in (0, 2]^l$ and $k \in K$. Here $(1-\log y)^m = (1-\log y_1)^m \cdots (1-\log y_l)^m$.

If supp $\beta_\mu f^\eta$ contains no inner points in G^d/P^d for all $\mu \in W\lambda$ which satisfy (3.30), then we can choose $\mu_0 = -\varepsilon\sum\alpha_i$ for a suitable positive number ε and therefore it follows from (3.26), (3.27) and (3.36) that $f \in L^2(G/H)$.

We will show

$$(3.37) \qquad |u(\bar{w}a(y))| \leq Cy^{\rho-\mu_0}(1-\log y)^m$$

for all $y \in (0, 2]^l$ with certain positive numbers C and m, which implies (3.36) because $\tilde{f}(k\bar{w}_0^{-1}a(y)\bar{w}_0H)=T(k^{-1})u(\bar{w}a(y))$. Since $[0, 2]^l$ is compact, it is sufficient to show that for any point $p \in [0, 2]^l$, there exists a neighborhood $U(p)$ of p such that (3.37) holds for all $y \in U(p) \cap (0, 2]^l$.

Consider in a small neighborhood of $\bar{w}P^d$. Then as we have seen, the expression (3.33) holds under the local coordinate system (3.32). The condition for μ_0 shows that $a_{\mu,i}=0$ for all $\mu \in W\lambda$ which satisfy (3.35). From (3.28) and (3.29) we have

$$u(\exp(\sum x_iX_i)\bar{w}a_ta(y)) = \sum_{\nu \in W\lambda} \sum_{i=1} a'_{\nu,i}(x, t, y)t^{\rho-\nu}y^{\rho-\nu}q'_{\nu,i}(\log(t), \log(y))$$

where $q'_{\nu,i}$ are polynomials and $a'_{\nu,i}$ are real analytic in $(-\varepsilon_1, 2\varepsilon_1)^{n+l'+l}$ with a certain positive number ε_1. Therefore by the same argument as before (c.f. (3.34)), we can find $h_0 \in H^d$, $C>0$, $\varepsilon_2>0$ and $m' \in \boldsymbol{N}$ such that $|u(h_0\bar{w}a(y))| \leq Cy^{\rho-\mu_0}(1-\log y)^{m'}$ for all $y \in (0, \varepsilon_2)^l$. Since $u(h_0\bar{w}a(y))= Tu(\bar{w}a(y))$ with an invertible matrix T, we have the estimate (3.37) in a neighborhood of $p=(0, \cdots, 0) \in [0, 2]^l$.

For any $y \in \boldsymbol{R}^l$, we define $y^*=(y_1^*, \cdots, y_{l'}^*) \in \boldsymbol{R}^{l'}$ by $y_i^*=1$ if $\tilde{\alpha}_i|_{\mathfrak{a}}=0$, and $y_i^*=t_j$ if there exists an $\alpha_j \in \Psi(\mathfrak{a})$ such that $\tilde{\alpha}_i|_{\mathfrak{a}}=\alpha_j$. Then it is easy to see that $a_{y^*}=a(y)$ for any $y \in (0, \infty)^l$ (c.f. (3.3) and (3.4)).

Let V_0 be the set of all $y \in [0, 2]^l$ such that (3.37) holds in a neighborhood of y with suitable positive numbers C and m. It is clear that V_0 is open. Suppose $V_0 \neq [0, 2]^l$ and choose $p=(p_1, \cdots, p_l) \in [0, 2]^l$ so that $p \notin V_0$. Put $p^*=(p_1^*, \cdots, p_{l'}^*)$. We may assume without loss of generality that $p_1=\cdots=p_k=0$, $p_{k+1}\neq 0$, $p_{k+2}\neq 0$, $\cdots$, $p_l\neq 0$, $p_1^*=\cdots=p_{k'}^*=0$, $p_{k'+1}^*\neq 0, \cdots, p_{l'}^*\neq 0$. Put $Y(k)=\{y \in [0, 2]^l \mid y_1=\cdots=y_k=0\}$ and $q= \pi((\bar{w}, p^*)) \in \tilde{X}$. Then as we have seen before, there exist differential operators S_i defined in a neighborhood of q such that each S_i has regular singularities (in the weak sense) along the hypersurface Y_i $(i=1, \cdots, k')$. Then the system $S_iu=0$ $(i=1, \cdots, k')$ has regular singularities (in the weak sense) along the set of walls $\{Y_1, \cdots, Y_{k'}\}$. Moreover the following statement is valid:

(3.38) $\quad u$ is ideally analytic in a neighborhood of $\pi((\bar{w}, p^*))$.

We will continue the proof of Proposition 2 and the proof of (3.38) is given after that. By the expression of the ideally analytic solution u in a neighborhood of q, we have an expression

$$u(a(y))=\sum_{j=1}^{k}\sum_{i_j=1}^{m}\sum_{\nu}a_\nu^i(y)y_1^{\nu_1}\cdots y_k^{\nu_k}(\log y_1)^{i_1}\cdots(\log y_k)^{i_k},$$

where $\nu=(\nu_1, \cdots, \nu_k)$ runs through a finite subset of C^k and a_ν^i are analytic in a neighborhood $U(p)$ of p. It follows from this expression that the assumption $p\notin V_0$ implies $p'\notin V_0$ for all $p'\in U(p)\cap Y(k)$, which means $V_0\cap Y(k)$ is closed and not equal to $Y(k)$. Since V_0 is open in $[0, 2]^l$ and contains $(0, \cdots, 0)\in[0, 2]^l$, $V_0\cap Y(k)$ is also open in $Y(k)$ and not empty, which leads a contradiction because $Y(k)$ is connected. Thus we can conclude $V_0=[0, 2]^l$.

Now we will prove (3.38). Put $a=a_{p^*}$ and $\Theta=\Theta_{p^*}$. Then $a\in A$ and $G^d\pi((\bar{w}, p^*))\simeq G^d/P_\Theta^d(K)$. Identify the tangent space of G^d at a point with $\mathfrak{g}^d$ by means of the right translation. Also identify the dual space of $\mathfrak{g}^d$ with itself by the Killing form $\langle\ ,\ \rangle$. Put $q=\bar{w}aP_\Theta^d(K)$. Let $\mathfrak{m}_\Theta^d$, $\mathfrak{m}_\Theta^d(K)$, $\mathfrak{a}_\Theta^d$, $\mathfrak{n}_\Theta^{+d}$ and $\mathfrak{p}_\Theta^d(K)$ be Lie algebras of M_Θ^d, $M_\Theta^d(K)$, A_Θ^d, N_Θ^{+d} and $P_\Theta^d(K)$, respectively, and put $\mathfrak{n}_\Theta^{-d}=\sigma(\mathfrak{n}_\Theta^{+d})$. Then the cotangent vector space $T_q^*(G^d/P_\Theta^d(K))$ at q is identified with $V(q)=\{X\in\mathfrak{g}^d \mid \langle X, Y\rangle=0$ for all $Y\in\mathfrak{p}_\Theta^d(K)\}$. By the direct sum decomposition $\mathfrak{g}^d=\mathfrak{n}_\Theta^{-d}\oplus(\mathfrak{p}^d\cap\mathfrak{m}_\Theta^d)\oplus\mathfrak{m}_\Theta^d(K)\oplus\mathfrak{a}_\Theta^d\oplus\mathfrak{n}_\Theta^{+d}$ we have $V(q)=\mathfrak{n}_\Theta^{+d}\oplus(\mathfrak{p}^d\cap\mathfrak{m}_\Theta^d)$. We will prove that the system (3.31) satisfies the condition

$$\text{(3.39)}\qquad \mathrm{SS}\,\mathscr{M}\cap T^*_{\pi((\bar{w},a))}\tilde{X}\subset T^*_{G^d\pi((\bar{w},a))}\tilde{X},$$

which implies u is ideally analytic by Theorem 5.2 in [19]. Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}_c$. Then the principal symbol of any differential operator on G^d is regarded as a $S(\mathfrak{g})$-valued function on G^d. Considering the system (3.31), if the system of the equation for $v^*\in V(q)$

$$\begin{cases}\langle \mathrm{Ad}(\bar{w}a)^{-1}X, v^*\rangle=0 & \text{for all } X\in\mathfrak{h}^d,\\ \langle v^*, v^*\rangle=0\end{cases}$$

means $v^*=0$, then (3.39) holds. (The second equation comes from the Casimir operator of $U(\mathfrak{g}^d)$.) Since $\langle\ ,\ \rangle|_{\mathfrak{n}_\Theta^{+d}}=0$ and $\langle\ ,\ \rangle|_{\mathfrak{m}_\Theta^d\cap\mathfrak{p}^d}$ is positive definite and $\langle X, Y\rangle=0$ for any $X\in\mathfrak{n}_\Theta^{+d}$ and $Y\in\mathfrak{m}_\Theta^d\cap\mathfrak{p}^d$, we have only to prove that if $v^*\in\mathfrak{n}_\Theta^{+d}$ satisfies $\langle\mathrm{Ad}(\bar{w}a)^{-1}X, v^*\rangle=0$ for all $X\in\mathfrak{h}^d$, then $v^*=0$. On the other hand $\bar{w}aP^d$ belongs to an open H^d-orbit in G^d/P^d, which means $\mathrm{Ad}(\bar{w}a)^{-1}\mathfrak{h}^d+\mathfrak{p}^d=\mathfrak{g}^d$. Since $v^*\in(\mathfrak{p}_\Theta^d)^\perp\cap(\mathrm{Ad}(\bar{w}a)^{-1}\mathfrak{h}^d)^\perp\subset(\mathfrak{p}^d)^\perp\cap(\mathrm{Ad}(\bar{w}a)^{-1}\mathfrak{h}^d)^\perp\subset(\mathfrak{p}^d+\mathrm{Ad}(\bar{w}a)^{-1}\mathfrak{h}^d)^\perp\subset(\mathfrak{g}^d)^\perp=\{0\}$, we have $v^*=0$. Thus we have completed the proof of Proposition 2. Q.E.D.

Thus we have replaced the L^2-estimate by the vanishing of certain boundary values. On the other hand, the map β_λ is bijective. Therefore in principle, if we know $\beta_\lambda(f^\eta)$, we can know supp $\beta_\mu f^\eta$ for all $\mu\in W\lambda$.

The following lemmas estimate supp $\beta_\mu f^\eta$ in terms of supp $\beta_\lambda f^\eta$:

Lemma 1. *Suppose* $f \in \mathcal{A}(G^d/K^d; \mathcal{M}_\lambda^d)$, $\mu \in W\lambda$ *and* $xP^d \in \operatorname{supp} \beta_\mu f$. *Let* Θ *be a subset of* $\Psi(\mathfrak{a}_\mathfrak{p}^d)$. *Then for every* $yP^d \in xM_\Theta^d P^d$, *there exists a* $\nu \in W\lambda$ *satisfying the following two conditions.*

(i) $\langle \nu-\mu, \tilde{\omega}_i \rangle \in \{0, 1, 2, \cdots\}$ *for all* i *satisfying* $\tilde{\alpha}_i \in \Psi(\mathfrak{a}_\mathfrak{p}^d)-\Theta$.

(ii) $yP^d \in \operatorname{supp} \beta_\nu f$.

Lemma 2. *Let* λ *be an element of* $(\mathfrak{a}_\mathfrak{p}^d)_c^*$ *satisfying* $\langle \lambda, \tilde{\alpha} \rangle > 0$ *for all* $\tilde{\alpha} \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ *and let* f *be an element of* $\mathcal{A}(G^d/K^d; \mathcal{M}_\lambda^d)$ *which satisfies* $\operatorname{supp} \beta_\lambda f \subset SP^d$ *with a subset* S *in* G^d. *Let* μ *be an element of* $W\lambda$. *For each* $w \in W$, *we fix an expression* $w = w_{\gamma_{L(w)}} \cdots w_{\gamma_1}$ *as a product of reflections with respect to simple roots in* $\Psi(\mathfrak{a}_\mathfrak{p}^d)$ *and put* $S(w) = SM_{\{\gamma_1\}}^d \cdots M_{\{\gamma_{L(w)}\}}^d P^d$. *For any* $\mu \in W\lambda$, *we put*

$$W(\mu) = \{w \in W \mid \langle w\lambda-\mu, \tilde{\omega}_i \rangle \in \{0, 1, 2, \cdots\} \text{ for } i=1, \cdots, l'\}.$$

Then we have

$$\operatorname{supp} \beta_\mu f \subset \bigcup_{w \in W(\mu)} S(w). \tag{3.40}$$

Before we prove these lemmas we give a rather general statement for solutions of differential equations:

(3.41) *Let* $\tau: X \to Y$ *be a smooth map between real analytic manifolds (i.e. the tangent map* $(\tau_*)_p: T_pX \to T_{\tau(p)}Y$ *is surjective for any point* p *in* X). *Then a system of differential equations on* X

$$\mathcal{M}: P_i u = 0 \qquad (i=1, \cdots, r)$$

is called elliptic along the fiber of τ *if*

$$\mathrm{SS}\, \mathcal{M} \subset T_{\tau^{-1}(q)}^* X \qquad \text{for any point } q \text{ in } Y. \tag{3.42}$$

Then the support of any hyperfunction solution of $\mathcal{M}$ *defined on* X *is a union of connected components of fibres of* τ.

Let u be a hyperfunction solution of $\mathcal{M}$. For any $q \in Y$, we will show that supp $u|_{\tau^{-1}(q)}$ is open in $\tau^{-1}(q)$, which clearly implies (3.41). Taking a local coordinate system it is sufficient to prove the following:

(3.43) Put $X = \{(x, y) \in \boldsymbol{R}^{m+n}; \sum x_i^2 < 1, \sum y_j^2 < 1\}$ and $Y = \{y \in \boldsymbol{R}^n; \sum y_j^2 < 1\}$ and let $\tau: X \to Y$ be the natural projection. Let $u \in \mathcal{B}(X)$ be a solution of a system $\mathcal{M}$ which is elliptic along the fibre of τ. If supp $u \not\ni (0, 0)$, then supp $u \not\ni (x, 0)$ for any $(x, 0) \in X$.

To prove (3.43), we suppose supp $u \not\ni (0, 0)$ and supp $u \cap \tau^{-1}(0) \neq \phi$. We choose a positive number $\varepsilon < 1$ so that

$$\text{supp } u \cap \{(x, y) \in X; \sum x_i^2 + \sum y_j^2 < 2\varepsilon\} = \phi.$$

Define polynomials $h_t = t \sum x_i^2 + \sum y_j^2 - \varepsilon$ for $t \in \boldsymbol{R}$ and put $H_t = \{(x, y) \in X; h_t = 0\}$. Then the assumption implies the existence of $C > 0$ such that $H_C \subset X$, $H_C \cap \text{supp } u \neq \phi$ and $H_t \cap \text{supp } u = \phi$ if $t > C$. Fix a point $p = (x^*, y^*) \in H_C \cap \text{supp } u$. Since $x^* \neq 0$, the set $T^*_{H_C} X \cap T^*_{\tau^{-1}(0)} X \cap T^*_p X$ equals $\{0\}$. Then by Holmgren's theorem for hyperfunctions and Sato's fundamental theorem for the solution u of $\mathcal{M}$, we conclude supp $u \not\ni p$ because $u = 0$ on the set defined by $h_C < 0$. This is a contradiction.

For a subset Θ of Ψ we defined a subgroup $P^d_\Theta(K)$. We may assume $\Theta = \{\tilde{\alpha}_1, \cdots, \tilde{\alpha}_k\}$. We identify the homogeneous space $G^d/P^d_\Theta(K)$ with a boundary component of G^d/K^d in $\tilde{X}$ by the map $G^d/P^d_\Theta(K) \ni gP^d_\Theta(K) \mapsto \pi((g, \varepsilon)) \in \tilde{X}$, where we put $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_{l'})$ and $\varepsilon_1 = \cdots = \varepsilon_k = 0$, $\varepsilon_{k+1} = \cdots = \varepsilon_{l'} = 1$. Fix a point $p \in G^d/P^d_\Theta(K) \subset \tilde{X}$ and a non-zero function $f \in \mathcal{A}_{H^d}(G^d/K^d; \mathcal{M}^d_\lambda)$. Since the system $S_j u = 0$ $(j = 1, \cdots, k)$ has regular singularities (in the weak sense) along the set of walls $\{Y_1, \cdots, Y_k\}$, as we have proved, we can define boundary values $\beta'_i(f)$ of f on a neighborhood V of p in $G^d/P^d_\Theta(K)$. These β'_i correspond to the characteristic exponents $\nu'_i = (\nu'_{i,1}, \cdots, \nu'_{i,k}) \in \boldsymbol{C}^k$ (including their multiplicities) of the system $(i = 1, \cdots, N)$. Fix a characteristic exponent $\mu' \in \{\nu'_1, \cdots, \nu'_N\}$ satisfying (1) $\beta'_i(f) = 0$ if $\mu' - \nu'_i \in \boldsymbol{N}^k - \{0\}$ and (2) $\beta'_i(f) \neq 0$ for a suitable i with $\mu' = \nu'_i$, Then at least one of $\beta'_i(f)$ with $\nu'_i = \mu'$, which may be assumed to be $\beta'_1(f)$, is non-zero and defines a hyperfunction valued section of a line bundle $(T^*_{Y_1}\tilde{X})^{\otimes \nu'_{1,1}} \otimes_V \cdots \otimes_V (T^*_{Y_k}\tilde{X})^{\otimes \nu'_{1,k}}$. Its definition does not depend on the choice of local coordinate systems.

Let Δ be the Laplace-Beltrami operator on G^d/K^d. We claim that the equation for $\beta'_1(f)$ induced by the equation $(\Delta - \chi^d_\lambda(\Delta))f = 0$ is elliptic along the fibre of the natural projection τ: $G^d/P^d_\Theta(K) \to G^d/P^d_\Theta$. In fact the induced equation is given by Theorem 6.1 ii) in [19]. It follows from the theorem that the principal symbol of the induced equation equals that of the operator $\tilde{\Delta}$ on G^d/P^d induced by the Casimir operator of $U(\mathfrak{g}^d)$. Then as in the last part of the proof of Proposition 2, we identify $T^*_p(G^d/P^d_\Theta(K))$ with $\mathfrak{n}^{+d}_\Theta \oplus (\mathfrak{m}^d_\Theta \cap \mathfrak{p}^d)$. Hence

$$T^*_{\tau^{-1}(\tau(p))}(G^d/P^d_\Theta(K)) \cap T^*_p(G^d/P^d_\Theta(K))$$

is identified with $\mathfrak{n}^{+d}_\Theta$ because $T_p P^d_\Theta \simeq \mathfrak{m}^d_\Theta \oplus \mathfrak{a}^d_\Theta \oplus \mathfrak{n}^{+d}_\Theta$. On the other hand, the zeros of the principal symbol of $\tilde{\Delta}$ in $T^*_p(G^d/P^d_\Theta(K))$ are given by

$$\{v^* \in \mathfrak{n}_\Theta^{+d} \oplus (\mathfrak{m}_\Theta^d \cap \mathfrak{p}^d) \,|\, \langle v^*, v^* \rangle = 0\}.$$

Since $\langle\ ,\ \rangle$ is positive definite on $(\mathfrak{m}_\Theta^d \cap \mathfrak{p}^d)$ and since $(\mathfrak{n}_\Theta^{+d})^\perp$ contains $\mathfrak{n}_\Theta^{+d}$ and $\mathfrak{m}_\Theta^d \cap \mathfrak{p}^d$, the condition (3.42) is clear in this case.

Applying (3.41) to our situation, we get the following: Let p and p' be points in $G^d/P_\Theta^d(K) \subset \tilde{X}$ which satisfy $\tau(p)=\tau(p')$. Fix a characteristic exponent $\nu' \in \{\nu_1', \cdots, \nu_N'\}$ of the system $S_i=0$ $(i=1, \cdots, k)$ so that (1) $p \notin \operatorname{supp} \beta_j'(f)$ if $\nu'-\nu_j' \in N^k-\{0\}$ and (2) $p \in \operatorname{supp} \beta_j'(f)$ for at least one j satisfying $\nu_j'=\nu'$. Then the same statements (1) and (2) also hold for p'.

Proof of Lemma 1. Retain the above notation and suppose $\operatorname{supp} \beta_\mu f \ni xP^d$. Fix a sufficiently small neighborhood U of p in $\tilde{X}$. Then Corollary 6.3 in [19] says the following. There exists a $\nu' \in \{\nu_1', \cdots, \nu_N'\}$ so that (0) $(\langle \rho-\mu, \tilde{\omega}_1 \rangle, \cdots, \langle \rho-\mu, \tilde{\omega}_k \rangle)-\nu' \in N^k$ and (1) $\beta_i'(f)=0$ on $U \cap G^d/P_\Theta^d(K)$ for all i satisfying $\nu'-\nu_i' \in N^k-\{0\}$ and (2) the closure of $\operatorname{supp} \beta_j'(f)$ in $\tilde{X}$ contains xP^d for at least one j with $\nu'=\nu_j'$. We may assume that we can choose the above j equals 1 and that $\beta_1'(f)$ is defined coordinate free.

