Advanced Studies in Pure Mathematics 4, 1984
Group Representations and Systems of Differential Equations
pp. 97–119

Irreducible Decomposition of Fundamental Modules for $A_l^{(1)}$ and $C_l^{(1)}$, and Hecke Modular Forms

Michio Jimbo and Tetsuji Miwa

§1.

In [1], [2] Kac and Peterson established a connection between the characters of irreducible highest weight modules over Euclidean Lie algebras and the classical theta and modular functions. In particular, they considered the generating functions of weight multiplicities along the direction of multiples of the null root (the string functions), and derived their automorphic properties.

In this paper, we treat a different sort of generating function which we encounter when we consider the decomposition of a highest weight module with regards to a subalgebra. To be specific, let us consider $A_{2l-1}^{(1)}$ and its subalgebra $C_l^{(1)}$. Let $\tilde{\Lambda}_j$ $(0 \leqq j \leqq 2l-1)$ denote the fundamental weights for $A_{2l-1}^{(1)}$ and let $L(\tilde{\Lambda}_j)$ denote the associated highest weight module. We use Λ_k and $L(\Lambda_k)$ $(0 \leqq k \leqq l)$ to denote those with respect to $C_l^{(1)}$. When we consider $L(\tilde{\Lambda}_j)$ as a $C_l^{(1)}$ module, it is no longer irreducible, but is decomposed into irreducible parts, each of which is isomorphic to one of $L(\Lambda_k)$'s. Thus, in terms of the characters, we have the identity of the form

$$\mathrm{ch}_{L(\tilde{\Lambda}_j)}|_{\mathfrak{h}} = \sum_{k=0}^{l} E_{jk}^l(q)\, \mathrm{ch}_{L(\Lambda_k)}.$$

Here $|_{\mathfrak{h}}$ means the restriction to the Cartan subalgebra of $C_l^{(1)}$, and $q = e^{-\delta}$ where δ is the null root of $C_l^{(1)}$. Our problem is to determine the power series $E_{jk}^l(q)$. If we set

$$e_{jk}^l(\tau) = q^{(j-k)/2 - j^2/4l + (k+1)^2/4(l+2)} E_{jk}^l(q)$$

with $q = e^{2\pi i \tau}$, it follows from the result of Kac and Peterson that $e_{jk}^l(\tau)$ is a modular form. On the other hand, through computer experiments we found the following asymptotic property as $l \to \infty$: there exist power series $F_{jk}^{(\nu)}(q)$ $(\nu = 0, 1, 2, \cdots)$ *independent of* l such that

Received February 10, 1983.

$$E^l_{jk}(q)=F^{(0)}_{jk}(q)+q^lF^{(1)}_{jk}(q)+q^{2l}F^{(2)}_{jk}(q)+\cdots$$

holds for all j, k and l. These facts tempted us to determine $E^l_{jk}(q)$ explicitly.

Let us state our result. We set $\varphi(q)=\prod_{n\geq 1}(1-q^n)$. Then we have for $j\leqq k$

$$\begin{aligned}\varphi(q)^2q^{(j-k)/2}E^l_{jk}(q)&=\varphi(q)^2E^l_{l-j,l-k}(q)\\ &=\Big(\sum_{\substack{m\geqq 0\\ n\geqq 0}}-\sum_{\substack{m<0\\ n<0}}\Big)(-)^nq^{n(n+1)/2+(k+1)m+(-j+k)n/2+(l+2)m(m+n)}\\ &\quad+\Big(\sum_{\substack{m\geqq 0\\ n>0}}-\sum_{\substack{m<0\\ n\leqq 0}}\Big)(-)^nq^{n(n+1)/2+(k+1)m+(j+k)n/2+(l+2)m(m+n)}.\end{aligned}\tag{1.1}$$

We note that $E^l_{jk}(q)\equiv 0$ for $j\not\equiv k \bmod 2$.

We can rewrite (1.1) so that the transformation property is more apparent. Let us consider the following quadratic form.

$$B\left(\begin{pmatrix}x\\ y\end{pmatrix},\begin{pmatrix}x\\ y\end{pmatrix}\right)=2(l+2)x^2-2ly^2.$$

We set $L=Z^2$ and $L^*=(1/2(l+2))Z\oplus(1/2l)Z$. Let us denote by G_0 the discrete group $\{g\in SO_0(B)\cap SL(2,Z)\,|\,g$ fixes each element of $L^*/L\}$. In fact, G_0 is generated by $\begin{pmatrix}l+1 & l\\ l+2 & l+1\end{pmatrix}^2$. For $\gamma=\begin{pmatrix}x\\ y\end{pmatrix}$ we set $\operatorname{sgn}\gamma=\operatorname{sgn}x$. Now we define

$$\theta^B_\mu(\tau)=\sum_{\substack{B(\gamma,\gamma)>0\\ \gamma\in G_0\backslash(L+\mu)}}\operatorname{sgn}\gamma\, e^{\pi i\tau B(\gamma,\gamma)}$$

for $\mu\in L^*$.

This is a modular form of weight 1, which Kac and Peterson called the Hecke indefinite modular form. They showed that the string functions for $A^{(1)}_1$ of arbitrary level m are nothing but Hecke indefinite modular forms. In their notations the result reads

$$\eta(\tau)^3c^{(m-k)\Lambda_0+k\Lambda_1}_{(m-j)\Lambda_0+j\Lambda_1}(\tau)=\theta^B_\mu(\tau),\qquad \mu=\begin{bmatrix}\dfrac{k+1}{2(m+2)}\\[2mm] \dfrac{j}{2m}\end{bmatrix}.\tag{1.2}$$

Here $\eta(\tau)=q^{1/24}\varphi(q)$ is the Dedekind eta function. Remarkably enough, our result is written in exactly the same expression.

$$(1.3) \qquad \eta(\tau)^2 e^l_{jk}(\tau) = \theta^B_\mu(\tau), \qquad \mu = \begin{bmatrix} \dfrac{k+1}{2(l+2)} \\ \dfrac{j}{2l} \end{bmatrix}.$$

Note that in (1.2) the rank of the algebra is restricted to 1 but the level m is arbitrary. On the other hand in (1.3) the rank l is arbitrary but the level is restricted to 1.

In [3] we considered a similar problem for the pair $A_2^{(1)} \supset A_2^{(2)}$, motivated by the connection with the soliton theory. Here we extend the result in [3] by showing that the decomposition rule for $A_{2l}^{(1)} \supset A_{2l}^{(2)}$ is included in that for $A_{4l+1}^{(1)} \supset C_{2l+1}^{(1)}$.

We also consider the case $C_{2l}^{(1)} \supset C_l^{(1)}$. In this case the decomposition rule is given in terms of the Hecke modular form with the quadratic form $B'(\gamma, \gamma) = 2(l+2)x^2 - 8(l+1)y^2$, for $\gamma = \begin{pmatrix} x \\ y \end{pmatrix}$.

In Section 2 we review the necessary ingredients of the representation theory of Euclidean Lie algebras. Section 3 is devoted to the derivation of (1.1) and (1.3). Our method is to utilize the principal specialization in order to find the inverse matrix to $(E^l_{jk})_{0 \leq j,k \leq l}$. Then the problem of inversion is solved by reducing it to that for a simpler one of infinite size $((-)^{(j-k)/2} q^{(j-k)(j-k+2)/8})_{j,k \in \mathbf{Z}}$. In Sections 4 and 5, we treat the cases $A_{2l}^{(1)} \supset A_{2l}^{(2)}$ and $C_{2l}^{(1)} \supset C_l^{(1)}$, respectively.

§ 2.

In this section we shall formulate the problem. As for the definition of Euclidean Lie algebras and their representation theory, the reader is referred to the literatures [2], [4], [5]. Throughout this paper, definitions and notations for basic concepts such as the Chevalley basis e_i, f_i, h_i $(0 \leq i \leq l)$, simple roots α_i, irreducible module $L(\Lambda)$ associated with the highest weight Λ, etc. will be based on [2]. In particular, we include the derivation d with the properties $[d, e_j] = \delta_{j0} e_j$, $[d, f_j] = -\delta_{j0} f_j$, $[d, h_j] = 0$ in the definition of Euclidean Lie algebras.

In this paper we shall be concerned mainly with two specific types of Euclidean Lie algebras, $A_{2l-1}^{(1)}$ and $C_l^{(1)}$.

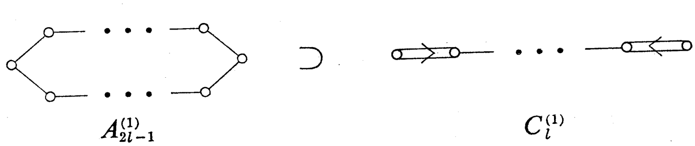

Fig. 1. Dynkin diagrams of $A_{2l-1}^{(1)}$ and $C_l^{(1)}$

We regard $C_l^{(1)}$ as a subalgebra embedded in $A_{2l-1}^{(1)}$ by a diagram automorphism of the latter. Namely, let $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{h}_i$ $(0 \leqq i \leqq 2l-1)$ denote the Chevalley basis of $\tilde{\mathfrak{g}} = A_{2l-1}^{(1)}$. Then the elements e_i, f_i, h_i $(0 \leqq i \leqq l)$ given by

$$x_0 = \tilde{x}_0, \quad x_i = \tilde{x}_i + \tilde{x}_{2l-i} \quad (1 \leqq i \leqq l-1), \quad x_l = \tilde{x}_l,$$
$$(x = e, f, h)$$

together with the derivation d, generate a subalgebra $\mathfrak{g}$ which can be (and is) identified with $C_l^{(1)}$. Under this identification, the two algebras share the central element

$$c = \sum_{i=0}^{2l-1} \tilde{h}_i = \sum_{i=0}^{l} h_i$$

in common. Denote by $\tilde{\mathfrak{h}} = (\oplus_{i=0}^{2l-1} C\tilde{h}_i) \oplus Cd$, $\mathfrak{h} = (\oplus_{i=0}^{l} Ch_i) \oplus Cd$ the Cartan subalgebras of $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$, respectively. Then the simple roots $\tilde{\alpha}_i \in \tilde{\mathfrak{h}}^*$ for $\tilde{\mathfrak{g}}$ and those $\alpha_i \in \mathfrak{h}^*$ for $\mathfrak{g}$ are related through

$$\tilde{\alpha}_0|_{\mathfrak{h}} = \alpha_0, \quad \tilde{\alpha}_i|_{\mathfrak{h}} = \tilde{\alpha}_{2l-i}|_{\mathfrak{h}} = \alpha_i \quad (1 \leqq i \leqq l-1),$$
$$\tilde{\alpha}_l|_{\mathfrak{h}} = \alpha_l.$$

In particular we have the relation of the null roots $\delta = \tilde{\delta}|_{\mathfrak{h}}$, where $\tilde{\delta} = \sum_{i=0}^{2l-1} \tilde{\alpha}_i$, $\delta = \alpha_0 + 2\sum_{i=1}^{l-1} \alpha_i + \alpha_l$. We note here the property of δ:

$$\lambda(h_i) = \lambda'(h_i) \quad (0 \leqq i \leqq l) \quad \text{if and only if } \lambda \equiv \lambda' \bmod C\delta \quad (\lambda, \lambda' \in \mathfrak{h}^*).$$

Denote also by $\tilde{\Lambda}_i \in \tilde{\mathfrak{h}}^*$, $\Lambda_i \in \mathfrak{h}^*$ the fundamental weights

$$\tilde{\Lambda}_i(\tilde{h}_j) = \delta_{ij}, \quad \tilde{\Lambda}_i(d) = 0 \qquad (0 \leqq i, j \leqq 2l-1),$$
$$\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0 \qquad (0 \leqq i, j \leqq l).$$

They are related through

$$\tilde{\Lambda}_0|_{\mathfrak{h}} = \Lambda_0, \quad \tilde{\Lambda}_i|_{\mathfrak{h}} = \tilde{\Lambda}_{2l-i}|_{\mathfrak{h}} = \Lambda_i \quad (1 \leqq i \leqq l+1), \quad \tilde{\Lambda}_l|_{\mathfrak{h}} = \Lambda_l.$$

Now let $\tilde{\Lambda}$ be a linear form on $\tilde{\mathfrak{h}}$ which is integral ($\tilde{\Lambda}(\tilde{h}_i) \in Z$ for all i) and dominant ($\tilde{\Lambda}(\tilde{h}_i) \geqq 0$ for all i). There corresponds an irreducible $\tilde{\mathfrak{g}}$-module $L(\tilde{\Lambda}) = \mathfrak{a}(\tilde{\mathfrak{g}}) v_{\tilde{\Lambda}}$ ($\mathfrak{a}(\tilde{\mathfrak{g}})$ = the universal enveloping algebra of $\tilde{\mathfrak{g}}$) such that $\tilde{e}_i v_{\tilde{\Lambda}} = 0$ (for all i) and $\tilde{h} v_{\tilde{\Lambda}} = \tilde{\Lambda}(\tilde{h}) v_{\tilde{\Lambda}}$ (for all $\tilde{h} \in \tilde{\mathfrak{h}}$). $L(\tilde{\Lambda})$ is uniquely determined up to isomorphisms by the set of non-negative integers $(\tilde{\Lambda}(\tilde{h}_0), \cdots, \tilde{\Lambda}(\tilde{h}_{2l-1}))$, irrespective of the value $\tilde{\Lambda}(d)$. On $L(\tilde{\Lambda})$, each $\tilde{f}_i$ is locally nilpotent (that is, for any $v \in L(\tilde{\Lambda})$, there exists an $M > 0$ such that $\tilde{f}_i^M v = 0$). Next regard $L(\tilde{\Lambda})$ as a $\mathfrak{g}$-module. Since $[\tilde{f}_i, \tilde{f}_{2l-i}] = 0$, $f_i = \tilde{f}_i +$

$\tilde{f}_{2l-i}$ is also locally nilpotent on $L(\tilde{\Lambda})$(*). By virtue of the complete reducibility theorem (Proposition 2.9) in [2], the $\mathfrak{g}$-module $L(\tilde{\Lambda})$ then decomposes into direct sums of the irreducible modules of the type $L(\Lambda)$, which are similarly defined via dominant integral forms $\Lambda \in \mathfrak{h}^*$. This decomposition is described more precisely in terms of the characters

$$\mathrm{ch}_{L(\tilde{\Lambda})} = \sum_{\tilde{\lambda} \in \tilde{P}} \mathrm{mult}_{\tilde{\Lambda}}(\tilde{\lambda}) e^{\tilde{\lambda}}$$

$$\mathrm{ch}_{L(\Lambda)} = \sum_{\lambda \in P} \mathrm{mult}_{\Lambda}(\lambda) e^{\lambda}.$$

Here $\tilde{P}$, P denote the lattice of integral weights for $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$, respectively. $\tilde{P}_+$, P_+ will designate the set of dominant integral weights. In the sequel we choose $\tilde{\Lambda}$, Λ so that $\tilde{\Lambda}(d)=0$, $\Lambda(d)=0$.

Proposition 1. *There exist power series $E^l_{\tilde{\Lambda}\Lambda}(q)$ in one variable q such that*

$$\mathrm{ch}_{L(\tilde{\Lambda})}|_{\mathfrak{h}} = \sum_{\substack{\Lambda \in P_+ \\ \Lambda(c)=\tilde{\Lambda}(c),\, \Lambda(d)=0}} E^l_{\tilde{\Lambda}\Lambda}(q)\, \mathrm{ch}_{L(\Lambda)}, \qquad q = e^{-\delta}. \tag{2.1}$$

They have the properties

$$E^l_{\tilde{\Lambda}\Lambda}(q) = 0 \qquad \text{if } \tilde{\Lambda}|_{\mathfrak{h}} \not\equiv \Lambda \bmod \bigoplus_{i=0}^{l} \mathbf{Z}\alpha_i \tag{2.2}$$

$$E^l_{\tilde{\Lambda}'\Lambda'}(q) = q^{(\tilde{\Lambda}(h^0) - \Lambda(h^0))/2} E^l_{\tilde{\Lambda}\Lambda}(q), \qquad h^0 = \sum_{j=0}^{l} j h_j \tag{2.3}$$

where in the last line $\tilde{\Lambda}'$, Λ' are defined by

$$\tilde{\Lambda}'(\tilde{h}_i) = \begin{cases} \tilde{\Lambda}(\tilde{h}_{l-i}) & (0 \leqq i \leqq l) \\ \tilde{\Lambda}(\tilde{h}_{3l-i}) & (l+1 \leqq i \leqq 2l-1), \end{cases}$$

$$\Lambda'(h_i) = \Lambda(h_{l-i}) \qquad (0 \leqq i \leqq l)$$

and $\tilde{\Lambda}'(d)=0$, $\Lambda'(d)=0$.

Proof. Set $V = \{v \in L(\tilde{\Lambda}) \mid e_i v = 0 \ (0 \leqq i \leqq l)\}$, and let $V = \oplus_{\lambda \in \mathscr{L}} V_\lambda$ ($\mathscr{L} = \{\lambda \in P \mid V_\lambda \neq 0\}$) be its weight space decomposition. It is known (Proposition 2.9 [2]) that $\mathscr{L} \subset P_+$ and that

$$L(\tilde{\Lambda}) \cong \bigoplus_{\lambda \in \mathscr{L}} (\underbrace{L(\lambda) \oplus \cdots \oplus L(\lambda)}_{\dim V_\lambda\text{-times}}).$$

(*) This can also be shown by using the vertex representation [3] [6].

For a $\lambda \in \mathscr{L}$, let $\Lambda \in P_+$ be such that $\Lambda(h_i)=\lambda(h_i)$ $(0 \leqq i \leqq l)$ and $\Lambda(d)=0$. Then λ can be written as

$$\lambda=\tilde{\Lambda}|_{\mathfrak{h}}-\sum_{i=0}^{l}\nu_i\alpha_i=\Lambda-n\delta$$

with some nonnegative integers ν_i and $n \in \boldsymbol{Z}$. This implies that $\tilde{\Lambda}|_{\mathfrak{h}} \equiv \Lambda \bmod \oplus_{i=0}^{l} \boldsymbol{Z}\alpha_i$, $\tilde{\Lambda}(c)=\lambda(c)=\Lambda(c)$ and $n=\nu_0 \geqq 0$. Setting $E^l_{\tilde{\Lambda}\Lambda}(q)=\sum_{n\geqq 0} \dim V_{\Lambda-n\delta}\cdot q^n$ we obtain (2.1) and (2.2).

It remains to show the symmetry (2.3). Define $\tilde{\xi} \in GL(\oplus_{i=0}^{2l-1} \boldsymbol{C}\tilde{\alpha}_i)$, $\xi \in GL(\oplus_{i=0}^{l} \boldsymbol{C}\alpha_i)$ by $\tilde{\xi}(\tilde{\alpha}_i)=\tilde{\alpha}_{l-i}$ $(0 \leqq i \leqq l)$, $=\tilde{\alpha}_{3l-i}$ $(l+1 \leqq i \leqq 2l-1)$ and $\xi(\alpha_i)=\alpha_{l-i}$ $(0 \leqq i \leqq l)$. Then the symmetry of Dynkin diagrams ensures the relations

$$\tilde{\xi}(e^{-\tilde{\Lambda}}\mathrm{ch}_{L(\tilde{\Lambda})})=e^{-\tilde{\Lambda}'}\mathrm{ch}_{L(\tilde{\Lambda}')}$$
$$\xi(e^{-\Lambda}\mathrm{ch}_{L(\Lambda)})=e^{-\Lambda'}\mathrm{ch}_{L(\Lambda')}.$$

(Note that these are formal power series in $e^{-\tilde{\alpha}_i}$ or $e^{-\alpha_i}$). Substitution into (1.1) yields

$$\xi(e^{-\tilde{\Lambda}|_{\mathfrak{h}}+\Lambda}E_{\tilde{\Lambda}\Lambda}(q))=e^{-\Lambda'|_{\mathfrak{h}}+\Lambda'}E_{\tilde{\Lambda}'\Lambda'}(q).$$

Using $\xi(\delta)=\delta$ and the identity

$$\xi(\Lambda_j-\Lambda_0)-(\Lambda_{l-j}-\Lambda_0)=\tfrac{1}{2}j\delta-(\Lambda_l-\Lambda_0)$$

we obtain (2.3).

In the next section we shall determine the functions $E^l_{\tilde{\Lambda}\Lambda}(q)$ when the level $\tilde{\Lambda}(c)=\Lambda(c)$ of $L(\tilde{\Lambda})$ is 1, i.e. for $\tilde{\Lambda}=\tilde{\Lambda}_j$. Write $E^l_{jk}(q)=E^l_{\tilde{\Lambda}_j\Lambda_k}(q)$, $0 \leqq j, k \leqq l$. Properties (2.2), (2.3) are rephrased as

(2.2)′ $$E^l_{jk}(q)=0 \qquad \text{if } j \not\equiv k \bmod 2,$$

(2.3)′ $$E^l_{l-j,l-k}(q)=q^{(j-k)/2}E^l_{jk}(q).$$

Since $E^l_{jk}(q)$'s are power series, we see that

(2.4) $$q^{(j-k)/2}E^l_{jk}(q)=\begin{cases} O(1) & (j \leqq k) \\ O(q^{(j-k)/2}) & (j \geqq k) \end{cases}$$

for $q \to 0$, as a consequence of (2.3)′.

An important property of $E^l_{\tilde{\Lambda}\Lambda}(q)$ is that, when multiplied by certain power of q, it becomes a modular form. As shown in [2], for Euclidean Lie algebras the characters $\mathrm{ch}_{L(\Lambda)}$ are expressible in terms of classical theta functions of several variables. Based on this observation, Kac-Peterson derived transformation formulas for these characters and the "string functions" [2]. From their results and (2.1), it follows that $E^l_{\tilde{\Lambda}\Lambda}(q)$'s also have automorphic properties. We give below their explicit forms for level 1 modules $L(\tilde{\Lambda}_j)$ and $L(\Lambda_k)$.

Introduce a coordinate on $\tilde{\mathfrak{h}}$ (resp. $\mathfrak{h}$) by setting

$$\tilde{h} = -2\pi i\left(\tau d + \sum_{i=1}^{2l-1} \tilde{z}_i \tilde{h}_i + tc\right), \qquad (\tau, \tilde{z}, t) \in \boldsymbol{C}\times\boldsymbol{C}^{2l-1}\times\boldsymbol{C}$$

$$(\text{resp. } h = -2\pi i\left(\tau d + \sum_{i=1}^{l} z_i h_i + tc\right), \qquad (\tau, z, t) \in \boldsymbol{C}\times\boldsymbol{C}^{l}\times\boldsymbol{C}).$$

We set

$$\tilde{\chi}_j(\tau, \tilde{z}, t) = (e^{-\tilde{s}_j\delta}\,\mathrm{ch}_{L(\tilde{\Lambda}_j)})(\tilde{h}), \qquad \tilde{s}_j = \frac{j(2l-j)}{4l} - \frac{2l-1}{24}$$

$$\chi_k(\tau, z, t) = (e^{-s_k\delta}\,\mathrm{ch}_{L(\Lambda_k)})(h), \qquad s_k = \frac{k(2l+2-k)}{4(l+2)} - \frac{l(2l+1)}{24(l+2)}.$$

These are holomorphic for $\mathrm{Im}\,\tau > 0$ ([2]). Transformation formulas for these characters read as follows:

$$\tilde{\chi}_j\left(-\frac{1}{\tau}, \frac{\tilde{z}}{\tau}, t + \frac{\langle \tilde{z}, \tilde{z}\rangle}{2\tau}\right) = \frac{1}{\sqrt{2l}} \sum_{j'=0}^{2l-1} \exp\left(\frac{2\pi i jj'}{2l}\right) \tilde{\chi}_{j'}(\tau, \tilde{z}, t)$$
$$(0 \leqq j \leqq 2l-1),$$

$$\chi_k\left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{\langle z, z\rangle}{2\tau}\right) = \sqrt{\frac{2}{l+2}} \sum_{k'=0}^{l} \sin\left(\frac{(k+1)(k'+1)\pi}{l+2}\right) \chi_{k'}(\tau, z, t)$$
$$(0 \leqq k \leqq l)$$

where

$$\langle \tilde{z}, \tilde{z}\rangle = 2\sum_{i=0}^{2l-1} \tilde{z}_i^2 - 2\sum_{i=0}^{2l-2} \tilde{z}_i \tilde{z}_{i+1}, \qquad \langle z, z\rangle = 2z_1^2 + 2\sum_{i=1}^{l-1} (z_i - z_{i+1})^2.$$

Proposition 2. *Let* $q = e^{2\pi i\tau}$, *and set* $e^l_{jk}(\tau) = q^{\tilde{s}_j - s_k} E^l_{jk}(q)$. *Then the matrix* $\mathscr{E}_l(\tau) = (e^l_{jk}(\tau))_{0\leqq j,k\leqq l}$ *obeys the transformation law*

$$\mathscr{E}_l(\tau+1) = \tilde{D}_l \mathscr{E}_l(\tau) D_l^{-1} \tag{2.5}$$
$$\mathscr{E}_l\left(-\frac{1}{\tau}\right) = \tilde{S}_l \mathscr{E}_l(\tau) S_l$$

where $\tilde{D}_l$, D_l, $\tilde{S}_l = \tilde{S}_l^{-1}$ *and* $S_l = S_l^{-1}$ *are constant matrices given by*

$$(\tilde{D}_l)_{jj'}=\delta_{jj'}e^{2\pi i\tilde{s}_j},\qquad (D_l)_{kk'}=\delta_{kk'}e^{2\pi i s_k},$$

$$\sqrt{2l}\,(\tilde{S}_l)_{jj'}=\begin{cases}2\cos(jj'\pi/l) & (1\leqq j'\leqq l-1)\\ \cos(jj'\pi/l) & (j'=0,l)\end{cases}$$

$$\sqrt{\frac{l+2}{2}}(S_l)_{kk'}=\sin((k+1)(k'+1)\pi/(l+2)).$$

Using (2.5) we can show that, by the substitution $\tau\mapsto 1/(2\tau+1)$, each of the blocks $(e^l_{jk}(\tau))_{j,k\,\text{even}}$ or $(e^l_{jk}(\tau))_{j,k\,\text{odd}}$ of $\mathscr{E}^l(\tau)$ separately undergoes a transformation of the form (2.5). Consequently, their determinants are modular forms with respect to the group $\Gamma_0(2)=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,Z)\,|\,c\equiv 0 \bmod 2\right\}$ which has $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\2&1\end{pmatrix}$ as its generators. In fact we can verify the following by an argument similar to the proof of Proposition 4.18 in [2]: For l odd,

$$\det(e^l_{jk}(\tau))_{j,k\,\text{even}}=\det(e^l_{jk}(\tau))_{j,k\,\text{odd}}=1,$$

for l even

$$\det(e^l_{jk}(\tau))_{j,k\,\text{even}}=\eta(\tau)\eta(2\tau)^{-1}$$
$$\det(e^l_{jk}(\tau))_{j,k\,\text{odd}}=\eta(\tau)^{-1}\eta(2\tau),$$

where $\eta(\tau)=q^{1/24}\varphi(q)$, $q=e^{2\pi i\tau}$ and $\varphi(q)=\prod_{n=0}^{\infty}(1-x^{n+1})$. For example, the formulas above for $l=2,3$ yield the following identities (cf. [3]), respectively:

$$K(q)^2+qL(q)^2=\varphi(q^2)^6/\varphi(q)^3\varphi(q^4)\varphi(q^8)^2$$
$$K(q)=\prod_{n=0}^{\infty}(1-q^{8n+1})^{-1}(1-q^{8n+7})^{-1},$$
$$L(q)=\prod_{n=0}^{\infty}(1-q^{8n+3})^{-1}(1-q^{8n+5})^{-1},$$
$$(2.6)\qquad G(q)G(q^9)+q^2H(q)H(q^9)=\varphi(q^3)^2/\varphi(q)\varphi(q^9)$$
$$G(q)=\prod_{n=0}^{\infty}(1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1},$$
$$H(q)=\prod_{n=0}^{\infty}(1-q^{5n+2})^{-1}(1-q^{5n+3})^{-1}.$$

Remark. For even l, $C^{(1)}_l$ contains $C^{(1)}_{l/2}$ as an invariant subalgebra by the diagram automorphism (see § 5). Restriction of the characters of the pair $A^{(1)}_{2l-1}\supset C^{(1)}_l$ to the Cartan subalgebra of $C^{(1)}_{l/2}$ gives rise to the matrix $(\tilde{e}^l_{jk}(\tau))_{0\leqq j,k\leqq l/2}$, with

$$\tilde{e}^l_{jk}(\tau)=\begin{cases} e^l_{jk}(\tau)+e^l_{jl-k}(\tau) & (0\leqq k<l/2) \\ e^l_{jk}(\tau) & (k=l/2). \end{cases}$$

For this matrix the following can be verified:

$$\det(\tilde{e}^l_{jk}(\tau))_{\substack{0\leqq j,k\leqq l/2\\ j,k\text{ even}}}=\begin{cases} \eta(\tau)^2\eta(2\tau)^{-1}\eta(\tau/2)^{-1} & (l\equiv 2 \bmod 4) \\ 1 & (l\equiv 0 \bmod 4), \end{cases}$$

$$\det(\tilde{e}^l_{jk}(\tau))_{\substack{0\leqq j,k\leqq l/2\\ j,k\text{ odd}}}=\begin{cases} \eta(2\tau)\eta(\tau)^{-1} & (l\equiv 2 \bmod 4) \\ \eta(\tau)\eta(\tau/2)^{-1} & (l\equiv 0 \bmod 4). \end{cases}$$

In the case $l=4$ or 6 they are reduced to

$$\prod_{n=0}^{\infty}(1+q^{6n+2})(1+q^{6n+3})^2(1+q^{6n+4})-2q\prod_{n=0}^{\infty}(1+q^{6n+1})(1+q^{6n+5})(1+q^{6n+6})^2$$
$$=\varphi(q)^2/\varphi(q^6)^2,$$
$$\varphi(q)\varphi(q^4)^2\varphi(q^{18})^3+q\varphi(q^2)^3\varphi(q^9)\varphi(q^{36})^2=\varphi(q^2)\varphi(q^3)^3\varphi(q^4)\varphi(q^{18})\varphi(q^{36})/\varphi(q^6),$$
$$K(q)K(q^9)+q^5L(q)L(q^9)=\varphi(q^2)\varphi(q^3)\varphi(q^{12})\varphi(q^{18})/\varphi(q)\varphi(q^8)\varphi(q^9)(q^{72})$$

where $K(q)$, $L(q)$ are given in (2.6).

§3.

We now proceed to determination of $E^l_{jk}(q)$. This is done by considering the following one-variable specialization (the principal specialization) of characters

$$e^{-\alpha_0}=e^{-\alpha_1}=\cdots=e^{-\alpha_l}=x.$$

It is known in general that the character $\mathrm{ch}_{L(\Lambda)}$ thus specialized reduces to a simple infinite product. In our case we have

$$e^{-\tilde{\Lambda}_j}\,\mathrm{ch}_{L(\tilde{\Lambda}_j)}|_{e^{-\tilde{\alpha}_0}=\cdots=e^{-\tilde{\alpha}_{2l-1}}=x}=\varphi(x^{2l})/\varphi(x) \qquad (0\leqq j\leqq 2l-1),$$
$$e^{-\Lambda_j}\,\mathrm{ch}_{L(\Lambda_j)}|_{e^{-\alpha_0}=\cdots=e^{-\alpha_l}=x}=F_{2l+4,2j+2}(x)/\varphi(x) \qquad (0\leqq j\leqq l),$$

where we set

$$F_{N,r}(x)=\prod_{n=0}^{\infty}(1-x^{Nn+r})(1-x^{Nn+N-r})(1-x^{Nn+N}).$$

Since $q=e^{-\delta}=x^{2l}$, specialization of (2.1) yields an identity among power series in $q^{1/2l}$.

Using Jacobi's triple product identity

$$\prod_{n=0}^{\infty}(1-x^{2n+2})(1-zx^{2n+1})(1-z^{-1}x^{2n+1})=\sum_{n\in Z}(-)^n z^n x^{n^2}$$

we can write down the series expansion of $F_{N,r}(x)$. Collecting like fractional powers we get

$$x^{j^2/2}F_{2l+4,2j+2}(x)=x^{j^2/2}\sum_{n\in Z}(-)^n x^{(l+2)n^2+(-l+2j)n}$$
$$=\sum_{\substack{k\equiv j \bmod 2\\ k \bmod 2l}} x^{k^2/2}P^l_{jk}(q),$$

where

$$P^l_{jk}(q)=\sum_{n\in Z}(-)^{nl+(j-k)/2}q^{Q(l,j,k;n)}$$

(3.1) $$Q(l,j,k;n)=\tfrac{1}{2}l(l+2)n^2-\tfrac{1}{2}(l(j+1)-(l+2)k)n+\tfrac{1}{8}(j-k)(j-k+2).$$

Introducing

$$P^\infty_{jk}(q)=(-)^{(j-k)/2}q^{(j-k)(j-k+2)/8}$$

we can rewrite (3.1) into a more convenient form

(3.2) $$P^l_{jk}(q)=q^{(j+1)^2/4(l+2)}\sum_{j'\equiv j \bmod 2(l+2)}q^{-(j'+1)^2/4(l+2)}P^\infty_{j'k}(q)$$
$$=q^{-k^2/4l}\sum_{k'\equiv k \bmod 2l}q^{k'^2/4l}P^\infty_{j'k}(q).$$

The following symmetry properties are now apparent:

(3.3) $P^l_{-j-2,-k}(q)=-P^l_{jk}(q)$, $\quad P^l_{l+1,k}(q)=P^l_{-l-3,k}(q)$,
$P^l_{j,l}(q)=P^l_{j,-l}(q)$, $\quad q^{-(j+1)^2/4(l+2)}P^l_{jk}(q)$
(resp. $q^{k^2/4l}P^l_{jk}(q)$) is periodic with respect to j (resp. k) with period $2(l+2)$ (resp. $2l$).

We set also

$$\tilde{P}^l_{jk}(q)=\begin{cases}P^l_{jk}(q)+P^l_{j,-k}(q) & (j\equiv k \bmod 2,\ 1\leqq k\leqq l-1)\\ P^l_{jk}(q) & (j\equiv k \bmod 2,\ k=0 \text{ or } l)\\ 0 & (\text{otherwise}).\end{cases}$$

For the moment assume that l is prime. Equating like fractional powers in the specialization of (2.1), we then obtain an equality

$$\varphi(q)\delta_{ik}=\sum_{j=0}^{l}q^{(i-j)/2}E^l_{ij}(q)\tilde{P}^l_{jk}(q).$$

In other words, the matrix $(q^{(i-j)/2}E^l_{ij}(q))$ in question is inverse to the matrix $(\varphi(q)^{-1}\tilde{P}^l_{ij}(q))$. We shall show (Proposition 3) that this is true for any (not necessarily prime) integer $l \geqq 2$.

For μ, $N \in Z$ with $\mu \equiv N \bmod 2$, we put

$$\theta^N_\mu(\tau) = \sum_{n \in Z} (-)^{nN} e^{\pi i N\tau(n+\mu/2N)^2}.$$

They have the transformation property

$$\theta^N_\mu\left(\frac{-1}{\tau}\right) = \sqrt{\frac{-i\tau}{N}} \sum_{\substack{\nu \equiv N \bmod 2 \\ \nu \bmod 2N}} e^{\pi i \mu\nu/2N} \theta^N_\nu(\tau). \tag{3.4}$$

In this notation (3.1) is written as

$$P^l_{jk}(q) = q^{-c_{jk}}(-)^{(j-k)/2}\theta^{l(l+2)}_{l(j+1)-(l+2)k}(\tau) \tag{3.5}$$

with $q = e^{2\pi i\tau}$ and

$$\begin{aligned} c_{jk} &= -\frac{(j+1)^2}{4(l+2)} + \frac{k^2}{4l} + \frac{1}{8} \\ &= s_j - \tilde{s}_k - \frac{1}{2}(j-k) + \frac{1}{24}. \end{aligned}$$

Set further $p^l_{jk}(\tau) = q^{c_{jk}}P^l_{jk}(q)$, $\tilde{p}^l_{jk}(\tau) = q^{c_{jk}}\tilde{P}^l_{jk}(q)$ and $\mathscr{P}_l(\tau) = (\eta(\tau)^{-1}\tilde{p}^l_{jk}(\tau))$, where $\eta(\tau) = q^{1/24}\varphi(q)$ signifies the Dedekind eta function.

Proposition 3. $1 = \mathscr{E}_l(\tau)\mathscr{P}_l(\tau)$.

Proof. Recall that the matrix $\mathscr{E}_l(\tau)$ obeys the transformation law (2.5) with respect to $SL(2, Z)$. We infer that $\mathscr{P}_l(\tau)$ has the corresponding properties

$$\begin{aligned} \mathscr{P}_l(\tau+1) &= D_l \mathscr{P}_l(\tau)\tilde{D}_l^{\,1} \\ \mathscr{P}_l\left(\frac{-1}{\tau}\right) &= S_l\mathscr{P}_l(\tau)\tilde{S}_l. \end{aligned} \tag{3.6}$$

The first one is obvious. To prove the second, we apply (3.4) to (3.5). Since $(l/2, (l+2)/2) = 1$ for even l and $(l, l+2) = 1$ for odd l, $\nu = l(j+1) - (l+2)k$ with $0 \leqq j+1 \leqq l+1$, $-(l-1) \leqq k \leqq l$, $j \equiv k \bmod 2$ (or $-(l+1) \leqq j \leqq 0$, $-l \leqq k \leqq l-1$, $j \equiv k \bmod 2$) runs over the set $\{\nu \in Z/2NZ \,|\, \nu \equiv N \bmod 2\}$, $N = l(l+2)$. This gives

$$\begin{aligned} &\sqrt{\frac{l(l+2)}{-i\tau}}\, p^l_{jk}\left(\frac{-1}{\tau}\right) \\ &= i \sum_{\substack{0 \leqq j'+1 \leqq l+1 \\ -(l-1) \leqq k' \leqq l \\ j' \equiv k' \bmod 2}} \exp\left(\left(-\frac{(j+1)(j'+1)}{l+2} + \frac{kk'}{l}\right)\pi i\right) p^l_{j'k'}(\tau) \end{aligned}$$

$$=i\sum_{\substack{-l-1\leq j'+1\leq 0\\ -l\leq k'\leq l-1\\ j'\equiv k' \bmod 2}}\exp\left(\left(-\frac{(j+1)(j'+1)}{l+2}+\frac{kk'}{l}\right)\pi i\right)p^{l}_{j'k'}(\tau).$$

Along with the symmetry (3.3) we obtain (3.6). Formulas (2.5) and (3.6) show that each of the matrix elements of $\mathscr{E}_l(\tau)\mathscr{P}_l(\tau)$ is invariant under the action of the full modular group $SL(2, Z)$. To prove $\mathscr{E}_l(\tau)\mathscr{P}_l(\tau)=$constant $(=1)$, it then suffices to show the estimate $(\mathscr{E}_l(\tau)\mathscr{P}_l(\tau))_{jk}-\delta_{jk}=0(1)$ as $\tau\to i\infty$. This can be checked by using (2.4), noting that only the terms with $n=0$ or $n=1$ in (3.1) have possible contributions to the lowest order.

The next step is to compute the inverse matrix of $\mathscr{P}_l(\tau)$. Namely we are to solve the linear equations of the form

$$\sum_{k=0}^{l}\tilde{P}^{l}_{jk}(q)V^{(i)}_k=\delta_{ji}\qquad(0\leq i,j\leq l)\tag{3.7}$$

with $V^{(i)}_k=\varphi(q)^{-1}q^{(k-i)/2}E^{l}_{ki}(q)$. Let us extend the range of suffixes of $V^{(i)}_k$ by requiring for $0\leq k\leq l-1$

$$V^{(i)}_{-k}=V^{(i)}_k.$$

The equation (3.7) then reads

$$\sum_{\substack{-l+1\leq k\leq l\\ k\equiv j \bmod 2}}P^{l}_{jk}(q)V^{(i)}_k=\delta_{ji}\qquad(0\leq i,j\leq l).$$

For $-l-3\leq j\leq -1$, we have, by using (3.3)

$$\begin{aligned}\sum_{\substack{-l+1\leq k\leq l\\ k\equiv j \bmod 2}}P^{l}_{jk}(q)V^{(i)}_k&=-\sum_{\substack{-l\leq k\leq l-1\\ k\equiv j \bmod 2}}P^{l}_{-j-2,-k}(q)V^{(i)}_k\\&=-\sum_{\substack{-l+1\leq k\leq l\\ k\equiv j \bmod 2}}P^{l}_{-j-2,-k}(q)V^{(i)}_k\\&=-\delta_{-j-2,i}.\end{aligned}$$

Note that, for $j=-1$,

$$\sum_{\substack{-l+1\leq k\leq l\\ k\equiv -1 \bmod 2}}P^{l}_{-1k}(q)V^{(i)}_k=-\sum_{\substack{-l+1\leq k\leq l\\ k\equiv -1 \bmod 2}}P^{l}_{-1-k}(q)V^{(i)}_k=0.$$

Likewise we have

$$\begin{aligned}\sum_{\substack{-l+1\leq k\leq l\\ k\equiv l+1 \bmod 2}}P^{l}_{l+1\,k}(q)V^{(i)}_k&=\sum_{\substack{-l+1\leq k\leq l\\ k\equiv -l-3 \bmod 2}}P^{l}_{-l-3\,k}(q)V^{(i)}_k\\&=-\sum_{\substack{-l+1\leq k\leq l\\ k\equiv l+1 \bmod 2}}P_{l+1\,k}(q)V^{(i)}_k=0.\end{aligned}$$

Thus we obtain

$$\sum_{\substack{-l+1\leqq k\leqq l\\ k\equiv j \bmod 2}} P^l_{jk}(q)V^{(i)}_k=\delta_{ji}-\delta_{-j-2,i} \qquad (0\leqq i\leqq l,\ -l-2\leqq j\leqq l+1).$$

By virtue of the quasi-periodicity of P^l_{jk} in j, this can be extended to all values of j as

$$\sum_{\substack{-l+1\leqq k\leqq l\\ k\equiv j \bmod 2}} q^{-(j+1)^2/4(l+2)}P^l_{jk}(q)V^{(i)}_k=q^{-(i+1)^2/4(l+2)}\sum_{i'\equiv i \bmod 2l+4}(\delta_{ji'}-\delta_{j,-i'-2})$$

$$(0\leqq i\leqq l,\ j\in \boldsymbol{Z}).$$

Finally, substituting (3.2) we obtain

$$\text{(3.8)}\qquad \sum_{\substack{k\in \boldsymbol{Z}\\ k\equiv j \bmod 2}} P^\infty_{jk}(q)V^{(i)}_k=q^{((j+1)^2-(i+1)^2/4(l+2)}\sum_{i'\equiv i \bmod 2l+4}(\delta_{ji'}-\delta_{j,-i'-2})$$

where $V^{(i)}_k$ is defined for all $k\in \boldsymbol{Z}$ through

$$q^{-k^2/4l}V^{(i)}_k=q^{-k'^2/4l}V^{(i)}_{k'} \qquad \text{if } k'\equiv k \bmod 2l.$$

In (3.8) the coefficients $P^\infty_{jk}(q)$ depend only on the difference $j-k$. Hence it can be easily solved by taking Fourier transforms. The result is as follows:

$$\sum_{k\in \boldsymbol{Z}} z^kV^{(i)}_k=\left(\prod_{n=0}^{\infty}(1-q^{n+1})(1-z^2q^{n+1})(1-z^{-2}q^n)\right)^{-1}$$
$$\times\sum_{\nu\in \boldsymbol{Z}} q^{(l+2)\nu^2+(i+1)\nu}(z^{2(l+2)\nu+i}-z^{-2(l+2)\nu-i-2}).$$

A residue calculus then gives $\varphi(q)^3V^{(i)}_k$ as a double series. Returning to the original notation, we obtain

$$\begin{aligned}\text{(3.9)}\qquad &\varphi(q)^2q^{(j-k)/2}E^l_{jk}(q)=\varphi(q)^2E^l_{l-j,l-k}(q)\\ &=\Big(\sum_{m\geqq 0,n\geqq 0}-\sum_{m<0,n<0}\Big)(-)^nq^{n(n+1)/2+(k+1)m+(-j+k)n/2+(l+2)m(m+n)}\\ &+\Big(\sum_{m\geqq 0,n>0}-\sum_{m<0,n\leqq 0}\Big)(-)^nq^{n(n+1)/2+(k+1)m+(j+k)n/2+(l+2)m(m+n)}\end{aligned}$$

for $j\equiv k \bmod 2$, $0\leqq j\leqq k\leqq l$. Replacing $m+n/2$ by m, and separating the sum over n into even and odd parts, we rewrite (3.9) as

$$\begin{aligned}\text{(3.10)}\qquad &\eta(\tau)^2e^l_{jk}(\tau)=\eta(\tau)^2e^l_{l-j,l-k}(\tau)\\ &=\Big(\sum_{m\geqq n\geqq 0}+\sum_{\substack{m\geqq 0>n\\ m+n\geqq 0}}-\sum_{0>n>m}-\sum_{\substack{n\geqq 0>m\\ 0>m+n}}\Big)q^{(l+2)(m+(k+1)/2(l+2))^2-l(n+j/2l)^2}\\ &+\Big(\sum_{0>n>m}+\sum_{\substack{n\geqq 0\geqq m\\ 0>m+n+1}}-\sum_{m\geqq n\geqq 0}-\sum_{\substack{m\geqq 0>n\\ m+n+1>0}}\Big)q^{(l+2)(m+1/2+(k+1)/2(l+2))^2-l(n+1/2+j/2l)^2}\end{aligned}$$

$$(j\equiv k \bmod 2,\ 0\leqq j\leqq k\leqq l)$$

As we have mentioned in the introduction, (3.10) is expressible in terms of a theta series associated with an indefinite binary quadratic form, which happens to be exactly the same for the string functions (1.2) for $A_1^{(1)}$ ([1], [2]).

Let $L \subset \boldsymbol{R}^2$ be a lattice of rank 2, and let $B(\gamma, \gamma)$ ($\gamma \in \boldsymbol{R}^2$) denote an indefinite binary quadratic form such that $B(\gamma, \gamma) \in 2\boldsymbol{Z}$ for $\gamma \in L$. Let $L^* = \{\gamma \in \boldsymbol{R}^2 \mid B(\gamma, \gamma') \in \boldsymbol{Z}$ for all $\gamma' \in L\}$ denote the dual lattice. Let further $G = \{g \in O(B) \mid gL \subset L\}$ and $G_0 = \{g \in G \cap SO_0(B) \mid g$ leaves L^*/L pointwise fixed$\}$. Finally, take a decomposition $B(\gamma, \gamma) = l_1(\gamma) l_2(\gamma)$ into real linear forms $l_i(\gamma)$. For $\mu \in L^*$ we set

$$\theta^B_{L,\mu}(\tau) = \sum_{\substack{\gamma \in G_0 \backslash (L+\mu) \\ B(\gamma,\gamma)>0}} \operatorname{sign} l_1(\gamma) \cdot e^{\pi i \tau B(\gamma,\gamma)}. \tag{3.11}$$

This type of series has been studied by Hecke [7], and is called in [2] a Hecke indefinite modular form. They have the following transformation properties:

$$\begin{aligned} &\theta^B_{L,\mu}(\tau+1) = e^{\pi i B(\mu,\mu)} \theta^B_{L,\mu}(\tau), \\ &\theta^B_{L,\mu}\left(\frac{-1}{\tau}\right) = \frac{-\tau}{\sqrt{|L^*/L|}} \sum_{\nu \in L^*/L} e^{2\pi i B(\mu,\nu)} \theta^B_{L,\nu}(\tau). \end{aligned} \tag{3.12}$$

Note that (3.11) are not all linearly independent, for we have the relation

$$\theta^B_{L,g\mu}(\tau) = \varepsilon(g) \theta^B_{L,\mu}(\tau) \qquad \text{for } g \in G \tag{3.13}$$

where $\varepsilon(g) = \pm 1$ is defined by $\operatorname{sign} l_1(g\gamma) = \varepsilon(g) \operatorname{sign} l_1(\gamma)$ for $B(\gamma, \gamma) > 0$.

Now we take $L = \boldsymbol{Z}^2$ and set

$$B(\gamma, \gamma) = 2(l+2)x^2 - 2ly^2 = l_1(\gamma) l_2(\gamma) \qquad \text{for } \gamma = \begin{pmatrix} x \\ y \end{pmatrix},$$

with $l_{\substack{1\\2}}(\gamma) = \sqrt{2(l+2)}\, x \pm \sqrt{2l}\, y$.

Then $L^* = (1/2(l+2))\boldsymbol{Z} \oplus (1/2l)\boldsymbol{Z}$. The group G is generated by

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} l+1 & l \\ l+2 & l+1 \end{pmatrix} \in G,$$

whereas G_0 is a cyclic group generated by a^2. If we set $F = \{(x, y) \mid -|x| < y \leqq |x|\}$, then F (resp. $F \cup a(F)$) is a fundamental region for $G \cap SO_0(B)$ (resp. G_0). Writing down the sum (3.11) explicitly and comparing with (3.10), we arrive at the conclusion

Theorem.

$$\eta(\tau)^2 e^l_{jk}(\tau) = \theta^B_{L,\mu}(\tau), \qquad \mu = \begin{bmatrix} \dfrac{k+1}{2(l+2)} \\ \dfrac{j}{2l} \end{bmatrix}.$$

§4.

In this section we consider the case $A^{(1)}_{2l} \supset A^{(2)}_{2l}$ by embedding it into the case $A^{(1)}_{4l+1} \supset C^{(1)}_{2l+1}$.

By a technical reason we need a slight modification of our formulation of the problem. So far, we included the derivation d corresponding to the 0-th node in the definition of algebras. For the case $A^{(1)}_{2l-1} \supset C^{(1)}_{l}$ (or $A^{(1)}_{2l} \supset A^{(2)}_{2l}$) we could identify d for both algebras. Now we need to consider the embedding $A^{(1)}_{4l+1} \supset A^{(1)}_{2l}$ and $C^{(1)}_{2l+1} \supset A^{(2)}_{2l}$, for which we cannot choose a common derivation. To avoid technical complexity we shall consider algebras without derivations in this and the next section. Then, we consider $e^{-\Lambda}\,\mathrm{ch}_{L(\Lambda)}$ as a formal power series of $q = e^{-\delta}$ and $e^{-\alpha_1}$, $e^{-\alpha_2}$, etc., rather than $\mathrm{ch}_{L(\Lambda)}$ as a function on the Cartan subalgebra. The decomposition reads as

$$e^{-\tilde{\Lambda}}\,\mathrm{ch}_{L(\tilde{\Lambda})}|_{\mathfrak{h}} = \sum_{\substack{\Lambda \in P_+ \\ \Lambda(c) = \tilde{\Lambda}(c)}} \check{E}_{\tilde{\Lambda}\Lambda} \cdot e^{-\Lambda}\,\mathrm{ch}_{L(\Lambda)}$$

with $E_{\tilde{\Lambda}\Lambda}(q) = e^{\tilde{\Lambda}|_{\mathfrak{h}} - \Lambda}\check{E}_{\tilde{\Lambda}\Lambda}$ depending only on q. Here $|_{\mathfrak{h}}$ means the appropriate specialization of variables. In the following we refer to $E_{\tilde{\Lambda}\Lambda}(q)$ as the coefficients of the decomposition of $(\cdots, \mathrm{ch}_{L(\tilde{\Lambda})}, \cdots)$ with regards to $(\cdots, \mathrm{ch}_{L(\Lambda)}, \cdots)$.

Let, $\iota_1, \iota_2 \in \mathfrak{S}_{4l+2}$ be involutive automorphisms of the Dynkin diagram of $A^{(1)}_{4l+1}$ given by

$$\iota_1(j) \equiv j + 2l + 1 \qquad (\mathrm{mod}\ 4l+2),$$
$$\iota_2(j) \equiv 4l + 2 - j \qquad (\mathrm{mod}\ 4l+2).$$

They induce involutive automorphisms of $A^{(1)}_{4l+1}$, which we also denote by ι_1 and ι_2. Then we have

$$\mathfrak{g}_1 \underset{\mathrm{def}}{=} \{X \in A^{(1)}_{4l+1} \mid \iota_1(X) = X\} \cong A^{(1)}_{2l},$$
$$\mathfrak{g}_2 \underset{\mathrm{def}}{=} \{X \in A^{(1)}_{4l+1} \mid \iota_2(X) = X\} \cong C^{(1)}_{2l+1},$$
$$\mathfrak{g}_3 \underset{\mathrm{def}}{=} \{X \in \mathfrak{g}_1 \mid \iota_2(X) = X\}$$
$$= \{X \in \mathfrak{g}_2 \mid \iota_1(X) = X\} \cong A^{(2)}_{2l}.$$

The Chevalley basis for $\mathfrak{g}_i$ $(i=1, 2, 3)$ is chosen by the following rule: Let $\mathfrak{g}$ be a subalgebra of $\tilde{\mathfrak{g}}$ induced by its involution ι. Let $\tilde{e}_j, \tilde{f}_j, \tilde{h}_j$ denote the Chevalley basis for $\tilde{\mathfrak{g}}$ and let e_j, f_j, h_j denote that for $\mathfrak{g}$. Then

1) $e_j=\tilde{e}_j, f_j=\tilde{f}_j, h_j=\tilde{h}_j$ if $\iota(j)=j$,

2) $e_j=\tilde{e}_j+\tilde{e}_{\iota(j)}, f_j=\tilde{f}_j+\tilde{f}_{\iota(j)}, h_j=\tilde{h}_j+\tilde{h}_{\iota(j)}$ if $\iota(j)\neq j$ and $\iota(j)$ is disconnected with j,

3) $e_j=\tilde{e}_j+\tilde{e}_{\iota(j)}, f_j=2(\tilde{f}_j+\tilde{f}_{\iota(j)}), h_j=2(\tilde{h}_j+\tilde{h}_{\iota(j)})$ if $\iota(j)\neq j$ and $\iota(j)$ is connected with j.

Thus we have the following diagram.

$$
\begin{array}{ccccc}
A^{(1)}_{4l+1}\supset C^{(1)}_{2l+1} & \quad & e^{-\tilde{\Lambda}_j}\,\mathrm{ch}_{L(\tilde{\Lambda}_j)} & \xrightarrow{\check{E}^{2l+1}_{jk}} & e^{-\Lambda_k}\,\mathrm{ch}_{L(\Lambda_k)} \\
\cup \quad \cup & & \Big| \check{G}^{2l}_{jj'} & & \Big| \check{H}^{l}_{kk'} \\
A^{(1)}_{2l}\supset A^{(2)}_{2l} & & e^{-\tilde{\Lambda}'_{j'}}\,\mathrm{ch}_{L(\tilde{\Lambda}'_{j'})} & \xrightarrow{\check{F}^{l}_{j'k'}} & e^{-\Lambda'_{k'}}\,\mathrm{ch}_{L(\Lambda'_{k'})}
\end{array}
\tag{4.1}
$$

Fig. 2. Commutative diagram and decomposition of characters

We remark that the numbering of nodes in the Dynkin diagram for $A^{(2)}_{2l}$ here is different from that in [2]. Namely, we have

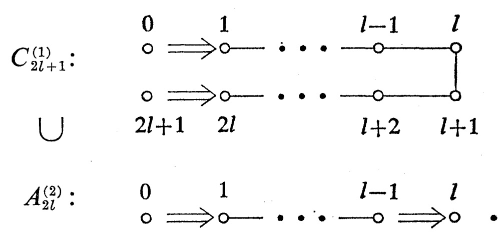

Fig. 3. Dynkin diagrams for $C^{(1)}_{2l+1}$ and $A^{(2)}_{2l}$

In the previous section we determined the coefficients of the decomposition of $(\mathrm{ch}_{L(\tilde{\Lambda}_0)}, \cdots, \mathrm{ch}_{L(\tilde{\Lambda}_{2l+1})})$ for $A^{(1)}_{4l+1}$ with regards to $(\mathrm{ch}_{L(\Lambda_0)}, \cdots, \mathrm{ch}_{L(\Lambda_{2l+1})})$ for $C^{(1)}_{2l+1}$, which we denoted by $E^{2l+1}_{jk}(q)$. Now denote by $\tilde{\Lambda}'_j$ $(j=0, \cdots, 2l)$ (resp. Λ'_j $(j=0, \cdots, l)$) the fundamental weights for $A^{(1)}_{2l}$ (resp. $A^{(2)}_{2l}$), and denote by $F^l_{jk}(q)$ $(0\leqq j, k\leqq l)$ the coefficients of the decomposition of $(\mathrm{ch}_{L(\tilde{\Lambda}'_0)}, \cdots, \mathrm{ch}_{L(\tilde{\Lambda}'_l)})$ with regards to $(\mathrm{ch}_{L(\Lambda'_0)}, \cdots, \mathrm{ch}_{L(\Lambda'_{l-1})}, \mathrm{ch}_{L(2\Lambda'_l)})$. We show that

Proposition 4.

$$F^l_{jk}(q)=E^{2l+1}_{j^*,k^*}(q^2) \qquad (j, k=0, \cdots, l).$$

where $j^*=j$ (*for* j *even*), $=2l+1-j$ (*for* j *odd*).

Let us denote by $G^{2l}_{jk}(q)$ $(0 \leqq j \leqq 2l+1,\ 0 \leqq k \leqq 2l)$ the coefficients of the decomposition of $(\mathrm{ch}_{L(\tilde{\Lambda}_0)}, \cdots, \mathrm{ch}_{L(\tilde{\Lambda}_{2l+1})})$ with regards to $(\mathrm{ch}_{L(\tilde{\Lambda}'_0)}, \cdots, \mathrm{ch}_{L(\tilde{\Lambda}'_{2l})})$, and by $H^{l}_{jk}(q)$ $(j, k=0, \cdots, l)$ those for the pair $(\mathrm{ch}_{L(\Lambda_0)}, \cdots, \mathrm{ch}_{L(\Lambda_l)})$ and $(\mathrm{ch}_{L(\Lambda'_0)}, \cdots, \mathrm{ch}_{L(\Lambda'_{l-1})}, \mathrm{ch}_{L(2\Lambda'_0)})$ (See Fig. 2). Then Proposition 4 follows from the diagram (4.1) and the following.

Proposition 5.

$$(4.2)\qquad G^{2l}_{jk}=\delta_{jk}\varphi(q^2)/\varphi(q) \qquad (j, k=0, \cdots, 2l),$$

$$G^{2l}_{2l+1,k}(q)=\delta_{0k}\varphi(q^2)/\varphi(q) \qquad (k=0, \cdots, 2l)$$

$$(4.3)\qquad H^{l}_{jk}(q)=\delta_{jk}\varphi(q^2)/\varphi(q) \qquad (j, k=0, \cdots, 2l).$$

Proof. The vanishing of the off diagonal elements of $(G^{2l}_{jk}(q))$ follows from the same reasoning with (2.2). Then by computing the principal specialization we can deduce (4.2). To prove (4.3) we exploit the vertex representation of $\mathfrak{gl}(\infty)$ [3], [6]. By considering $C^{(1)}_{2l+1}$ and $A^{(2)}_{2l}$ as subalgebras of $\mathfrak{gl}(\infty)$, the highest weight modules $(L(\Lambda_0), \cdots, L(\Lambda_l))$ and $(L(\Lambda'_0), \cdots, L(\Lambda'_{l-1}), L(2\Lambda'_l))$ are realized in the polynomial ring $\boldsymbol{C}[x_1, x_2, x_3, \cdots]$. The vertex representation of $C^{(1)}_{2l+1}$ contains the multiplication by x_n with odd n. On the other hand, that of $A^{(1)}_{2l}$ does not contain operators depending on x_n or $\partial/\partial x_n$ with $n=2l+1, 2(2l+1), 3(2l+1), \cdots$. Thus $L(\Lambda_j)$ $(j=0, \cdots, l)$ realized in $\boldsymbol{C}[x_1, x_2, x_3, \cdots]$ contains a subspace $V \cong \boldsymbol{C}[x_{2l+1}, x_{3(2l+1)}, x_{5(2l+1)}, \cdots] \otimes L(\Lambda'_j)$ $(j \neq l)$ (or $\cong \boldsymbol{C}[x_{2l+1}, x_{3(2l+1)}, x_{5(2l+1)}, \cdots] \otimes L(2\Lambda'_l)$ $(j=l)$). Note that $\sum_{n=0}^{\infty} \dim V_n x^n = \varphi(x^{4l+2})/\varphi(x^{2l+1})$ where $V_n=\{f \in V \mid \deg f=n\}$. Identifying q with x^{2l+1}, denote by $\chi_j(x)$, $\chi'_j(x)$ the principally specialized characters $\mathrm{ch}_{L(\Lambda_j)}$, $\mathrm{ch}_{L(\Lambda'_j)}$ (or $\mathrm{ch}_{L(2\Lambda'_l)}$ if $j=l$) respectively. Then considerations above imply

$$(4.4)\qquad \chi_j(x) \gg H^{l}_{jj}(q)\chi'_j(x) \gg \frac{\varphi(q^2)}{\varphi(q)}\chi'_j(x).$$

(The notation $F(x) \gg G(x)$ signifies that all the coefficients of the series $F(x)-G(x)$ are non-negative). On the other hand, a direct computation shows

$$(4.5)\qquad \chi_j(x)=\frac{\varphi(q^2)}{\varphi(q)}\chi'_j(x).$$

(4.4) and (4.5) prove the equality (4.3).

§ 5.

This section is devoted to the case $C^{(1)}_{2l} \supset C^{(1)}_{l}$ (Fig. 4). We denote by $\iota \in \mathfrak{S}_{2l+1}$ the following involution of the Dynkin diagram of $C^{(1)}_{2l}$:

$$\iota(j)=2l-j \qquad (j=0,\cdots,2l).$$

Then we have

(5.1) $$C_{2l}^{(1)}\supset\{X\in C_{2l}^{(1)}\,|\,\iota(X)=X\}\cong C_l^{(1)}$$

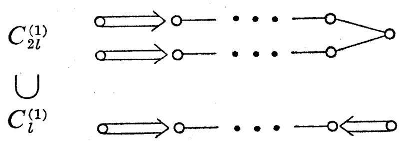

Fig. 4. Dynkin diagrams for $C_{2l}^{(1)}$ and $C_l^{(1)}$

We denote by δ the null root in $C_l^{(1)}$ and set $q=e^{-\delta}$. Note that the specialization of the null root in $C_{2l}^{(1)}$ with respect to the embedding (5.1) gives rise to 2δ.

We denote by $J_{jk}^l(q)$ $(0\le j,k\le l)$ the coefficients of the decomposition of $(\mathrm{ch}_{L(\tilde{\Lambda}_0)},\cdots,\mathrm{ch}_{L(\tilde{\Lambda}_l)})$ with regards to $(\mathrm{ch}_{L(\Lambda_0)},\cdots,\mathrm{ch}_{L(\Lambda_l)})$, where $\tilde{\Lambda}_j$'s (resp. Λ_k's) denote the fundamental weights of $C_{2l}^{(1)}$ (resp. $C_l^{(1)}$). Note that $J_{jk}^l(q)=0$ if $j\not\equiv k \bmod 2$.

We introduce a quadratic form B' and the associated modular forms: We set

$$B'(\gamma,\gamma)=2(l+2)x^2-8(l+1)y^2, \qquad \gamma=\begin{pmatrix}x\\y\end{pmatrix},$$

and

$$l_{\substack{1\\2}}(\gamma)=\sqrt{2(l+2)}x\pm\sqrt{8(l+1)}y\,.$$

If we set $L=\boldsymbol{Z}\oplus\boldsymbol{Z}$, then the dual lattice is given by $L^*=\boldsymbol{Z}/2(l+2)\oplus\boldsymbol{Z}/8(l+1)$. Let

$$a=\begin{pmatrix}2l+3 & 4l+4\\ l+2 & 2l+3\end{pmatrix}\in SO(B'),$$

and

$$G_0=\{a^{2n}\,|\,n\in\boldsymbol{Z}\}.$$

Then G_0 fixes L^*/L elementwise. Now we define $\theta_{L,\mu}^{B'}(\tau)$ by (3.11). Identifying q with $e^{2\pi i\tau}$, we obtain the following explicit formula for $J_{jk}(q)$.

Theorem 2.

(5.2) $$\eta(2\tau)\eta(\tau)q^{1/8+(j-k)/2-(j+1)^2/4(l+1)+(k+1)^2/4(l+2)}J_{jk}^l(q)=\theta_{L,\mu}^{B'}(\tau),$$
$$(j,k=0,\cdots,l)$$

where

$$\mu=\begin{bmatrix}\dfrac{1}{2}+\dfrac{k+1}{2(l+2)}\\ \dfrac{1}{4}+\dfrac{j+1}{4(l+1)}\end{bmatrix}.$$

Proof. We exploit the following commutative diagram.

$$\begin{array}{ccc}A^{(1)}_{4l-1} & \supset & C^{(1)}_{2l}\\ \cup & & \cup\\ A^{(1)}_{2l-1} & \supset & C^{(1)}_{l}\end{array}$$

Fig. 5. Commutative diagram $A^{(1)}_{4l-1}\supset C^{(1)}_{2l}$ and $A^{(1)}_{2l-1}\supset C^{(1)}_{l}$

Since we know the decompositions for $A^{(1)}_{4l-1}\supset A^{(1)}_{2l-1}$, $A^{(1)}_{4l-1}\supset C^{(1)}_{2l}$ and $A^{(1)}_{2l-1}\supset C^{(1)}_{l}$, by a similar argument as in Section 4, we have

$$q^{(j-i)/2}J^{l}_{ji}(q)=\sum_{k=0}^{2l}\tilde{P}^{2l}_{jk}(q^2)V^{l}_{ki}(q),\qquad (5.3)$$

$$(j=0,\cdots,2l,\ i=0,\cdots,l),$$

where $V^{l}_{ki}(q)$ denotes $V^{(i)}_{k}$ in (3.7). Let us define $J^{l}_{ji}(q)$ for arbitrary j, i by the right hand side of (5.3). Then we have

$$\begin{aligned}&\sum_{j\in Z}z^{j}\varphi(q^2)\varphi(q)q^{(j-i)/2}J^{l}_{ji}(q)\\ &=\frac{\prod_{n=1}^{\infty}(1-q^{2n})^2}{\prod_{n=0}^{\infty}(1-z^2q^{2n+1})(1-z^{-2}q^{2n+1})}\\ &\quad\times\sum_{m\in Z}q^{(l+2)m^2+(i+1)m}(z^{2(l+2)m+i}-z^{-2(l+2)m-i-2}).\end{aligned}$$

A residue calculus gives the following sum: For $0\leqq j\leqq k\leqq l$ (resp. $0\leqq k<j\leqq l$)

$$\begin{aligned}&\varphi(q^2)\varphi(q)q^{(j-k)/2}J_{jk}(q)\\ (5.4)\quad &=\Big(\sum_{\substack{m\geqq 0\\ n\geqq 0(\text{resp.}\,n>0)}}-\sum_{\substack{m<0\\ n<0(\text{resp.}\,n\leqq 0)}}\Big)(-)^{n}q^{n(n+1)+(l+2)m^2+(k+1)m+(2n+1)((l+2)m+(k-j)/2)}\\ &\quad-\Big(\sum_{\substack{m\geqq 0\\ n\geqq 0}}-\sum_{\substack{m<0\\ n<0}}\Big)(-)^{n}q^{n(n+1)+(l+2)m^2+(k+1)m+(2n+1)((l+2)m+(k+j+2)/2)}\end{aligned}$$

Finally, by rearranging the summation in (5.4), we obtain (5.2).

As for the determinant of (J_{jk}) we have the following.

Proposition 6.

$$\left.\begin{aligned}\det(J_{jk})_{j,k=0,2,\cdots,l}&=\varphi(q)^{-l/2}\varphi(q^2)^{l/2}\\ \det(J_{jk})_{j,k=1,3,\cdots,l-1}&=\varphi(q)^{-l/2}\varphi(q^2)^{-l/2}\end{aligned}\right\}\quad l\text{: even,}$$

$$\left.\begin{aligned}\det(J_{jk})_{j,k=0,2,\cdots,l-1}&=\varphi(q)^{(1-l)/2}\varphi(q^2)^{(l-3)/2}\varphi(q^4)\\ \det(J_{jk})_{j,k=1,3,\cdots,l}&=\varphi(q)^{(-1-l)/2}\varphi(q^2)^{(3-l)/2}\varphi(q^4)^{-1}\end{aligned}\right\}\quad l\text{: odd.}$$

For the proof, we set

$$\theta^l_{m,n}(\tau)=\theta^{B'}_{L,\mu}(\tau),\ \mu=\begin{bmatrix}\dfrac{m}{2(l+2)}\\ \dfrac{n}{8(l+1)}\end{bmatrix}.$$

In the present case the coefficients of the decomposition $J_{jk}(q)$ ($j, k=0, \cdots, l$) corresponds only to those $\theta_{m,n}(\tau)$ with even n. