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Symbols and Formal Symbols of Pseudodifferential Operators

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Introduction

In this paper, we present a symbol theory of pseudodifferential operators in analytic category. A pseudodifferential operator is, by definition, an integral operator

$$u(x) \longmapsto Pu(x) = \int K(x, x')u(x')dx' \tag{0.1}$$

with a holomorphic microfunction kernel $K(x, x')$ defined on the conormal bundle supported by $x=x'$. The sheaf of rings of pseudodifferential operators is denoted by $\mathscr{E}^R$ ([5], [6], [8]). It follows from Cauchy's integral formula that $\mathscr{E}^R$ contains all linear differential operators with analytic coefficients. Moreover, $\mathscr{E}^R$ includes the sheaf $\mathscr{E}^\infty$ of microdifferential operators ([10]). Needless to say, those classes of operators are very important in the investigations of various problems. We emphasize that the classes contain operators of infinite order and that the use of such operators is crucial in many cases (cf. [6], [10], [12]).

Symbols of pseudodifferential operators are defined by Kataoka [7]. He defines symbols from the cohomological definition of $\mathscr{E}^R$ by the aid of Radon transformations. On the other hand, Boutet de Monvel [4] introduces analytic pseudodifferential operators by using oscillatory integrals for given symbol classes and shows that standard symbolic calculus is valid as well as in C^∞-category (see [11], for example). We note that pseudodifferential operators in the sense of [4] are contained in $\mathscr{E}^R$ by virtue of Kataoka's theory.

The aim of this paper is to develop and to complete the symbol theory of $\mathscr{E}^R$ from the standpoint of [7]. The advantages of the viewpoint are related to the invariance of the cohomological definition of $\mathscr{E}^R$. The sheaf itself is defined independently of a choice of local coordinate systems. The cohomology group which defines $\mathscr{E}^R$ can be represented elementarily by the method of the Radon transformation ([7], [8]); we shall make full use of the method. One of our main contributions is introducing the

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notion of formal symbols, which is generalization of a definition introduced in [4]. Such generalization enables us to deal with operators of infinite order in general (cf. [2], [3]).

The plan of this paper is as follows. In Section 1, we recall the algebraic definitions of pseudodifferential operators and microdifferential operators. Concrete descriptions of pseudodifferential operators are given in Section 2. In Section 3, we recall the theory of Radon transformations of Kataoka. Section 4 gives a definition of symbols of pseudodifferential operators. In Sections 5 and 6, we introduce formal symbols and double formal symbols and study infinite sums of symbols. By using the formalism developed in those sections, we establish symbolic calculus of operators in Section 7.

§ 1. Preliminaries

In this section we recall the algebraic definition of $\mathscr{E}^{R}$, $\mathscr{E}^{\infty}$, $\mathscr{D}^{\infty}$, etc. (Cf. [5]–[10]).

Let X be an n-dimensional complex manifold, $\mathcal{O}_X$ the sheaf of holomorphic functions on X. Let Y be a d-codimensional complex submanifold in X. The conormal vector bundle of Y in X is denoted by $T^*_Y X$. We identify the real comonoidal transformation of X with center Y with $\pi_{Y|X}\colon (X-Y)\amalg T^*_Y X \to X$.

The sheaf $\mathscr{C}^{R}_{Y|X}$ on $T^*_Y X$ is defined by

$$\mathscr{C}^{R}_{Y|X} = \mathscr{H}^{d}_{T^*_Y X}(\pi^{-1}_{Y|X}\mathcal{O}_X)^{a}, \tag{1.1}$$

where $a\colon T^*_Y X \to T^*_Y X$ is the antipodal mapping. The sections of $\mathscr{C}^{R}_{Y|X}$ are called holomorphic microfunctions. The sheaf $\mathscr{C}^{R}_{Y|X}$ is locally constant along the orbit of the action of $\boldsymbol{R}^{+}$ on $T^*_Y X$. The restriction of the sheaf to $T^*_X X$ is $\mathscr{B}^{\infty}_{Y|X} = \mathscr{H}^{d}_{Y}(\mathcal{O}_X)$.

Let us identify X with the diagonal of $X\times X$. Then the cotangent vector bundle T^*X is identified with $T^*_X(X\times X)$ by the first projection. The sheaf $\mathscr{E}^{R}_X$ is defined by

$$\mathscr{E}^{R}_X = \mathscr{C}^{R}_{X|X\times X} \underset{p_2^{-1}\mathcal{O}_X}{\otimes} p_2^{-1}\Omega^{n}_X, \tag{1.2}$$

where Ω^{n}_X is the sheaf of holomorphic n-forms on X and $p_2\colon X\times X \to X$ is the second projection. The sections of $\mathscr{E}^{R}_X$ are called pseudodifferential operators.

By the definition, a pseudodifferential operator P is written in the form $P=K(x, x')dx'$, where $K(x, x')$ is a section of $\mathscr{C}^{R}_{X|X\times X}$. The holomorphic microfunction $K(x, x')$ is called the kernel function of P. The product (composition) of two operators $P_1 = K_1(x, x')dx'$ and $P_2 =$

$K_2(x, x')dx'$ is defined by $P_1P_2=\left(\int K_1(x, x'')K_2(x'', x')dx''\right)dx'$, where the integral is taken as a holomorphic microfunction (cf. [10], Chap. II). Then $\mathscr{E}_X^R$ becomes the sheaf of (non-commutative) rings on T^*X.

Let us denote by γ the projection from $T^*X - T_X^*X$ to the cotangential projective bundle $P^*X=(T^*X-T_X^*X)/C^\times$. The sheaf $\mathscr{E}_X^\infty$ is defined by

$$\mathscr{E}_X^\infty|_{T^*X-T_X^*X}=\gamma^{-1}\gamma_*\mathscr{E}_X^R,$$
$$\mathscr{E}_X^\infty|_{T_X^*X}=\mathscr{E}_X^R|_{T_X^*X}.$$

The sections of $\mathscr{E}_X^\infty$ are called microdifferential operators (of finite or infinite order). The sheaf of microdifferential operators of finite order is denoted by $\mathscr{E}_X$. The sheaf $\mathscr{D}_X^\infty$ of differential operators on X is defined by

$$\mathscr{D}_X^\infty=\mathscr{B}_{X|X\times X}^\infty \underset{p_2^{-1}\mathcal{O}_X}{\otimes} p_2^{-1}\Omega_X^n.$$

Let us denote by $\mathscr{D}_X$ the sheaf of differential operators of finite order. There are the following canonical injective homomorphisms of sheaves of rings:

$$\begin{array}{ccccc} \pi^{-1}\mathscr{D}_X & \hookrightarrow & \mathscr{E}_X & & \\ \downarrow & & \downarrow & & \\ \pi^{-1}\mathscr{D}_X^\infty & \hookrightarrow & \mathscr{E}_X^\infty & \hookrightarrow & \mathscr{E}_X^R, \end{array}$$

where $\pi: T^*X \to X$ is the projection. Hence microdifferential operators and differential operators (of finite order or of infinite order) are pseudo-differential operators.

Remark. In [1], [2], [8], the sections of $\mathscr{E}_X^R$ are called holomorphic microlocal operators. On the other hand, in [10], the sections of $\mathscr{E}_X^\infty$ (denoted by $\mathscr{P}_X$ there) are called pseudodifferential operators. As we shall see later (Theorem 4.5), each operator in $\mathscr{E}^R$ is represented as a modulo class of temperate functions (=symbols). Therefore it seems to be natural to call operators in $\mathscr{E}^R$ pseudodifferential operators (cf. the definition of pseudodifferential operators in C^∞-category; see [11]). This is the reason why we use the naming "pseudodifferential operators" instead of "holomorphic microlocal operators".

§ 2. Defining functions of the kernel functions

We shall give concrete description of kernel functions of pseudo-differential operators by holomorphic functions ([10], Chap. II, § 1.4).

Hereafter X denotes an open set in C^n with coordinate system $x=(x_1, \cdots, x_n)$. Then we have the following identifications:

$$T^*X \simeq X\times C^n \ni x^*=(x, \xi)=(x_1, \cdots, x_n, \xi_1, \cdots, \xi_n),$$
$$TX \simeq X\times C^n \ni x_*=(x, v)=(x_1, \cdots, x_n, v_1, \cdots, v_n).$$

Here TX is the tangent vector bundle of X. The inner product of $(TX)_x \ni v$ and $(T^*X)_x \ni \xi$ is given by $\mathrm{Re}\langle v, \xi\rangle=\mathrm{Re}(v_1\xi_1+\cdots+v_n\xi_n)$. Let us fix a point $\mathring{x}^*=(\mathring{x}, \mathring{\xi})$ in T^*X. We can assume without loss of generality that $(\mathring{x}, \mathring{\xi})=(0; \lambda, 0, \cdots, 0)$ $(\lambda \in C)$. Conic neighborhoods of $\mathring{x}^*$ in T^*X are denoted by $\Omega, \Omega', \cdots$, etc. Here a subset $V\subset T^*X$ is said to be conic if $rV=\{(x, r\xi); (x, \xi) \in V\}\subset V$ for any $r\geqq 1$.

We shall consider the stalk $\mathscr{E}^R_{X,\mathring{x}^*}=\mathscr{E}^R_{\mathring{x}^*}$ of $\mathscr{E}^R_X$ at $\mathring{x}^*$. If $\lambda=0$, then the stalk coincides with $\mathscr{D}^\infty_{X,\mathring{x}}$, which is well known. Hence from now on we assume $\lambda\neq 0$. By the definition we have

$$\mathscr{E}^R_{\mathring{x}^*}=\mathscr{H}^n_{T^*X}(\pi^{-1}_{X|X\times X}\mathscr{O}^{(0,n)}_{X\times X})_{a(\mathring{x}^*)}, \tag{2.1}$$

where $\mathscr{O}^{(0,n)}_{X,X}=\mathscr{O}_{X\times X}\otimes p_2^{-1}\Omega^n_X$, $a(\mathring{x}^*)=(0; -\lambda, 0, \cdots, 0)$. For $c>0$, $\varepsilon>0$ we set

$$U_c=\{(x, x') \in X\times X; |x|<c, |x'|<c\},$$
$$Z_{c,\varepsilon}=\{(x, x') \in U_c; \mathrm{Re}(\lambda(x_1-x_1'))\geqq\varepsilon|\mathrm{Im}(\lambda(x_1-x_1'))|,$$
$$|x_1-x_1'|\geqq\varepsilon|x_j-x_j'|, j=2, \cdots, n\}.$$

Then it follows from Proposition 1.2.3 in [10], Chapter I that the right-hand side of (2.1) becomes the inductive limit

$$\varinjlim H^n_{Z_{c,\varepsilon}}(U_c; \mathscr{O}^{(0,n)}_{X\times X}) \qquad \text{as} \quad c, \varepsilon \longrightarrow 0.$$

Let us fix c, $\varepsilon>0$ and calculate the cohomology. The open set $U_c-Z_{c,\varepsilon}$ is covered by holomorphically convex sets $V^{(\nu)}$ $(\nu=1, \cdots, n)$ defined by

$$V^{(1)}=V^{(1)}_{c,\varepsilon}=\{(x, x') \in U_c; \mathrm{Re}(\lambda(x_1-x_1'))<\varepsilon|\mathrm{Im}(\lambda(x_1-x_1'))|\},$$
$$V^{(\nu)}=V^{(\nu)}_{c,\varepsilon}=\{(x, x') \in U_c; |x_1-x_1'|<\varepsilon|x_\nu-x_\nu'|\}, \qquad \nu=2, \cdots, n.$$

Set

$$V=V_{c,\varepsilon}=\bigcup_{\nu=1}^n V^{(\nu)}, \qquad \hat{V}^{(\nu)}=\bigcap_{\mu\neq\nu} V^{(\mu)}. \tag{2.2}$$

We have the following exact sequence:

$$\bigoplus_{\nu=1}^n \Gamma(\hat{V}^{(\nu)}; \mathscr{O}^{(0,n)}_{X\times X})\longrightarrow\Gamma(V; \mathscr{O}^{(0,n)}_{X\times X})\longrightarrow H^n_{Z_{c,\varepsilon}}(U_c: \mathscr{O}^{(0,n)}_{X\times X})\longrightarrow 0.$$

Hence any $P=K(x, x')dx' \in \mathscr{E}^{R}_{\mathring{x}^*}$ can be represented as an equivalence class of a holomorphic form $\psi(x, x')dx' \in \Gamma(V; \mathscr{O}^{(0,n)}_{X\times X})$ for some $c, \varepsilon>0$. We may write as follows:

$$P=K(x, x')dx'=[\psi(x, x')dx']. \tag{2.3}$$

We call $\psi(x, x')dx'$ (or $\psi(x, x')$) a defining function of the pseudodifferential operator $P \in \mathscr{E}^{R}_{\mathring{x}^*}$.

Example 2.1. Let us define a holomorphic function of one variable τ with parameter μ by

$$\Phi_\mu(\tau)=\frac{1}{2\pi\sqrt{-1}}\frac{\Gamma(1+\mu)}{(-\tau)^{1+\mu}} \qquad (\mu\neq-1, -2, \cdots),$$

$$\Phi_{-m}(\tau)=\frac{1}{2\pi\sqrt{-1}(m-1)!}\tau^{m-1}\Big\{\log\tau-\Big(\sum_{\nu=1}^{m-1}\frac{1}{\nu}-\gamma\Big)\Big\}, \quad m=1, 2, \cdots.$$

Here γ is the Euler constant. Let $\alpha=(\alpha_1, \cdots, \alpha_n)$ be a multi-index ($\alpha_2, \cdots, \alpha_n \in Z_+=\{0, 1, 2, \cdots\}$). Set

$$\Phi_\alpha(x-x')dx'=\Phi_{\alpha_1}(x_1-x_n')\cdot\cdots\cdot\Phi_{\alpha_n}(x_n-x_n')dx_1'\cdot\cdots\cdot dx_n'.$$

Then $\Phi_\alpha(x-x')dx'$ defines an operator in $\mathscr{E}^{R}_{\mathring{x}^*}$; we shall write it in the form $D_x^\alpha=D_{x_1}^{\alpha_1}\cdot\cdots\cdot D_{x_n}^{\alpha_n}$ ($D_{x_j}=\partial/\partial x_j$, $j=1, 2, \cdots, n$). If $\alpha_1 \in Z$ (resp. $\alpha_1 \in Z_+$) then the operator belongs to $\mathscr{E}_{\mathring{x}^*}$ (resp. $\mathscr{D}_{\mathring{x}}$).

§ 3. Radon transformations

Let us define W, Y by

$$W=\{(x, \xi, p);\, x \in X,\, \xi \in C^n-\{0\},\, p \in C\},$$
$$Y=\{(x, \xi, p) \in W;\, p=0\}.$$

Let $P=K(x, x')dx'$ be an operator $\in \mathscr{E}^{R}_{\mathring{x}^*}$. By the definition $K(x, x')$ is a section of $\mathscr{C}^{R}_{X|X\times X}$ over an open (conic) neighborhood Ω of $\mathring{x}^*=(\mathring{x}, \mathring{\xi})=(0; \lambda, 0, \cdots, 0)(\lambda\neq 0)$ identified with $(\mathring{x}, \mathring{x}, \mathring{\xi}, -\mathring{\xi}) \in T^*_X(X\times X)$. We set

$$f(x, \xi, p)=\frac{(n-1)!}{(-2\pi\sqrt{-1})^n}\int\frac{K(x, x')}{(p-\langle x-x', \xi\rangle)^n}dx'. \tag{3.1}$$

It follows from the theory of integration for holomorphic microfunctions (cf. [10], Chap. II) that $f(x, \xi, p)$ is a section of $\mathscr{C}^{R}_{Y|W}$ over a neighborhood of $\{(x, \xi, p; tdp) \in T^*_Y W;\, p=0,\, t>0,\, (x, \xi) \in \Omega\}$. It is clear that $f(x, \xi, p)$ is homogeneous of order $(-n)$ with respect to (x, ξ). Conversely, if such a holomorphic microfunction $f(x, \xi, p)$ is given, then a section $K(x, x')$ of

$\mathscr{C}^R_{X|X\times X}$ can be defined by the theory of Radon transformations (cf. [8]). The holomorphic microfunction $K(x, x')$ is formally defined by

$$K(x, x')=\int f(x, \xi, \langle x-x', \xi\rangle)\omega(\xi), \tag{3.2}$$

where $\omega(\xi)=\sum_{j=1}^n(-1)^{j-1}\xi_j d\xi_1\wedge\cdots\wedge\widehat{d\xi_j}\wedge\cdots\wedge d\xi_n$. The correspondence $f\mapsto K$ is reciprocal to (3.1). Since Y is of codimension 1 in W, $\mathscr{C}^R_{Y|W}$ is easily represented; we have the following:

Theorem 3.1 ([8], Theorem 3.2.3, Definition 3.2.4). *Let $\mathscr{T}$ and $\mathscr{A}$ be two sheaves on T^*X defined as follows: For each $x_0^*=(x_0, \xi_0)\in T^*X$,*

$$\mathscr{T}_{x_0^*}=\{f(x, \xi, p)\omega(\xi); f \text{ is holomorphic on } \{(x, \xi, p)\in W;$$
$$|\xi-\xi_0|<\varepsilon,\ |p|<\varepsilon,\ -\mathrm{Re}\,p>0,\ |x-x_0|<\varepsilon\} \text{ for some } \varepsilon>0$$
$$\text{and is homogeneous of order } (-n) \text{ with respct to } (\xi, p)\},$$
$$\mathscr{A}_{x_0^*}=\{f(x, \xi, p)\omega(\xi)\in\mathscr{T}_{x_0^*}; f \text{ is holomorphic at } (x, \xi, p)=(x_0, \xi_0, 0)\}.$$

Then the correspondences (3.1) *and* (3.2) *yield the following linear isomorphism*:

$$\mathscr{E}^R_X\simeq\mathscr{T}/\mathscr{A}.$$

Let us remark that neither $\mathscr{T}$ nor $\mathscr{A}$ is invariant under the change of variables. By the preceding theorem we may write

$$P=[f(x, \xi, p)\omega(\xi)],$$

where $f(x, \xi, p)\omega(\xi)\in\mathscr{T}_{\mathring{x}^*}$. We call $f(x, \xi, p)\omega(\xi)$ or $f(x, \xi, p)$ the Radon transformation of $K(x, x')$ (or P). If $P=K(x, x')dx'$ is represented by a defining function $\psi(x, x')\in\Gamma(V_{c,\varepsilon}; \mathscr{O}_{X\times X})$ (for some c, $\varepsilon>0$) as (2.3), then the correspondence (3.1) is written in the form

$$f(x, \xi, p)=\frac{(n-1)!}{(2\pi\sqrt{-1})^n}\int_{\beta_0}^{\beta_1}dy_1\oint dy_2\cdots\oint dy_n\frac{\psi(x, x-y)}{(p-\langle y, \xi\rangle)^n} \tag{3.3}$$

as a holomorphic function. Here β_0, β_1 are sufficiently small complex numbers so that

$$0<\mathrm{Re}\,\lambda\beta_0<\varepsilon\,\mathrm{Im}\,\lambda\beta_0,$$
$$0<\mathrm{Re}\,\lambda\beta_1<-\varepsilon\,\mathrm{Im}\,\lambda\beta_1,$$

$\oint dy_j$ means the contour integral along the cycle $\{y_j; |y_j|=\varepsilon^{-1}|y_1|+\delta\}$ with $0<\delta\ll 1, j=2, \cdots, n$ (cf. [6], Chap. III).

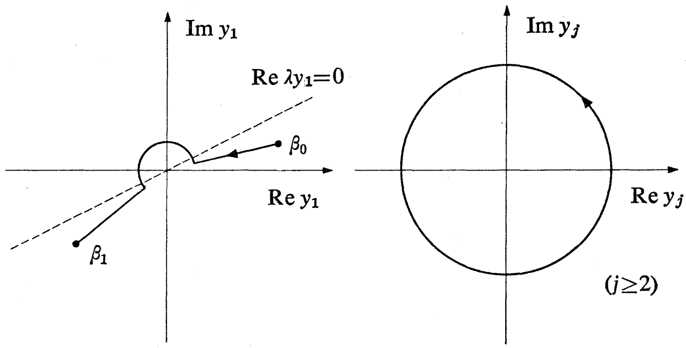

Fig. 3.1.

§ 4. Symbols

We shall define the symbol of a pseudodifferential operator $P\in\mathscr{E}^{R}_{\mathring{x}^*}$. Let $f(x,\xi,p)$ be the Radon transformation of P. We may assume that P is holomorphic on

$$\{(x,\xi,p)\in W;\,|x-\mathring{x}|<c,\,|\xi-\mathring{\xi}|<c,\,|p|<c,\,\mathrm{Re}\,p<0\}$$

for some $c>0$. It follows from the homogeneity of f that f becomes holomorphic on

$$\{(x,\xi,p)\in W;\,|x-\mathring{x}|<c,\,|\xi-\mathring{\xi}|<c,\,|p|<c,\,\mathrm{Re}\,p<\varepsilon|\mathrm{Im}\,p|\}$$

for some $\varepsilon>0$. Let s_0, s_1 be holomorphic functions of ξ homogeneous of order 1 with respect to ξ so that $|s_0(\xi)|<c$, $|s_1(\xi)|<c$, $0<\mathrm{Re}\,s_0(\xi)<-\varepsilon\,\mathrm{Im}\,s_0(\xi)$, $0<\mathrm{Re}\,s_1(\xi)<\varepsilon\,\mathrm{Im}\,s_1(\xi)$ for $|\xi-\mathring{\xi}|<c$. Let $\Sigma=\Sigma(\xi)$ be a path starting from s_0, ending at s_1 and around the origin clockwise.