Let c be a sufficiently small positive number so that $\pi((x, s\varepsilon)) \in U$ for all $s \in [0, c]$. We remark that $\operatorname{supp} \beta_1'(f)$ contains $\pi((x, c\varepsilon))$. In fact, if $\operatorname{supp} \beta_1'(f) \not\ni \pi((x, c\varepsilon))$, there exists a neighborhood V of x in G^d satisfying $\operatorname{supp} \beta_1'(f) \cap \{(\pi((g, c\varepsilon)) \,|\, g \in V\} = \phi$ and therefore it follows from the unique continuation property of $\beta_1'(f)$ along the fibre of τ that

$$\operatorname{supp} \beta_1'(f) \cap \{\pi((g, s\varepsilon)) \,|\, g \in V \text{ and } s \in (0, c]\} = \phi,$$

which contradicts the fact that the closure of $\operatorname{supp} \beta_1'(f)$ in $\tilde{X}$ contains xP^d. The unique continuation property also shows that $\operatorname{supp} \beta_1'(f) \supset xM_\Theta^d P_\Theta^d(K)$ and $\beta_i'(f)=0$ on a neighborhood of $xM_\Theta^d P_\Theta^d(K)$ in $G^d/P_\Theta^d(K)$ if $\nu'-\nu_i' \in N^k-\{0\}$. Choose any $y \in G^d$ so that $yP^d \subset xM_\Theta^d(K)P^d$ and fix a sufficiently small neighborhood U_0 of $\pi((y, 0))$ in $\tilde{X}$. We remark that $\operatorname{supp} \beta_1'(f) \ni \pi((y, s\varepsilon))$ for all $s>0$ and $\beta_i'(f)=0$ on $U_0 \cap G^d/P_\Theta^d(K)$ if $\nu'-\nu_i' \in N^k-\{0\}$.

Let f' be the column vector formed by $\{\beta_j'(f)|_{U_0 \cap G^d/P_\Theta^d(K)} \,|\, \nu_j'=\nu'\}$. Then Theorem 6.2 and Corollary 6.3 in [19] say the following: The vector f' satisfies a system of differential equations with regular singularities along the set of walls $\{Y_{k+1} \cap G^d/P_\Theta^d(K), \cdots, Y_{l'} \cap G^d/P_\Theta^d(K)\}$ with the edge Y. Put $W(\nu')=\{w \in W \,|\, (\langle \rho-w\lambda, \tilde{\omega}_1 \rangle, \cdots, \langle \rho-w\lambda, \tilde{\omega}_k \rangle)=\nu'\}$. Then corresponding to any $w \in W(\nu')$, we can define a boundary value $\beta_w^*(f')$ of f' so that $\beta_\lambda^w(f)=\beta_w^*(f')$ (c.f. (3.12)). Moreover if $\beta_w^*(f')=0$ for all $w \in W(\nu')$, then $f'=0$ in a neighborhood of $U_0 \cap G^d/P^d$. This means especially $W(\nu') \neq \phi$. Since the point $\pi((y, s\varepsilon))$ converges into yP^d when

$s \to 0$, we can conclude supp $\beta_\lambda^w(f) \ni yP^d$ with at least one w in $W(\nu')$, from which Lemma 1 follows. Q.E.D.

Proof of Lemma 2. We may assume SP^d is compact in G^d/P^d because supp $\beta_\lambda u$ is compact.

First we will prove Lemma 2 under the different assumption

$$\text{(3.44)} \qquad \langle w\lambda - \lambda, \tilde{\omega}_i \rangle \notin \boldsymbol{Z} \quad \text{for all } w \in W - \{1\} \text{ and } i = 1, \cdots, l'.$$

In this case, $W(\mu) = \{w\}$ with the element $w \in W$ satisfying $\mu = w\lambda$ and it is clear that it is sufficient to prove Lemma 2 when $L(w) = 1$. So we assume $w = w_{\tilde{\alpha}_i}$. Now we use Proposition 6.1 in [7], which says

$$\text{(3.45)} \qquad c_{w,-\lambda}\beta_{w\lambda}\mathscr{P}_\lambda = c_{w\lambda}\mathscr{T}_w^\lambda,$$

where $c_{w,-\lambda}$ and $c_{w\lambda}$ are non-zero constants and $\mathscr{T}_w^\lambda$ is the normalized intertwining operator from $\mathscr{B}(G^d/P^d; L_\lambda)$ to $\mathscr{B}(G^d/P^d; L_{w\lambda})$. Since $\mathscr{P}_\lambda(\beta_\lambda(f))$ is a non-zero multiple of f, we see that $\beta_{w\lambda}(f) = C\mathscr{T}_w^\lambda\beta_\lambda(f)$ with a constant number C. Now we recall the intertwining operator. It is an integral transformation

$$\text{(3.46)} \qquad \begin{array}{ccc} \mathscr{T}_w^\lambda\colon \mathscr{B}(G^d/P^d; L_\lambda) & \longrightarrow & \mathscr{B}(G^d/P^d; L_{w\lambda}) \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ \psi & \longmapsto & (\mathscr{T}_w^\lambda\psi)(g) = \int_K \psi(k) T_w^\lambda(k^{-1}g)dk \end{array}$$

with a kernel function $T_w^\lambda \in \mathscr{B}(G^d/P^d; L_{w\lambda})$. We will use the identification

$$\text{(3.47)} \qquad \mathscr{B}(G^d/P^d; L_\lambda) \xrightarrow{\sim} \mathscr{B}(K^d/M^d)$$

by the restriction. Then $T_w^\lambda = \mathscr{T}_w^\lambda(\delta)$ with the Dirac's delta function δ on K^d/M^d whose support is M^d. The function T_w^ν is meromorphic with respect to the parameter $\nu \in (\mathfrak{a}_\mathfrak{p}^d)_c^*$ and if $\operatorname{Re}\langle \nu, \alpha \rangle < 0$ for all $\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$, then

$$(\mathscr{T}_w^\nu\psi)(g) = \int_{N_w^{+d}} \psi(gn\bar{w})dn \qquad \text{for all } \psi \in \mathscr{C}^\infty(G^d/P^d; L_\nu).$$

Here $N_w^{+d} = N^{+d} \cap \bar{w}N^{-d}\bar{w}^{-1}$ and dn is a Haar measure on N_w^{+d}. Hence it is clear that supp T_w^λ is contained in the closure of $P^d\bar{w}P^d$, which we denote by P_w^d. Then $P_w^d = P_{\{\tilde{\alpha}_i\}}^d = M_{\{\tilde{\alpha}_i\}}^d P^d$. Suppose supp $\beta_\lambda f \subset SP^d$. Then it follows from (3.46) that if $xP^d \in$ supp $\beta_{w\lambda}f$, there exists $k \in K$ such that $k \in SP^d$ and $k^{-1}x \in P_w^d$. Therefore $x \in SP^dP_w^d = SM_{\{\tilde{\alpha}_i\}}^d P^d$ and we have (3.40).

Now we will consider the original lemma. Choose $\varepsilon > 0$ so that

$$\langle \nu - (\lambda + z\rho), \tilde{\omega}_i \rangle \notin \boldsymbol{Z}$$

for all $\nu \in W(\lambda+z\rho)$, $i=1, \cdots, l'$ and $z \in \boldsymbol{C}$ which satisfy $\nu \neq \lambda+z\rho$ and $0<|z|<\varepsilon$.

We put $Z=\{z \in \boldsymbol{C};\ |z|<\varepsilon\}$. Identifying $\mathscr{B}(K^d/M^d)$ with $\mathscr{B}(G^d/P^d; L_{\lambda+z\rho})$, we put $\tilde{f}_z=C\mathscr{P}_{\lambda+z\rho}\beta_\lambda(f)$. We determine the constant C so that $\tilde{f}_0=f$. Then $\tilde{f}_z$ defines a function on $Z\times(G^d/K^d)$ and the function is holomorphic with respect to z. Moreover $D\tilde{f}_z=\chi^d_{\lambda+z\rho}(D)\tilde{f}_z$ for all $D \in \boldsymbol{D}(G^d/K^d)$. Then for a small open set V of G^d/P^d, we can define linear maps (c.f. Definition 4.3 in [19]):

$$\tilde{\beta}_w:\ {}_Z\mathscr{A}(G^d/K^d;\ \mathscr{M}^d)\longrightarrow {}_Z\mathscr{B}(V) \qquad (w \in W)$$

which correspond to characteristic exponents $(\langle\rho-w(\lambda+z\rho), \tilde{\omega}_1\rangle, \cdots, \langle\rho-w(\lambda+z\rho), \tilde{\omega}_{l'}\rangle)$, respectively. We denote by ${}_Z\mathscr{A}(G^d/K^d)$ the space of real analytic functions on G^d/K^d with the holomorphic parameter $z \in Z$ and by ${}_Z\mathscr{B}(V)$ the space of hyperfunctions on V with the holomorphic parameter $z \in Z$. Then ${}_Z\mathscr{A}(G^d/K^d;\ \mathscr{M}^d)=\{\tilde{u} \in {}_Z\mathscr{A}(G^d/K^d)\,|\,D\tilde{u}=\chi^d_{\lambda+z\rho}(D)\tilde{u}$ for all $D \in \boldsymbol{D}(G^d/K^d)\}$. The maps $\tilde{\beta}_w$ have the following property (c.f. Theorem 4.5 and Lemma 4.6 in [19]): Fix a $w'' \in W$. If $\tilde{\beta}_w(\tilde{u})=0$ for all $w \in W$ satisfying $(\langle w\lambda-w''\lambda, \tilde{\omega}_1\rangle, \cdots, \langle w\lambda-w''\lambda, \tilde{\omega}_{l'}\rangle) \in \boldsymbol{N}^{l'}-\{0\}$, then $\tilde{\beta}_{w''}(\tilde{u})|_{z=z''}$ is a non-zero constant multiple of $\beta^{w''}_{\lambda+z''\rho}(\tilde{u}|_{z=z''})$ for any $z'' \in Z$.

Let xP^d be any point in G^d/P^d which is not contained in the compact subset $U(\mu)=\bigcup_{w\in W(\mu)}S(w)$ of G^d/P^d. Choose a neighborhood V of xP^d in G^d/P^d so that $U(\mu)\cap V=\phi$. Then we have $\beta_{w(\lambda+z''\rho)}(\tilde{f}|_{z=z''})|_V=0$ for all $z'' \in Z-\{0\}$ and all $w \in W(\mu)$ because $\lambda+z''\rho$ satisfies the condition (3.44) if $z'' \in Z-\{0\}$. Hence for all $w \in W(\mu)$, we obtain $\tilde{\beta}_w(\tilde{f})|_{Z-\{0\}}=0$ and therefore $\tilde{\beta}_w(\tilde{f})=0$ by the analytic continuation, which means $\beta^w_\lambda(f)|_V$ equal identically zero because they are constant multiples of $\tilde{\beta}_w(\tilde{f})|_{z=0}$, respectively. Thus we can conclude supp $\beta_\mu f\cap V=\phi$ and we finish the proof of Lemma 2. Q.E.D.

§ 4. Proof of Theorem 1 (First reduction)

First we review the results on H^d-orbits on G^d/P^d according to [9]. Let $\Sigma_{\mathfrak{a}}(\mathfrak{a}^d_{\mathfrak{p}})$ denote the subset in $\Sigma(\mathfrak{a}^d_{\mathfrak{p}})$ defined by

$$\Sigma_{\mathfrak{a}}(\mathfrak{a}^d_{\mathfrak{p}})=\{\alpha \in \Sigma(\mathfrak{a}^d_{\mathfrak{p}})\,|\,\langle\alpha, \mathfrak{a}_1\rangle=\{0\}\}$$

where $\mathfrak{a}_1=\mathfrak{a}^d_{\mathfrak{p}}\cap\mathfrak{h}^d$. Put $\mathfrak{q}^{da}=\mathfrak{k}^d\cap\mathfrak{q}^d+\mathfrak{p}^d\cap\mathfrak{h}^d$. Then a normalized $\mathfrak{q}^{da}$-orthogonal system Q of $\Sigma_{\mathfrak{a}}(\mathfrak{a}^d_{\mathfrak{p}})$ is a set of root vectors $\{X_{\beta_1}, \cdots, X_{\beta_k}\}$ satisfying the following three conditions.

(i) $\beta_i \in \Sigma_{\mathfrak{a}}(\mathfrak{a}^d_{\mathfrak{p}})$ and $X_{\beta_i} \in \mathfrak{g}^d(\mathfrak{a}^d_{\mathfrak{p}};\ \beta_i)\cap\mathfrak{q}^{da}$ for $i=1, \cdots, k$.

(ii) $[X_{\beta_i}, X_{\beta_j}]=[X_{\beta_i}, \sigma X_{\beta_j}]=0$ for $i\neq j$.

(iii) $2\langle\beta_i, \beta_i\rangle B(X_{\beta_i}, \sigma X_{\beta_i})=-1$ for $i=1, \cdots, k$

where $B(\ ,\ )$ is the Killing form on $\mathfrak{g}^d$ and $\langle\ ,\ \rangle$ is the bilinear form on $\mathfrak{a}_\mathfrak{p}^{d*}$ induced from $B(\ ,\ )$.

Let S denote the set of normalized $\mathfrak{q}^{da}$-orthogonal systems of $\Sigma_\mathfrak{a}(\mathfrak{a}_\mathfrak{p}^d)$ and S' the subset of S consisting of $Q=\{X_{\beta_1}, \cdots, X_{\beta_k}\}$ such that $k<l$ ($l=\dim \mathfrak{a}$). For a $Q=\{X_{\beta_1}, \cdots, X_{\beta_k}\} \in S$, we put

$$c(Q)=\exp \frac{\pi}{2}(X_{\beta_1}+\sigma X_{\beta_1})\cdots \exp \frac{\pi}{2}(X_{\beta_k}+\sigma X_{\beta_k}).$$

Put $W_\theta=W_\theta(\mathfrak{a}_\mathfrak{p}^d)=\{w \in W \mid \theta w=w\theta\}$.

Proposition 3 ([9]). (i) *For every* $x \in G^d$, *there exist* $Q \in S$ *and* $w \in W$ *such that*

$$H^d x P^d=H^d c(Q) w P^d.$$

(ii) *If* $\operatorname{rank}(G/H)>\operatorname{rank}(K/K\cap H)$, then $S'=S$.

(iii) *Let* Q *and* w *be elements of* S *and* W, *respectively. Then* $H^d c(Q) w P^d$ *is open in* G^d *if and only if* $Q=\phi$ *and* $w \in W_\theta$.

(iv) *Let* Q *and* w *be elements of* S *and* W, *respectively, and suppose that* $\operatorname{rank}(G/H)=\operatorname{rank}(K/K\cap H)$. *Then* $H^d c(Q) w P^d$ *is closed in* G^d *if and only if* $Q \in S\backslash S'$. *Moreover let* $Q_0=\{X_{\beta_1}, \cdots, X_{\beta_l}\}$ *be an element of* $S\backslash S'$. *Then every closed* H^d*-orbit on* G^d/P^d *can be written as* $H^d c(Q_0) w P^d$ *with some* $w \in W$.

Proposition 3 is an easy consequence of Theorem 2, Theorem 3, Proposition 1 and Proposition 2 in [9].

Now we prepare notations in the case of $\operatorname{rank}(G/H)=\operatorname{rank}(K/K\cap H)$. Let Q_0 be an element of $S\backslash S'$ and put $\mathfrak{a}'_\mathfrak{p}=\operatorname{Ad}(c(Q_0))\mathfrak{a}_\mathfrak{p}^d$. Then $\mathfrak{a}'_\mathfrak{p}$ is a maximal abelian subspace of $\mathfrak{p}^d$ contained in $\mathfrak{p}^d\cap\mathfrak{h}^d$ ([9], Theorem 2). By Proposition 3 (iv), we can choose a complete set of representatives $\{x_1, \cdots, x_m\}$ of closed H^d-orbits on G^d/P^d such that $\operatorname{Ad}(x_j)\mathfrak{a}_\mathfrak{p}^d=\mathfrak{a}'_\mathfrak{p}$ for $j=1, \cdots, m$. For a $\mu \in (\mathfrak{a}_\mathfrak{p}^d)_c^*$ and for $j=1, \cdots, m$, we define an element $\mu^j \in (\mathfrak{a}'_\mathfrak{p})_c^*$ by $\mu^j=\mu\circ\operatorname{Ad}(x_j)^{-1}$. For each $j=1, \cdots, m$, we put $\Sigma(\mathfrak{a}'_\mathfrak{p})_j^+=\{\alpha^j \mid \alpha \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+\}$ and $\mathfrak{n}^{+j}=\operatorname{Ad}(x_j)\mathfrak{n}^{+d}$. A root α in $\Sigma(\mathfrak{a}'_\mathfrak{p})$ is said to be a compact (resp. noncompact) root if $\mathfrak{g}^d(\mathfrak{a}'_\mathfrak{p};\alpha)\subset\mathfrak{h}^d$ (resp. $\mathfrak{g}^d(\mathfrak{a}'_\mathfrak{p};\alpha)\not\subset\mathfrak{h}^d$).

Theorem 1. *Let* λ *be an element of* $(\mathfrak{a}_\mathfrak{p}^d)_c^*$ *satisfying* $\operatorname{Re}\langle\lambda, \tilde{\alpha}\rangle\geq 0$ *for all* $\tilde{\alpha} \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$. *Suppose that there exists a nonzero function* f *in*

$$\mathscr{A}_K(G/H; \mathscr{M}_\lambda)\cap L^2(G/H).$$

Then

(i) $\operatorname{rank}(G/H)=\operatorname{rank}(K/K\cap H)$ *and*

(ii) $\operatorname{supp}\beta_\lambda f^\eta$ *is contained in the union of closed* H^d*-orbits on* G^d/P^d.

Now we suppose a further condition that supp $\beta_\lambda f^\eta \subset H^d x_j P^d$. *Then we have*

(iii) $\mathrm{Re}\langle\lambda^j, \alpha\rangle > 0$ *for every noncompact simple root* α *in* $\Sigma(\mathfrak{a}'_\mathfrak{p})_j^+$.

Remark. (i) Let (π, V) be a discrete series for G/H. By [18] p. 463, every formally selfadjoint operator in $D(G/H)$ extends to a selfadjoint operator. Thus $L^2(G/H)$ has a spectral decomposition for $D(G/H)$. It follows from the irreducibility of (π, V) that V is realized in a simultaneous eigenspace for $D(G/H)$ in $L^2(G/H)$.

Let V_K be the subspace of K-finite elements in V. Let f be an element in V_K. Realizing f as a function on G, f is proved to be analytic on G by [17], Vol. II, p. 177, Appendix. Thus V_K is realized in

$$\mathscr{A}_K(G/H; \mathscr{M}_\lambda) \cap L^2(G/H)$$

for some $\lambda \in (\mathfrak{a}_\mathfrak{p}^d)_c^*$ such that $\mathrm{Re}\langle\lambda, \tilde{\alpha}\rangle \geq 0$ for all $\tilde{\alpha} \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ (since $\chi_{w\lambda}^d = \chi_\lambda^d$ for $w \in W$).

(ii) The regularity of λ^j for compact simple roots will be proved in Theorem 3.

The proof of this theorem is reduced to Lemma 1 in Section 3 and to the following lemma which is proved in the following sections.

Lemma 3. *Let* λ *be an element of* $(\mathfrak{a}_\mathfrak{p}^d)_c^*$ *satisfying* $\mathrm{Re}\langle\lambda, \tilde{\alpha}\rangle \geq 0$ *for all* $\tilde{\alpha} \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ *and let* x *be an element of* G^d. *Suppose that one of the following three conditions is satisfied.*

(i) $\mathrm{rank}(G/H) \neq \mathrm{rank}(K/K \cap H)$.

(ii) $H^d x P^d$ *is not closed in* G^d.

(iii) $\mathrm{rank}(G/H) = \mathrm{rank}(K/K \cap H)$ *and there is a* j $(1 \leq j \leq m)$ *such that* $H^d x P^d = H^d x_j P^d$ *and that* $\mathrm{Re}\langle\lambda^j, \alpha\rangle = 0$ *for a noncompact simple root* α *in* $\Sigma(\mathfrak{a}'_\mathfrak{p})_j^+$.

Then there exist subsets $\Theta_1, \cdots, \Theta_N$ *in* Ψ *satisfying the following two conditions.*

(a) *The set* $H^d x M_{\Theta_1}^d \cdots M_{\Theta_N}^d P^d$ *contains inner points in* G^d.

