Advanced Studies in Pure Mathematics 4, 1984
Group Representations and Systems of Differential Equations
pp. 243–253

Regular Holonomic Systems and their Minimal Extensions II

Application to the Multiplicity Formula for Verma Modules

Jiro Sekiguchi

§ 0. Introduction

(0.1) This note together with [10] is an introduction to a part of Professor Kashiwara's lectures at RIMS in 1981. As explained in [10], the contents of the lectures were concerned with the recent development of the regular holonomic systems and its application to the representation theory of a semisimple Lie algebra.

In this note, we give a report on the part of "its application to the representation theory".

(0.2) The multiplicity formula for Verma modules conjectured by Kazhdan-Lusztig [7] was proved by Brylinski-Kashiwara [3] and Beilinson-Bernstein [1].

In this note we give an outline of a proof of the multiplicity formula based on Kashiwara's lectures. Needless to say, we do not give a complete proof of all the results stated in [3]. We mainly give a summary and describe in detail the theorems whose proofs are slightly different from the ones in [3]. They are Theorems (3.3) and (3.4) in this note. We use the Beilinson-Bernstein Theorem in [1] to prove Theorem (3.3) and simplify the proof of Theorem (3.4) by using the Bernstein-Gelfand-Gelfand Theorem stated in (1.11.1).

(0.3) We frequently use the notation and results in [10] without notice. Accordingly we recommend the reader to consult the report [10] in reading this note. On the other hand, the report written by Tanisaki [12] contains topics related to the text of this note and further applications of the $\mathscr{D}$-Modules to the representation theory.

§ 1. The category $\tilde{\mathcal{O}}$

(1,1) Let $\mathfrak{g}$ be a semisimple Lie algebra over $\boldsymbol{C}$ and let $\mathfrak{t}$ be a Cartan

Received April 30, 1983.

subalgebra of $\mathfrak{g}$. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. Then $\mathfrak{b}=\mathfrak{t}+\mathfrak{n}$ is a direct sum decomposition, where $\mathfrak{n}=[\mathfrak{b}, \mathfrak{b}]$ is the nilpotent radical of $\mathfrak{b}$.

For a Lie algebra $\mathfrak{a}$, $U(\mathfrak{a})$ denotes the universal enveloping algebra of $\mathfrak{a}$.

(1.2) Let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$ and let Δ^+ be the positive system of Δ corresponding to $\mathfrak{b}$. For later use, we define $\rho=1/2 \sum_{\alpha\in\Delta^+}\alpha$ as usual. Let W be the Weyl group.

Definition (1.3). Let $\tilde{\mathcal{O}}$ be the category of $U(\mathfrak{g})$-modules defined as follows.

A $U(\mathfrak{g})$-module M is an object of $\tilde{\mathcal{O}}$ if and only if M satisfies the following properties:

(1.3.1) M is a finitely generated $U(\mathfrak{g})$-module.

(1.3.2) For any $u \in M$, we have $\dim U(\mathfrak{b})u < \infty$.

A morphism in this category is a $\mathfrak{g}$-homomorphism.

(1.4) If M is a $\mathfrak{g}$-module, then for any $\lambda \in \mathfrak{t}^*$ we define

$$M_\lambda=\{u \in M;\ (H-\lambda(H))^N u=0 \quad \text{for any } H \in \mathfrak{t},\ N \gg 0\}.$$

The category $\mathcal{O}$ introduced in [2] is a full subcategory of $\tilde{\mathcal{O}}$. An object M of $\tilde{\mathcal{O}}$ is contained in $\mathcal{O}$ if and only if $(H-\lambda(H))M_\lambda=0$ for any $\lambda \in \mathfrak{t}^*$ and $H \in \mathfrak{t}$.

(1.5) Let θ be an involution of $\mathfrak{g}$ such that $\theta|\mathfrak{t}=-\mathrm{Id}$. We note that such a θ exists unique up to inner automorphisms of $\mathfrak{g}$. Using θ, we induce a $\mathfrak{g}$-module structure on $\mathrm{Hom}_C(M, C)$ in the following way: For any $A \in \mathfrak{g}$, the action of A on $\mathrm{Hom}_C(M, C)$ is defined by

(1.5.1) $\langle Af, u\rangle=-\langle f, \theta(A)u\rangle$ for $f \in \mathrm{Hom}_C(M, C)$ and $u \in M$.

Let M be an object of $\tilde{\mathcal{O}}$. Then we define M^* by

$$M^*=\{f \in \mathrm{Hom}_C(M, C);\ f(M_\lambda)=0 \text{ except a finite number of } \lambda\}.$$

(1.6) We now state some fundamental properties of the category $\tilde{\mathcal{O}}$. Most of the results are shown by arguments similar to those for the results to the category $\mathcal{O}$. For this reason, we omit their proof. The reader may consult [2, 4].