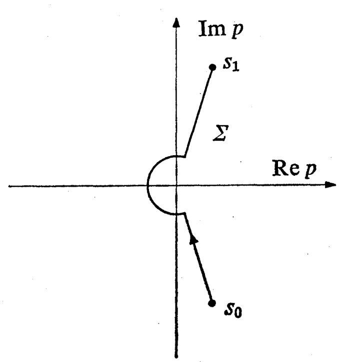

Fig. 4.1

We set

$$(4.1) \qquad P(x, \xi)=(2\pi\sqrt{-1})^{n-1}\int_{\Sigma} f(x, \xi, p)e^{-p}\,dp.$$

Then we have

Proposition 4.1 (Cf. [7], § 3.3; [1], Theorem 2.1.1). a) *$P(x, \xi)$ is holomorphic on a conic neighborhood Ω of $\mathring{x}^*$ and satisfies the following estimate: For each $\Omega' \Subset \Omega$ and for every $h>0$ there is a constant $C>0$ such that*

$$|P(x, \xi)| \leq C \exp(h|\xi|)$$

for any $(x, \xi) \in \Omega'$. Here $\Omega' \Subset \Omega$ means Ω' is a compactly generated conic subset in Ω.

b) *If $f \in \mathscr{A}_{\mathring{x}^*}$, then for each $\Omega' \Subset \Omega$ there are constants $h>0$, $C>0$ such that*

$$|P(x, \xi)| \leq C \exp(-h|\xi|)$$

for any $(x, \xi) \in \Omega'$.

Now we introduce a symbol class and its subclass.

Definition 4.2. Let Ω be a conic open set in T^*X. Then we denote by $S(\Omega)$ the set of all $P(x, \xi) \in \mathcal{O}_{T^*X}(\Omega)$ such that for every $\Omega' \Subset \Omega$ and for each $h>0$

$$|P(x, \xi)| \leq C_h \exp(h|\xi|), \qquad (x, \xi) \in \Omega'$$

is valid for some constant C_h. The elements of $S(\Omega)$ are called symbols defined in Ω or temperate (holomorphic) functions. If for each $\Omega' \Subset \Omega$

$$|P(x, \xi)| \leq C \exp(-h|\xi|), \qquad (x, \xi) \in \Omega'$$

holds for some positive constants h, C, then the symbol $P(x, \xi) \in S(\Omega)$ is said to be a null-symbol or rapidly decreasing in Ω. We denote by $R(\Omega)$ the set of all null-symbols defined in Ω.

It is clear that $R(\Omega)$ is an ideal of the commutative ring $S(\Omega)$. Combining Proposition 4.1 with Definition 4.2 yields at once the following

Theorem 4.3. *The correspondence*

$$\mathscr{T}_{\mathring{x}^*} \ni f(x, \xi, p)\omega(\xi) \longmapsto P(x, \xi)$$

defined by (4.1) *induces the following linear homomorphism*

$$\sigma\colon \mathscr{E}^R_{\mathring{x}^*} \longrightarrow \varinjlim S(\Omega)/R(\Omega),$$

where Ω runs on the family of conic neighborhoods of $\mathring{x}^*$. The homomorphism σ is independent of the choices of s_0, s_1.

Definition 4.4 (Cf. [7], §3.3). The mapping σ defined in Theorem 4.3 is called the symbol mapping. The image (or its representative) $\sigma(P)=P(x, \xi)$ of an operator $P \in \mathscr{E}^R_{\mathring{x}^*}$ is said to be the symbol of P.

Remark. We use the same letters to represent the operators and their symbols. As a rule the variables (x, ξ) are designated when the letters denote the symbols.

We shall show that the symbol mapping σ is a linear isomorphism. Let Ω be a conic neighborhood of $\mathring{x}^*=(\mathring{x}, \mathring{\xi})=(0; \lambda, 0, \cdots, 0)$ $(\lambda \neq 0)$, $P(x, \xi) \in S(\Omega)$. Let r be a sufficiently large number >0. We set

$$g(x, \xi, p)=\frac{1}{(2\pi\sqrt{-1})^n}\int_r^\infty P(x, \tau\xi)e^{\tau p}\tau^{n-1}d\tau \tag{4.2}$$

for $\xi_1=\lambda$ and extend $g(x, \xi, p)$ as a homogeneous function of order $(-n)$ with respect to (ξ, p). Then we have

Theorem 4.5. *There is a linear homomorphism*

$$\varpi\colon \varinjlim_{\Omega \ni \mathring{x}^*} S(\Omega)/R(\Omega) \longrightarrow \mathscr{T}_{\mathring{x}^*}/\mathscr{A}_{\mathring{x}^*} \simeq \mathscr{E}^R_{\mathring{x}^*}$$

induced from $P(x, \xi) \mapsto g(x, \xi, p)\omega(\xi)$ defined by (4.2). *The homomorphism ϖ is independent of the choice of r. Moreover $\varpi \circ \sigma = \mathrm{id}$ and $\sigma \circ \varpi = \mathrm{id}$ hold.*

Proof. It is clear that $g(x, \xi, p)\omega(\xi)$ defined by (4.2) belongs to $\mathscr{T}_{\mathring{x}^*}$ for each $P(x, \xi) \in S(\Omega)$, Ω being a conic neighborhood of $\mathring{x}^*$. Let r' be a sufficiently large positive number. Then the integral

$$\int_{r'}^r P(x, \tau\xi)e^{\tau p}\tau^{n-1}d\tau$$

is holomorphic at $p=0$. Hence ϖ does not depend on r modulo $\mathscr{A}_{\mathring{x}^*}$. We shall show $\varpi \circ \sigma = \mathrm{id}$. Let us assume $P(x, \xi)=\sigma([f(x, \xi, p)\omega(\xi)])$, i.e., (4.1) holds. Because of the homogeneity, it is sufficient to prove $\varpi \circ \sigma|_{\xi_1=\lambda} = \mathrm{id}$. Let $g(x, \xi, p)\omega(\xi)$ be the image of $P(x, \xi)$ by ϖ. Then we have

$$\begin{aligned} g(x, \xi, p)\omega(\xi) &= \frac{1}{2\pi\sqrt{-1}}\int_r^\infty \int_{\Sigma(\tau\xi)} f(x, \tau\xi, q)e^{-q}dq\, e^{\tau p}\tau^{n-1}d\tau\omega(\xi) \\ &= \frac{1}{2\pi\sqrt{-1}}\int_r^\infty \int_{\Sigma(\xi)} f(x, \xi, q)e^{-\tau q}dq\, e^{\tau p}d\tau\omega(\xi) \end{aligned}$$

$$=\frac{1}{2\pi\sqrt{-1}}\int_{\Sigma(\xi)} f(x, \xi, q)\frac{e^{(p-q)r}}{q-p}dq\,\omega(\xi)$$

$$\equiv f(x, \xi, p)\omega(\xi) \quad \text{modulo } \mathscr{A}_{\hat{x}^*}.$$

Thus we have $\varpi\circ\sigma=\mathrm{id}$. Next we prove $\sigma\circ\varpi=\mathrm{id}$. Let us suppose that the image of $P(x, \xi)\in S(\Omega)$ by ϖ is $g(x, \xi, p)\omega(\xi)$. If $\xi_1\neq 0$ we have by the definition

$$
\begin{aligned}
(4.3)\qquad g(x, \xi, p) &= g(x, \lambda\xi/\xi_1, \lambda p/\xi_1)\cdot(\lambda/\xi_1)^n \\
&= \frac{1}{(2\pi\sqrt{-1})^n}\int_r^\infty P(x, \tau\lambda\xi/\xi_1)\exp(\tau\lambda p/\xi_1)\tau^{n-1}d\tau\cdot(\lambda/\xi_1)^n.
\end{aligned}
$$

The integral converges locally uniformly and defines a holomorphic function if $\mathrm{Re}(\lambda p/\xi_1)<0$. Let us denote by $\gamma_\pm$ the paths starting from r, tending to ∞ along $\mathrm{Im}\,\tau=\pm\varepsilon'\,\mathrm{Re}\,\tau$ $(0<\varepsilon'\ll 1)$.

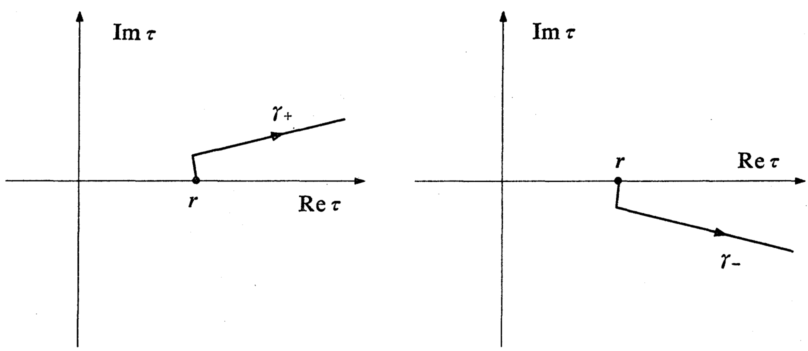

Fig. 4.2

If the path of integration in (4.3) is replaced by $\gamma_\pm$ then $g(x, \xi, p)$ is holomorphically continued to $\mathrm{Re}\,p<\varepsilon''|\mathrm{Im}\,p|$ $(0<\varepsilon''\ll 1)$. We set $Q(x, \xi)=\sigma([g(x, \xi, p)\omega(\xi)])$, that is,

$$(4.4)\qquad Q(x, \xi)=(2\pi\sqrt{-1})^{n-1}\int_\Sigma g(x, \xi, p)e^{-p}dp,$$

where Σ is a path like Fig. 4.1 (s_0, s_1 may be replaced). Let us decompose $\Sigma=\Sigma_+ + \Sigma_-$ where $\Sigma_\pm=\Sigma\cap\{\mathrm{Im}\,p\gtrless 0\}$. Then $\mathrm{Re}(\tau p)<0$ for any $\tau\in\gamma_\pm$, $p\in\Sigma_\pm$. Now combining (4.3) and (4.4) yields

$$
\begin{aligned}
(4.5)\qquad Q(x, \xi) &= \frac{1}{2\pi\sqrt{-1}}\int_{\Sigma_+}dp\int_{\gamma_+}d\tau P(x, \tau\lambda\xi/\xi_1) \\
&\quad\times\exp((\tau\lambda/\xi_1-1)p)\tau^{n-1}(\lambda/\xi_1)^n
\end{aligned}
$$

$$+\frac{1}{2\pi\sqrt{-1}}\int_{\Sigma_-}dp\int_{\gamma_-}d\tau P(x,\tau\lambda\xi/\xi_1)$$
$$\times\exp((\tau\lambda/\xi_1-1)p)\tau^{n-1}(\lambda/\xi_1)^n.$$

We denote by I_+ (resp. I_-) the first term (resp. the second term) of the right-hand side of (4.5). Then we write

$$\begin{aligned}I_+&=\frac{1}{2\pi\sqrt{-1}}\int_{\gamma+}d\tau P(x,\tau\lambda\xi/\xi_1)\frac{\lambda^n\xi_1^{1-n}\tau^{n-1}}{\tau\lambda-\xi_1}\exp((\tau\lambda/\xi_1-1)s_1)\\&\quad-\frac{1}{2\pi\sqrt{-1}}\int_{\gamma+}d\tau P(x,\tau\lambda\xi/\xi_1)\frac{\lambda^n\xi_1^{1-n}\tau^{n-1}}{\tau\lambda-\xi_1}\exp((\tau\lambda/\xi_1-1)p_0)\\&=I_+^{(1)}-I_-^{(2)},\end{aligned}$$

where $p_0=\Sigma\cap\{\operatorname{Im}p=0\}$. Similarly, we have

$$\begin{aligned}I_-&=\frac{1}{2\pi\sqrt{-1}}\int_{\gamma-}d\tau P(x,\tau\lambda\xi/\xi_1)\frac{\lambda^n\xi_1^{1-n}\tau^{n-1}}{\tau\lambda-\xi_1}\exp((\tau\lambda/\xi_1-1)p_0)\\&\quad-\frac{1}{2\pi\sqrt{-1}}\int_{\gamma-}d\tau P(x,\tau\lambda\xi/\xi_1)\frac{\lambda^n\xi_1^{1-n}\tau^{n-1}}{\tau\lambda-\xi_1}\exp((\tau\lambda/\xi_1-1)s_0)\\&=I_-^{(1)}-I_-^{(2)}.\end{aligned}$$

It follows from the choice of s_0, s_1 that $I_+^{(1)}$ and $I_-^{(2)}$ are rapidly decreasing as $|\xi|\to\infty$. Hence $Q(x,\xi)\equiv I_-^{(1)}-I_+^{(2)}$ modulo rapidly decreasing functions. We shall calculate the right-hand side:

$$I_-^{(1)}-I_+^{(2)}=\frac{1}{2\pi\sqrt{-1}}\int_{-\gamma_++\gamma_-}d\tau P(x,\tau\lambda\xi/\xi_1)\frac{(\tau\lambda/\xi_1)^{n-1}}{\tau-\xi_1/\lambda}\exp((\tau\lambda/\xi_1-1)p_0).$$

Since $(\tau-\xi_1/\lambda)^{-1}\exp((\tau\lambda/\xi_1-1)p_0)$ is the Cauchy kernel with damping factor, the right-hand side coincides with $P(x,\xi)$ in a conic neighborhood of $\mathring{x}^*$. Therefore we have $\sigma\circ\varpi=\text{id}$.

Definition 4.6. The image $P=\varpi(P(x,\xi))\in\mathscr{E}^R_{\mathring{x}^*}$ of a symbol $P(x,\xi)\in S(\Omega)$ (or its equivalence class) by ϖ is called the normal product of $P(x,\xi)$ and denoted by

$$P=:P(x,\xi):.$$

Remarks. a) The notation is an analogue of that in quantum field theory, for the commutation relations of differential operators are the same as those of free bosons.

b) $:P(x,\xi):$ is denoted by $P(x,D_x)$ in [1], [7].

c) The normal products of x_j, ξ_j, $x_j\xi_j$ $(j=1,\cdots,n)$ are x_j, $D_{x_j}=$

$\partial/\partial x_j$, $x_jD_{x_j}$ respectively. Moreover, if $P(x, \xi)=\Sigma a_\alpha(x)\xi^\alpha$ is a polynomial in ξ with analytic coefficients, then $:P(x, \xi):=\Sigma a_\alpha(x)D_x^\alpha$.

If the Radon transformation $f(x, \xi, p)$ of $P \in \mathscr{E}_{\mathring{x}^*}^R$ is written as (3.3), then the symbol is

$$\begin{aligned}P(x, \xi)&=(2\pi\sqrt{-1})^{n-1}\int_\Sigma f(x, \xi, p)e^{-p}dp\\&=\frac{(n-1)!}{2\pi\sqrt{-1}}\int_{\beta_0}^{\beta_1} dy_1\oint dy_2\cdots\oint dy_n\int_\Sigma dp\frac{\psi(x, x-y)}{(p-\langle y, \xi\rangle)^n}e^{-p}\\&\equiv(-1)^n\int_{\beta_0}^{\beta_1} dy_1\oint dy_2\cdots\oint dy_n\psi(x, x-y)e^{-\langle y, \xi\rangle}.\end{aligned}$$

Here the last identity makes sense modulo rapidly decreasing functions. Hence we have

Proposition 4.7. *If $P \in \mathscr{E}_{\mathring{x}^*}^R$ is represented by a defining function $\psi(x, x') \in \mathcal{O}(V_{c,\varepsilon})$ (for some c, $\varepsilon>0$;* cf. § 2), *then the symbol $P(x, \xi)$ of P is calculated by*

$$P(x, \xi)=(-1)^n\int_{\beta_0}^{\beta_1} dy_1\oint dy_2\cdots\oint dy_n\psi(x, x-y)e^{-\langle y, \xi\rangle}. \tag{4.6}$$

The symbol obtained in the preceding proposition is an entire function of ξ of exponential type. Of course the symbol is temperate in a conic neighborhood of $\mathring{x}^*$.

Now we introduce the orders of symbols.

Definition 4.8. Let m be a real number. A symbol $P(x, \xi)$ defined in a conic open set $\Omega\subset T^*X$ is said to be of order at most m (resp. $m-0$) in Ω if for every $\Omega'\Subset\Omega$, $P(x, \xi)|\xi|^{-m}$ is bounded in Ω' (resp. for every, $\Omega'\Subset\Omega$, $P(x, \xi)|\xi|^{-m}\to 0$ as $|\xi|\to\infty$, $(x, \xi) \in \Omega'$).

If there is no m satisfying the above condition for a symbol $P(x, \xi)$, then the symbol is called of infinite order in Ω. For the symbols of infinite order, the orders of their logarithms are important.

Definition 4.9. Let ρ be a real number such that $0\leq\rho<1$ (resp. $0<\rho\leq 1$). A symbol $P(x, \xi)$ defined in a conic open set Ω is said to be of growth at most ρ (resp. $\rho-0$) if for any $\Omega'\Subset\Omega$, there exist positive constants h, C (resp. for any $\Omega'\Subset\Omega$, $h>0$ there is a positive constant C) such that

$$|P(x, \xi)|\leq C\exp(h|\xi|^\rho)$$

is valid for each $(x, \xi) \in \Omega'$.

Remarks. a) By the definition, any symbol is always of growth at most 1-0 (cf. Proposition 4.1).

b) In [1], symbols of growth at most ρ (resp. $\rho-0$) are called of growth order at most (ρ) (resp. $\{\rho\}$).

§ 5. Formal symbols

We shall generalize a definition introduced in a particular case by Boutet de Monvel [4] (cf. [1], [2]).

Let Ω denote a conic open set in T^*X.

Definition 5.1. A formal power series $P(t; x, \xi)=\sum_{j=0}^{\infty} t^j P_j(x, \xi)$ in t with coefficients in $S(\Omega)$ is called a formal symbol (defined in Ω) if for every $\Omega' \Subset \Omega$ there are constants r, A satisfying the following conditions:

a) $0<r$, $0<A<1$.

b) For each $h>0$ there is a positive constant C such that

$$|P_j(x, \xi)| \leq CA^j \exp(h|\xi|) \tag{5.1}$$

is valid for any $(x, \xi) \in \Omega' \cap \{|\xi| \geq (j+1)r\}$, $j=0, 1, 2, \cdots$. We denote by $\hat{S}(\Omega)$ the set of all formal symbols defined in Ω.

We introduce the sum and the product in $\hat{S}(\Omega)$ as formal power series in t. Then $\hat{S}(\Omega)$ becomes a commutative ring. The space of symbols $S(\Omega)$ introduced in Section 4 is identified with a subring of $\hat{S}(\Omega)$:

$$S(\Omega) \simeq \hat{S}(\Omega)|_{t=0} = \{P=\sum t^j P_j;\ P_j \equiv 0 \text{ for all } j>0\}. \tag{5.2}$$

Sometimes $\sum_{j=0}^{\infty} t^j P_j(x, \xi)$ is abbreviated to $\sum_{j=0}^{\infty} P_j(x, \xi)$.

Definition 5.2. A formal symbol $P(t; x, \xi)=\sum_{j=0}^{\infty} t^j P_j(x, \xi)$ defined in Ω is said to be equivalent to zero (in Ω) and written $P(t; x, \xi) \sim 0$ if for every $\Omega' \Subset \Omega$ there are constants r, A such that

a) $0<r$, $0<A<1$.

b) For every $h>0$ there exists a constant $C>0$ so that

$$\left|\sum_{j=0}^{m-1} P_j(x, \xi)\right| \leq CA^m \exp(h|\xi|) \tag{5.3}$$

holds for all $(x, \xi) \in \Omega' \cap \{|\xi| \geq mr\}$, $m=1, 2, \cdots$. We denote by $\hat{R}(\Omega)$ the set of all formal symbols equivalent to zero in Ω. Two formal symbols $P, Q \in \hat{S}(\Omega)$ are said to be equivalent if $P-Q \in \hat{R}(\Omega)$; then we write $P \sim Q$.

Proposition 5.3 (Cf. [4]). *Under identification* (5.2), $S(\Omega) \cap \hat{R}(\Omega) = R(\Omega)$ *holds.*

Proof. Suppose $P(x, \xi) \in S(\Omega) \cap \hat{R}(\Omega)$. Then for each $\Omega' \Subset \Omega$, there exist constants r, A satisfying the following conditions:

a) $0<r$, $0<A<1$.

b) For each $h>0$ there is a positive constant C such that

$$|P(x, \xi)| \leq CA^m \exp(h|\xi|) \tag{5.4}$$

holds for every $(x, \xi) \in \Omega' \cap \{|\xi| \geq mr\}$, $m=1, 2, \cdots$.

For each ξ, we take m as the integral part of $1+|\xi|/r$. Then from (5.4) we have

$$|P(x, \xi)| \leq CA^{|\xi|/r} \exp(h|\xi|).$$

Since $0<A<1$, we can choose $h>0$ as $h+r^{-1}\log A<0$. Hence $P(x, \xi) \in R(\Omega)$.

Conversely, let $P(x, \xi)$ be an element of $R(\Omega)$. Then by the definition there are constants C, $h>0$ for each $\Omega' \Subset \Omega$ so that

$$|P(x, \xi)| \leq C\exp(-h|\xi|), \qquad (x, \xi) \in \Omega'$$

holds. If $|\xi| \geq m$, then $\exp(-h|\xi|) \leq \exp(-hm) = B^m$. Here we set $B= e^{-h}$; hence $0<B<1$. Therefore we have

$$|P(x, \xi)| \leq CB^m$$

for $(x, \xi) \in \Omega' \cap \{|\xi| \geq m\}$. This completes the proof.

Proposition 5.4. *$\hat{R}(\Omega)$ is an ideal of $\hat{S}(\Omega)$.*

Proof. It is clear that $\hat{R}(\Omega)$ is an additive subgroup of $\hat{S}(\Omega)$. Let $P(t; x, \xi) = \sum t^j P_j(x, \xi) \in \hat{R}(\Omega)$, $Q(t; x, \xi) = \sum t^k Q_k(x, \xi) \in \hat{S}(\Omega)$. It suffices to show $P(t; x, \xi)Q(t; x, \xi) \in \hat{R}(\Omega)$. We may assume that for each $\Omega' \Subset \Omega$ there exist constants $r>0$, $0<A<1$ so that for every $h>0$ there is a positive constant C satisfying

$$\begin{cases} |P_j(x, \xi)| \leq CA^j \exp(h|\xi|), & |\xi| \geq (j+1)r, \\ |Q_k(x, \xi)| \leq CA^k \exp(h|\xi|), & |\xi| \geq (k+1)r, \\ \left|\sum_{j=0}^{m-1} P_j(x, \xi)\right| \leq CA^m \exp(h|\xi|), & |\xi| \geq mr \end{cases} \tag{5.5}$$

for $(x, \xi) \in \Omega'$, $j, k=0, 1, 2, \cdots$, $m=1, 2, \cdots$. We set

$$W(t; x, \xi) = P(t; x, \xi)Q(t; x, \xi) = \sum_{l=0}^{\infty} t^l W_l(x, \xi).$$

That is,

$$(5.6)\qquad W_l(x,\xi)=\sum_{j+k=l} P_j(x,\xi)Q_k(x,\xi).$$

We shall estimate $|\sum_{l=0}^{m-1} W_l(x,\xi)|$ $(m=1,2,\cdots)$. Combining (5.5) and (5.6) yields

$$(5.7)\qquad \begin{aligned}\left|\sum_{j=0}^{m-1} W_l(x,\xi)\right| &\le \left|\sum_{j=0}^{m-1} P_j(x,\xi)\sum_{k=0}^{m-1} Q_k(x,\xi)\right| \\ &\quad + \left|\sum_{l=m}^{2m-2}\sum_{\substack{j+k=l\\ j,k\le m-1}} P_j(x,\xi)Q_k(x,\xi)\right| \\ &\le \frac{C^2}{1-A}A^m \exp(2h|\xi|) \\ &\quad + C^2A^m\sum_{l=0}^{m-2}(l+m+1)A^l\exp(2h|\xi|)\end{aligned}$$

for $(x,\xi)\in\Omega'\cap\{|\xi|\ge mr\}$, $m=1,2,\cdots$. Since $0<A<1$, we can choose B so that $A<B<1$. Then it follows from (5.7) that there is C' for every $h>0$ such that

$$\left|\sum_{l=0}^{m-1} W_l(x,\xi)\right| \le C'B^m \exp(2h|\xi|)$$

is valid for $(x,\xi)\in\Omega'\cap\{|\xi|\ge mr\}$, $m=1,2,\cdots$. Hence $W(t;x,\xi)\in\hat{R}(\Omega)$.

We consider the commutative ring $\hat{S}(\Omega)/\hat{R}(\Omega)$. By Proposition 5.3, the inclusion $S(\Omega)\subsetneq\hat{S}(\Omega)$ (see (5.2)) induces the injective homomorphism $\iota_{01}: S(\Omega)/R(\Omega)\to\hat{S}(\Omega)/\hat{R}(\Omega)$. Conversely we have the following

Theorem 5.5. *Let Ω be a conic neighborhood of $\mathring{x}^*=(\mathring{x};\lambda,0,\cdots,0)$ in T^*X. For every formal symbol $P(t;x,\xi)=\sum_{j=0}^{\infty} t^jP_j(x,\xi)$ defined in Ω, there are a conic neighborhood $\Omega_1\subset\Omega$ of $\mathring{x}^*$ and a symbol $P(x,\xi)$ defined in Ω_1 so that $P(x,\xi)\sim P(t;x,\xi)$ in Ω_1.*

Proof. We may assume that $P_j(x,\xi)$ satisfies the estimate in Definition 5.1. Let us define $f_k(x,\xi,p)$ $(k=0,1,2,\cdots)$ as follows: If $\xi_1=\lambda$, we set

$$(5.8)\qquad \begin{aligned} f_k(x,\xi,p) &= \frac{1}{(2\pi\sqrt{-1})^n}\int_{(k+1)r}^{\infty} P_k(x,\tau\xi)e^{\tau p}\tau^{n-1}d\tau \\ &= \frac{(k+1)^n}{(2\pi\sqrt{-1})^n}\int_r^{\infty} P_k(x,(k+1)\tau\xi)e^{(k+1)\tau p}\tau^{n-1}d\tau.\end{aligned}$$

We extend $f_k(x,\xi,p)$ as a homogeneous function of degree $(-n)$ with respect to (ξ,p). Then we have $f_k(x,\xi,p)\omega(\xi)\in\mathscr{T}_{\mathring{x}^*}$. By Theorem 4.5, we have $:P_k(x,\xi):=[f_k(x,\xi,p)\omega(\xi)]$. Let us define $\tilde{P}_k(x,\xi)$ by

$$(5.9)\qquad \tilde{P}_k(x, \xi)=(2\pi\sqrt{-1})^{n-1}\int_{\Sigma} f_k(x, \xi, p)e^{-p}dp, \qquad k=0, 1, \cdots.$$

Here Σ is a path defined in Section 4. Then $\tilde{P}_k(x, \xi)$ is a symbol defined in a neighborhood Ω_1 of $\mathring{x}^*$. By virtue of Theorem 4.5, $\tilde{P}_k(x, \xi)$ is equivalent to $P_k(x, \xi)$ in Ω_1. Since we have assumed (5.1), we obtain for $\xi_1=\lambda$, $h|\xi|+\mathrm{Re}\, p<0$

$$|f_k(x, \xi, p)|\leq \frac{(k+1)^n}{(2\pi)^n}\int_r^{\infty} CA^k \exp\{(k+1)\tau(h|\xi|+\mathrm{Re}\, p)\}\tau^{n-1}d\tau$$
$$\leq C'A^k|r(h|\xi|+\mathrm{Re}\, p)|^{-n}\exp\{(k+1)r(h|\xi|+\mathrm{Re}\, p)\}.$$

Here C' is a constant depending on h. Hence Σf_k converges locally uniformly in $\mathrm{Re}\, p<0$, for h is an arbitrary positive constant. Similar estimate is still valid when the path of integration in the last term in (5.8) is replaced by $\gamma_{\pm}$ (introduced in the proof of Theorem 4.5). Therefore Σf_k converges locally uniformly in $\mathrm{Re}\, p<\varepsilon''|\mathrm{Im}\, p|$ $(0<\varepsilon''\ll 1)$. Since $\tilde{P}_k(x, \xi)$ is defined by (5.9), it satisfies the following estimate: For each $\Omega_1'\Subset\Omega_1$, there is a constant $0<A<1$ such that for every $h>0$,

$$|\tilde{P}_k(x, \xi)|\leq C''A^k\exp(h|\xi|), \qquad (x, \xi)\in\Omega_1'$$

is valid for some $C''>0$. Hence $\sum_{k=0}^{\infty}\tilde{P}_k(x, \xi)$ converges locally uniformly in Ω_1. We now set

$$P(x, \xi)=\sum_{k=0}^{\infty}\tilde{P}_k(x, \xi),$$
$$\tilde{P}(t; x, \xi)=\sum_{k=0}^{\infty}t^k\tilde{P}_k(x, \xi).$$

It is clear that $P(x, \xi)\in S(\Omega_1)$, $\tilde{P}(t; x, \xi)\in\hat{S}(\Omega_1)$ and that $P(x, \xi)\sim\tilde{P}(t; x, \xi)$. We shall prove that $P(x, \xi)\sim P(t; x, \xi)$. It suffices to show $\tilde{P}(t; x, \xi)\sim P(t; x, \xi)$. We use the same notation as in the proof of Theorem 4.5. The right-hand side of (5.9) is calculated in the same way as (4.5), that is, we have

$$P_k(x, \xi)=I_+ + I_-.$$

Here we set

$$I_{\pm}=\frac{(k+1)^n}{2\pi\sqrt{-1}}\int_{\Sigma_{\pm}}dp\int_{\gamma_{\pm}}d\tau P_k(x, (k+1)\lambda\tau\xi/\xi_1)$$
$$\times\exp\{((k+1)\lambda\tau/\xi_1-1)p\}\tau^{n-1}(\lambda/\xi_1)^n.$$

We calculate $I_{\pm}$ by changing the order of the integrations.

$$I_+ = \frac{1}{2\pi\sqrt{-1}}\left(\frac{(k+1)\lambda}{\xi_1}\right)^n \int_{\gamma+} d\tau P_k(x, (k+1)\lambda\tau\xi/\xi_1) \times \frac{\exp\{((k+1)\lambda\tau/\xi_1 - 1)s_1\}}{(k+1)\lambda\tau/\xi_1 - 1}\tau^{n-1}$$
$$- \frac{1}{2\pi\sqrt{-1}}\left(\frac{(k+1)\lambda}{\xi_1}\right)^n \int_{\gamma+} d\tau P_k(x, (k+1)\lambda\tau\xi/\xi_1) \times \frac{\exp\{((k+1)\lambda\tau/\xi_1 - 1)p_0\}}{(k+1)\lambda\tau/\xi_1 - 1}\tau^{n-1}$$
$$= I_+^{(1)} - I_+^{(2)},$$

$$I_- = \frac{1}{2\pi\sqrt{-1}}\left(\frac{(k+1)\lambda}{\xi_1}\right)^n \int_{\gamma-} d\tau P_k(x, (k+1)\lambda\tau\xi/\xi_1) \times \frac{\exp\{((k+1)\lambda\tau/\xi_1 - 1)p_0\}}{(k+1)\lambda\tau/\xi_1 - 1}\tau^{n-1}$$
$$- \frac{1}{2\pi\sqrt{-1}}\left(\frac{(k+1)\lambda}{\xi_1}\right)^n \int_{\gamma-} d\tau P_k(x, (k+1)\lambda\tau\xi/\xi_1) \times \frac{\exp\{((k+1)\lambda\tau/\xi_1 - 1)s_0\}}{(k+1)\lambda\tau/\xi_1 - 1}\tau^{n-1}$$
$$= I_-^{(1)} - I_-^{(2)}.$$

We have $\tilde{P}_k(x, \xi) = I_+^{(1)} - I_-^{(2)} + I_-^{(1)} - I_+^{(2)}$. Since $\exp\{((k+1)\lambda\tau/\xi_1 - 1)p_0\}\cdot((k+1)\lambda\tau/\xi_1 - 1)^{-1}$ is the Cauchy kernel (up to constant multiple) with damping factor, $I_-^{(1)} - I_+^{(2)} = \int_{\gamma_+ - \gamma_-}$ coincides with $P_k(x, \xi)$ if $\xi_1/(k+1)\lambda$ is contained in the inside of $\gamma_+ - \gamma_-$. Therefore, if (x, ξ) is contained in a sufficiently small neighborhood of $\mathring{x}^*$ and if $|\xi| \geq (k+1)r''$ ($r \gg 1$), then $P_k(x, \xi) - \tilde{P}_k(x, \xi) = I_-^{(2)} - I_+^{(1)}$ holds. Let us recall that s_0, s_1 are homogeneous function of ξ of order 1, Then $|I_-^{(2)}|$ and $|I_+^{(1)}|$ are estimated (locally uniformly) by $C_0(k+1)^n A^k \exp(-h'|\xi|)$ for some C_0, $h' > 0$. Hence there are positive constants C, h such that

$$\left|\sum_{k=0}^{m-1} (P_k(x, \xi) - \tilde{P}_k(x, \xi))\right| \leq C \exp(-h|\xi|)$$

holds for $|\xi| \geq mr''$, $m = 1, 2, \cdots$. This completes the proof.

Let us remark that the above construction of $P(x, \xi)$ is independent of the choice of r modulo rapidly decreasing functions. Hence we have

Theorem 5.6. *The correspondence* $P(t; x, \xi) \mapsto P(x, \xi)$ *obtained in Theorem* 5.5 *induces the ring isomorphism*

$$\rho_{10}\colon \varinjlim_{\Omega\ni \mathring{x}^*} \hat{S}(\Omega)/\hat{R}(\Omega) \xrightarrow{\sim} \varinjlim_{\Omega\ni \mathring{x}^*} S(\Omega)/R(\Omega)$$

such that $\rho_{10}\circ \iota_{01}=\mathrm{id}$, $\iota_{01}\circ\rho_{10}=\mathrm{id}$. *Here* Ω *runs on the family of conic neighborhoods of* $\mathring{x}^*$ *in* T^*X.

Combining this with Theorem 4.5 yields the following

Theorem 5.7. *Set* $\varpi_1=\varpi\circ\rho_{10}$, *then*

$$\varpi_1\colon \varinjlim_{\Omega\ni \mathring{x}^*} \hat{S}(\Omega)/\hat{R}(\Omega) \longrightarrow \mathscr{E}^R_{\mathring{x}^*}$$

is a linear isomorphism.

Definition 5.8. The image of a formal symbol (or its equivalence class) $P(t; x, \xi)=\sum_{j=0}^{\infty} t^j P_j(x, \xi) \in \hat{S}(\Omega)$ by ϖ_1 is denoted by

$$:P(t; x, \xi):=:\sum_{j=0}^{\infty} t^j P_j(x, \xi):$$

and called the normal product of $P(t; x, \xi)$.

Remark. Sometimes we abbreviate $:\sum_{j=0}^{\infty} t^j P_j(x, \xi):$ to

$$:\sum_{j=0}^{\infty} P_j(x, \xi):.$$

Of course, Definition 4.6 is included in the preceding definition.

Definition 5.9. Let m be a real number, $P(t; x, \xi)=\sum_{j=0}^{\infty} t^j P_j(x, \xi)$ a formal symbol defined in a conic open set Ω in T^*X. We call $P(t; x, \xi)$ a formal symbol of order at most m in Ω if for every $\Omega'\Subset\Omega$ there exist positive constants r, C, A such that $0<A<1$ and that

$$|P_j(x, \xi)|\leq CA^j|\xi|^m$$

holds for any $(x, \xi)\in \Omega'\cap\{|\xi|\geq (j+1)r\}$, $j=0, 1, 2, \cdots$.

Definition 5.10. Let ρ be a real number such that $0\leq\rho<1$ (resp. $0<\rho\leq 1$), $P(t; x, \xi)=\sum_{j=0}^{\infty} t^j P_j(x, \xi)$ a formal symbol defined in a conic open set Ω in T^*X. The formal symbol $P(t; x, \xi)$ is said to be of growth at most ρ (resp. $\rho-0$) if for every $\Omega'\Subset\Omega$ there exist positive constants r, C, A, h such that $0<A<1$ and that (resp. for every $\Omega'\Subset\Omega$ there are positive constants r, A such that $0<A<1$ and that for each $h>0$ there exists a constant $C>0$ for which)

$$|P_j(x, \xi)| \leq CA^j \exp(h|\xi|^\rho)$$

holds for any $(x, \xi) \in \Omega' \cap \{|\xi| \geq (j+1)r\}$, $j=0, 1, 2, \cdots$.

The preceding definitions are natural extensions of Definitions 4.8 and 4.9, for we have

Proposition 5.11 (Cf. [4], (1.14)). *Let Ω be a conic open set in T^*X, $P(x, \xi)$ a symbol $\in S(\Omega)$, $P(t; x, \xi)$ a formal symbol $\in \hat{S}(\Omega)$ of order at most m (resp. of growth at most ρ, of growth at most $\rho-0$). If $P(x, \xi) \sim P(t; x, \xi)$, then $P(x, \xi)$ is of order at most m (resp. of growth at most ρ, of growth at most $\rho-0$).*

The proposition can be proved in the same manner as Proposition 5.3.

We give some examples of formal symbols and symbols equivalent to them. We suppose $n=1$ and write $\xi_1=\xi$.

Example 5.12. a) $\sum_{j=0}^{\infty} t^j 2^{-j} \sim 2$. This is trivial one. But we remark that the left-hand side is not a formal symbol in the sense of [4] (cf. [1]).

b) $\sum_{j=0}^{\infty} t^j \xi^{-j} \sim \xi \cdot (\xi-1)^{-1} \sim \xi \cdot (1-e^{1-\xi}) \cdot (\xi-1)^{-1}$ (Re $\xi>0$). The last term is an entire function.

c) $\sum_{j=0}^{\infty} t^j j! (-\xi)^{-j} \sim \int_{\xi}^{\infty} e^{\xi-s} \xi s^{-1} ds$ (Re $\xi>0$).

d) $\sum_{j=0}^{\infty} t^j (j!)^{-1} \xi^{j/2} \sim \sum_{j=0}^{\infty} (j!)^{-1} \xi^{j/2} = \exp \sqrt{\xi}$. Both members are of growth 1/2.

§ 6. Double formal symbols

It is sometimes convenient to deal with certain double formal series of symbols rather than formal symbols (cf. [3]). We introduce the following

Definition 6.1. Let Ω be a conic open set in T^*X. Let $P(t_1, t_2; x, \xi) = \sum_{j,k=0}^{\infty} t_1^j t_2^k P_{jk}(x, \xi)$ be a formal power series in (t_1, t_2) with coefficients in $S(\Omega)$. We call $P(t_1, t_2; x, \xi)$ a double formal symbol defined in Ω if for any $\Omega' \Subset \Omega$ there exist positive constants r, A satisfying the following conditions:

a) $0<A<1$,

b) For each $h>0$ there is a constant $C>0$ such that

$$|P_{jk}(x, \xi)| \leq CA^{j+k} \exp(h|\xi|) \tag{6.1}$$

holds for any $(x, \xi) \in \Omega' \cap \{|\xi| \geq (j+k+1)r\}$, $j, k=0, 1, 2, \cdots$. The set of all double formal symbols defined in Ω is denoted by $\hat{S}_2(\Omega)$.

Null-class in the space of double formal symbols $\hat{S}_2(\Omega)$ is defined as follows.

Definition 6.2. A double formal symbol

$$P(t_1, t_2; x, \xi) = \sum_{j,k=0}^{\infty} t_1^j t_2^k P_{jk}(x, \xi)$$

defined in Ω is said to be equivalent to zero if for any $\Omega' \Subset \Omega$ there exist positive constants r, A satisfying the following:

a) $0 < A < 1$,

b) For each $h > 0$ there is a constant $C > 0$ such that

$$\Big| \sum_{j+k \leq m-1} P_{jk}(x, \xi) \Big| \leq C A^m \exp(h|\xi|) \tag{6.2}$$

is valid for any $(x, \xi) \in \Omega' \cap \{|\xi| \geq mr\}$, $m = 1, 2, \cdots$. The set of all double formal symbols $\in \hat{S}_2(\Omega)$ equivalent to zero is denoted by $\hat{R}_2(\Omega)$. Two double formal symbols defined in Ω are said to be equivalent if the difference belongs to $\hat{R}_2(\Omega)$.

We introduce the sum and the product in $\hat{S}_2(\Omega)$ as formal power series in (t_1, t_2). Then $\hat{S}_2(\Omega)$ becomes a commutative ring. The ring of formal symbols $\hat{S}(\Omega)$ is identified with a subring of $\hat{S}_2(\Omega)$ by setting $t = t_1$. Hereafter we always consider $t = t_1$. By the definition, the following proposition is trivial.

Proposition 6.3. $\hat{S}(\Omega) \cap \hat{R}_2(\Omega) = \hat{R}(\Omega)$.

Hence we may write $P(t_1, t_2; x, \xi) \sim Q(t_1, t_2; x, \xi)$ if $P(t_1, t_2; x, \xi) \in \hat{S}_2(\Omega)$ is equivalent to $Q(t_1, t_2; x, \xi) \in \hat{S}_2(\Omega)$ (cf. Definition 5.2).

Proposition 6.4. *$\hat{R}_2(\Omega)$ is an ideal of $\hat{S}_2(\Omega)$.*

The preceding proposition is proved in the same way as proposition 5.4.

Now, we have the following injections:

$$\begin{array}{ccccc} R(\Omega) & \hookrightarrow & \hat{R}(\Omega) & \hookrightarrow & \hat{R}_2(\Omega) \\ \cap\downarrow & & \cap\downarrow & & \cap\downarrow \\ S(\Omega) & \hookrightarrow & \hat{S}(\Omega) & \hookrightarrow & \hat{S}_2(\Omega). \end{array}$$

By virtue of Proposition 6.3, there is an injective homomorphism

$$\iota_{12}\colon \hat{S}(\Omega)/\hat{R}(\Omega) \longrightarrow \hat{S}_2(\Omega)/\hat{R}_2(\Omega)$$

defined by $\iota_{12}(P(t; x, \xi)) = P(t_1 \colon x, \xi)$.

Remark. If $P(t; x, \xi) \in \hat{S}(\Omega)$ then $P(t_1; x, \xi) \sim P(t_2; x, \xi)$.

Let us define ρ_{21}: $\hat{S}_2(\Omega) \to S(\Omega)$ by setting $\rho_{21}(P(t_1, t_2; x, \xi)) = P(t, t; x, \xi)$. Then we have

Theorem 6.5. *The mapping ρ_{21} induces the homomorphism*

$$\rho_{21}: \hat{S}_2(\Omega)/\hat{R}_2(\Omega) \longrightarrow \hat{S}(\Omega)/\hat{R}(\Omega)$$

such that $\rho_{21} \circ \iota_{12} = \mathrm{id}$, $\iota_{12} \circ \rho_{21} = \mathrm{id}$.

Proof. If $P(t_1, t_2; x, \xi) \in \hat{R}_2(\Omega)$, then, by the definition, we have $\rho_{21}(P(t_1, t_2; x, \xi)) \in \hat{R}(\Omega)$. It is clear that $\rho_{21} \circ \iota_{12} = \mathrm{id}$. We shall show $\iota_{12} \circ \rho_{21} = \mathrm{id}$. It suffices to show that $P(t_1, t_2; x, \xi) \sim P(t_1, t_1; x, \xi)$ holds for any $P(t_1, t_2; x, \xi) \in \hat{S}_2(\Omega)$. Suppose $P(t_1, t_2; x, \xi) = \sum_{j,k=0}^{\infty} t_1^j t_2^k P_{jk}(x, \xi)$. We write $P(t_1, t_1; x, \xi) = \sum_{j,k=0}^{\infty} t_1^j t_2^k \tilde{P}_{jk}(x, \xi)$. Then we have $\tilde{P}_{j0}(x, \xi) = \sum_{k=0}^{j} P_{j-k,k}(x, \xi)$, $j = 0, 1, 2, \cdots$ and $\tilde{P}_{jk}(x, \xi) \equiv 0$, $k > 0$, $j = 0, 1, \cdots$. Hence we have

$$\begin{aligned}
&\sum_{j+k \le m-1} (P_{jk}(x, \xi) - \tilde{P}_{jk}(x, \xi)) \\
&= \sum_{j+k \le m-1} P_{jk}(x, \xi) - \sum_{j=0}^{m-1} \tilde{P}_{j0}(x, \xi) \\
&= \sum_{j+k \le m-1} P_{jk}(x, \xi) - \sum_{j=0}^{m-1} \sum_{k=0}^{j} P_{j-k,k}(x, \xi) \\
&= 0.
\end{aligned}$$

This completes the proof.

Combining this with Theorem 5.7 yields

Theorem 6.6. *Set* $\varpi_2 = \varpi_1 \circ \rho_{21}$, *then*

$$\varpi_2: \varinjlim_{\Omega \ni \mathring{x}^*} \hat{S}_2(\Omega)/\hat{R}_2(\Omega) \longrightarrow \mathscr{E}^R_{\mathring{x}^*}$$

is a linear isomorphism. Here Ω runs on the family of conic neighborhoods of $\mathring{x}^*$ in T^*X.

Definition 6.7. The image of $P(t_1, t_2: x, \xi) \in \hat{S}_2(\Omega)$ (or its equivalence class) by ϖ_2 is denoted by $:P(t_1, t_2; x, \xi):$ and called the normal product of $P(t_1, t_2; x, \xi)$.

§ 7. Symbolic calculus

In this section we establish some calculation rules concerning pseudo-differential operators, which are expressed in terms of formal symbols and of double formal symbols.

We first prepare the following lemma. Let Ω denote a conic neighborhood of $\mathring{x}^*=(x; \lambda, 0, \cdots, 0)$ in T^*X.

Lemma 7.1. *Let $M=(a_{ij})$ be an $n\times n$ matrix $(a_{ij}\in \mathbf{C}, 1\leq i,j\leq n)$, $P(t; x, \xi)$ a formal symbol defined in Ω. Then* $\exp(t_2\partial_\xi\cdot M\partial_x)P(t_1; x, \xi)$ *is a double formal symbol defined in Ω. Hence* $\exp(t\partial_\xi\cdot M\partial_x)P(t; x, \xi)$ *is a formal symbol defined in Ω. Moreover, if $P(t; x, \xi)\sim 0$, then* $\exp(t_2\partial_\xi\cdot M\partial_x)P(t_1; x, \xi)\sim 0$ *and* $\exp(t\partial_\xi\cdot M\partial_x)P(t; x, \xi)\sim 0$. *Here* $\partial_\xi\cdot M\partial_x=\sum_{i,j} a_{ij}\partial_{\xi_i}\partial_{x_j}$.

Proof. It suffices to show in the case $M=\begin{pmatrix}1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0\end{pmatrix}$. We assume that the formal symbol $P(t; x, \xi)=\sum_{j=0}^{\infty} t^j P_j(x, \xi)$ satisfies the condition of Definition 5.1. We have to estimate the coefficients of

$$\exp(t_2\partial_\xi\cdot M\partial_x)P(t_1; x, \xi)=\sum_{j,k=0}^{\infty} t_1^j t_2^k \frac{1}{k!}\partial_{\xi_1}^k\partial_{x_1}^k P_j(x, \xi).$$

Let Ω' and Ω'' be compactly generated conic subsets in Ω such that $\Omega\supset\Omega'\supset\Omega''$. By using Cauchy's formula, we have for $(x, \xi)\in\Omega''$

$$\partial_{\xi_1}^k\partial_{x_1}^k P_j(x, \xi)=\frac{k!^2}{(2\pi\sqrt{-1})^2}\iint_{\substack{|z-x_1|=\varepsilon\\ |\zeta-\xi_1|=\varepsilon|\xi|}} dzd\zeta\frac{P_j(z, x', \zeta, \xi')}{(\zeta-\xi_1)^{k+1}(z-x_1)^{k+1}}$$

where $x'=(x_2, \cdots, x_n)$, $\xi'=(\xi_2, \cdots, \xi_n)$, $\varepsilon>0$ is a sufficiently small number, $j, k=0, 1, 2, \cdots$. Hence we have

$$\begin{aligned}|\partial_{\xi_1}^k\partial_{x_1}^k P_j(x, \xi)| &\leq k!^2|\xi|^{-k}\varepsilon^{-2k}\sup_{\substack{|z-x_1|=\varepsilon\\ |\zeta-\xi_1|=\varepsilon|\xi|}}|P_j(z, x', \zeta, \xi')|\\ &\leq k!^2|\xi|^{-k}\varepsilon^{-2k}C'A^j\exp(h|\xi|).\end{aligned}$$

Here h is an arbitrary constant >0, C' is some constant >0 depending on h. Let r' be a positive number. Then we have

$$\frac{1}{k!}|\partial_{\xi_1}^k\partial_{x_1}^k P_j(x, \xi)|\leq k!(j+k+1)^{-k}r'^{-k}\varepsilon^{-2k}C'A^j\exp(h|\xi|) \tag{7.1}$$

for every $(x, \xi)\in\Omega''\cap\{|\xi|\geq(j+k+1)r'\}$, $j, k=0, 1, 2, \cdots$. If r' is taken as $r'>\varepsilon^{-2}A^{-1}$, then (7.1) becomes

$$\frac{1}{k!}|\partial_{\xi_1}^k\partial_{x_1}^k P_j(x, \xi)|\leq C'A^{j+k}\exp(h|\xi|).$$

Thus $\exp(t_2\partial_{\xi_1}\partial_{x_1})P(t_1; x, \xi) \in \hat{S}_2(\Omega')$. We conclude that

$$\exp(t_2\partial_{\xi_1}\partial_{x_1})P(t_1; x, \xi) \in \hat{S}_2(\Omega),$$

for $\Omega' \Subset \Omega$ is arbitrary.

Next, let us suppose $P(t; x, \xi) \sim 0$. We assume the estimate in Definition 5.2. In the same way as above, we obtain for $(x, \xi) \in \Omega' \cap \{|\xi| \geq mr'\}$, $m=1, 2, \cdots$,

$$(7.2) \qquad \left|\sum_{j+k\leq m-1} \frac{1}{k!}\partial_{\xi_1}^k \partial_{x_1}^k P_j(x, \xi)\right|$$

$$\leq \sum_{k=0}^{m-1} \frac{1}{k!}\left|\partial_{\xi_1}^k \partial_{x_1}^k \sum_{j=0}^{m-k-1} P_j(x, \xi)\right|$$

$$\leq C' \sum_{k=0}^{m-1} k!\varepsilon^{-2k}|\xi|^{-k}A^{m-k}\exp(h|\xi|)$$

$$\leq C''A^m \exp(h|\xi|),$$

where $C'' = C'\varepsilon^2 r'A(\varepsilon^2 r'A-1)^{-1}$. Hence $\exp(t_2\partial_{\xi_1}\partial_{x_1})P(t_1; x, \xi) \in \hat{R}_2(\Omega)$.

Theorem 7.2 (Composition). *Let $P(t; x, \xi)$ and $Q(t; x, \xi)$ be formal symbols defined in Ω. Set*

$$(7.3) \qquad W(t_1, t_2; x, \xi) = \exp(t_2\partial_\xi \cdot \partial_x)P(t_1; x, \xi)Q(t_1; y, \eta)|_{\eta=\xi}^{y=x}.$$

Then $W(t_1, t_2; x, \xi)$ is a double formal symbol defined in Ω such that

$$:P(t; x, \xi): \, :Q(t; x, \xi): = :W(t_1, t_2; x, \xi):$$

holds in $\mathscr{E}_{\hat{x}^*}^{R}$.

Since $W(t_1, t_2; x, \xi) \sim W(t, t; x, \xi)$ (cf. Theorem 6.5), the preceding theorem is equivalent to the following:

Theorem 7.2′. *Let $P(t; x, \xi)$ and $Q(t; x, \xi)$ be formal symbols defined in Ω. Set*

$$(7.4) \qquad W(t; x, \xi) = \exp(t\partial_\xi \cdot \partial_y)P(t; x, \xi)Q(t; y, \eta)|_{\eta=\xi}^{y=x}.$$

Then $W(t; x, \xi)$ is a formal symbol defined in Ω so that

$$:P(t; x, \xi): \, :Q(t; x, \xi): = :W(t; x, \xi):$$

holds in $\mathscr{E}_{\hat{x}^*}^{R}$.

Proof. Let $P^{(1)}(x, \xi)$ and $Q^{(1)}(x, \xi)$ be symbols equivalent respectively to $P(t; x, \xi)$ and $Q(t; x, \xi)$ (cf. Theorem 5.5). It follows from Lemma 7.1 that the formal symbol

$$W^{(1)}(t; x, \xi)=\exp(t\partial_\xi\cdot\partial_y)P^{(1)}(x, \xi)Q^{(1)}(y, \eta)|_{\eta=\xi}^{y=x}$$

is equivalent to $W(t; x, \xi)$. Hence it suffices to show

$$:P^{(1)}(x, \xi):\ :Q^{(1)}(x, \xi):=\ :W^{(1)}(t; x, \xi):.$$

Let $K(x, x')$ and $L(x, x')$ (resp. $f(x, \xi, p)$ and $g(x, \xi, p)$) denote the kernel functions (resp. the Radon transformations) of $:P^{(1)}(x, \xi):$ and $:Q^{(1)}(x, \xi):$, respectively. As holomorphic microfunctions, we have

$$f(x, \xi, p)=\frac{(n-1)!}{(-2\pi\sqrt{-1})^n}\int\frac{K(x, x')}{(p-\langle x-x', \xi\rangle)^n}dx',$$
$$g(x, \xi, p)=\frac{(n-1)!}{(-2\pi\sqrt{-1})^n}\int\frac{L(x, x')}{(p-\langle x-x', \xi\rangle)^n}dx'.$$

The kernel function of the composite operator $:P^{(1)}(x, \xi):\ :Q^{(1)}(x, \xi):$ is $H(x, x')=\int K(x, x'')L(x'', x')dx''$. Hence the Radon transformation $h(x, \xi, p)$ of $H(x, x')$ is

$$\begin{aligned}h(x, \xi, p)&=\frac{(n-1)!}{(-2\pi\sqrt{-1})^n}\int\frac{H(x, x')}{(p-\langle x-x', \xi\rangle)^n}dx'\\&=\frac{(n-1)!}{(-2\pi\sqrt{-1})^n}\iint\frac{K(x, x'')L(x'', x')}{(p-\langle x-x', \xi\rangle)^n}dx''dx'\\&=\int K(x, x'')g(x'', \xi, p-\langle x-x'', \xi\rangle)dx''.\end{aligned}$$

We may assume that $K(x, x')$ is represented by a defining function $\psi(x, x') \in \mathcal{O}(V_{c,\varepsilon})$ for some $c, \varepsilon>0$ (cf. § 2). Then the defining function of $h(x, \xi, p)$ is

$$h(x, \xi, p)=(-1)^n\int_{\beta_0}^{\beta_1}dy_1\oint dy'\psi(x, x-y)g(x-y, \xi, p-\langle y, \xi\rangle),$$

where the paths of the integrations are taken as in (3.3). (We use the same letters h, g to represent the defining functions of holomorphic microfunctions h, g, respectively). The symbol $W(x, \xi)$ of $h(x, \xi, p)$, hence of $:P^{(1)}(x, \xi):\ :Q^{(1)}(x, \xi):$, is

$$W(x, \xi)=(2\pi\sqrt{-1})^{n-1}\int_\Sigma h(x, \xi, p)e^{-p}dp.$$

(Cf. (4.1) and Definition 4.2). We have

$$\int_\Sigma h(x, \xi, p)e^{-p}dp$$

$$=(-1)^n \int_\Sigma dp \int_{\beta_0}^{\beta_1} dy_1 \oint dy' \psi(x, x-y) g(x-y, \xi, p-\langle y, \xi\rangle)e^{-p}$$

$$=(-1)^n \int_{\beta_0}^{\beta_1} dy_1 \oint dy' \psi(x, x-y)e^{-\langle y, \xi\rangle} \int_\Sigma g(x-y, \xi, q)e^{-q}dq$$

$$=(-1)^n \int_{\beta_0}^{\beta_1} dy_1 \oint dy' \sum_\alpha \frac{(-y)^\alpha}{\alpha!} \psi(x, x-y)e^{-\langle y, \xi\rangle} \int_\Sigma \partial_x^\alpha g(x, \xi, q)e^{-q}dq$$

$$=\sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha((-1)^n \int_{\beta_0}^{\beta_1} dy_1 \oint dy' \psi(x, x-y)e^{-\langle y, \xi\rangle})\partial_x^\alpha \int_\Sigma g(x, \xi, q)e^{-q}dq.$$

Now we set

$$P^{(2)}(x, \xi)=(-1)^n \int_{\beta_0}^{\beta_1} dy_1 \oint dy' \psi(x, x-y)e^{-\langle y, \xi\rangle},$$

$$Q^{(2)}(x, \xi)=(2\pi\sqrt{-1})^{n-1} \int_\Sigma dq\, g(x, \xi, q)e^{-q}.$$

Then we have $P^{(1)}(x, \xi) \sim P^{(2)}(x, \xi)$, $Q^{(1)}(x, \xi) \sim Q^{(2)}(x, \xi)$ and $W(x, \xi)= \sum_\alpha (1/\alpha!)\partial_\xi^\alpha P^{(2)}(x, \xi)\cdot \partial_x^\alpha Q^{(2)}(x, \xi)$, where the right-hand side converges if $|\beta_0|, |\beta_1| \ll 1$, $|\xi| \gg 1$. If we set

$$W^{(2)}(t; x, \xi)=\exp(t\partial_\xi \cdot \partial_y)P^{(2)}(x, \xi)Q^{(2)}(y, \eta)|_{\eta=\xi}^{y=x},$$

then we have immediately $W(x, \xi) \sim W^{(2)}(t; x, \xi)$. This implies $W(x, \xi) \sim W^{(1)}(t; x, \xi)$, for $W^{(1)}(t; x, \xi) \sim W^{(2)}(t; x, \xi)$. Hence we have $W(x, \xi) \sim W(t; x, \xi)$; this completes the proof.

Theorem 7.3 (Formal adjoint). *Let $P(t; x, \xi)$ be a formal symbol defined in Ω. Then*

$$P^*(t_1, t_2; x, \xi)=\exp(t_2\partial_\xi \cdot \partial_x)P(t_1; x, -\xi)$$

is a double formal symbol defined in Ω^a satisfying

$$(:P(t; x, \xi):)^* = :P^*(t_1, t_2; x, \xi): \qquad in\ \mathscr{E}^R_{\alpha(\hat{x}^*)}.$$

Here the left-hand side means the formal adjoint operator of $:P(t: x, \xi):$.

Proof. By Theorem 6.5, it suffices to show the following

Theorem 7.3′. *Let $P(t; x, \xi)$ be a formal symbol defined in Ω. Then*

$$P^*(t; x, \xi)=\exp(t\partial_\xi \cdot \partial_x)P(t; x, -\xi)$$

is a formal symbol defined in Ω^a such that

$$(:P(t; x, \xi):)^* = :P^*(t; x, \xi): \qquad in\ \mathscr{E}^R_{\alpha(\mathring{x}^*)}.$$

Proof of Theorem 7.3′. Let $P(x, \xi)$ be a symbol equivalent to $P(t; x, \xi)$. We may assume that $P(x, \xi)$ can be written in the form (cf. Proposition 4.7)

$$(7.5) \qquad P(x, \xi) = (-1)^n \int_{\beta_0}^{\beta_1} dy_1 \oint dy' \psi(x, x-y) e^{-\langle y, \xi \rangle}$$

where $\psi(x, x') \in \mathcal{O}(V_{c,\varepsilon})$ ($c, \varepsilon > 0$; cf. § 2) is a defining function of $:P(t; x, \xi):$. Then the formal adjoint P^* of $P = :P(t; x, \xi):$ is defined by

$$P^* = [(-1)^n \psi(x', x) dx'].$$

Hence the symbol of P^* is

$$(7.6) \qquad P^*(x, \xi) = \int_{\beta_0'}^{\beta_1'} dy_1 \oint dy' \psi(x-y, x) e^{-\langle y, \xi \rangle},$$

where $\beta_0' = -\beta_0$, $\beta_1' = -\beta_1$. We set $\varphi(x, y) = \psi(x, x-y)$. Then (7.5) and (7.6) become respectively

$$(7.5)' \qquad P(x, \xi) = (-1)^n \int_{\beta_0}^{\beta_1} dy_1 \oint dy' \varphi(x, y) e^{-\langle y, \xi \rangle},$$

$$(7.6)' \qquad P^*(x, \xi) = \int_{\beta_0'}^{\beta_1'} dy_1 \oint dy' \varphi(x-y, -y) e^{-\langle y, \xi \rangle}.$$

Combining (7.5)′ with (7.6)′ yields

$$\begin{aligned} P^*(x, \xi) &= (-1)^n \int_{\beta_0}^{\beta_1} dy_1 \oint dy' \varphi(x+y, y) e^{-\langle y, -\xi \rangle} \\ &= (-1)^n \int_{\beta_0}^{\beta_1} dy_1 \oint dy' \sum_\alpha \frac{y^\alpha}{\alpha!} \partial_x^\alpha \varphi(x, y) e^{-\langle y, -\xi \rangle} \\ &= \sum_\alpha \frac{(-1)^{|\alpha|+n}}{\alpha!} \partial_x^\alpha \partial_\eta^\alpha \int_{\beta_0}^{\beta_1} dy_1 \oint dy' \varphi(x, y) e^{-\langle y, \eta \rangle} |_{\eta=-\xi} \\ &= \sum_\alpha \frac{1}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha P(x, -\xi). \end{aligned}$$

The last term converges if $|\beta_0|, |\beta_1| \ll 1$, $|\xi| \gg 1$. Then it is clear that $P^*(t; x, \xi) \sim P^*(x, \xi)$. This completes the proof.

Theorem 7.4 (Change of variables). *Let $P(t; x, \xi)$ be a formal symbol $\in \hat{S}(\Omega)$ in a coordinate system (x). Let (y) be another coordinate system and denote by (y, η) the corresponding coordinate system on T^*X,* i.e., $\eta_j =$

$\sum_{k=1}^{n} (\partial x_k/\partial y_j)\xi_k$. *Let* $M(x, x')$ *be a matrix defined by* $y(x)-y(x')=M(x, x')(x-x')$. *Then*

$$\tilde{P}(t_1, t_2; y, \eta)=\exp(t_2\partial_{x'}\cdot\partial_{\xi'})P(t_1; x, \xi'+{}^tM(x, x')\eta)|_{\substack{x'=x,\\ \xi'=0}}$$

is a double formal symbol in the coordinate system (y) *such that*

$$:P(t; x, \xi):=:\tilde{P}(t_1, t_2; y, \eta):.$$

Here the right-hand side means the operator $\in \mathscr{E}^R_{\mathfrak{X}^*}$ *corresponding to* $\tilde{P}(t_1, t_2; y, \eta)$ *with respect to the coordinate system* (y, η).

Theorem 7.4′. *Under the same notation as in Theorem* 7.4, *set*

$$\tilde{P}(t; y, \eta)=\exp(t\partial_{x'}\cdot\partial_{\xi'})P(t; x, \xi'+{}^tM(x, x')\eta)|_{\substack{x'=x.\\ \xi'=0}}$$

Then $\tilde{P}(t; y, \eta)$ *is a formal symbol defined in* Ω *such that*

$$:P(t; x, \xi):=:\tilde{P}(t; y, \eta):.$$

Proofs of Theorems 7.4 *and* 7.4′. It suffices to show Theorem 7.4′. Let $P(x, \xi)$ be a symbol equivalent to $P(t; x, \xi)$. We may assume that $P(x, \xi)$ is written in the form

$$P(x, \xi)=(-1)^n\int dy\psi(x, x-y)e^{-\langle y,\xi\rangle}$$
$$=\int dx'\psi(x, x')e^{-\langle x-x',\xi\rangle},$$

where $\psi(x, x')\in\mathcal{O}(V_{c,\varepsilon})$ is a defining function of $:P(t; x, \xi):$ $(c, \varepsilon>0$; cf. § 2 and Proposition 4.7). Then the symbol of $P=[\psi(x, x')dx']$ with respect to the coordinate system (y, η) is

$$\tilde{P}(y, \eta)=\int dx''\psi(x, x'')e^{-\langle y-y'',\eta\rangle},$$

where $y''=y(x'')$. We shall give formal calculus only; the justification is the same as in the proofs of Theorems 7.2′ and 7.3′.

$$\tilde{P}(y, \eta)=\int dx''\psi(x, x'')e^{-\langle M(x,x'')(x-x''),\eta\rangle}$$
$$=\int dx''\psi(x, x'')e^{-\langle x-x'',{}^tM(x,x'')\eta\rangle}$$
$$=\int dx''\psi(x, x'')\sum_\alpha\frac{(x''-x)^\alpha}{\alpha!}\partial_{x'}^\alpha e^{-\langle x-x'',{}^tM(x,x')\eta\rangle}|_{x'=x}$$

$$=\int dx''\psi(x, x'') \sum_{\alpha} \frac{\partial_{\xi'}^{\alpha}\partial_{x'}^{\alpha}}{\alpha!} e^{-\langle x-x'', {}^tM(x,x')\eta+\xi'\rangle}\Big|_{\substack{x'=x\\ \xi'=0}}$$

$$=\exp(\partial_{x'}\cdot\partial_{\xi'})P(x, {}^tM(x, x')\eta+\xi')\Big|_{\substack{x'=x\\ \xi'=0}}.$$

Hence we have

$$\tilde{P}(y, \eta)=\exp(\partial_{x'}\cdot\partial_{\xi'})P(x, {}^tM(x, x')\eta+\xi')\Big|_{\substack{x'=x\\ \xi'=0}}$$

$$\sim\exp(t\partial_{x'}\cdot\partial_{\xi'})P(x, {}^tM(x, x')\eta+\xi')\Big|_{\substack{x'=x\\ \xi'=0}}$$

$$\sim\exp(t\partial_{x'}\cdot\partial_{\xi'})P(t; x, {}^tM(x, x')\eta+\xi')\Big|_{\substack{x'=x\\ \xi'=0}}.$$

This implies $\tilde{P}(y, \eta)\sim\tilde{P}(t; y, \eta)$.

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