(b) *Define subsets* $\Lambda_0, \cdots, \Lambda_N$ *in* $W\lambda$ *inductively by* $\Lambda_0 = \{\lambda\}$ *and* $\Lambda_i = \{\nu \in W\lambda \mid$ *There exists a* $\mu \in \Lambda_{i-1}$ *such that* $\mathrm{Re}\langle\nu, \tilde{\omega}_j\rangle \geq \mathrm{Re}\langle\mu, \tilde{\omega}_j\rangle$ *for all* j *with* $\tilde{\alpha}_j \in \Psi \backslash \Theta_i\}$ $(i = 1, \cdots, N)$. *Then*

$$(\mathrm{Re}\langle\mu, \omega_1\rangle, \cdots, \mathrm{Re}\langle\mu, \omega_l\rangle) \notin (-\infty, 0)^l$$

for all $\mu \in \Lambda_N$.

Proof of Theorem 1. Since η and β_λ are bijective (Remark in § 3), we have $\beta_\lambda f^\eta \neq 0$. Suppose first that (i) is not true. Then we will get a contradiction. Let xP^d be a point in supp $\beta_\lambda f^\eta$. Since the assumption

(i) in Lemma 3 holds, there exist $\Theta_1, \cdots, \Theta_N \subset \Psi$ satisfying (a) and (b) in Lemma 3. By (a), there exist $m_i \in M^d_{\Theta_i}$ $(i=1, \cdots, N)$ such that $H^d x m_1 \cdots m_N P^d$ is open in G^d. Applying Lemma 1, we see that there exists a $\mu \in \Lambda_i$ such that

$$x m_1 \cdots m_i P^d \in \operatorname{supp} \beta_\mu f^\eta$$

for every $i=0, \cdots, N$. Thus there exists a $\mu \in \Lambda_N$ such that $\operatorname{supp} \beta_\mu f^\eta$ contains inner points in G^d/P^d. By (b) and Proposition 2, f is not contained in $L^2(G/H)$. Thus we have a contradiction to $f \in L^2(G/H)$ and we have proved (i).

Next suppose that (ii) is not true. Then there is an $x \in \operatorname{supp} \beta_\lambda f^\eta$ such that $H^d x P^d$ is not closed in G^d/P^d. Since the assumption (ii) in Lemma 3 holds, we can easily get a contradiction by the same argument as in the proof of (i).

The proof of (iii) is similar to the above ones. Q.E.D.

§ 5. Proof of Lemma 3 (Second reduction)

An orthogonal system $\bar{Q}$ in $\Sigma_\mathfrak{a}(\mathfrak{a}^d_\mathfrak{p})$ is by definition a subset $\{\beta_1, \cdots, \beta_k\}$ in $\Sigma_\mathfrak{a}(\mathfrak{a}^d_\mathfrak{p})$ such that $\langle \beta_i, \beta_j \rangle = 0$ for $i \neq j$. Let $\bar{S}$ denote the set of orthogonal systems in $\Sigma_\mathfrak{a}(\mathfrak{a}^d_\mathfrak{p})$ and $\bar{S}'$ the subset in $\bar{S}$ consisting of orthogonal systems with less than l elements. If $Q=\{X_{\beta_1}, \cdots, X_{\beta_k}\}$ is a normalized $\mathfrak{q}^{da}$-orthogonal system of $\Sigma_\mathfrak{a}(\mathfrak{a}^d_\mathfrak{p})$, then $\bar{Q}=\{\beta_1, \cdots, \beta_k\}$ is an element of $\bar{S}$.

Lemma 4. *Let λ be an element of $\mathfrak{a}^{d*}_\mathfrak{p}$ such that $\langle \lambda, \tilde{\alpha} \rangle \geq 0$ for all $\tilde{\alpha} \in \Sigma(\mathfrak{a}^d_\mathfrak{p})^+$. Then for every $\bar{Q} \in \bar{S}'$ and $w \in W$, there exist a $w_0 \in W_\theta$, an integer $N \geq 1$ and subsets $\Theta_0, \Theta_1, \cdots, \Theta_{N-1}$ in Ψ satisfying the following three conditions.*

(i) $w_0 \bar{Q} \in \langle \Theta_0 \rangle$.

(ii) $w^{-1} w_0^{-1} \in W_{\Theta_1} \cdots W_{\Theta_{N-1}}$.

(iii) *Put $\Theta_N = \Theta_0$. Then $\lambda, \Theta_1, \cdots, \Theta_N$ satisfy the condition* (b) *in Lemma* 3.

Here $\langle \Theta_0 \rangle$ is the subspace in $\mathfrak{a}^{d*}_\mathfrak{p}$ spanned by Θ_0.

This lemma is proved in the following sections. So in the rest of this section we will prove Lemma 3 assuming Lemma 4.

Proof of Lemma 3. Let λ be an element of $(\mathfrak{a}^d_\mathfrak{p})^*_c$. Then we can write uniquely that $\lambda = \operatorname{Re} \lambda + \operatorname{Im} \lambda$ where $\operatorname{Re} \lambda$(resp. $\operatorname{Im} \lambda$) is an element in $(\mathfrak{a}^d_\mathfrak{p})^*_c$ which is real-valued (resp. pure imaginary-valued) on $\mathfrak{a}^d_\mathfrak{p}$. Since $\operatorname{Im} \lambda$ has no contribution to the statement of Lemma 3, we may assume that $\lambda \in \mathfrak{a}^{d*}_\mathfrak{p}$.

(I) The cases of (i) and (ii). Assume the condition (i) in Lemma 3. Then by Proposition 3 (i) and (ii), we can write

$$H^d x P^d = H^d c(Q) w P^d \quad \text{with some } Q \in S' \text{ and } w \in W. \tag{5.1}$$

When the condition (ii) in Lemma 3 holds, we also have (5.1) by Proposition 3 (i) and (iv).

Applying Lemma 4 to λ, $\bar{Q}$ and w, we have $w_0, \Theta_0, \cdots, \Theta_{N-1}$ satisfying the three conditions in Lemma 4. We have only to prove (a) in Lemma 3 ($\Theta_N = \Theta_0$). We have

$$\begin{aligned} H^d x M^d_{\Theta_1} \cdots M^d_{\Theta_N} P^d &= H^d x P^d M^d_{\Theta_1} \cdots M^d_{\Theta_N} P^d \\ &= H^d c(Q) w M^d_{\Theta_1} \cdots M^d_{\Theta_N} P^d \\ &\supset H^d c(Q) w w^{-1} w_0^{-1} M^d_{\Theta_N} P^d \quad \text{(by Lemma 4 (ii))} \\ &= H^d w_0^{-1} (w_0 c(Q) w_0^{-1}) M^d_{\Theta_N} P^d \\ &\supset H^d w_0^{-1} P^d \quad \text{(by Lemma 4 (i)).} \end{aligned}$$

Since $w_0 \in W_\theta$, $H^d w_0^{-1} P^d$ is open in G^d by Proposition 3 (iii).

(II) The case of (iii). Let $\tilde{\alpha}_i$ be the root in Ψ given by $\tilde{\alpha}_i = \alpha \circ \mathrm{Ad}(x_j)$. Since the Lie algebra of $x_j M^d_{\{\tilde{\alpha}_i\}} x_j^{-1}$ contains $\mathfrak{g}^d(\mathfrak{a}'_\mathfrak{p}; -\alpha)$ and since $\mathfrak{g}^d(\mathfrak{a}'_\mathfrak{p}; -\alpha) \not\subset \mathfrak{h}^d$, we have $\dim H^d x_j M^d_{\{\tilde{\alpha}_i\}} P^d > \dim H^d x_j P^d$. Now we can apply the result obtained in (I) to an element $y \in H^d x_j M^d_{\{\tilde{\alpha}_i\}} P^d$ not contained in $H^d x_j P^d$ and obtain $\Theta_1, \cdots, \Theta_N \subset \Psi$ satisfying conditions (a) and (b) for y and λ. It is clear that (a) is satisfied for x_j if $\Theta_1, \cdots, \Theta_N$ are replaced by $\{\tilde{\alpha}_i\}, \Theta_1, \cdots, \Theta_N$. Thus we have only to prove (b) for the sequence $\{\tilde{\alpha}_i\}, \Theta_1, \cdots, \Theta_N$. Put $\Lambda' = \{\mu \in W\lambda \mid \langle \mu, \tilde{\omega}_k \rangle \geq \langle \lambda, \tilde{\omega}_k \rangle \text{ for } k \neq i\}$. Then we have only to prove that $\Lambda' = \Lambda_0$. Let μ be an element in Λ'. Then μ can be written as $\mu = \lambda - \sum_{k=1}^{l'} c_k \tilde{\alpha}_k$ with some nonnegative real numbers $c_1, \cdots, c_{l'}$. Since $\langle \mu, \tilde{\omega}_h \rangle = \langle \lambda - \sum_{k=1}^{l'} c_k \tilde{\alpha}_k, \tilde{\omega}_h \rangle = \langle \lambda, \tilde{\omega}_h \rangle - c_h$ for $h = 1, \cdots, l'$, it follows from the definition of Λ' that $\mu = \lambda - c_i \tilde{\alpha}_i$. Since $\langle \lambda, \tilde{\alpha}_i \rangle = 0$, we have $\langle \mu, \mu \rangle = \langle \lambda - c_i \tilde{\alpha}_i, \lambda - c_i \tilde{\alpha}_i \rangle = \langle \lambda, \lambda \rangle + c_i^2 \langle \tilde{\alpha}_i, \tilde{\alpha}_i \rangle$. Since $\langle \mu, \mu \rangle = \langle \lambda, \lambda \rangle$, we have $c_i = 0$ and therefore we have proved that $\Lambda' = \Lambda_0$. Q.E.D.

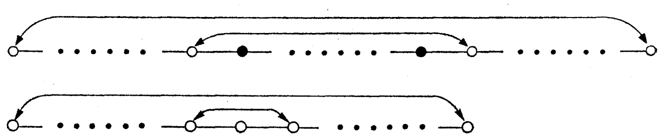
§ 6. Proof of Lemma 4 (Third reduction)

Since $\mathfrak{a}^d_\mathfrak{p}$ is θ-stable, θ acts on $\Sigma(\mathfrak{a}^d_\mathfrak{p})$. We call such a pair $(\Sigma(\mathfrak{a}^d_\mathfrak{p}), \theta)$ a θ-system. A θ-system $(\Sigma(\mathfrak{a}^d_\mathfrak{p}), \theta)$ is decomposed to irreducible ones and it is clear that we have only to prove Lemma 4 for an irreducible θ-system $(\Sigma(\mathfrak{a}^d_\mathfrak{p}), \theta)$.

In order to classify irreducible θ-systems, we can use (generalized) Satake diagrams as in [17], Vol. 1, p. 30. A list of root systems for all the irreducible semisimple symmetric spaces is given in [13] (c.f. [20]). Following the list, we can write the Satake diagrams for all the irreducible θ-systems corresponding to semisimple symmetric spaces as follows.

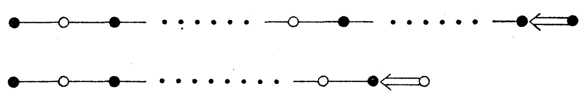
(1) AI 

(2) AII 

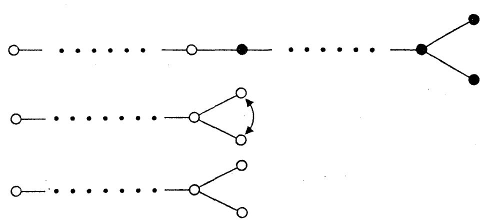
(3) AIII 

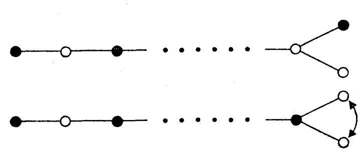
(4) BI 

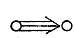
CI  $(l \neq l' \Rightarrow *)$

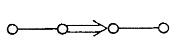
(5) CII 

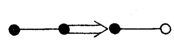
BIII* 

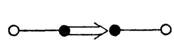
(6) DI 

(7) DIII 

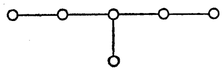
(8) G 

(9) FI 

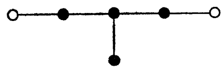
FII 

FIII* 

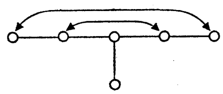
(10) EI



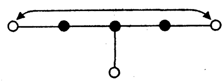
EIV



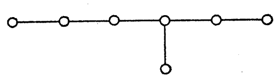
(11) EII



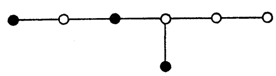
EIII



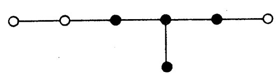
(12) EV



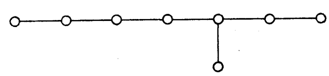
EVI



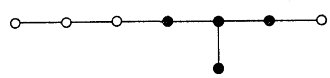
EVII



(13) EVIII



EIX



(14) $\Sigma(\mathfrak{a}_{\mathfrak{p}}^{d})=\Sigma_1 \coprod \theta\Sigma_1$ (Σ_1 is a connected Dynkin diagram)

Here the diagrams with asterisks do not exist in the original Satake diagrams.

Let w^* be the unique element in W satisfying $w^*\Sigma(\mathfrak{a}_{\mathfrak{p}}^{d})^+=-\Sigma(\mathfrak{a}_{\mathfrak{p}}^{d})^+$. Then $w^*\theta$ is an involutive automorphism of $\Sigma(\mathfrak{a}_{\mathfrak{p}}^{d})$ such that $w^*\theta\Sigma(\mathfrak{a})^+=\Sigma(\mathfrak{a})^+$. We will first give a proof of Lemma 4 in the cases of (1) with $l=2$ and (14).

(I) *Proof of Lemma* 4 *in the case of* (1) *with* $l=2$. Put $\Psi=\{\tilde{\alpha}_1, \tilde{\alpha}_2\}$. When $w^{-1}\bar{Q}=\phi$, $\{\tilde{\alpha}_1\}$, $\{-\tilde{\alpha}_1\}$, $\{\tilde{\alpha}_2\}$ or $\{-\tilde{\alpha}_2\}$, we put $w_0=w^{-1}$ and $N=1$. Then it is clear that Θ_0 and w_0 satisfy the conditions (i), (ii) and (iii) in Lemma 4 if we put $\Theta_0=\{\tilde{\alpha}_1\}$ or $\{\tilde{\alpha}_2\}$.

Thus we may assume that $w^{-1}\bar{Q}=\{\tilde{\alpha}_1+\tilde{\alpha}_2\}$ or $\{-\tilde{\alpha}_1-\tilde{\alpha}_2\}$. Since $w^*\theta$ is an automorphism of $\Sigma(\mathfrak{a}_{\mathfrak{p}}^{d})$ satisfying $w^*\theta\tilde{\alpha}_1=\tilde{\alpha}_2$ and $w^*\theta\tilde{\alpha}_2=\tilde{\alpha}_1$, we may assume that $\langle\lambda, \omega_1\rangle\geq\langle\lambda, \omega_2\rangle$. Put $w_0^{-1}=ww_{\tilde{\alpha}_2}$, $\Theta_0=\{\tilde{\alpha}_1\}$, $\Theta_1=\{\tilde{\alpha}_2\}$ and $N=2$. Then (i) and (ii) in Lemma 4 are clear. (iii) is proved as follows. If

$\nu \in \Lambda_1$, then $\langle \nu, \omega_2\rangle = \langle w_{\tilde{\alpha}_2}\nu, w_{\tilde{\alpha}_2}\omega_2\rangle = \langle w_{\tilde{\alpha}_2}\nu, \omega_1 - \omega_2\rangle$. Since $w_{\tilde{\alpha}_2}\nu \in \Lambda_1$ and since λ is dominant for $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$, we have $\langle w_{\tilde{\alpha}_2}\nu, \omega_1\rangle \geq \langle \lambda, \omega_1\rangle$ and $\langle w_{\tilde{\alpha}_2}\nu, \omega_2\rangle \leq \langle \lambda, \omega_2\rangle$. Hence we have

$$\langle \nu, \omega_2\rangle = \langle w_{\tilde{\alpha}_2}\nu, \omega_1 - \omega_2\rangle \geq \langle \lambda, \omega_1\rangle - \langle \lambda, \omega_2\rangle \geq 0.$$

On the other hand if $\mu \in \Lambda_2$, then $\langle \mu, \omega_2\rangle \geq \langle \nu, \omega_2\rangle$ for some $\nu \in \Lambda_1$ by the definition of Λ_2. Thus we have $\langle \mu, \omega_2\rangle \geq 0$ for every $\mu \in \Lambda_2$ and therefore we have proved (iii). Q.E.D.

(II) *Proof of Lemma* 4 *in the case of* (14). Let $\tilde{\omega}$ be an element in $\{\tilde{\omega}_1, \cdots, \tilde{\omega}_{l'}\}$ which is orthogonal to $\theta\Sigma_1$. Then $\omega = \tilde{\omega} - \theta\tilde{\omega}$ is an element in $\{\omega_1, \cdots, \omega_l\}$. Since $w^*\theta$ is an involutive automorphism of $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$ commuting with θ, We may assume that

$$\langle \lambda, \tilde{\omega}\rangle \geq \langle \lambda, w^*\theta\tilde{\omega}\rangle.$$

Since $\Sigma_a(\mathfrak{a}_\mathfrak{p}^d) = \phi$, we have $\bar{S}' = \{\phi\}$. For a $w \in W$, we put $N=2$, $\Theta_0 = \phi$ and $\Theta_1 = \Psi \cap \theta\Sigma_1$. $w_0 \in W_\theta$ is defined as follows. There exists a unique pair (u, v) of elements in W such that $w = uv$ and that u(resp. v) acts trivially on $\theta\Sigma_1$ (resp. Σ_1). Put $w_0^{-1} = u(\theta u\theta)$. Then (i) and (ii) in Lemma 4 is clear. Let ν be an element in Λ_1. Then we have $\langle \nu, \tilde{\omega}\rangle \geq \langle \lambda, \tilde{\omega}\rangle$ by the definition of Λ_1 and we have $\langle \nu, -\theta\tilde{\omega}\rangle \geq \langle w^*\lambda, -\theta\tilde{\omega}\rangle$ since $-\theta\tilde{\omega} \in \{\tilde{\omega}_1, \cdots, \tilde{\omega}_{l'}\}$ and since $-w^*\lambda$ is dominant for $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$. Thus we have

$$\langle \nu, \omega\rangle = \langle \nu, \tilde{\omega} - \theta\tilde{\omega}\rangle \geq \langle \lambda, \tilde{\omega}\rangle - \langle \lambda, w^*\theta\tilde{\omega}\rangle \geq 0,$$

proving (iii) in Lemma 4 ($\Lambda_2 = \Lambda_1$). Q.E.D.

Put $\mathfrak{a}_{\mathfrak{p}+}^d = \{Y \in \mathfrak{a}_\mathfrak{p}^d \mid \tilde{\alpha}(Y) \geq 0 \text{ for all } \tilde{\alpha} \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+\}$ and $\boldsymbol{R}_- = -\boldsymbol{R}_+ = \{t \in \boldsymbol{R} \mid t \leq 0\}$.

Lemma 5. *Suppose that* $(\Sigma(\mathfrak{a}_\mathfrak{p}^d), \theta)$ *is irreducible and is neither of type* (1) *with* $l=2$ *nor of type* (14). *Let* λ *be an element of* $\mathfrak{a}_\mathfrak{p}^{d*}$ *such that* $\langle \lambda, \tilde{\alpha}\rangle \geq 0$ *for all* $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$. *Then for every* $\bar{Q} \in \bar{S}'$ *there exists an* $\omega \in \{\omega_1, \cdots, \omega_l\}$ *satisfying the following two conditions.*

(i) *Put* $\Theta_0' = \{\tilde{\alpha} \in \Psi \mid \langle \tilde{\alpha}, \omega\rangle = \langle \tilde{\alpha}, w^*\theta\omega\rangle = 0\}$. *Then there exists a* $w_0' \in W_\theta$ *such that*

$$w_0'\bar{Q} \subset \langle \Theta_0'\rangle.$$

(ii) *For every* $w' \in W$ *satisfying* $\langle w'\omega, \mathfrak{a}_{\mathfrak{p}+}^d\rangle \not\subset \boldsymbol{R}_-$ *and* $\langle w'\omega, \lambda\rangle \geq 0$, *there exist an integer* $N \geq 1$, *subsets* $\Theta_1, \cdots, \Theta_{N-1}$ *of* Ψ *and elements* $\omega^{(1)}, \cdots, \omega^{(N)}$ *in* $W\omega$ *satisfying the following four conditions.*

(a) $\omega^{(N)} = \omega$ *and* $\omega^{(i)} \in W_{\Theta_i}\omega^{(i+1)}$ *for* $i = 1, \cdots, N-1$.

(b) $\langle \lambda, \omega^{(1)}\rangle \geq 0$.

(c) *There exists a sequence of simple roots* $\gamma_1, \cdots, \gamma_k \in \Psi$ *satisfying* $w_{\gamma_k} \cdots w_{\gamma_1} w' \omega = \omega^{(1)}$ *and*

$$w_{\gamma_{i-1}} \cdots w_{\gamma_1} w' \omega - w_{\gamma_i} \cdots w_{\gamma_1} w' \omega = c_i \gamma_i$$

for some $c_i > 0$ $(i = 1, \cdots, k)$.

(d) *For every* $i = 1, \cdots, N-1$ *and* $j = 1, \cdots, l'$,

$$\text{if} \quad \langle \tilde{\alpha}_j, \omega^{(i)} \rangle > 0, \quad \text{then } \tilde{\alpha}_j \notin \Theta_i;$$
$$\text{if} \quad \langle \tilde{\alpha}_j, \omega^{(i)} \rangle < 0, \quad \text{then } \tilde{\alpha}_j \notin \Theta_1 \cup \cdots \cup \Theta_{i-1}.$$

This lemma is proved in Section 7. Assuming this lemma, we prove Lemma 4 in the rest of this section.

Now we review two well-known facts about the Bruhat ordering in the Weyl group W. These facts will be also used in Section 8. For a w in W, let $w = w_1 \cdots w_n$ be a reduced (minimal) expression of w by the reflections $w_1, \cdots, w_n$ with respect to simple roots in $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$. Then we put $l(w) = n$.

Proposition 4 ([3]). *Let* w *and* w' *be two elements in* W. *Then the following two conditions are equivalent.*

(i) *Let* $w = w_1 \cdots w_n$ *be a reduced expression of* w *by the reflections* $w_1, \cdots, w_n$ *with respect to simple roots in* $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$. *Then* w' *can be written as*

$$w' = w_{i_1} \cdots w_{i_r} \quad (1 \leq i_1 < \cdots < i_r \leq n).$$

(ii) *There exist elements* $w^{(0)}, \cdots, w^{(k)}$ *in* W *satisfying the following three conditions.*

(a) $w^{(0)} = w$, $w^{(k)} = w'$.

(b) $w^{(i)}(w^{(i-1)})^{-1}$ *is a reflection with respect to some root in* $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$ *for* $i = 1, \cdots, k$.

(c) $l(w^{(i)}) < l(w^{(i-1)})$ *for* $i = 1, \cdots, k$.

Proposition 5 ([4], p. 250, 7.7.2 Lemme). *Let* μ *be an element in* $\mathfrak{a}_\mathfrak{p}^d$ *satisfying* $\langle \mu, \tilde{\alpha} \rangle \geq 0$ *for all* $\tilde{\alpha} \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ *and let* w *and* w' *be elements in* W. *Suppose that* ww'^{-1} *is a reflection with respect to a root* β *in* $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ *and that* $l(w') < l(w)$. *Then* $w'\mu - w\mu \in \boldsymbol{R}_+ \beta$.

Proof of Lemma 4. By the first part of this section, we may assume that $(\Sigma(\mathfrak{a}_\mathfrak{p}^d), \theta)$ is irreducible and is neither of type (1) with $l = 2$ nor of type (14).

Let $\bar{Q}$ and w be arbitrary elements in $\bar{S}'$ and W, respectively. By Lemma 5 (i), there are a w_0' in W_θ and an ω in $\{\omega_1, \cdots, \omega_l\}$ satisfying $w_0' Q$

$\subset\langle\Theta_0'\rangle$. If $\langle w^{-1}w_0'^{-1}\omega, \mathfrak{a}_{\mathfrak{p}+}^d\rangle\not\subset \boldsymbol{R}_-$ and $\langle w^{-1}w_0'^{-1}\omega, \lambda\rangle\geq 0$, then we put $w'=w^{-1}w_0'^{-1}$ and $w_0=w_0'$. Otherwise we put $w'=w^{-1}w_0'^{-1}w^*$ and $w_0=w^*w_0'$. Then we have $w_0\overline{Q}\subset\langle\Theta_0'\rangle$. It is clear that the condition $\langle w'\omega, \mathfrak{a}_{\mathfrak{p}+}^d\rangle\not\subset\boldsymbol{R}_-$ and $\langle w'\omega, \lambda\rangle\geq 0$ or the condition $\langle w'\omega', \mathfrak{a}_{\mathfrak{p}+}^d\rangle\not\subset\boldsymbol{R}_-$ and $\langle w'\omega', \lambda\rangle\geq 0$ is satisfied. Here $\omega'=w^*\theta\omega=-w^*\omega$ is an element in $\{\omega_1, \cdots, \omega_l\}$. Since there is an involutive automorphism $-w^*$ of $(\Sigma(\mathfrak{a}_{\mathfrak{p}}^d), \theta)$ such that $-w^*\omega=\omega'$, we may assume that $\langle w'\omega, \mathfrak{a}_{\mathfrak{p}+}^d\rangle\not\subset\boldsymbol{R}_-$ and $\langle w'\omega, \lambda\rangle\geq 0$.

Put $\Theta_0=\{\tilde{\alpha}\in\Psi\mid\langle\tilde{\alpha}, \omega\rangle=0\}$. Then (i) in Lemma 4 is clear from the above argument. By Lemma 5 (ii) (a), there exist $w^{(1)}\in W_{\Theta_1}, \cdots, w^{(N-1)}\in W_{\Theta_{N-1}}$ such that $\omega^{(1)}=w^{(1)}\cdots w^{(N-1)}\omega$. Put $w''=w_{\gamma_k}\cdots w_{\gamma_1}w'$ as in (*c*). Then $w''\omega=\omega^{(1)}$. Clearly there is a $w^{(0)}$ in W_{Θ_0} such that $w''=w^{(1)}\cdots w^{(N-1)}w^{(0)}$. Since a reduced expression of w'' can be obtained as a subexpression from an arbitrary expression of w'', it follows from (c), Proposition 4 and Proposition 5 that w' can be expressed by a subexpression of w''. Thus we have $w'\in W_{\Theta_1}\cdots W_{\Theta_{N-1}}W_{\Theta_0}$ and (ii) in Lemma 4 is proved.

Lastly we will prove (iii) in Lemma 4. We have only to prove for $i=1, \cdots, N$ that

(6.1) For every $\mu\in\Lambda_i$, there exists a $\nu\in\Lambda_{i-1}$ such that

$$\langle\mu, \omega^{(i+1)}\rangle\geq\langle\nu, \omega^{(i)}\rangle.$$

Here we put $\omega^{(N+1)}=\omega$. If (6.1) is proved, then for every $\kappa\in\Lambda_N$ we have

$$\langle\kappa, \omega\rangle=\langle\kappa, \omega^{(N+1)}\rangle\geq\langle\lambda, \omega^{(1)}\rangle\geq 0$$

and we have proved (iii).

We will prove (6.1). Since $\omega^{(i+1)}=w^{(i)}\omega^{(i)}$, we have $\langle\mu, \omega^{(i+1)}\rangle=\langle\mu, w^{(i)}\omega^{(i)}\rangle=\langle(w^{(i)})^{-1}\mu, \omega^{(i)}\rangle$. Since $(w^{(i)})^{-1}\mu\in\Lambda_i$, there is a $\nu\in\Lambda_{i-1}$ such that $\langle(w^{(i)})^{-1}\mu, \tilde{\omega}_j\rangle\geq\langle\nu, \tilde{\omega}_j\rangle$ for $\tilde{\alpha}_j\in\Psi\backslash\Theta_i$ by the definition of Λ_i. On the other hand, since $\langle(w^{(i)})^{-1}\mu, -\tilde{\omega}_k\rangle\geq\langle\lambda, -\tilde{\omega}_k\rangle$ and $\langle\nu, -\tilde{\omega}_k\rangle=\langle\lambda, -\tilde{\omega}_k\rangle$ for $\tilde{\alpha}_k\in\Psi\backslash(\Theta_1\cup\cdots\cup\Theta_{i-1})$, we have

$$\langle(w^{(i)})^{-1}\mu, -\tilde{\omega}_k\rangle\geq\langle\nu, -\tilde{\omega}_k\rangle\quad\text{if } \tilde{\alpha}_k\in\Psi\backslash(\Theta_1\cup\cdots\cup\Theta_{i-1}).$$

Since $\omega^{(i)}=\sum_{j=1}^{l'}\langle\tilde{\alpha}_j, \omega^{(i)}\rangle\tilde{\omega}_j$, we have

$$\langle\mu, \omega^{(i+1)}\rangle=\langle(w^{(i)})^{-1}\mu, \omega^{(i)}\rangle\geq\langle\nu, \omega^{(i)}\rangle$$

by (d) in Lemma 5 for $i=1, \cdots, N-1$. When $i=N$, (6.1) is clear from the definition of Λ_N since $\Theta_N=\Theta_0$. Q.E.D.

§ 7. Proof of Lemma 5

In the following lemma a part of Lemma 5 is proved.

Lemma 6. *Suppose $(\Sigma(\mathfrak{a}_{\mathfrak{p}}^{d}), \theta)$ is irreducible and is not of the type* (1), (2), (6), (10) *nor* (14). *Then we have the followings.*

(i) *Let θ' denote the restriction of θ to $\mathfrak{a}_{\mathfrak{p}}^{d}$. Then $\theta' \in W_{\theta}$.*

(ii) *Let β be a root in $\Sigma_{\mathfrak{a}}(\mathfrak{a}_{\mathfrak{p}}^{d})$. Then the maximum root α in $W\beta$ with respect to the order $\Sigma(\mathfrak{a}_{\mathfrak{p}}^{d})^{+}$ is contained in $W_{\theta}\beta(\subset \Sigma_{\mathfrak{a}}(\mathfrak{a}_{\mathfrak{p}}^{d}))$ and $-w^{*}\alpha=\alpha$.*

(iii) *Let α be as in* (ii). *Then there is an i $(1 \leq i \leq l)$ such that $\alpha = c_{\alpha}\omega_{i}$. Here c_{α} is a constant given by $c_{\alpha}=\frac{1}{2}\langle\alpha, \alpha\rangle$ if $\langle\alpha, \alpha\rangle \geq \langle\alpha_{i}, \alpha_{i}\rangle$, $c_{\alpha}=\langle\alpha, \alpha\rangle$ if $\langle\alpha, \alpha\rangle=\frac{1}{2}\langle\alpha_{i}, \alpha_{i}\rangle$ and $c_{\alpha}=\frac{3}{2}\langle\alpha, \alpha\rangle$ if $\langle\alpha, \alpha\rangle=\frac{1}{3}\langle\alpha_{i}, \alpha_{i}\rangle$.*

(iv) *For every $\bar{Q} \in \bar{S}'$, there is an α given in* (ii) *satisfying $w_{0}(\bar{Q}) \subset \langle\Theta_{0}\rangle$ for some $w_{0} \in W_{\theta}$ where $\Theta_{0}=\{\tilde{\alpha} \in \Psi \mid \langle\tilde{\alpha}, \alpha\rangle=0\}$.*

Proof. (i) If $(\Sigma(\mathfrak{a}_{\mathfrak{p}}^{d}), \theta)$ is the same root system with an involution as that of a real simple Lie algebra, then there is a strongly orthogonal system $\{\gamma_{1}, \cdots, \gamma_{l}\}$ in $\Sigma_{\mathfrak{a}}(\mathfrak{a}_{\mathfrak{p}}^{d})$ $(l=\dim \mathfrak{a})$ when $(\Sigma(\mathfrak{a}_{\mathfrak{p}}^{d}), \theta)$ is not of the type (1), (2), (6), (10) nor (14). Thus $\theta'=w_{\gamma_{1}} \cdots w_{\gamma_{l}} \in W_{\theta}$. When $(\Sigma(\mathfrak{a}_{\mathfrak{p}}^{d}), \theta)$ is of type BIII or CI, (i) is clear from the cases of CII or BI, respectively. The proof in the case of FIII is easy.

(ii) Put $\bar{\mathfrak{a}}_{+}=\{Y \in \mathfrak{a} \mid \alpha(Y) \geq 0$ for all $\alpha \in \Sigma(\mathfrak{a})^{+}\}$. Then there is a unique root α in $W_{\theta}\beta \cap \bar{\mathfrak{a}}_{+}$ since $\bar{\mathfrak{a}}_{+}$ is a fundamental domain for $W(\mathfrak{a})=W_{\theta}|_{\mathfrak{a}}$. Since $\bar{\mathfrak{a}}_{+} \subset \bar{\mathfrak{a}}_{\mathfrak{p}+}^{d}$, α is the maximum root in $W\beta$. Since $-w^{*}\alpha$ has the same length as α, we have $-w^{*}\alpha \in W\alpha$. Thus $-w^{*}\alpha=\alpha$ since $-w^{*}\alpha \in \bar{\mathfrak{a}}_{+}$.

(iii) First we will prove that there exists a $\bar{Q}=\{\beta_{1}, \cdots, \beta_{l-1}\} \in \bar{S}'$ such that $\langle\alpha, \beta_{i}\rangle=0$ for $i=1, \cdots, l-1$. Let $\bar{Q}=\{\beta_{1}, \cdots, \beta_{k}\}$ be a maximal element in $\bar{S}'$ satisfying $\langle\alpha, \beta_{i}\rangle=0$ for $i=1, \cdots, k$. Suppose that $k<l-1$. Then $w=w_{\alpha}w_{\beta_{1}} \cdots w_{\beta_{k}}\theta' \in W_{\theta}$ fixes the subspace E of $\mathfrak{a}_{\mathfrak{p}}^{d*}$ which is generated by $\alpha, \beta_{1}, \cdots, \beta_{k}$ and $\mathfrak{a}_{1}$. Put $W_{E}=\{w \in W \mid w\mu=\mu$ for all $\mu \in E\}$. Since w is not the identity and since W_{E} is generated by the reflections contained in W_{E}, there is a root $\beta \in \Sigma_{\mathfrak{a}}(\mathfrak{a}_{\mathfrak{p}}^{d})$ which is orthogonal to $\alpha, \beta_{1}, \cdots, \beta_{k}$. Thus we have a contradiction to the maximality of $\bar{Q}$ and therefore $k=l-1$.

Since α is dominant for $\Sigma(\mathfrak{a})^{+}$, the subgroup

$$W(\mathfrak{a})_{\alpha}=\{w \in W(\mathfrak{a}) \mid w\alpha=\alpha\}$$

of $W(\mathfrak{a})$ is generated by simple reflections (i.e. reflections with respect to simple roots) contained in $W(\mathfrak{a})_{\alpha}$. Since $w_{\beta_{1}}|_{\mathfrak{a}}, \cdots, w_{\beta_{l-1}}|_{\mathfrak{a}}$ is contained in $W(\mathfrak{a})_{\alpha}$, the roots $\beta_{1}, \cdots, \beta_{l-1}$ can be written as linear combinations of the simple roots contained in $W(\mathfrak{a})_{\alpha}$. Thus the number of such simple roots must be $l-1$ and so there is an i $(1 \leq i \leq l)$ such that α is a constant multiple of ω_{i}. (The constant can be easily calculated.)

(iv) Let $\bar{Q}=\{\beta_{1}, \cdots, \beta_{k}\}$ be an element in $\bar{S}'$. Then there is a $\beta \in \Sigma_{\mathfrak{a}}(\mathfrak{a}_{\mathfrak{p}}^{d})$ which is orthogonal to $\beta_{1}, \cdots, \beta_{k}$ by the same argument in (iii).

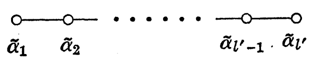
By (ii), there is a $w_0 \in W_\theta$ such that $\alpha = w_0\beta$ is maximum in $W\beta$. Then it is clear that α and w_0 satisfy (iv). Q.E.D.

Proof of Lemma 5. Suppose $(\Sigma(\mathfrak{a}_\mathfrak{p}^d), \theta)$ is not of type (1), (2), (6) nor (10). Then we put $\omega = \omega_i$ which is a constant multiple of α in Lemma 6 (ii). Hence Lemma 5 (i) is proved in Lemma 6 (iv). Let w' be an element in W satisfying $\langle w'\omega, \mathfrak{a}_{\mathfrak{p}+}^d\rangle \not\subset \boldsymbol{R}_-$. Then it is clear that $w'\omega \in c_\alpha^{-1}\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ and therefore $\langle w'\omega, \lambda\rangle \geq 0$. It is easy to show that the condition (c) in Lemma 5 (ii) is satisfied for some $\omega^{(1)} \in c_\alpha^{-1}\Psi$. Hence we have only to give for every $\omega^{(1)} \in \Psi$ a list of $N, \Theta_1, \cdots, \Theta_{N-1}, \omega^{(2)}, \cdots, \omega^{(N-1)}$ satisfying the conditions (a) and (d) in Lemma 5 (ii) in these cases. (The constant c_α has no effect on the proof.)

In the following we will prove Lemma 5 for each $(\Sigma(\mathfrak{a}_\mathfrak{p}^d), \theta)$ of type from (1) to (13).

In the cases of (1), (2) and (3), we take an orthonormal basis $\{e_1, \cdots, e_{l'+1}\}$ in $\boldsymbol{R}^{l'+1}$ and represent Ψ as $\tilde{\alpha}_1 = e_1 - e_2, \cdots, \tilde{\alpha}_{l'} = e_{l'} - e_{l'+1}$.

(1) AI $(l = l' \neq 2)$



Suppose that $l \geq 3$. Put $\omega = \tilde{\omega}_2 = e_1 + e_2 \pmod{\boldsymbol{R}(e_1 + \cdots + e_{l'+1})}$. Then $\omega' = \tilde{\omega}_{l'-1} = -(e_{l'} + e_{l'+1}) \pmod{\boldsymbol{R}(e_1 + \cdots + e_{l'+1})}$.

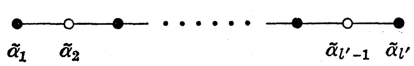
Proof of (i). Let $\bar{Q} = \{\beta_1, \cdots, \beta_k\}$ be an element in $\bar{S}' = \bar{S}$. Since $W_\theta = W$ is the group of all the permutations of $\{1, \cdots, l'+1\}$, there is a $w_0 \in W_\theta$ such that $w_0\beta_1 = \tilde{\alpha}_1$ and $w_0\beta_2 = \tilde{\alpha}_{l'}$. Then it is clear that $w_0\bar{Q} \subset \langle\Theta_0\rangle$.

Proof of (ii). $W\omega = \{e_i + e_j \mid 1 \leq i < j \leq l'+1\}$. If $w'\omega = e_i + e_j$ $(i<j)$, then we put $N=3$,

$$\Theta_1 = \{\tilde{\alpha}_1, \cdots, \tilde{\alpha}_{i-1}\}, \qquad \Theta_2 = \{\tilde{\alpha}_2, \cdots, \tilde{\alpha}_{j-1}\},$$
$$\omega^{(1)} = w'\omega = e_i + e_j, \qquad \omega^{(2)} = e_1 + e_j.$$

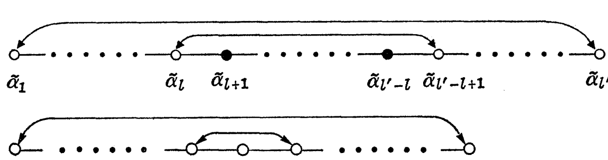
If $l = 1$, then we put $\omega = \omega' = \frac{1}{2}\tilde{\alpha}_1$, $\Theta_0 = \phi$ and $N = 1$.

(2) AII



$\bar{S} = \{\phi\}$ and the others are the same as (1).

(3) AIII

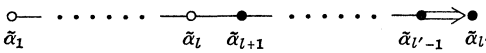


$\omega=e_1-e_{l'+1}$. For every i $(1\leq i\leq l')$, we put $N=2$,

$$\Theta_1=\{\tilde{\alpha}_1,\cdots,\tilde{\alpha}_{i-1},\tilde{\alpha}_{i+1},\cdots,\tilde{\alpha}_{l'}\},\qquad \omega^{(1)}=\tilde{\alpha}_i.$$

In the cases of (4) and (5), we take an orthonormal basis $\{e_1,\cdots,e_{l'}\}$ in $\boldsymbol{R}^{l'}$ and represent Ψ as $\tilde{\alpha}_1=e_1-e_2,\cdots,\tilde{\alpha}_{l'-1}=e_{l'-1}-e_{l'}$, $\tilde{\alpha}_{l'}=e_{l'}$ if $\Sigma(\mathfrak{a}_{\mathfrak{p}}^d)$ is of type $B_{l'}$. ($\tilde{\alpha}_{l'}=2e_{l'}$ if $\Sigma(\mathfrak{a}_{\mathfrak{p}}^d)$ is of type $C_{l'}$.)

(4) BI



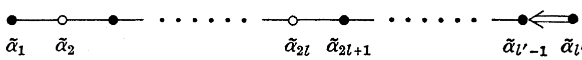
CI

(a) If $\omega=e_1$, then we put $N=2$, $\omega^{(1)}=e_{l'}$, $\Theta_1=\{\tilde{\alpha}_1,\cdots,\tilde{\alpha}_{l'-1}\}$.

(b) Suppose that $\omega=e_1+e_2$. Then for every $\omega^{(1)}=\tilde{\alpha}_i$ $(1\leq i\leq l'-1)$, we put $N=3$,

$$\Theta_1=\{\tilde{\alpha}_1,\cdots,\tilde{\alpha}_{i-1},\tilde{\alpha}_{i+1},\cdots,\tilde{\alpha}_{l'}\},\qquad \omega^{(2)}=e_1+e_{i+1},$$
$$\Theta_2=\{\tilde{\alpha}_2,\cdots,\tilde{\alpha}_i\}.$$

(5) CII

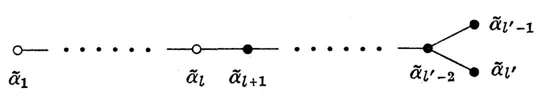


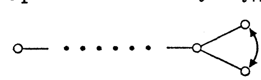
BIII

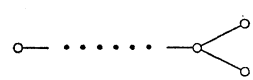
$\omega=e_1+e_2$ and the others are the same as (4) (b).

In the cases of (6) and (7), we put $\tilde{\alpha}_1=e_1-e_2,\cdots,\tilde{\alpha}_{l'-1}=e_{l'-1}-e_{l'}$, $\tilde{\alpha}_{l'}=e_{l'-1}+e_{l'}$ where $\{e_1,\cdots,e_{l'}\}$ is an orthonormal basis in $\boldsymbol{R}^{l'}$.

(6) DI







Proof of (i). (a) If l is odd, then we put $\omega=\omega'=e_1$ and it is easy to show (i).

(b) Suppose that l is even. Since there is a strongly orthogonal system with l elements in $\Sigma_{\mathfrak{a}}(\mathfrak{a}_{\mathfrak{p}}^d)$, Lemma 6 holds in this case.

Proof of (ii). (a) Since $\langle\mathfrak{a}_{\mathfrak{p}+}^d,-e_i\rangle\subset\boldsymbol{R}_-$ for $i=1,\cdots,l'-1$, we have $w'\omega=e_i$ for some $i=1,\cdots,l'$ or $w'\omega=-e_{l'}$. We put $\omega^{(1)}=e_{l'}$ if

$\langle\lambda, e_{l'}\rangle \geq 0$ and we put $\omega^{(1)}=-e_{l'}$ otherwise. Then it is easy to see that Lemma 4 (ii) (c) is satisfied. Put $N=2$.

If $\omega^{(1)}=e_{l'}$, then we put $\Theta_1=\{\tilde{\alpha}_1, \cdots, \tilde{\alpha}_{l'-1}\}$.

If $\omega^{(1)}=-e_{l'}$ then we put $\Theta_1=\{\tilde{\alpha}_1, \cdots, \tilde{\alpha}_{l'-2}, \tilde{\alpha}_{l'}\}$.

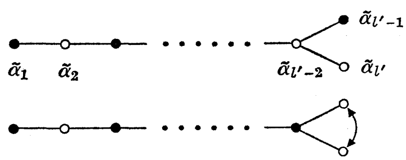
(b) $\omega=e_1+e_2$. If $\omega^{(1)}=\tilde{\alpha}_i$ $(1\leq i\leq l'-2)$, then we put $N=3$,

$$\Theta_1=\{\tilde{\alpha}_1, \cdots, \tilde{\alpha}_{i-1}, \tilde{\alpha}_{i+1}, \cdots, \tilde{\alpha}_{l'}\}, \qquad \omega^{(2)}=e_1+e_{i+1},$$
$$\Theta_2=\{\tilde{\alpha}_2, \cdots, \tilde{\alpha}_i\}.$$

If $\omega^{(1)}=\tilde{\alpha}_{l'-1}$, then we put $N=2$, $\Theta_1=\{\tilde{\alpha}_1, \cdots, \tilde{\alpha}_{l'-2}, \tilde{\alpha}_{l'}\}$.

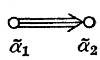
If $\omega^{(1)}=\alpha_{l'}$, then we put $N=2$, $\Theta_1=\{\tilde{\alpha}_1, \cdots, \tilde{\alpha}_{l'-1}\}$.

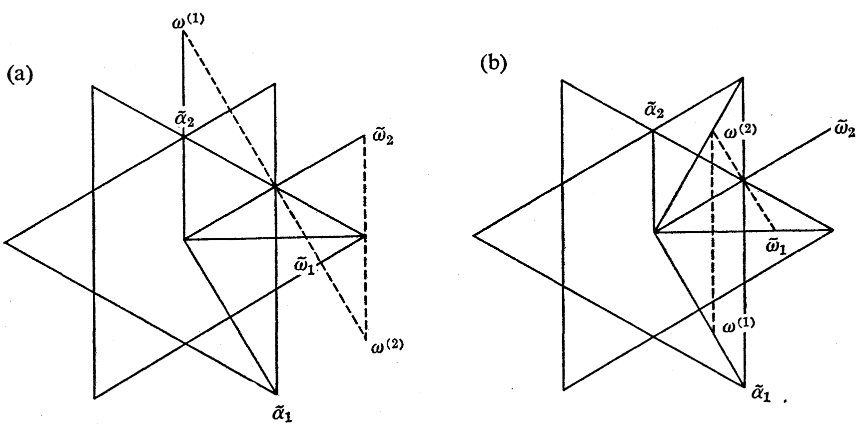
(7) DIII



$\omega=e_1+e_2$ and the others are the same as (6) (b).

(8) G





Suppose $\langle\tilde{\alpha}_1, \tilde{\alpha}_1\rangle=3$ and $\langle\tilde{\alpha}_2, \tilde{\alpha}_2\rangle=1$. Put $N=3$.

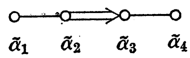
(a) If $\omega=\tilde{\omega}_2$, then we put

$$\omega^{(1)}=2\tilde{\alpha}_2,\ \Theta_1=\{\tilde{\alpha}_1\},$$
$$\omega^{(2)}=3\tilde{\omega}_1-\tilde{\omega}_2,\ \Theta_2=\{\tilde{\alpha}_2\}.$$

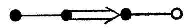
(b) If $\omega=\tilde{\omega}_1$, then we put

$$\omega^{(1)}=\tfrac{2}{3}\tilde{\alpha}_1,\ \Theta_1=\{\tilde{\alpha}_2\},$$
$$\omega^{(2)}=\tilde{\omega}_2-\tilde{\omega}_1,\ \Theta_2=\{\tilde{\alpha}_1\}.$$

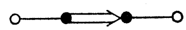
(9) FI



FII



FIII



We put $\tilde{\alpha}_1=e_1-e_2$, $\tilde{\alpha}_2=e_2-e_3$, $\tilde{\alpha}_3=e_3$, $\tilde{\alpha}_4=\frac{1}{2}(e_4-e_1-e_2-e_3)$ where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis in $\boldsymbol{R}^4$. Put $N=3$.

(a) $\omega=\tilde{\omega}_4=2e_4$.

If $\omega^{(1)}=2\tilde{\alpha}_4$, then we put $\Theta_1=\{\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3\}$, $\omega^{(2)}=e_1+e_2+e_3+e_4=\tilde{\omega}_3-\tilde{\omega}_4$, $\Theta_2=\{\tilde{\alpha}_4\}$.

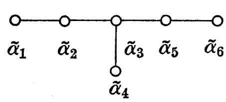
If $\omega^{(1)}=2\tilde{\alpha}_3$, then we put $\Theta_1=\{\tilde{\alpha}_1, \tilde{\alpha}_2\}$, $\omega^{(2)}=2e_1=2\tilde{\omega}_1-\tilde{\omega}_4$, $\Theta_2=\{\tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4\}$.

(b) $\omega=\tilde{\omega}_1=e_1+e_4$ (if $\Sigma(\mathfrak{a}_\mathfrak{p}^d, \theta)$ is of type FI or FIII).

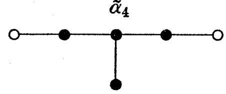
If $\omega^{(1)}=\tilde{\alpha}_1$, then we put $\Theta_1=\{\tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4\}$, $\omega^{(2)}=e_2+e_4=\tilde{\omega}_2-\tilde{\omega}_1$, $\Theta_2=\{\tilde{\alpha}_1\}$.

If $\omega^{(1)}=\tilde{\alpha}_2$, then we put $\Theta_1=\{\tilde{\alpha}_3, \tilde{\alpha}_4\}$, $\omega^{(2)}=e_4-e_1=\tilde{\omega}_4-\tilde{\omega}_1$, $\Theta_2=\{\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3\}$.

(10) EI



EIV



Put $\omega=\tilde{\omega}_6$ and $\omega'=\tilde{\omega}_1$. Then (i) is clear from [14], p. 417 and p. 419. If $\langle\tilde{\alpha}_i, \tilde{\alpha}_i\rangle=2$ $(1\leq i\leq 6)$, then

$$\tilde{\omega}_6=\tfrac{1}{3}(2\tilde{\alpha}_1+4\tilde{\alpha}_2+6\tilde{\alpha}_3+3\tilde{\alpha}_4+5\tilde{\alpha}_5+4\tilde{\alpha}_6).$$

For every $w\in W$, we write $w\tilde{\omega}_6$ as

$$\begin{matrix}a_1a_2a_3a_5a_6\\ a_4\end{matrix}$$

for the sake of simplicity if $w\tilde{\omega}_6=\frac{1}{3}(a_1\tilde{\alpha}_1+a_2\tilde{\alpha}_2+a_3\tilde{\alpha}_3+a_4\tilde{\alpha}_4+a_5\tilde{\alpha}_5+a_6\tilde{\alpha}_6)$. Using this notation, $W\tilde{\omega}_6$ is described as Fig. 1 where $\xrightarrow{(i)}$ denotes the reflection with respect to $\tilde{\alpha}_i$.

By Lemma 5 (ii) (c), we have only to consider the case of $\omega^{(1)}=\begin{matrix}21021\\ 0\end{matrix}$

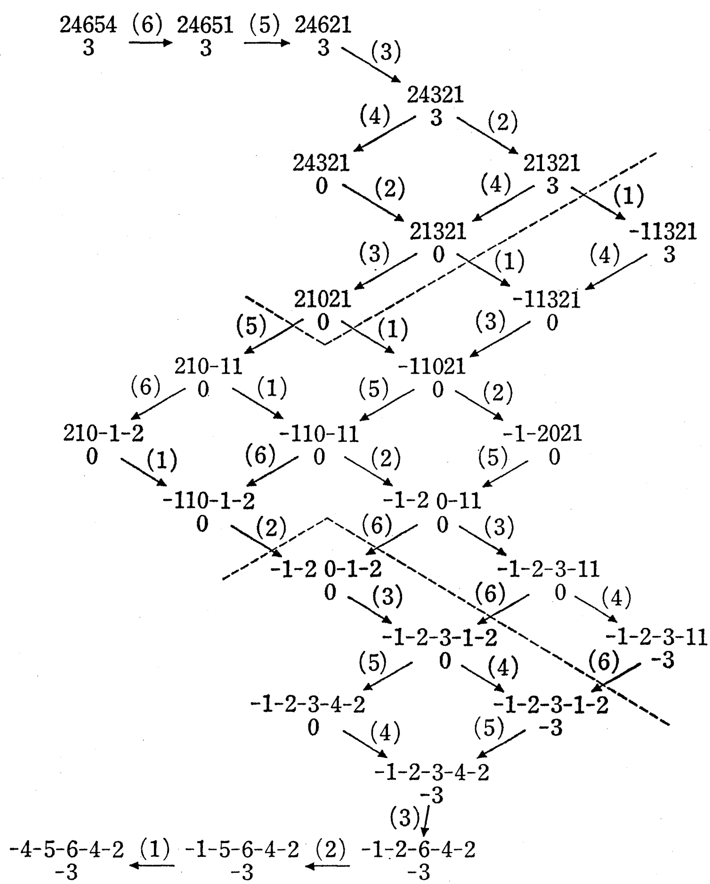


Fig. 1.

and the cases that $\omega^{(1)}$ is contained in the middle domain of Fig. 1. In the followings we use the abbreviation such as

$$\tilde{\omega}_6 \xrightarrow{(56)} \tilde{\omega}_3 - \tilde{\omega}_5 \xrightarrow{(234)} \tilde{\omega}_1 + \tilde{\omega}_5 - \tilde{\omega}_3 = \begin{matrix} 21021 \\ 0 \end{matrix}$$

which means that we put $N=3$,

$$\omega^{(2)} = \tilde{\omega}_3 - \tilde{\omega}_5, \qquad \omega^{(1)} = \tilde{\omega}_1 + \tilde{\omega}_5 - \tilde{\omega}_3 = \tfrac{1}{3}(2\tilde{\alpha}_1 + \tilde{\alpha}_2 + 2\tilde{\alpha}_5 + \tilde{\alpha}_6),$$

$$\Theta_2 = \{\tilde{\alpha}_5, \tilde{\alpha}_6\}, \qquad \Theta_1 = \{\tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4\}.$$

$$\tilde{\omega}_6 \xrightarrow{(56)} \tilde{\omega}_3 - \tilde{\omega}_5 \xrightarrow{(234)} \tilde{\omega}_1 + \tilde{\omega}_5 - \tilde{\omega}_3 = \begin{matrix} 21021 \\ 0 \end{matrix}$$

$$\tilde{\omega}_6 \xrightarrow{(6)} \tilde{\omega}_5-\tilde{\omega}_6 \xrightarrow{(2345)} \tilde{\omega}_1+\tilde{\omega}_6-\tilde{\omega}_5=\begin{matrix}210-11\\0\end{matrix}$$

$$\xrightarrow{(1)} \tilde{\omega}_2+\tilde{\omega}_6-\tilde{\omega}_1-\tilde{\omega}_5=\begin{matrix}-110-11\\0\end{matrix}$$

$$\xrightarrow{(12)} \tilde{\omega}_3+\tilde{\omega}_6-\tilde{\omega}_2-\tilde{\omega}_5=\begin{matrix}-1-20-11\\0\end{matrix}$$

$$\xrightarrow{(123)} \tilde{\omega}_4+\tilde{\omega}_6-\tilde{\omega}_3=\begin{matrix}-1-2-3-11\\0\end{matrix}$$

$$\xrightarrow{(1234)} \tilde{\omega}_6-\tilde{\omega}_4=\begin{matrix}-1-2-3-11\\-3\end{matrix}$$

$$\tilde{\omega}_6 \xrightarrow{(23456)} \tilde{\omega}_1-\tilde{\omega}_6=\begin{matrix}210-1-2\\0\end{matrix}$$

$$\xrightarrow{(1)} \tilde{\omega}_2-\tilde{\omega}_1-\tilde{\omega}_6=\begin{matrix}-110-1-2\\0\end{matrix}$$

$$\tilde{\omega}_6 \xrightarrow{(12356)} \tilde{\omega}_4-\tilde{\omega}_1=\begin{matrix}-11321\\3\end{matrix}$$

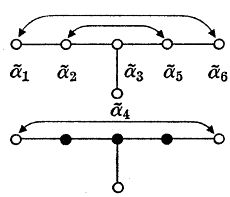
$$\xrightarrow{(4)} \tilde{\omega}_3-\tilde{\omega}_1-\tilde{\omega}_4=\begin{matrix}-11321\\0\end{matrix}$$

$$\xrightarrow{(34)} \tilde{\omega}_2+\tilde{\omega}_5-\tilde{\omega}_1-\tilde{\omega}_3=\begin{matrix}-11021\\0\end{matrix}$$

$$\xrightarrow{(234)} \tilde{\omega}_5-\tilde{\omega}_2=\begin{matrix}-1-2021\\0\end{matrix}$$

(11) EII

EIII



$\omega=\tilde{\omega}_4$. In this case we use the abbreviation such as

$$\begin{matrix}\begin{matrix}12321\\2\end{matrix} & \xrightarrow{(4)} & \begin{matrix}12321\\1\end{matrix} & \xrightarrow{(12356)} & \begin{matrix}00000\\1\end{matrix}\\ \| & & \| & & \| \\ \tilde{\omega}_4 & \longrightarrow & \tilde{\omega}_3-\tilde{\omega}_4 & \longrightarrow & 2\tilde{\omega}_4-\tilde{\omega}_3\end{matrix}$$

which means that we put $N=3$,

$$\omega^{(2)}=\tilde{\omega}_3-\tilde{\omega}_4=\tilde{\alpha}_1+2\tilde{\alpha}_2+3\tilde{\alpha}_3+\tilde{\alpha}_4+2\tilde{\alpha}_5+\tilde{\alpha}_6,$$

$$\omega^{(1)}=2\tilde{\omega}_4-\tilde{\omega}_3=\tilde{\alpha}_4,$$

$$\Theta_2=\{\tilde{\alpha}_4\},\qquad \Theta_1=\{\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_5, \tilde{\alpha}_6\}.$$

$$\begin{array}{ccccc} 12321 & \xrightarrow{(4)} & 12321 & \xrightarrow{(12356)} & 00000\\ 2 & & 1 & & 1 \\ \| & & \| & & \| \\ \tilde\omega_4 & \longrightarrow & \tilde\omega_3-\tilde\omega_4 & \longrightarrow & 2\tilde\omega_4-\tilde\omega_3\end{array}$$

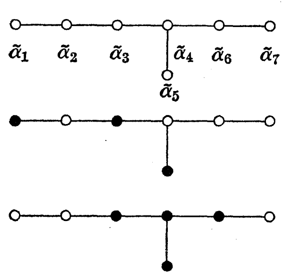
$$\begin{array}{ccccc} 12321 & \xrightarrow{(2345)} & 11111 & & \\ 2 & & 0 & & \\ \| & & \| & & \\ \tilde\omega_4 & \longrightarrow & \tilde\omega_1+\tilde\omega_6-\tilde\omega_4 & & \end{array}$$

$$\begin{array}{ccc} 11111 & & \\ 0 & \xrightarrow{(2356)} & 10000 \\ & & 0 \\ & & \| \\ \tilde\omega_1+\tilde\omega_6-\tilde\omega_4 & \longrightarrow & 2\tilde\omega_1-\tilde\omega_2 \\ & & \\ 0 & \xrightarrow{(1356)} & 01000 \\ & & 0 \\ & & \| \\ \tilde\omega_1+\tilde\omega_6-\tilde\omega_4 & \longrightarrow & 2\tilde\omega_2-\tilde\omega_1-\tilde\omega_3 \\ & & \\ 0 & \xrightarrow{(1256)} & 00100 \\ & & 0 \\ & & \| \\ \tilde\omega_1+\tilde\omega_6-\tilde\omega_4 & \longrightarrow & 2\tilde\omega_3-\tilde\omega_2-\tilde\omega_4-\tilde\omega_5 \\ & & \\ 0 & \xrightarrow{(1236)} & 00010 \\ & & 0 \\ & & \| \\ \tilde\omega_1+\tilde\omega_6-\tilde\omega_4 & \longrightarrow & 2\tilde\omega_5-\tilde\omega_3-\tilde\omega_6 \\ & & \\ 0 & \xrightarrow{(1235)} & 00001 \\ & & 0 \\ & & \| \\ \tilde\omega_1+\tilde\omega_6-\tilde\omega_4 & \longrightarrow & 2\tilde\omega_6-\tilde\omega_5 \end{array}$$

(12) EV

EVI

EVII



$\omega=\tilde\omega_7$ and we use the same abbreviation as in (11).

$$\begin{array}{ccccc} 123432 & \xrightarrow{(7)} & 123431 & \xrightarrow{(1\text{–}6)} & 000001\\ 2 & & 2 & & 0 \\ \| & & \| & & \| \\ \tilde\omega_7 & \longrightarrow & \tilde\omega_6-\tilde\omega_7 & \longrightarrow & 2\tilde\omega_7-\tilde\omega_6\end{array}$$

$$\begin{array}{ccc} \begin{matrix}123432\\2\end{matrix} & \xrightarrow{(2\text{–}7)} & \begin{matrix}100000\\0\end{matrix} \\ \| & & \| \\ \tilde\omega_7 & \longrightarrow & 2\tilde\omega_1-\tilde\omega_2 \end{array}$$

$$\begin{array}{ccc} \begin{matrix}123432\\2\end{matrix} & \xrightarrow{(3\text{–}7)} & \begin{matrix}122210\\1\end{matrix} \\ \| & & \| \\ \tilde\omega_7 & \longrightarrow & \tilde\omega_2-\tilde\omega_7 \end{array}$$

$$\begin{array}{ccccc} \begin{matrix}122210\\1\end{matrix} & \xrightarrow{(2)} & \begin{matrix}112210\\1\end{matrix} & \xrightarrow{(13456)} & \begin{matrix}010000\\0\end{matrix} \\ \| & & \| & & \| \\ \tilde\omega_2-\tilde\omega_7 & \longrightarrow & \tilde\omega_1+\tilde\omega_3-\tilde\omega_2-\tilde\omega_7 & \longrightarrow & 2\tilde\omega_2-\tilde\omega_1-\tilde\omega_3 \end{array}$$

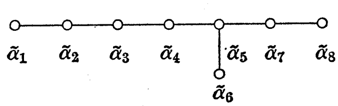
$$\begin{array}{ccccc} \begin{matrix}122210\\1\end{matrix} & \xrightarrow{(23)} & \begin{matrix}111210\\1\end{matrix} & \xrightarrow{(12456)} & \begin{matrix}001000\\0\end{matrix} \\ \| & & \| & & \| \\ \tilde\omega_2-\tilde\omega_7 & \longrightarrow & \tilde\omega_1+\tilde\omega_4-\tilde\omega_3-\tilde\omega_7 & \longrightarrow & 2\tilde\omega_3-\tilde\omega_2-\tilde\omega_4 \end{array}$$

$$\begin{array}{ccccc} \begin{matrix}122210\\1\end{matrix} & \xrightarrow{(234)} & \begin{matrix}111110\\1\end{matrix} & \xrightarrow{(12356)} & \begin{matrix}000100\\0\end{matrix} \\ \| & & \| & & \| \\ \tilde\omega_2-\tilde\omega_7 & \longrightarrow & \tilde\omega_1+\tilde\omega_5+\tilde\omega_6-\tilde\omega_4-\tilde\omega_7 & \longrightarrow & 2\tilde\omega_4-\tilde\omega_3-\tilde\omega_5-\tilde\omega_6 \end{array}$$

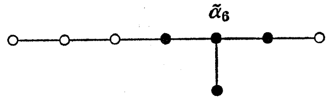
$$\begin{array}{ccc} \begin{matrix}122210\\1\end{matrix} & \xrightarrow{(12346)} & \begin{matrix}000000\\1\end{matrix} \\ \| & & \| \\ \tilde\omega_2-\tilde\omega_7 & \longrightarrow & 2\tilde\omega_5-\tilde\omega_4 \end{array}$$

$$\begin{array}{ccc} \begin{matrix}122210\\1\end{matrix} & \xrightarrow{(12345)} & \begin{matrix}000010\\0\end{matrix} \\ \| & & \| \\ \tilde\omega_2-\tilde\omega_7 & \longrightarrow & 2\tilde\omega_6-\tilde\omega_4-\tilde\omega_7 \end{array}$$

(13) EVIII



EIX



$\omega=\tilde\omega_1$ and we use the same abbreviation as in (11).

$$\begin{array}{ccccc} \begin{matrix}2345642\\3\end{matrix} & \xrightarrow{(1)} & \begin{matrix}1345642\\3\end{matrix} & \xrightarrow{(2\text{–}8)} & \begin{matrix}1000000\\0\end{matrix} \\ \| & & \| & & \| \\ \tilde\omega_1 & \longrightarrow & \tilde\omega_2-\tilde\omega_1 & \longrightarrow & 2\tilde\omega_1-\tilde\omega_2 \end{array}$$

$$\begin{array}{ccccc} \begin{matrix}2345642\\3\end{matrix} & \xrightarrow{(1\text{–}7)} & \begin{matrix}0123432\\2\end{matrix} & \xrightarrow{(3\text{–}8)} & \begin{matrix}0100000\\0\end{matrix} \\ \| & & \| & & \| \\ \tilde\omega_1 & \longrightarrow & \tilde\omega_8-\tilde\omega_1 & \longrightarrow & 2\tilde\omega_2-\tilde\omega_1-\tilde\omega_3 \end{array}$$

$$\begin{array}{ccccc}
2345642 & \xrightarrow{(1\text{–}7)} & 0123432 & \xrightarrow{(4\text{–}8)} & 0122210 \\
3 & & 2 & & 1 \\
\| & & \| & & \| \\
\tilde{\omega}_1 & \longrightarrow & \tilde{\omega}_8-\tilde{\omega}_1 & \longrightarrow & \tilde{\omega}_3-\tilde{\omega}_1-\tilde{\omega}_8
\end{array}$$

and from 0122210 ($\tilde{\omega}_3-\tilde{\omega}_1-\tilde{\omega}_8$) the branches:

$$\begin{array}{ccccc}
\xrightarrow{(3)} & 0112210 & \xrightarrow{(24567)} & 0010000 \\
 & 1 & & 0 \\
 & \| & & \| \\
\longrightarrow & \tilde{\omega}_2+\tilde{\omega}_4-\tilde{\omega}_1-\tilde{\omega}_3-\tilde{\omega}_8 & \longrightarrow & 2\tilde{\omega}_3-\tilde{\omega}_2-\tilde{\omega}_4
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{(34)} & 0111210 & \xrightarrow{(23567)} & 0001000 \\
 & 1 & & 0 \\
 & \| & & \| \\
\longrightarrow & \tilde{\omega}_2+\tilde{\omega}_5-\tilde{\omega}_1-\tilde{\omega}_4-\tilde{\omega}_8 & \longrightarrow & 2\tilde{\omega}_4-\tilde{\omega}_3-\tilde{\omega}_5
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{(345)} & 0111110 & \xrightarrow{(23467)} & 0000100 \\
 & 1 & & 0 \\
 & \| & & \| \\
\longrightarrow & \tilde{\omega}_2+\tilde{\omega}_6+\tilde{\omega}_7-\tilde{\omega}_1-\tilde{\omega}_5-\tilde{\omega}_8 & \longrightarrow & 2\tilde{\omega}_5-\tilde{\omega}_4-\tilde{\omega}_6-\tilde{\omega}_7
\end{array}$$

$$\begin{array}{cc}
\xrightarrow{(23457)} & 0000000 \\
 & 1 \\
 & \| \\
\longrightarrow & 2\tilde{\omega}_6-\tilde{\omega}_5
\end{array}$$

$$\begin{array}{cc}
\xrightarrow{(23456)} & 0000010 \\
 & 0 \\
 & \| \\
\longrightarrow & 2\tilde{\omega}_7-\tilde{\omega}_5-\tilde{\omega}_8
\end{array}$$

$$\begin{array}{ccccccc}
2345642 & \xrightarrow{(1\text{–}7)} & 0123432 & \xrightarrow{(8)} & 0123431 & \xrightarrow{(2\text{–}7)} & 0000001 \\
3 & & 2 & & 2 & & 0 \\
\| & & \| & & \| & & \| \\
\tilde{\omega}_1 & \longrightarrow & \tilde{\omega}_8-\tilde{\omega}_1 & \longrightarrow & \tilde{\omega}_7-\tilde{\omega}_1-\tilde{\omega}_8 & \longrightarrow & 2\tilde{\omega}_8-\tilde{\omega}_7
\end{array}$$

Thus we have proved Lemma 5. Q.E.D.

§ 8. Proof of Theorem 2

Theorem 2. *Suppose that* $\operatorname{rank}(G/H)=\operatorname{rank}(K/K\cap H)$ *and let* λ *be an element of* $(\mathfrak{a}_\mathfrak{p}^d)_c^*$ *satisfying* $\operatorname{Re}\langle\lambda,\tilde{\alpha}\rangle>0$ *for all* $\tilde{\alpha}\in\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$. *Then we have the following* $\mathfrak{g}_c$*-isomorphism*

$$\eta^{-1}\circ\mathscr{P}_\lambda\colon \bigoplus_{j=1}^{m}\mathscr{B}_{H^d}^j(G^d/P^d;L_\lambda)\xrightarrow{\sim}\mathscr{A}_K(G/H;\mathscr{M}_\lambda)\cap L^2(G/H)$$

by Flensted-Jensen's isomorphism η *and the Poisson transform* $\mathscr{P}_\lambda$.

The proof of Theorem 2 is reduced to Lemma 2 in Section 3 and to the following lemma.

Lemma 7. *Let λ be an element of $\mathfrak{a}_{\mathfrak{p}}^{d*}$ satisfying $\langle\lambda, \tilde{\alpha}\rangle > 0$ for all $\tilde{\alpha} \in \Psi$ and suppose that $H^d x P^d$ is a closed subset in G^d. Suppose that $w \in W$ satisfies $\langle w\lambda, \omega_k\rangle \geq 0$ for some k $(1 \leq k \leq l)$. Let $w = w_{\gamma_N} \cdots w_{\gamma_1}$ be a reduced expression of w by the reflections with respect to roots in Ψ. Then the subset*

$$H^d x M^d_{\{\gamma_1\}} \cdots M^d_{\{\gamma_N\}} P^d$$

has no inner points in G^d.

Using these lemmas we will first prove Theorem 2. By Theorem 1 (ii) and by the fact that $\beta_\lambda = c\mathscr{P}_\lambda^{-1}$ with some constant c (§ 3), we have only to prove the following.

(8.1) If $g \in \mathscr{B}^j_{H^d}(G^d/P^d; L_\lambda)$, then $\eta^{-1} \circ \mathscr{P}_\lambda(g) \in L^2(G/H)$.

(8.1) is proved as follows. Put $f = \eta^{-1} \circ \mathscr{P}_\lambda(g)$. Then $g = c^{-1}\beta_\lambda f^\eta$. By Proposition 2 in Section 3, we have only to prove that supp $\beta_{w\lambda} f^\eta$ has no inner points if Re $\langle w\lambda, \omega_k\rangle \geq 0$ for some k $(1 \leq k \leq l)$. Let w' be an element of $W(w\lambda)$ (§ 3, Lemma 2). Then it is clear from the definition of $W(w\lambda)$ that Re $\langle w'\lambda, \omega_k\rangle \geq 0$. Let $w' = w_{\gamma_N} \cdots w_{\gamma_1}$ be a reduced expression of w' and put $S = H^d x_j$. Then by Lemma 7, $S(w') = H^d x_j M^d_{\{\gamma_1\}} \cdots M^d_{\{\gamma_N\}} P^d$ has no inner points in G^d. (We may assume that Re $\lambda = \lambda$.) On the other hand, we have supp $\beta_{w\lambda} f^\eta \subset \bigcup_{w' \in W(w\lambda)} S(w')$ by Lemma 2. Thus we have proved the theorem. Q.E.D.

Though the following fact seems to be well-known, we will give a proof for the sake of completeness.

Lemma 8. *Let P' be a minimal parabolic subgroup of G^d and $A'_{\mathfrak{p}} = \exp \mathfrak{a}'_{\mathfrak{p}}$ a split component of P'. Let $\Sigma(\mathfrak{a}'_{\mathfrak{p}})^+$ be the positive system of $\Sigma(\mathfrak{a}'_{\mathfrak{p}})$ corresponding to P'. Let $\gamma'_1, \cdots, \gamma'_n$ be simple roots in $\Sigma(\mathfrak{a}'_{\mathfrak{p}})^+$ and $w'_1, \cdots, w'_n$ be reflections with respect to $\gamma'_1, \cdots, \gamma'_n$, respectively. Put $W'_i = \{1, w'_i\}$ and $P'_i = P'W'_iP'$. Let $P'_i = M'_iA'_iN'_i$ be a Langlands decomposition of P'_i $(1 \leq i \leq n)$. Then for every $x \in P'M'_1 \cdots M'_nP'$, there exist $i_1, \cdots, i_r$ satisfying $1 \leq i_1 < \cdots < i_r \leq n$ and*

$$x \in P'w'_{i_1} \cdots w'_{i_r} P'.$$

Proof. We will prove it by induction on n. Since $P'M'_1 \cdots M'_nP' = P'M'_1 \cdots M'_{n-1}P'M'_n$, there is a $y \in P'M'_1 \cdots M'_{n-1}P'$ such that $x \in yM'_n$. By the assumption of induction, there exist $i_1, \cdots, i_s$ satisfying $1 \leq i_1 < \cdots < i_s \leq n-1$ and $y \in P'wP'$ with $w = w'_{i_1} \cdots w'_{i_s}$. Hence

$$x \in P'wP'M'_n = w(w^{-1}P'w)M'_nP'.$$

By the Bruhat decomposition $M'_n=(w^{-1}P'w\cap M'_n)W'_n(P'\cap M'_n)$ of M'_n, we have

$$x\in w(w^{-1}P'w)W'_nP'=P'wW'_nP',$$

proving the lemma. Q.E.D.

Proof of Lemma 7. We write $w_{\gamma_i}=w_i$ and $M^d_{\{\gamma_i\}}=M^d_i$ for the sake of simplicity. Since $H^dxM^d_1\cdots M^d_NP^d=H^dx_jM^d_1\cdots M^d_NP^d$ if $H^dxP^d=H^dx_jP^d$, we may assume that $x=x_j$ for some $1\leq j\leq m$.

Suppose that $H^dxM^d_1\cdots M^d_NP^d$ has inner points in G^d. Then we will get a contradiction. By Proposition 3 (iii) in Section 4, there is a $\tilde{w}_0\in N_{K^d}(\mathfrak{a}^d_\mathfrak{p})$ such that $\mathrm{Ad}(\tilde{w}_0)\in W_\theta$ and that $\tilde{w}_0=hxm_1\cdots m_Np$ for some $h\in H^d$, $m_i\in M^d_i$ $(i=1,\cdots,N)$ and $p\in P^d$. Put $m'_i=xm_ix^{-1}$, $M'_i=xM^d_ix^{-1}$, $p'=xpx^{-1}$ and $P'=xP^dx^{-1}$. Then $\tilde{w}_0x^{-1}=hm'_1\cdots m'_Np'$. We have

$$\theta(\tilde{w}_0x^{-1})=h\theta(m'_1)\cdots\theta(m'_N)\theta(p')\in hM'_1\cdots M'_NP'$$

since $\theta M'_i=M'_i$ and $\theta P'=P'$. (Note that $\mathfrak{a}'_\mathfrak{p}\subset\mathfrak{h}^d$.) Thus we have

$$x\tilde{w}_0^{-1}\theta(\tilde{w}_0)\theta(x^{-1})\in P'M'_N\cdots M'_1M'_1\cdots M'_NP'.$$

For every $Y\in\mathfrak{a}'_\mathfrak{p}$, we have

$$\begin{aligned}\mathrm{Ad}(x\tilde{w}_0^{-1}\theta(\tilde{w}_0)\theta(x^{-1}))Y&=\mathrm{Ad}(x\tilde{w}_0^{-1}\theta(\tilde{w}_0))\theta(\mathrm{Ad}(x^{-1})Y)\\&=(\mathrm{Ad}(x)\circ\theta\circ\mathrm{Ad}(x^{-1}))Y\end{aligned}\tag{8.2}$$

since $\mathrm{Ad}(\tilde{w}_0)\theta(Z)=\theta\,\mathrm{Ad}(\tilde{w}_0)(Z)$ and since $\mathrm{Ad}(\theta(\tilde{w}_0))(Z)=\theta\,\mathrm{Ad}(\tilde{w}_0)\theta^{-1}(Z)$ $=\mathrm{Ad}(\tilde{w}_0)(Z)$ for $Z\in\mathfrak{a}^d_\mathfrak{p}$. Thus we have

$$x\tilde{w}_0^{-1}\theta(\tilde{w}_0)\theta(x^{-1})\in N_{K^d}(\mathfrak{a}'_\mathfrak{p}).$$

Applying Lemma 8 to $x\tilde{w}_0^{-1}\theta(\tilde{w}_0)\theta(x^{-1})$, there are $i_1,\cdots,i_s,j_1,\cdots,j_t$ $(1\leq i_1<\cdots<i_s\leq N,\ 1\leq j_1<\cdots<j_t\leq N)$ such that

$$\mathrm{Ad}(x\tilde{w}_0^{-1}\theta(\tilde{w}_0)\theta(x^{-1}))|_{\mathfrak{a}'_\mathfrak{p}}=w'_{i_s}\cdots w'_{i_1}w'_{j_1}\cdots w'_{j_t}$$

where w'_i is the reflection with respect to $\gamma'_i=\gamma_i\circ\mathrm{Ad}(x)^{-1}$. Hence we have by (8.2)

$$\theta(Z)=w_{i_s}\cdots w_{i_1}w_{j_1}\cdots w_{j_t}(Z)\qquad\text{for } Z\in\mathfrak{a}^d_\mathfrak{p}.$$

Since $\omega_k\in\mathfrak{a}$, we have $\theta\omega_k=-\omega_k$ and therefore we have

$$(w_{i_1}\cdots w_{i_s})\omega_k=-(w_{j_1}\cdots w_{j_t})\omega_k.\tag{8.3}$$

Since λ is regular, we may assume that $\langle w\lambda, \omega_k\rangle > 0$ by taking a small shift of λ. By Proposition 4 and Proposition 5 in Section 6, we have

$$(w_{i_1}\cdots w_{i_s})\omega_k - w^{-1}\omega_k \in \sum_{i=1}^{l'} \boldsymbol{R}_+\tilde{\alpha}_i$$

and

$$(w_{j_1}\cdots w_{j_t})\omega_k - w^{-1}\omega_k \in \sum_{i=1}^{l'} \boldsymbol{R}_+\tilde{\alpha}_i.$$

Hence we have

$$\langle \lambda, (w_{i_1}\cdots w_{i_s})\omega_k\rangle \geq \langle \lambda, w^{-1}\omega_k\rangle = \langle w\lambda, \omega_k\rangle > 0$$

and

$$\langle \lambda, (w_{j_1}\cdots w_{j_t})\omega_k\rangle > 0.$$

But these contradict to (8.3). Thus the lemma is proved. Q.E.D.

§ 9. Theorem 3

In this section we assume that $\operatorname{rank}(G/H) = \operatorname{rank}(K/K\cap H)$ and fix j $(1 \leq j \leq m)$. Let L_- denote the semilattice in $\mathfrak{a}_\mathfrak{p}'^*$ generated by the roots $\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p}')_j^-$ satisfying $\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}'; \alpha) \not\subset \mathfrak{h}^d$. Let $L_{K/K\cap H}$ (resp. L_{G_c/H_c}) denote the semilattice in $\mathfrak{a}_\mathfrak{p}'^*$ consisting of highest weights with respect to the order $\Sigma(\mathfrak{a}_\mathfrak{p}')_j^+$ of finite-dimensional representations of K (resp. holomorphic representations of G_c) with $K\cap H$-fixed vectors (resp. H_c-fixed vectors). (Note that $\sqrt{-1}\,\mathfrak{a}_\mathfrak{p}'$ is a maximal abelian subspace of $\mathfrak{k}\cap\mathfrak{q} = \sqrt{-1}(\mathfrak{p}^d\cap\mathfrak{h}^d)$ and of $\mathfrak{q}$. Let ρ_t^j be an element in $\mathfrak{a}_\mathfrak{p}'^*$ defined by

$$\rho_t^j(Y) = \tfrac{1}{2}\operatorname{trace}(\operatorname{ad}(Y)|_{\mathfrak{n}+j\cap\mathfrak{h}^d})$$

for $Y \in \mathfrak{a}_\mathfrak{p}'$. For a $\lambda \in (\mathfrak{a}_\mathfrak{p}^d)_c^*$, we put $\mu_\lambda^j = \lambda^j + \rho^j - 2\rho_t^j$.

Let $\mathfrak{a}_\mathfrak{k}^d$ be a maximal abelian subspace of $\mathfrak{m}^d$ and put $\mathfrak{a}_\mathfrak{g}^d = \mathfrak{a}_\mathfrak{k}^d + \mathfrak{a}_\mathfrak{p}^d$. Let $\Sigma(\mathfrak{a}_\mathfrak{g}^d)$ be the root system of the pair $(\mathfrak{g}_c, \mathfrak{a}_{\mathfrak{g}c}^d)$. For every $\alpha \in \Sigma(\mathfrak{a}_\mathfrak{g}^d)$ let $\bar{\alpha}$ denote the restriction of α to $\mathfrak{a}_\mathfrak{p}^d$. Choose a positive system $\Sigma(\mathfrak{a}_\mathfrak{g}^d)^+$ of $\Sigma(\mathfrak{a}_\mathfrak{g}^d)$ so that $\Sigma(\mathfrak{a}_\mathfrak{g}^d)^+$ is compatible with $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ (i.e. the condition $\alpha \in \Sigma(\mathfrak{a}_\mathfrak{g}^d)^+$ and $\bar{\alpha} \neq 0$ implies $\bar{\alpha} \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$). Put $\rho_\mathfrak{m} = \frac{1}{2}\sum \alpha$ where the sum is taken over all $\alpha \in \Sigma(\mathfrak{a}_\mathfrak{g}^d)^+$ such that $\bar{\alpha} = 0$.

Theorem 3. *Suppose that* $\operatorname{rank}(G/H) = \operatorname{rank}(K/K\cap H)$ *and let* λ *be an element of* $(\mathfrak{a}_\mathfrak{p}^d)_c^*$ *such that* $\operatorname{Re}\langle \lambda, \tilde{\alpha}\rangle \geq 0$ *for all* $\tilde{\alpha} \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$. *Suppose that* $\mathscr{B}_{H^d}^j(G^d/P^d; L_\lambda) \neq \{0\}$. *Then we have the followings.*

(i) *Suppose that* $\mathscr{B}^j_{H^d}(G^d/P^d; L_\lambda)$ *contains a* H^d*-type* (τ, E) *with lowest weight* $\nu \in \mathfrak{a}'^*_\mathfrak{p}$. *(i.e. There exists a vector* $v \in E$ *such that* $\tau(Y)v = \nu(Y)v$ *for* $Y \in \mathfrak{a}'_\mathfrak{p}$ *and that* $\tau(Z)v = 0$ *for* $Z \in \mathfrak{n}^{-j} \cap \mathfrak{h}^d$.) *Then* $-\nu \in \mu^j_\lambda - L_-$. *Especially* λ *is real-valued on* $\mathfrak{a}^d_\mathfrak{p}$.

(ii) *If* $\langle \lambda + \rho_\mathfrak{m}, \alpha \rangle \geq 0$ *for all* $\alpha \in \Sigma(\mathfrak{a}^d_\mathfrak{g})^+$, *then* $\mathscr{B}^j_{H^d}(G^d/P^d; L_\lambda)$ *is an irreducible* $\mathfrak{g}_c$*-module.*

(iii) μ^j_λ *is contained in the lattice in* $\mathfrak{a}'_\mathfrak{p}$ *generated by* $L_{K/K\cap H}$.

(iv) *Let* α *be a compact simple root in* $\Sigma(\mathfrak{a}'_\mathfrak{p})^+_j$. *Then* $\langle \lambda^j - \rho^j, \alpha \rangle \geq 0$.

Put $P^j = x_j P^d x_j^{-1}$, $N^{-j} = x_j N^{-d} x_j^{-1}$, $A'_\mathfrak{p} = \exp \mathfrak{a}'_\mathfrak{p}$ and $\tilde{G} = A'_\mathfrak{p}(N^{-j} \cap H^d) \times P^j$. Then $\tilde{G}$ acts on G^d on the right by

$$z \cdot (y, p) = y^{-1} z p$$

for $z \in G^d$, $y \in A'_\mathfrak{p}(N^{-j} \cap H^d)$ and $p \in P^j$. Let V be the $\tilde{G}$-orbit on G^d containing the identity. Then V is an open dense subset of the closed set $\bar{V} = H^d P^j$ because of the Bruhat decomposition

$$H^d = \bigcup_w (N^{-j} \cap H^d) w (P^j \cap H^d)$$

where w is taken over $N_{H^d}(\mathfrak{a}'_\mathfrak{p})/Z_{H^d}(\mathfrak{a}'_\mathfrak{p})$. Let ν be a character of $A'_\mathfrak{p}$. Then we can define a character χ of $\tilde{G}$ by

$$\chi(a_1 n_1, m_2 a_2 n_2) = a_1^\nu a_2^{\lambda^j - \rho^j} \tag{9.1}$$

for $a_1 \in A'_\mathfrak{p}$, $n_1 \in N^{-j} \cap H^d$, $m_2 \in M'$, $a_2 \in A'_\mathfrak{p}$ and $n_2 \in N^{+j}$ where $a_2^{\lambda^j - \rho^j} = \exp \langle \lambda^j - \rho^j, \log a_2 \rangle$. Put

$$\begin{aligned}\mathscr{B}(\bar{V}, \chi) = \{v \in \mathscr{B}(G^d) \mid & \operatorname{supp} v \subset \bar{V} \text{ and } v(y^{-1} z p) \\ & = \chi((y, p)) v(z) \text{ for } (y, p) \in \tilde{G} \text{ and } z \in G^d\}\end{aligned}$$

and

$$\mathscr{B}(V, \chi) = \mathscr{B}(\bar{V}, \chi) / \mathscr{B}(\bar{V} \backslash V) \cap \mathscr{B}(\bar{V}, \chi)$$

where $\mathscr{B}(\bar{V} \backslash V)$ is the set of all hyperfunctions on G^d with supports in $\bar{V} \backslash V$. Consider the following two conditions.

(C_1) There exists a Y in $\mathfrak{a}'_\mathfrak{p}$ such that $\beta(Y) > 0$ for all $\beta \in \Sigma(\mathfrak{a}'_\mathfrak{p})^-_j$ and that

$$-\mu(Y) + 2(\rho^j - \rho^j_t)(Y) + d\chi(Y) \neq 0$$

for all $\mu \in L_-$.

(C_2) The condition (C_1) holds except for the case $\mu = 0$. Then the following proposition is an easy consequence of Lemma in [7] Appendix I.

Proposition 6. *If the condition* (C_1) *holds, then* $\mathscr{B}(V, \chi)=\{0\}$. *If the condition* ($C_2$) *holds, then the dimension of the vector space* $\mathscr{B}(V, \chi)$ *is at most one and* $\mathscr{B}(V, \chi)$ *consists of elements of the form* $\phi\delta_V$ *with a real analytic function* ϕ *on* V *and a delta function* δ_V *with support* V. (*Note that the quotient of tangent spaces* T_1G^d/T_1V *is naturally identified with* $\mathfrak{n}^{-j}\cap\mathfrak{q}^d$.)

Proof of Theorem 3 (i). Suppose that $\mathscr{B}^j_{H^d}(G^d/P^d; L_\lambda)\neq\{0\}$. Then we may assume that there is an $f\neq 0$ in $\mathscr{B}(G^d)$ satisfying

$$f(n_1^{-1}a_1^{-1}xman)=a_1^\nu a^{\lambda-\rho}f(x)$$

for $x\in G^d$, $n_1\in N^{-j}\cap H^d$, $a_1\in A'_{\mathfrak{p}}$, $m\in M^d$, $a\in A^d_{\mathfrak{p}}$ and $n\in N^{+d}$. Here ν is a lowest weight of an H^d-type in $\mathscr{B}^j_{H^d}(G^d/P^d; L_\lambda)$ and f is a lowest weight function in the H^d-type. Put $v(x)=f(xx_j)$ $(x\in G^d)$. Then it is clear that $v\in\mathscr{B}(\bar{V}, \chi)$ where χ is defined by (9.1) and that $v\notin\mathscr{B}(\bar{V}\backslash V)$. Thus $\mathscr{B}(V, \chi)\neq\{0\}$.

We claim that

$$-\mu+2(\rho^j-\rho^j_t)+d\chi=0$$

on $\mathfrak{a}'_{\mathfrak{p}}$ for some $\mu\in L_-$. In fact, if $-\mu+2(\rho^j-\rho^j_t)+d\chi\neq 0$ on $\mathfrak{a}'_{\mathfrak{p}}$ for all $\mu\in L_-$, then we can choose $Y\in\mathfrak{a}'_{\mathfrak{p}}$ such that $\beta(Y)>0$ for all $\beta\in\Sigma(\mathfrak{a}'_{\mathfrak{p}})^-_j$ and that $-\mu(Y)+2(\rho^j-\rho^j_t)(Y)+d\chi(Y)\neq 0$ for all $\mu\in L_-$ since

$$-\mu+2(\rho^j-\rho^j_t)+d\chi=0$$

defines a hyperplane in $\mathfrak{a}'_{\mathfrak{p}}$ for every $\mu\in L_-$. Then by Proposition 6 we have $\mathscr{B}(V, \chi)=\{0\}$, a contradiction to $\mathscr{B}(V, \chi)\neq\{0\}$. Since $d\chi(Y)=(\nu+\lambda^j-\rho^j)(Y)$ for $Y\in\mathfrak{a}'_{\mathfrak{p}}$, we have $-\mu+\rho^j-2\rho^j_t+\nu+\lambda^j=0$ and therefore we have $\mu^j_\lambda-\mu=-\nu$, proving Theorem 3 (i). Q.E.D.

§ 10. An application of Vogan's result and the proof of Theorem 3

Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$ containing $\sqrt{-1}\,\mathfrak{a}'_{\mathfrak{p}}$ $(\subset\sqrt{-1}\,\mathfrak{p}^d\cap\mathfrak{h}^d=\mathfrak{k}\cap\mathfrak{q})$ and $\mathfrak{a}'_{\mathfrak{g}}$ a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. Fix a positive system $\Sigma(\mathfrak{a}'_{\mathfrak{g}})^+$ of the root system $\Sigma(\mathfrak{a}'_{\mathfrak{g}})$ of the pair $(\mathfrak{g}_c, \mathfrak{a}'_{\mathfrak{g}c})$ such that $\theta\Sigma(\mathfrak{a}'_{\mathfrak{g}})^+=\Sigma(\mathfrak{a}'_{\mathfrak{g}})^+$ and that $\Sigma(\mathfrak{a}'_{\mathfrak{g}})^+$ is compatible with $\Sigma(\mathfrak{a}'_{\mathfrak{p}})^-_j$. (i.e. The condition $\alpha\in\Sigma(\mathfrak{a}'_{\mathfrak{g}})^+$ and $\alpha|_{\mathfrak{a}'_{\mathfrak{p}}}\neq 0$ implies that $\alpha|_{\mathfrak{a}'_{\mathfrak{p}}}\in\Sigma(\mathfrak{a}'_{\mathfrak{p}})^-_j$.) Let $\Sigma(\mathfrak{t})^+$ be the restriction of $\Sigma(\mathfrak{a}'_{\mathfrak{g}})^+$ to $\mathfrak{t}$ and let R denote the subset in $\sqrt{-1}\,\mathfrak{t}^*$ defined by

$$R=\sum_{\alpha\in\Sigma(\mathfrak{t})^+}\boldsymbol{R}_+\alpha.$$

Let $\Sigma(\mathfrak{k}, \mathfrak{t})$ be the root system of the pair $(\mathfrak{k}_c, \mathfrak{t}_c)$ and $\Sigma(\mathfrak{k}, \mathfrak{t})^+$ be the positive system of $\Sigma(\mathfrak{k}, \mathfrak{t})$ defined by $\Sigma(\mathfrak{k}, \mathfrak{t})^+ = \Sigma(\mathfrak{k}, \mathfrak{t}) \cap \Sigma(\mathfrak{t})^+$. Put $\tilde{\rho} = \frac{1}{2}\sum_{\alpha \in \Sigma(\mathfrak{a}'_{\mathfrak{g}})^+}\alpha$ and $\rho_c = \frac{1}{2}\sum_{\beta \in \Sigma(\mathfrak{k},\mathfrak{t})^+}\beta$. For every $\mu \in \mathfrak{t}_c^*$, the real part Re μ is defined by Re $\mu = \mu_1$ if $\mu = \mu_1 + \sqrt{-1}\,\mu_2$ and μ_1, μ_2 are real-valued on $\sqrt{-1}\,\mathfrak{t}$.

We will first prove a lemma which is an application of Vogan's lowest $\mathfrak{k}$-type theory ([15]).

Lemma 9. *Let X be an irreducible Harish-Chandra module of $\mathfrak{g}$ with an infinitesimal character parametrized by $\nu \in (\mathfrak{a}'_{\mathfrak{g}})_c^*$ such that* $\mathrm{Re}\,\langle \nu, \alpha\rangle \geq 0$ *for $\alpha \in \Sigma(\mathfrak{a}'_{\mathfrak{g}})^+$. Let $\mu \in \sqrt{-1}\,\mathfrak{t}^*$ be the highest weight with respect to $\Sigma(\mathfrak{k}, \mathfrak{t})^+$ of a lowest $\mathfrak{k}$-type* (*in the sense of* [15]) *in X. Then*

$$\mu + 2\rho_c - \tilde{\rho} \in \mathrm{Re}\,(\nu|_{\sqrt{-1}\mathfrak{t}}) - R.$$

Proof. Let $W(\mathfrak{t})$ and $W(\mathfrak{a}'_{\mathfrak{g}})$ be the Weyl groups of $\Sigma(\mathfrak{t})$ and $\Sigma(\mathfrak{a}'_{\mathfrak{g}})$, respectively, and $W_\theta(\mathfrak{a}'_{\mathfrak{g}})$ the subgroup in $W(\mathfrak{a}'_{\mathfrak{g}})$ defined by

$$W_\theta(\mathfrak{a}'_{\mathfrak{g}}) = \{w \in W(\mathfrak{a}'_{\mathfrak{g}}) \mid w\theta = \theta w\}.$$

Then $W(\mathfrak{t})$ is the restriction of $W_\theta(\mathfrak{a}'_{\mathfrak{g}})$ to $\mathfrak{t}$. Thus we can choose $w \in W_\theta(\mathfrak{a}'_{\mathfrak{g}})$ so that $\mu + 2\rho_c$ is dominant for $w\Sigma(\mathfrak{t})^+$. According to Proposition 4.1 in [15], we can choose roots $\beta_1, \cdots, \beta_r$ in $w\Sigma(\mathfrak{a}'_{\mathfrak{g}})^+$ and real numbers $c_1, \cdots, c_r$ ($0 \leq c_i \leq 1$ for all i) so that

(10.1) $\tilde{\mu} = \mu + 2\rho_c - w\tilde{\rho} + \frac{1}{2}c_i\beta_i$ is dominant for $w\Sigma(\mathfrak{a}'_{\mathfrak{g}})^+$, $\beta_1, \cdots, \beta_r$ are orthogonal to each other and $c_i = -2\langle \mu + 2\rho_c - w\tilde{\rho}, \beta_i\rangle / \langle \beta_i, \beta_i\rangle$.

Let $\mathfrak{l}$ be the subalgebra of $\mathfrak{g}_c$ defined by $\mathfrak{l} = \mathfrak{z}_{\mathfrak{g}_c}(\mathfrak{t}) + \sum_{\alpha \in \Sigma(\mathfrak{l},\mathfrak{t})}\mathfrak{g}_c(\mathfrak{t}, \alpha)$ where $\Sigma(\mathfrak{l}, \mathfrak{t}) = \{\alpha \in \Sigma(\mathfrak{t}) \mid \langle \tilde{\mu}, \alpha\rangle = 0\}$. Put $\mathfrak{t}^+ = \mathfrak{t} \cap (\text{center of } \mathfrak{l})$ and $\mathfrak{t}^- = \mathfrak{t} \cap [\mathfrak{l}, \mathfrak{l}]$. Then $\mathfrak{t} = \mathfrak{t}^+ + \mathfrak{t}^-$ is a direct sum. By Proposition 5.8 in [15], there is a $w_1 \in W(\mathfrak{a}'_{\mathfrak{g}})$ such that

$$\tilde{\mu}|_{\mathfrak{t}^+} = w_1(\nu)|_{\mathfrak{t}^+}.$$

(Note that $\beta_i|_{\mathfrak{t}^+} = 0$ for $i = 1, \cdots, r$ by (10.1).)

It is clear that $\tilde{\mu}|_{\mathfrak{t}^-} = 0$ and we can choose an element w_2 in the Weyl group of $\Sigma(\mathfrak{l}, \mathfrak{t})$ such that $\mathrm{Re}\,(w_2w_1(\nu)|_{\mathfrak{t}^-})$ is dominant for $\Sigma(\mathfrak{l}, \mathfrak{t}) \cap w\Sigma(\mathfrak{t})^+$. Then it is clear that

(10.2) $$\mathrm{Re}\,(w_2w_1(\nu)|_{\sqrt{-1}\mathfrak{t}}) \in \tilde{\mu} + wR.$$

On the other hand, we have

(10.3) $$\mathrm{Re}(w_2w_1(\nu)) \in \mathrm{Re}(w(\nu)) - w\sum_{\alpha \in \Sigma(\mathfrak{a}'_{\mathfrak{g}})^+} \boldsymbol{R}_+\alpha$$

since $\mathrm{Re}(w(\nu))$ is dominant for $w\Sigma(\mathfrak{a}'_{\mathfrak{g}})^+$. It follows from (10.1), (10.2) and (10.3) that

$$\mu+2\rho_c-w\tilde{\rho}\in \mathrm{Re}(w(\nu)|_{\sqrt{-1}\mathfrak{t}})-wR.$$

Thus we have

$$w^{-1}(\mu+2\rho_c)-\tilde{\rho}\in \mathrm{Re}(\nu|_{\sqrt{-1}\mathfrak{t}})-R.$$

On the other hand, we have $w^{-1}(\mu+2\rho_c)-(\mu+2\rho_c)\in R$ since $w^{-1}(\mu+2\rho_c)$ is dominant for $\Sigma(\mathfrak{a}'_{\mathfrak{g}})^+$. Thus we have

$$\mu+2\rho_c-\tilde{\rho}\in \mathrm{Re}(\nu|_{\sqrt{-1}\mathfrak{t}})-R,$$

proving the lemma. Q.E.D.

Proof of Theorem 3 (ii). Put $\rho^j_{\mathfrak{m}}=\frac{1}{2}\sum\alpha$ where the sum is taken over $\alpha\in\Sigma(\mathfrak{a}'_{\mathfrak{g}})^+$ such that $\alpha|_{\mathfrak{a}'_{\mathfrak{p}}}=0$. Then the infinitesimal character of the Harish-Chandra module $X=\mathscr{B}^j_{H^d}(G^d/P^d;L_\lambda)$ is parametrized by $-\lambda^j+\rho^j_{\mathfrak{m}}\in\mathfrak{t}^*_c$. Then it is clear from the assumption on λ that $-\lambda^j+\rho^j_{\mathfrak{m}}$ is dominant for $\Sigma(\mathfrak{a}'_{\mathfrak{g}})^+$. Let $\mu\in\sqrt{-1}\,\mathfrak{t}^*$ be the highest weight with respect to $\Sigma(\mathfrak{t})^+$ of a lowest $\mathfrak{k}$-type in an irreducible component of X. Then by Lemma 9 we have

$$\mu+2\rho_c-\tilde{\rho}\in -\lambda^j+\rho^j_{\mathfrak{m}}-R.$$

Taking its restriction to $\mathfrak{a}'_{\mathfrak{p}}$, we have

$$\mu|_{\mathfrak{a}'_{\mathfrak{p}}}-2\rho^j_t+\rho^j\in -\lambda^j-R|_{\mathfrak{a}'_{\mathfrak{p}}}$$

since $\Sigma(\mathfrak{t})^+$ is compatible with $\Sigma(\mathfrak{a}'_{\mathfrak{p}})^-_j$. Thus we have

$$\mu^j_\lambda+\mu|_{\mathfrak{a}'_{\mathfrak{p}}}=\lambda^j+\rho^j-2\rho^j_t+\mu|_{\mathfrak{a}'_{\mathfrak{p}}}\in -R|_{\mathfrak{a}'_{\mathfrak{p}}}.$$

On the other hand, we have

$$\mu^j_\lambda+\mu|_{\mathfrak{a}'_{\mathfrak{p}}}\in L_-$$

by (i). Since $-R|_{\mathfrak{a}'_{\mathfrak{p}}}\cap L_-=\{0\}$, we have

$$\mu^j_\lambda+\mu|_{\mathfrak{a}'_{\mathfrak{p}}}=0.$$

Then it follows from Proposition 6 that $\dim\mathscr{B}(V,\chi)$ (χ is defined for $\mu|_{\mathfrak{a}'_{\mathfrak{p}}}$ and λ^j) is at most one and therefore the multiplicity of the $\mathfrak{k}$-type with highest weight μ is at most one in X. Thus the $\mathfrak{g}_c$-module X is irreducible.

Q.E.D.

Proof of (iii). Suppose first that λ satisfies the assumption in (ii). Let $\mu \in \sqrt{-1}\,\mathfrak{t}^*$ be as in the proof of (ii). Then by Proposition 6 there exists a function f in $\mathscr{B}^j_{H^d}(G^d/P^d; L_\lambda)$ which is unique up to constant multiple such that $f(n^{-1}a^{-1}x)=a^\mu f(x)$ for $n \in N^{-j}\cap H^d$ and $a \in A'_\mathfrak{p}$. Furthermore f is of the form $c\delta(X_2)$ with some constant c on an open set $\exp(\mathfrak{n}^{-j}\cap\mathfrak{h}^d)\exp(\mathfrak{n}^{-j}\cap\mathfrak{q}^d)x_jP^d$ if we take a coordinate

$$(X_1, X_2) \longrightarrow \exp X_1 \exp X_2 x_j P^d \qquad \text{for } X_1 \in \mathfrak{n}^{-j}\cap\mathfrak{h}^d,\ X_2 \in \mathfrak{n}^{-j}\cap\mathfrak{q}^d$$

by Proposition 6. Here $\delta(X_2)$ is the Dirac delta function on $\mathfrak{n}^{-j}\cap\mathfrak{q}^d$ with support $\{0\}$. We claim that f is $M'\cap H^d$-invariant. For let m be an element in $M'\cap H^d$. If

$$x=\exp X_1 \exp X_2 x_j (X_1 \in \mathfrak{n}^{-j}\cap\mathfrak{h}^d,\ X_2 \in \mathfrak{n}^{-j}\cap\mathfrak{q}^d),$$

then

$$\begin{aligned} f(mx) &= f(\exp(\mathrm{Ad}(m)X_1)\exp(\mathrm{Ad}(m)X_2)x_j x_j^{-1}mx_j) \\ &= f(\exp X_1 \exp X_2 x_j) = f(x) \end{aligned}$$

since $M'\cap H^d$ is compact, $\mathrm{Ad}(m)(\mathfrak{n}^{-j}\cap\mathfrak{h}^d)=\mathfrak{n}^{-j}\cap\mathfrak{h}^d$, $\mathrm{Ad}(m)(\mathfrak{n}^{-j}\cap\mathfrak{q}^d)=\mathfrak{n}^{-j}\cap\mathfrak{q}^d$ and $x_j^{-1}mx_j \in M^d$. Hence f is $M'\cap H^d$-invariant in a neighborhood of x_jP^d. Since $\dim \mathscr{B}(V, \chi)=1$, f must be $M'\cap H^d$-invariant. Then it follows from [17], Vol. I, p. 211 that $\mu \in -L_{K/K\cap H}$. Thus we have $\mu^j_\lambda=-\mu \in L_{K/K\cap H}$.

Next suppose that λ does not satisfy the assumption in (ii). Let (τ, E) be an irreducible finite-dimensional holomorphic representation of G_c with H_c-fixed vectors. Then there exists a vector $v \in E$ such that $\tau(\mathrm{man})v=a^\Lambda v$ for $m \in M^d$, $a \in A^d_\mathfrak{p}$, $n \in N^{+d}$ where Λ is the highest weight of (τ, E). We choose (τ, E) so that $\lambda+\Lambda$ satisfies the assumption in (ii). (λ is replaced by $\lambda+\Lambda$.) Consider an analytic function ϕ on G^d given by $\phi(x)=\langle u, \tau(x)v\rangle$ for some $u \in E^*$. Then ϕ satisfies $\phi(xman)=a^\Lambda\phi(x)$ ($m \in M^d$, $a \in A^d_\mathfrak{p}$, $n \in N^{+d}$) and is H^d-finite of type in $\hat{K}$.

Let f be a nontrivial function in $\mathscr{B}^j_{H^d}(G^d/P^d; L_\lambda)$. Considering the left G^d-action to ϕ, if necessary, we may assume that $\phi(x)\neq 0$ for some point x in $\mathrm{supp}\, f$. Then the product ϕf of functions is a nonzero element in $\mathscr{B}^j_{H^d}(G^d/P^d; L_{\lambda+\Lambda})$. Thus $\mathscr{B}^j_{H^d}(G^d/P^d; L_{\lambda+\Lambda})\neq\{0\}$, so we have

$$\mu^j_\lambda+\Lambda^j \in L_{K/K\cap H}\ (\Lambda^j=\Lambda\circ\mathrm{Ad}(x_j)^{-1})$$

by the preceding argument. Since $\Lambda^j \in L_{K/K\cap H}$, λ^j is contained in the lattice in $\mathfrak{a}'_\mathfrak{p}$ generated by $L_{K/K\cap H}$.

Proof of (iv). Let $\Psi(\mathfrak{a}'_\mathfrak{q})$ be the set of simple roots in $\Sigma(\mathfrak{a}'_\mathfrak{q})^+$ and let

Ψ_c be the subset in $\Psi(\mathfrak{a}_\mathfrak{g}')$ consisting of $\alpha \in \Psi(\mathfrak{a}_\mathfrak{g}')$ such that $\alpha|_{\mathfrak{a}_\mathfrak{p}'}=0$ or that $\alpha|_{\mathfrak{a}_\mathfrak{p}'}$ is a compact simple root in $\Sigma(\mathfrak{a}_\mathfrak{p}')_j^-$. Let $W(\Psi_c)$ be the subgroup in $W(\mathfrak{a}_\mathfrak{g}')$ generated by the reflections with respect to the roots in Ψ_c. Then there exists a w in $W(\Psi_c)$ such that $w(-\lambda^j+\rho_\mathfrak{m}^j)$ is dominant for Ψ_c.

Suppose first that $w(-\lambda^j+\rho_\mathfrak{m}^j)$ is dominant for $\Psi(\mathfrak{a}_\mathfrak{g}')$. Let μ be the highest weight of a lowest $\mathfrak{k}$-type in $X=\mathscr{B}_{H^d}^j(G^d/P^d; L_\lambda)$. Then by Lemma 9, we have

$$\mu+2\rho_c-\tilde{\rho} \in w(-\lambda^j+\rho_\mathfrak{m}^j)-R. \tag{10.4}$$

Let $\mathfrak{a}_0$ be the subspace in $\mathfrak{a}_\mathfrak{p}'$ consisting of elements orthogonal to Ψ_c. Restricting (10.4) to $\mathfrak{a}_0$, we have

$$\mu|_{\mathfrak{a}_0}-2\rho_t^j|_{\mathfrak{a}_0}+\rho^j|_{\mathfrak{a}_0} \in -\lambda^j|_{\mathfrak{a}_0}-R|_{\mathfrak{a}_0}.$$

On the other hand, we have

$$\mu|_{\mathfrak{a}_\mathfrak{p}'}-2\rho_t^j+\rho^j \in -\lambda^j+L_-$$

by (i). Since every nonzero element in L_- has nonzero restriction to $\mathfrak{a}_0$, we have

$$\mu|_{\mathfrak{a}_\mathfrak{p}'}-2\rho_t^j+\rho^j=-\lambda^j.$$

Hence if α is a compact simple root in $\Sigma(\mathfrak{a}_\mathfrak{p}')_j^+$, then

$$\begin{aligned}\langle\lambda^j-\rho^j, \alpha\rangle &= \langle\lambda^j+\rho^j-2\rho_t^j, \alpha\rangle \\ &= \langle-\mu|_{\mathfrak{a}_\mathfrak{p}'}, \alpha\rangle \geq 0.\end{aligned}$$

When $w(-\lambda^j+\rho_\mathfrak{m}^j)$ is not dominant for $\Psi(\mathfrak{a}_\mathfrak{g}')$, we proceed as follows. Choose an element Λ^j in L_{G_c/H_c} such that $\langle\Lambda^j, \alpha\rangle=0$ for compact simple roots α in $\Sigma(\mathfrak{a}_\mathfrak{p}')_j^+$ and that $w(-\lambda^j+\rho_\mathfrak{m}^j)+\Lambda^j$ is dominant for $\Psi(\mathfrak{a}_\mathfrak{g}')$. Let $f \in \mathscr{B}_{H^d}^j(G^d/P^d; L_\lambda)$ and $\phi \in \mathscr{A}_{H^d}(G^d/P^d; L_{\Lambda+\rho})$ be as in the proof of (ii). Then the product ϕf of functions is a nonzero element in $\mathscr{B}_{H^d}^j(G^d/P^d; L_{\lambda+\Lambda})$. Hence by the preceding argument, we have

$$\langle\lambda^j-\rho^j, \alpha\rangle=\langle\lambda^j+\Lambda^j-\rho^j, \alpha\rangle\geq 0$$

for all compact simple roots α in $\Sigma(\mathfrak{a}_\mathfrak{p}')_j^+$. Q.E.D.

Lemma 10. *Suppose that all the irreducible components of the root system* $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$ *are of type* A_n, D_n *or* E_n $(n\geq 2)$. *Let* λ *be an element of* $\mathfrak{a}_\mathfrak{p}^{d*}$ *such that* $\langle\lambda, \alpha\rangle\geq 0$ *for all* $\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ *and that* μ_λ^j *is contained in the lattice in* $\mathfrak{a}_\mathfrak{p}'^*$ *generated by* $L_{K/K\cap H}$. *Then the following two conditions are equivalent.*

(i) $\langle \mu_\lambda^j, \alpha\rangle \geq 0$ *for all compact simple roots* α *in* $\Sigma(\mathfrak{a}_\mathfrak{p}')_j^+$.

(ii) $\mu_\lambda^j \in L_{K/K\cap H}$.

(*We have* $\langle \mu_\lambda^j, \alpha\rangle = \langle \lambda^j - \rho^j, \alpha\rangle$ *for all compact simple roots* α *in* $\Sigma(\mathfrak{a}_\mathfrak{p}')_j^+$ *since* $\langle \rho^j, \alpha\rangle = \langle \rho_t^j, \alpha\rangle$.)

Proof. Clearly (ii) implies (i). Thus we have only to prove that (i) implies (ii). We may assume that $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$ is irreducible. Then it is easy to see that either of the following two conditions holds.

(a) $\dim(\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}'; \alpha)\cap\mathfrak{h}^d) = \dim(\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}'; \alpha)\cap\mathfrak{q}^d)$ for all $\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p}')$.

(b) For every $\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p}')$, $\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}'; \alpha)\subset\mathfrak{h}^d$ or $\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}'; \alpha)\subset\mathfrak{q}^d$.

Let $\Sigma(\mathfrak{h}^d, \mathfrak{a}_\mathfrak{p}')$ denote the root system of the pair $(\mathfrak{h}^d, \mathfrak{a}_\mathfrak{p}')$ and put $\Sigma(\mathfrak{h}^d, \mathfrak{a}_\mathfrak{p}')^+ = \Sigma(\mathfrak{h}^d, \mathfrak{a}_\mathfrak{p}')\cap\Sigma(\mathfrak{a}_\mathfrak{p}')_j^+$. If the condition (a) holds, then $\mu_\lambda^j = \lambda^j + 2\rho_t^j - \rho^j = \lambda^j$ is dominant for $\Sigma(\mathfrak{h}^d, \mathfrak{a}_\mathfrak{p}')^+$. Consider the case (b). Let β be a simple root in $\Sigma(\mathfrak{h}^d, \mathfrak{a}_\mathfrak{p}')^+$. If β is a simple root in $\Sigma(\mathfrak{a}_\mathfrak{p}')_j^+$, then it follows from (i) that

$$\langle \mu_\lambda^j, \beta\rangle \geq 0.$$

Thus we may assume that $\beta = \alpha_1 + \cdots + \alpha_k$ where $\alpha_1, \cdots, \alpha_k$ are simple roots in $\Sigma(\mathfrak{a}_\mathfrak{p}')_j^+$ and $k \geq 2$. Then we have

$$\begin{aligned}\langle \mu_\lambda^j, \beta\rangle &= \langle \lambda^j + \rho^j - 2\rho_t^j, \beta\rangle \\ &= \langle \lambda^j, \beta\rangle + \langle \rho^j, \alpha_1\rangle + \cdots + \langle \rho^j, \alpha_k\rangle - 2\langle \rho_t^j, \beta\rangle.\end{aligned}$$

Since the multiplicities of the roots in $\Sigma(\mathfrak{a}_\mathfrak{p}')$ are the same, we have $\langle \rho^j, \alpha_1\rangle = \cdots = \langle \rho^j, \alpha_k\rangle = \langle \rho_t^j, \beta\rangle$ and therefore we have $\langle \mu_\lambda^j, \beta\rangle \geq \langle \lambda^j, \beta\rangle \geq 0$. Since μ_λ^j is contained in the lattice generated by $L_{K/K\cap H}$, we have proved $\mu_\lambda^j \in L_{K/K\cap H}$. Q.E.D.

Lemma 11. *Let* λ *be an element of* $\mathfrak{a}_\mathfrak{p}^{d*}$ *such that* $\langle \lambda + \rho_\mathfrak{m}, \alpha\rangle \geq 0$ *for all* $\alpha \in \Sigma(\mathfrak{a}_\mathfrak{g}^d)^+$ *and that* μ_λ^j *is contained in the lattice in* $\mathfrak{a}_\mathfrak{p}'^*$ *generated by* $L_{K/K\cap H}$. *Then the conditions* (i) *and* (ii) *in Lemma* 10 *are equivalent.*

Proof. Clearly we have only to prove that (i) implies (ii). Let $\mathfrak{t}$, $\Sigma(\mathfrak{t})^+$, $\Sigma(\mathfrak{k}, \mathfrak{t})^+$, $\tilde{\rho}$, ρ_c and $\rho_\mathfrak{m}^j$ be as in the first part of this section. Let $\Sigma(\mathfrak{m}')$ and $\Sigma(\mathfrak{m}'\cap\mathfrak{h}^d)$ be the root systems of the pair $(\mathfrak{m}_c', \mathfrak{t}_c)$ and $(\mathfrak{m}_c'\cap\mathfrak{k}_c, \mathfrak{t}_c)$, respectively. Put $\Sigma(\mathfrak{m}')^+ = \Sigma(\mathfrak{m}')\cap\Sigma(\mathfrak{t})^+$, $\Sigma(\mathfrak{m}'\cap\mathfrak{h}^d)^+ = \Sigma(\mathfrak{m}'\cap\mathfrak{h}^d)\cap\Sigma(\mathfrak{t})^+$ and $\rho_{\mathfrak{m}c}^j = \frac{1}{2}\sum_{\alpha\in\Sigma(\mathfrak{m}'\cap\mathfrak{h}^d)^+}\alpha$. Let β be a simple root in $\Sigma(\mathfrak{h}^d, \mathfrak{a}_\mathfrak{p}')^+$ and $\tilde{\beta}$ a simple root in $\Sigma(\mathfrak{k}, \mathfrak{t})^+$ such that $\tilde{\beta}|_{\mathfrak{a}_\mathfrak{p}'} = -\beta$. Then we have

$$\begin{aligned}\langle \mu_\lambda^j, \beta\rangle &= -\langle \lambda^j + \rho^j - 2\rho_t^j, \tilde{\beta}\rangle \\ &= \langle -\lambda^j + \rho_\mathfrak{m}^j + \tilde{\rho} - 2\rho_c, \tilde{\beta}\rangle + \langle 2\rho_{\mathfrak{m}c}^j - 2\rho_\mathfrak{m}^j, \tilde{\beta}\rangle.\end{aligned} \tag{10.5}$$

From the facts $\langle -\lambda^j + \rho_\mathfrak{m}^j, \tilde{\beta}\rangle \geq 0$ (by the assumption), $\langle \tilde{\rho}, \tilde{\beta}\rangle \geq \frac{1}{2}\langle \tilde{\beta}, \tilde{\beta}\rangle$,

$\langle \rho_c, \tilde{\beta}\rangle = \frac{1}{2}\langle \tilde{\beta}, \tilde{\beta}\rangle$ and $\langle \mu_\lambda^j, \tilde{\beta}\rangle - \langle 2\rho_{\mathfrak{m}c}^j - 2\rho_{\mathfrak{m}}^j, \tilde{\beta}\rangle \in \frac{1}{2}\langle \tilde{\beta}, \tilde{\beta}\rangle \boldsymbol{Z}$ (by the assumption) it follows that

(10.6) $\langle -\lambda^j + \rho_{\mathfrak{m}}^j + \tilde{\rho} - 2\rho_c, \tilde{\beta}\rangle = -\frac{1}{2}\langle \tilde{\beta}, \tilde{\beta}\rangle$ if $\tilde{\beta}$ is a compact (i.e. $\mathfrak{g}_c(\mathfrak{t}; \tilde{\beta}) \subset \mathfrak{k}_c$) simple root in $\Sigma(\mathfrak{t})^+$ and $\langle -\lambda^j + \rho_{\mathfrak{m}}^j + \tilde{\rho} - 2\rho_c, \tilde{\beta}\rangle \geq 0$ otherwise.

(If there exists another simple root $\tilde{\beta}'$ in $\Sigma(\mathfrak{k}, \mathfrak{t})^+$ such that $\tilde{\beta}'|_{\mathfrak{a}_\mathfrak{p}'} = -\beta$, then we also have the same result for $\tilde{\beta}'$.)

Let $\Sigma(\Theta)$ be the subset of $\Sigma(\mathfrak{k}, \mathfrak{t})$ defined by $\{\alpha \in \Sigma(\mathfrak{k}, \mathfrak{t});\ \alpha|_{\mathfrak{a}_\mathfrak{p}'} \in \boldsymbol{Z}\beta\}$. If α is a simple root in $\Sigma(\mathfrak{m}' \cap \mathfrak{h}^d)^+$, then

$$\langle -\lambda^j + \rho_{\mathfrak{m}}^j + \tilde{\rho} - 2\rho_c, \alpha\rangle = \langle 2\rho_{\mathfrak{m}}^j - 2\rho_{\mathfrak{m}c}^j, \alpha\rangle \geq 0.$$

Hence if $\tilde{\beta}$ (and $\tilde{\beta}'$) is not compact simple in $\Sigma(\mathfrak{t})^+$, then $-\lambda^j + \rho_{\mathfrak{m}}^j + \tilde{\rho} - 2\rho_c$ is dominant for $\Sigma(\Theta) \cap \Sigma(\mathfrak{k}, \mathfrak{t})^+$ by (10.6). Therefore $\langle \mu_\lambda^j, \beta\rangle \geq 0$ since $\beta \in \sum \boldsymbol{R}_+\alpha$ where the sum is taken over all α in $\Sigma(\Theta) \cap \Sigma(\mathfrak{k}, \mathfrak{t})^+$.

When $\tilde{\beta}$ (or $\tilde{\beta}'$) is compact simple in $\Sigma(\mathfrak{t})^+$, we proceed as follows. (We may assume that $\tilde{\beta}$ is compact simple in $\Sigma(\mathfrak{t})^+$.) Suppose that $\langle \mu_\lambda^j, \beta\rangle < 0$. Then we will get a contradiction. Since $\langle \alpha, \tilde{\beta}\rangle \leq 0$ for $\alpha \in \Sigma(\mathfrak{m}')^+$, we have $\langle 2\rho_{\mathfrak{m}c}^j - 2\rho_{\mathfrak{m}}^j, \tilde{\beta}\rangle \in \frac{1}{2}\langle \tilde{\beta}, \tilde{\beta}\rangle \boldsymbol{Z}_+$. Thus it follows from (10.5) and (10.6) that

$$\langle \alpha, \tilde{\beta}\rangle = 0 \qquad \text{for all } \alpha \in \Sigma(\mathfrak{m}') \text{ such that } \mathfrak{g}_c(\mathfrak{t}; \alpha) \not\subset \mathfrak{k}_c.$$

Hence if we put $E = \sum \boldsymbol{R}\alpha$ where the sum is taken over all $\alpha \in \Sigma(\mathfrak{m}')$ such that $\mathfrak{g}_c(\mathfrak{t}; \alpha) \not\subset \mathfrak{k}_c$, then every compact root γ in $\Sigma(\mathfrak{m}')$ is contained in E or orthogonal to E. Note that every element δ in $\Sigma(\mathfrak{t})$ satisfying $\delta|_{\mathfrak{a}_\mathfrak{p}'} = -\beta$ can be written as a sum of $\tilde{\beta}$ (or $\sigma\tilde{\beta}$) and elements in $\Sigma(\mathfrak{m}')$. Then by the above result, δ can be written as a sum of $\tilde{\beta}$ and compact roots in $\Sigma(\mathfrak{m}')$ (or as a sum of $\sigma\tilde{\beta}$ and compact roots in $\Sigma(\mathfrak{m}')$ since $\sigma\mathfrak{k}_c = \mathfrak{k}_c$). Thus we have $\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}'; \beta) \subset \mathfrak{h}^d$. By the condition (i), we have $\langle \mu_\lambda^j, \beta\rangle \geq 0$ a contradiction. Q.E.D.

Added in proof (August 25, 1984)

(i) To prove Theorem in this paper we do not use the assumption that the connected real semisimple Lie group G has a complexification G_c. Therefore Theorem is valid without this assumption. But if G has infinite center, we must change the definition of "discrete series" as in [5].

(ii) E. P. van den Ban pointed out that the proof of Remark following Lemma 9 in [18] is incomplete, which is quoted in Remark in §4. The missing ingredients are given in his preprint "Invariant differential operators on a semisimple symmetric space and finite multiplicities in a Plancherel formula".

(iii) We have obtained a simpler proof of Theorem 1 which does not require in another paper.

(iv) We would like to thank H. Schlichtkrull who pointed us out some errors in the original manuscript.

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