(1.6.1) Let $0\to M'\to M\to M''\to 0$ be an exact sequence of $\mathfrak{g}$-modules.

Then the following statements hold.

(i) M is an object of $\tilde{\mathcal{O}}$ if and only if M' and M'' are objects of $\tilde{\mathcal{O}}$.

(ii) If M is an object of $\tilde{\mathcal{O}}$, then for any $\lambda \in \mathfrak{t}^*$ we have an exact sequence $0 \to M'_\lambda \to M_\lambda \to M''_\lambda \to 0$. Moreover, $\dim M_\lambda < \infty$.

(1.6.2) Each object of $\tilde{\mathcal{O}}$ has a composition series of finite length.

(1.6.3) (i) If M is an object of $\tilde{\mathcal{O}}$, so is M^*.

(ii) For any object M of $\tilde{\mathcal{O}}$, we have $(M^*)^* = M$.

(iii) If we define a functor $*$ of $\tilde{\mathcal{O}}$ to $\tilde{\mathcal{O}}$ by $M \to M^*$, then $*$ is a contravariant exact functor.

We here note that (1.6.1) (i) is particular to $\tilde{\mathcal{O}}$ and does not hold for $\mathcal{O}$.

(1.7) For any $M \in \tilde{\mathcal{O}}$, we define

$$\mathrm{ch}\, M = \sum_{\lambda \in \mathfrak{t}^*} \dim (M_\lambda) e^\lambda$$

and call it the character of M (cf. [4]).

We here give two basic properties of $\mathrm{ch}\, M$ which are consequences of (1.6).

(1.7.1) If M is an object of $\tilde{\mathcal{O}}$, then

$$\mathrm{ch}\, M = \mathrm{ch}\, M^*.$$

(1.7.2) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of objects of $\tilde{\mathcal{O}}$, then we have

$$\mathrm{ch}\, M = \mathrm{ch}\, M' + \mathrm{ch}\, M''.$$

(1.8) For any $\lambda \in \mathfrak{t}^*$, we define a $\mathfrak{g}$-module $M(\lambda)$ by

$$M(\lambda) = U(\mathfrak{g}) / U(\mathfrak{g})\mathfrak{n} + \sum_{H \in \mathfrak{t}} U(\mathfrak{g})(H - (\lambda - \rho)(H))$$

and call it the Verma module with the highest weight $\lambda - \rho$. There exists a unique maximal proper $\mathfrak{g}$-submodule K of $M(\lambda)$. We set $L(\lambda) = M(\lambda)/K$. The $\mathfrak{g}$-module $L(\lambda)$ is a simple module.

Proposition (1.8.1) (cf. [4]). *For any $\lambda \in \mathfrak{t}^*$, we have*

$$\mathrm{ch}\, M(\lambda) = \frac{e^{\lambda - \rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}.$$

(1.9) Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Then it is known (cf. [4]) that for any $\lambda \in \mathfrak{t}^*$, there exists an algebra homomorphism χ_λ of $Z(\mathfrak{g})$ to $\boldsymbol{C}$ such that $(P-\chi_\lambda(P))|M(\lambda)=0$ for any $P \in Z(\mathfrak{g})$. Noting this, we define

$$\tilde{\mathcal{O}}_{[\lambda]}=\{M \in \tilde{\mathcal{O}}\ ;\ (P-\chi_\lambda(P))|M=0 \quad \text{for any } P \in Z(\mathfrak{g})\}.$$

Clearly $M(\lambda)$ and $L(\lambda)$ are contained in $\tilde{\mathcal{O}}_{[\lambda]}$. We note that $\tilde{\mathcal{O}}_{[w\lambda]}=\tilde{\mathcal{O}}_{[\lambda]}$ for any $w \in W$ and that any simple module contained in $\tilde{\mathcal{O}}_{[\lambda]}$ is isomorphic to some $L(w\lambda)$ ($w \in W$).

(1.10) It follows from (1.6.2) that for any $M \in \tilde{\mathcal{O}}$, there exists a sequence of $\mathfrak{g}$-modules $\{M_j\}_{j=0}^r$ such that $M_0=M$, $M_r=0$, $M_j \supset M_{j+1}$ and M_j/M_{j+1} is a simple module. Then we put $r(M)=r$ and call it the length of M. For a simple object S of $\tilde{\mathcal{O}}$, we denote by $[M: S]$ the number of times of the appearance of S in the composition factor series of M. It follows from (1.6.2) that $[M: S]$ is always finite.

(1.11) We introduce a Bruhat order $\leq$ on W. Let w and w' be in W. Then $w \leqq w'$ if and only if $\bar{X}_w \subseteqq \bar{X}_{w'}$. Here X_w and $X_{w'}$ denote the Bruhat cells defined in (2.3) of the next section.

We state a fundamental result due to Bernstein-Gelfand-Gelfand [2].

Theorem (1.11.1) (cf. [2]). *For any two elements w, w' of W, we have* $[M(-w\rho): L(-w'\rho)] \neq 0$ *if and only if* $w \geqq w'$.

(1.12) For later use, we define $K(\tilde{\mathcal{O}}_{[\rho]})=\sum_{w \in W} \boldsymbol{Z}(\mathrm{ch}\, L(w\rho))$. Then it is clear from the definition that for any $M \in \tilde{\mathcal{O}}_{[\rho]}$, $\mathrm{ch}\, M$ is contained in $K(\tilde{\mathcal{O}}_{[\rho]})$.

Lemma (1.13). *Let f be a map of $\tilde{\mathcal{O}}_{[\rho]}$ to $K(\tilde{\mathcal{O}}_{[\rho]})$ satisfying the conditions* (i), (ii):

(i) $f(M(w\rho))=0$ *for any* $w \in W$.

(ii) *If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence of objects in* $\tilde{\mathcal{O}}_{[\rho]}$, *then* $f(M)=f(M')+f(M'')$.

Then we have $f=0$.

This lemma is easy to prove.

§ 2. Localization of $\mathfrak{g}$-modules

(2.1) Let G be a connected and simply connected Lie group with the Lie algebra $\mathfrak{g}$. Let B be the Borel subgroup of G with the Lie algebra $\mathfrak{b}$ and let T be the maximal torus of G with the Lie algebra $\mathfrak{t}$.

Let $X=G/B$ be the flag manifold. Since any two Borel subgroups of G are conjugate each other and since the normalizer of B coincides with B

itself, X is regarded as the totality of Borel subgroups of G.

(2.2) For any $A \in \mathfrak{g}$, we define a vector field D_A on X as follows. If $f(x)$ is a function on X, then

$$(D_A f)(x) = \frac{d}{dt} f(e^{-tA}x)|_{t=0}.$$

By definition, we have $[D_A, D_{A'}] = D_{[A,A']}$ for any $A, A' \in \mathfrak{g}$. Therefore the map $\mathfrak{g} \to \Gamma(X, \mathscr{D}_X)$ defined by $A \to D_A$ induces an algebra homomorphism of $U(\mathfrak{g})$ to $\Gamma(X, \mathscr{D}_X)$.

Since $D_{A,x}(A \in \mathfrak{g})$ span the tangent space $T_x X$, the map $\phi: \mathfrak{g} \to T_x X$ defined by $\phi(A) = D_{A,x}$ is surjective. If $x = gB$, we define $\mathfrak{b}(x) = \mathrm{Ad}(g)\mathfrak{b}$ and $\mathfrak{n}(x) = \mathrm{Ad}(g)\mathfrak{n}$. Then $\mathrm{Ker}\,\phi = \mathfrak{b}(x)$. This implies that $T_x X = \mathfrak{g}/\mathfrak{b}(x)$. Accordingly we have $T_x^* X = \mathfrak{b}(x)^\perp$. Since $\mathfrak{b}(x)^\perp \subset \mathfrak{g}^*$, we obtain a map $T^*X \to X \times \mathfrak{g}^*$. Composing this map and the projection from $X \times \mathfrak{g}^*$ to $\mathfrak{g}^*$, we can define a map $\gamma: T^*X \to \mathfrak{g}^*$ by $\gamma(x, \lambda) = \lambda$ for any $x \in X$, $\lambda \in \mathfrak{b}(x)^\perp$. Then it is easy to see that $\sigma_1(D_A) = A \circ \gamma$ for any $A \in \mathfrak{g}$.

(2.3) Let $W = N_G(T)/T$ be the Weyl group. For any $w \in W$, we take a representative $\bar{w}$ of $N_G(T)$. Then we define a point $\bar{w}B$ of X. Since this point does not depend on the choice of a representative of w, we can define an injective map of W to X by $w \to \bar{w}B$. Due to this map, we may regard W as a finite subset of X. In the sequel, we frequently regard W as the finite subset of X. For any $w \in W$, we put $X_w = BwCX$. Then according to Bruhat, we see that $X = \coprod_{w \in W} X_w$ is a disjoint union. Define $l(w) = \dim X_w$ for any $w \in W$.

(2.4) Let $M'_{\mathrm{coh}}(\mathscr{D}_X)$ be the category of coherent $\mathscr{D}_X$-Modules which have global good filtrations. Let $D_f(R)$ be the category of finitely generated R-modules. Here we set $R = \Gamma(X, \mathscr{D}_X)$. Then we have the following theorem due to Beilinson-Bernstein [1]. This plays an important role in the subsequent discussion.

Theorem (2.5). *The functor taking the global sections* $\Gamma: \mathscr{M} \to \Gamma(X, \mathscr{M})$ *is an equivalence of the categories* $M'_{\mathrm{coh}}(\mathscr{D}_X)$ *and* $D_f(R)$. *The inverse functor is given by* $M \to \mathscr{D}_X \otimes_R M (M \in D_f(R))$.

Remark (2.6). It is not known if any coherent $\mathscr{D}_X$-Module $\mathscr{M}$ on X has a global good filtration. But if $\mathscr{M}$ is regular holonomic, it follows from [6] that $\mathscr{M}$ has a global good filtration. In the following sections, we restrict our attention mainly to regular holonomic systems. Therefore Theorem (2.5) is sufficient for our purpose.

§ 3. The category $\mathfrak{M}$

(3.1) We define a category $\mathfrak{M}$ of coherent $\mathscr{D}_X$-Modules on X. An object $\mathscr{M}$ of $\mathfrak{M}$ is a regular holonomic system on X with the condition $\mathrm{Ch}(\mathscr{M}) \subseteqq \overline{\bigcup_{w \in W} T^*_{X_w} X}$. It follows from [6] that $\mathfrak{M}$ is a subcategory of $M'_{\mathrm{coh}}(\mathscr{D}_X)$.

(3.2) It follows from [9] that $\mathscr{H}^k_{[X_w]}(\mathscr{O}_X)=0$ $(k \neq n-l(w))$ for any $w \in W$. Noting this, we define

$$\mathscr{M}_w = (\mathscr{H}^{n-l(w)}_{[X_w]}(\mathscr{O}_X))^*.$$

Here we put $n=\dim X$. The following statements hold (cf. [9, 3]).

(3.2.1) $\mathscr{M}_w,\ \mathscr{M}^*_w \in \mathfrak{M}$.

(3.2.2) $\mathrm{Supp}\,\mathscr{M}_w = \mathrm{Supp}\,\mathscr{M}^*_w = \overline{X}_w$.

(3.2.3) $H^k(X, \mathscr{M}^*_w)=0 \qquad (\forall k>0)$.

(3.2.4) $\mathscr{H}^k_{[\partial X_w]}(\mathscr{M}^*_w)=0 \qquad (\forall k)$.

(3.2.5) $\mathfrak{t}$ acts semisimply on $\Gamma(X, \mathscr{M}^*_w)$.

(3.2.6) $\mathrm{ch}(\Gamma(X, \mathscr{M}^*_w)) = \mathrm{ch}(M(-w\rho))$.

Theorem (3.3). *The functor in Theorem* (2.5) *induces an equivalence between the categories* $\mathfrak{M}$ *and* $\tilde{\mathscr{O}}_{[\rho]}$.

Proof. Let $\mathscr{M}$ be an object of $\mathfrak{M}$. We put $M=\Gamma(X, \mathscr{M})$. Then it follows from Theorem (2.5) that M is a finitely generated $U(\mathfrak{g})$-module and that $(P-\chi_\rho(P))|M=0$ for any $P \in Z(\mathfrak{g})$. Hence to prove that M is an object of $\tilde{\mathscr{O}}_{[\rho]}$, it suffices to show that $\dim U(\mathfrak{b})u<\infty$ for any $u \in M$. Since $\mathscr{M}$ is regular holonomic, it follows from [10] that there exists a global good filtration $\{\mathscr{M}(j)\}_{j \in Z}$ satisfying the condition:

(3.3.1) For any j and m, if $P \in \mathscr{D}_X(m)$ satisfies that $\sigma_m(P)|\mathrm{Ch}(\mathscr{M})=0$, then $P\mathscr{M}(j) \subseteqq \mathscr{M}(j+m-1)$.

We put $M_j=\Gamma(X, \mathscr{M}(j))$ for any $j \in Z$. It follows from the definition of a global good filtration that each $\mathscr{M}(j)$ is a coherent $\mathscr{O}_X$-Module. Since $\sigma_1(D_A)|\mathrm{Ch}(\mathscr{M})=0$ for any $A \in \mathfrak{b}$, we find that $D_A\mathscr{M}(j) \subseteqq \mathscr{M}(j)$ $(\forall j \in Z)$. Accordingly we see that $AM_j \subseteqq M_j (\forall A \in \mathfrak{b})$ and moreover that $U(\mathfrak{b})M_j \subseteqq M_j$ for any $j \in Z$. Take an element u of M. Since $M=\bigcup_{j \in Z} M_j$, u is contained in M_j for some $j \in Z$. Then $U(\mathfrak{b})u \subseteqq U(\mathfrak{b})M_j \subseteqq M_j$. Since X is projective algebraic, owing to Serre's theorem we conclude that $\dim U(\mathfrak{b})u \leqq$

$\dim M_j < \infty$. We have thus proved that $M = \Gamma(X, \mathscr{M})$ is an object of $\tilde{\mathcal{O}}_{[\rho]}$.

We next show that for any $M \in \tilde{\mathcal{O}}_{[\rho]}$, $\mathscr{D}_X \otimes_R M$ is an object of $\mathfrak{M}$. The proof of this is rather long. Therefore we accomplish the proof by decomposing into three steps.

Step 1. For any $w \in W$, $\mathrm{Ch}(\mathscr{D}_X \otimes_R M(-w\rho)) \subseteqq \bigcup_{y \in W} \overline{T^*_{X_y} X}$.

Proof. It follows from (1.8) that

$$\mathscr{D}_X \otimes_R M(-w\rho) = \mathscr{D}_X / \sum_{A \in \mathfrak{n}} \mathscr{D}_X D_A + \sum_{H \in \mathfrak{t}} \mathscr{D}_X (D_H + (w\rho + \rho)(H)).$$

Then we find that $\mathrm{Ch}(\mathscr{D}_X \otimes_R M(-w\rho)) \subseteqq \gamma^{-1}(\mathfrak{b})$. Noting that $\gamma^{-1}(\mathfrak{b}) = \bigcup_{y \in W} \overline{T^*_{X_y} X}$, we obtain the claim in Step 1.

Step 2. For any $w \in W$, $\mathscr{D}_X \otimes_R L(-w\rho)$ is an object of $\mathfrak{M}$.

Proof. Since $M(-w\rho) \to L(-w\rho) \to 0$ is exact, it follows from Theorem (2.5) that $\mathscr{D}_X \otimes_R M(-w\rho) \to \mathscr{D}_X \otimes_R L(-w\rho) \to 0$ is exact. Then we conclude from [10] that $\mathrm{Ch}(\mathscr{D}_X \otimes_R L(-w\rho)) \subseteqq \gamma^{-1}(\mathfrak{b})$. Next we show that $\mathscr{D}_X \otimes_R L(-w\rho)$ is regular holonomic. We set $M = \Gamma(X, \mathscr{M}^*_w)$. Then it follows from (3.2.6) that $-w\rho - \rho$ is a highest weight of M. This means that there exists an element $u (\neq 0)$ of M such that $\mathfrak{n}u = 0$ and $Hu = -(w\rho + \rho)(H)u$ for any $H \in \mathfrak{t}$. Since Theorem (2.5) implies that $\mathscr{M}^*_w = \mathscr{D}_X \otimes_R M$ and in particular this is regular holonomic and since $L(-w\rho)$ is a subquotient module of M, we conclude from [10] that $\mathscr{D}_X \otimes_R L(-w\rho)$ is regular holonomic. Hence the claim in Step 2 is proved.

Step 3. For any $M \in \tilde{\mathcal{O}}_{[\rho]}$, $\mathscr{D}_X \otimes_R M$ is an object of $\mathfrak{M}$.

Proof. We prove the statement by induction on the length $r(M)$ of M.

If $r(M) = 1$, it follows from (1.9) that there exists an element w of W such that $M = L(-w\rho)$. Then we conclude from Step 2 that $\mathscr{D}_X \otimes_R M$ is an object of $\mathfrak{M}$.

We fix a positive integer r. Assuming that for any $M' \in \tilde{\mathcal{O}}_{[\rho]}$ such that $r(M') \leqq r-1$, $\mathscr{D}_X \otimes_R M'$ is an object of $\mathfrak{M}$, we show that if M is an object of $\tilde{\mathcal{O}}_{[\rho]}$ such that $r(M) = r$, then $\mathscr{D}_X \otimes_R M$ is in $\mathfrak{M}$. It is clear that there are a $\mathfrak{g}$-submodule M' of M and $w \in W$ such that $r(M') = r-1$ and that

$$0 \to M' \to M \to L(-w\rho) \to 0$$

is an exact sequence. Then it follows from Theorem (2.5) that

$$0 \to \mathscr{D}_X \otimes_R M' \to \mathscr{D}_X \otimes_R M \to \mathscr{D}_X \otimes_R L(-w\rho) \to 0$$

is an exact sequence. Owing to the hypothesis of the induction and Step 2, we conclude that $\mathscr{D}_X \otimes_R M \in \mathfrak{M}$.

We have thus proved the claim in Step 3 by induction.

From the above discussion, Theorem (3.3) is completely proved.

Theorem (3.4). *For any $w \in W$, the following statements hold.*

(i) *We put $\mathscr{L}_w = \mathscr{D}_X \otimes_R L(-w\rho)$. Then $\mathscr{L}_w = \mathscr{L}(\bar{X}_w, X)$. Here $\mathscr{L}(\bar{X}_w, X)$ is the minimal extension of $\mathscr{H}^{n-l(w)}_{[X_w]}(\mathscr{O}_X)|X-\partial X_w$ to X.*

(ii) $\Gamma(X, \mathscr{M}_w) = M(-w\rho)$, $\Gamma(X, \mathscr{M}_w^*) = M(-w\rho)^*$.

Proof. (i) We prove by induction on $l(w)$.

First we assume that $l(w)=0$. Then $w=e$. Since X_e consists of one point, we conclude from [10] that $\mathscr{L}_e = \mathscr{M}_e^*$ is a minimal extension.

Next we assume that $\mathscr{L}_y = \mathscr{L}(\bar{X}_y, X)$ for any $y \in W$ such that $l(y) < l(w)$ and show that $\mathscr{L}_w = \mathscr{L}(\bar{X}_w, X)$. Putting $M = \Gamma(X, \mathscr{M}_w^*)$, we see from (3.2.6) that $L(-w\rho)$ is a subquotient of M, that is, there are $\mathfrak{g}$-submodules N_1, N_2 of M such that $N_1 \subset N_2$ and that $N_2/N_1 = L(-w\rho)$. Then owing to Theorem (1.11.1), we find that each composition factor of M/N_2 and N_1 is isomorphic to $L(-w'\rho)$ for some $w' \in W$ such that $w' < w$. Therefore it follows from Theorem (2.5) that $\mathscr{L}_w|_{X-\partial X_w} = \mathscr{M}_w^*|_{X-\partial X_w}$. Noting this, we need only to prove the following:

(3.4.1) If $\mathscr{M}$ is a coherent $\mathscr{D}_X$-Module such that $\mathscr{M}$ is a subquotient of $\mathscr{L}_w$ and that $\operatorname{Supp} \mathscr{M} \subseteqq \partial X_w$, then $\mathscr{M}=0$.

Let $\mathscr{M}$ be such a $\mathscr{D}_X$-Module. Then $\Gamma(X, \mathscr{M})$ is a subquotient of $L(-w\rho)$. But $L(-w\rho)$ is simple. Therefore we find that $\Gamma(X, \mathscr{M})$ is $L(-w\rho)$ or 0. If $\Gamma(X, \mathscr{M}) = L(-w\rho)$, then $\mathscr{M} = \mathscr{L}_w$ and in particular $\operatorname{Supp} \mathscr{M} = \bar{X}_w$. This is a contradiction. On the other hand, if $\Gamma(X, \mathscr{M}) = 0$, then it follows from Theorem (2.5) that $\mathscr{M}=0$. Hence (3.4.1) and therefore (i) is proved.

(ii) We put $N = (\Gamma(X, \mathscr{M}_w^*))^*$. Then $\operatorname{ch} N = \operatorname{ch} N^* = \operatorname{ch} M(-w\rho)$. Hence $\operatorname{Hom}_{\mathfrak{g}}(M(-w\rho), N) \neq 0$. Let f be a non-trivial $\mathfrak{g}$-homomorphism of $M(-w\rho)$ to N and put $N' = N/f(M(-w\rho))$. Then we obtain an exact sequence $0 \to N'^* \to N^* \to M(-w\rho)^*$. This combined with Theorem (1.11.1) implies that each composition factor of N'^* is isomorphic to $L(-w'\rho)$ for some $w' \in W$, $w' < w$. Therefore we find that $\operatorname{Supp}(\mathscr{D}_X \otimes_R N'^*) \subseteqq \partial X_w$. On the other hand, we have an inclusion $\mathscr{D}_X \otimes_R N'^* \hookrightarrow \mathscr{D}_X \otimes_R N^* = \mathscr{M}_w^*$. It follows from (3.2.4) and [10] that $\mathscr{H}^0_{\partial X_w}(\mathscr{M}_w^*) = \mathscr{H}^0_{[\partial X_w]}(\mathscr{M}_w^*) = 0$. These imply that $\mathscr{D}_X \otimes_R N'^* = 0$. Therefore we find from Theorem (2.5) that $N'^* = 0$. This means that N^* is a submodule of $M(-w\rho)^*$. But the characters of these modules coincide. Accordingly we conclude that $\Gamma(X, \mathscr{M}_w^*) = M(-w\rho)^*$.

Next we put $M=\Gamma(X, \mathscr{M}_w)$. Since (i) shows that $\mathscr{L}_y^*=\mathscr{L}_y$ for any $y \in W$, the composition factors of $\mathscr{M}_w$ coincide with those of $\mathscr{M}_w^*$ including their multiplicities. Hence it follows from Theorem (2.5) that the composition factors of M coincide with those of $M(-w\rho)^*=\Gamma(X, \mathscr{M}_w^*)$. In particular we find that $\mathrm{ch}\, M=\mathrm{ch}\, M(-w\rho)$. Therefore we have an exact sequence $M(-w\rho)\to M\to N'\to 0$ such that $\mathrm{Im}(M(-w\rho)\to M)\neq 0$. Then it follows from Theorem (2.5) that $\mathscr{D}_X\otimes_R M(-w\rho)\to\mathscr{M}_w\to\mathscr{D}_X\otimes_R N'\to 0$ is an exact sequence. We see that $(\mathscr{D}_X\otimes_R N')^*$ is a coherent $\mathscr{D}_X$-sub-Module of $\mathscr{M}_w^*$ and also see from Theorem (1.11.1) that $\mathrm{Supp}(\mathscr{D}_X\otimes_R N')^*\subseteqq\partial X_w$. Hence we can show $(\mathscr{D}_X\otimes_R N')^*=0$ by an argument similar to the discussion above. Therefore $N'=0$ and $M(-w\rho)\to M\to 0$ is exact. Since $\mathrm{ch}\, M(-w\rho)=\mathrm{ch}\, M$, we conclude that $M(-w\rho)=M$. q.e.d.

Conjecture (3.5). For any $\mathscr{M} \in \mathfrak{M}$, the following holds:

$$\Gamma(X, \mathscr{M}^*)\simeq\Gamma(X, \mathscr{M})^*.$$

§ 4. The multiplicity formula for Verma modules

(4.1) Let w, w' be elements of W. Then we can define Kazhdan-Lusztig polynomial $P_{w,w'}(q) \in \boldsymbol{Z}[q]$ (cf. [8]). We do not give here a precise definition of $P_{w,w'}(q)$ but only note that the following theorem holds.

Theorem (4.1.1) (cf. [8]). *For any* $w, w' \in W$, *we have*

(i) $\sum_k \dim \mathscr{H}^{2k}(\pi_{\bar{X}_{w'}})_w q^k=P_{w,w'}(q)$.

(ii) $\mathscr{H}^{2k-1}(\pi_{\bar{X}_w})=0$ *for any* k.

One finds in [11] an alternative proof of this theorem due to MacPherson.

Theorem (4.2). *For any* $w \in W$, *we have the following statements.*

(i) $\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{L}_w)=\pi_{\bar{X}_w}[-(n-l(w))]$.

(ii) $\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{M}_w)=\boldsymbol{C}_{X_w}[-(n-l(w))]$.

Proof. The claim (i) follows from a general result (cf. [3, 10]).
We prove (ii).

$$\begin{aligned}
&\mathscr{D}\mathscr{R}((\mathscr{H}^{n-l(w)}_{[X_w]}(\mathscr{O}_X))^*)\\
&\quad=(\mathscr{D}\mathscr{R}(\mathscr{H}^{n-l(w)}_{[X_w]}(\mathscr{O}_X)))^*\\
&\quad=(\mathscr{D}\mathscr{R}(\boldsymbol{R}\Gamma_{[X_w]}(\mathscr{O}_X)[n-l(w)]))^* \qquad \text{(cf. (3.2))}\\
&\quad=(\boldsymbol{R}\Gamma_{X_w}(\mathscr{D}\mathscr{R}(\mathscr{O}_X))[n-l(w)])^*\\
&\quad=(\boldsymbol{R}\Gamma_{X_w}(\boldsymbol{C}_X))^*[-(n-l(w))]\\
&\quad=(\boldsymbol{R}\mathscr{H}om_{\boldsymbol{C}_X}(\boldsymbol{C}_{X_w}, \boldsymbol{C}_X))^*[-(n-l(w))]
\end{aligned}$$

$$=C_{X_w}[-(n-l(w))].$$ q e.d.

(4.3) We review the index of a holonomic system (cf. [5]).

Definition (4.3.1). Let $\mathcal{M}$ be a holonomic system on X. Then we define

$$\chi_x(\mathcal{M})=\sum_j(-1)^j \dim(\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})_x) \qquad (\forall x \in X).$$

A fundamental property of $\chi_x(\mathcal{M})$ is the following.

Lemma (4.3.2). *If* $0\to\mathcal{M}'\to\mathcal{M}\to\mathcal{M}''\to 0$ *is an exact sequence of holonomic systems on* X, *then we have*

$$\chi_x(\mathcal{M})=\chi_x(\mathcal{M}')+\chi_x(\mathcal{M}'') \qquad \textit{for any } x \in X.$$

Theorem (4.4). *For any* $M \in \tilde{\mathcal{O}}_{[\rho]}$, *we have*

$$\mathrm{ch}(M)=\sum_{w\in W}(-1)^{n-l(w)}\chi_w(\mathcal{D}_X\underset{R}{\otimes}M)\,\mathrm{ch}(M(-w\rho)).$$

Proof. We define a map f of $\tilde{\mathcal{O}}_{[\rho]}$ to $K(\tilde{\mathcal{O}}_{[\rho]})$ by

$$f(M)=\mathrm{ch}(M)-\sum_{w\in W}(-1)^{n-l(w)}\chi_w(\mathcal{D}_X\underset{R}{\otimes}M)\,\mathrm{ch}(M(-w\rho)).$$

Then it follows from Theorem (4.2) (ii), (1.7.2) and Lemma (4.3.2) that the conditions (i) and (ii) in Lemma (1.13) hold for $f(M)$. Therefore Lemma (1.13) implies the theorem. q.e.d.

(4.5) We are in a position to prove the theorem conjectured by Kazhdan-Lusztig [7] and proved by Brylinski-Kashiwara [3] and Beilinson-Bernstein [1].

Theorem (4.6). $[M(-w\rho)\colon L(-w'\rho)]=P_{w_0w,w_0w'}(1)$ $(\forall w, w' \in W)$. *Here* w_0 *is the unique element of* W *such that* X_{w_0} *is open in* X.

Proof. By an inversion formula for Kazhdan-Lusztig polynomials (cf. [7]), the theorem is equivalent to the following identity:

(4.6.1) $$\mathrm{ch}(L(-w\rho))=\sum_{y\in W}(-1)^{l(w)-l(y)}P_{y,w}(1)\,\mathrm{ch}(M(-y\rho)) \quad (\forall w \in W).$$

We are going to prove (4.6.1). It follows from Theorem (4.4) that

$$\mathrm{ch}(L(-w\rho))=\sum_{y\in W}(-1)^{n-l(y)}\chi_y(\mathcal{L}_w)\,\mathrm{ch}(M(-y\rho)).$$

On the other hand, owing to Theorems (4.1.1) and (4.2), we find that

$$\begin{aligned}\chi_y(\mathscr{L}_w) &= \sum_j (-1)^j \dim \mathscr{H}^j(\pi_{X_w}[-(n-l(w))])_y \\ &= \sum_j (-1)^{j+n-l(w)} \dim \mathscr{H}^j(\pi_{X_w})_y \\ &= (-1)^{n-l(w)} P_{y,w}(1)\,.\end{aligned}$$

Combining these equalities, we obtain (4.6.1) and therefore have shown Theorem (4.6). q.e.d.

References

[1] Beilinson, A. and Bernstein, J., Localisation de $\mathfrak{g}$-modules, Comptes Rendus **292**A (1981), 15–18.
[2] Bernstein, I. N., Gelfand, I. M. and Gelfand, S. I., Differential operators on the base affine space and a study of $\mathfrak{g}$-modules, in: Lie groups and their representations, edited by Gelfand, I. N., London (1975), 21–64.
[3] Brylinski, J. L. and Kashiwara, M., Kazhdan-Lusztig conjecture and holonomic systems, Invent. Math., **64** (1981), 387–410.
[4] Dixmier, J., Enveloping algebras, North-Holland (1977).
[5] Kashiwara, M., Systems of microdifferential equations, Notes by J. M. Fernandes, Birkhäuser, 1983.
[6] Kashiwara, M. and Kawai, T., On holonomic systems of micro-differential equations, III, —Systems with regular singularities—, Publ. RIMS, Kyoto Univ., **17** (1981), 813–979.
[7] Kazhdan, D. and Lusztig, G., Representations of Coxeter groups and Hecke algebras, Invent. Math., **53** (1979), 165–184.
[8] ——, Schubert varieties and Poincaré duality, Proc. Symp. in Pure Math., **36** (1980), 185–203.
[9] Kempf, G., The Grothendieck-Cousin complex of an induced representation, Adv. in Math., **29** (1978), 310–396.
[10] Noumi, M., Regular holonomic systems and their minimal extensions I, this volume.
[11] Springer, T. A., Quelques applications de la cohomologie d'intersection, Séminaire Bourbaki 34e année, no. 589 (1981/1982).
[12] Tanisaki, T., Representation theory of complex semisimple Lie algebras and $\mathscr{D}$-Modules, In: Reports of the fifth seminar on Algebra II, (1983), 67–163 (in Japanese).

Department of Mathematics
Tokyo Metropolitan University
Setagaya-ku, Tokyo 158
Japan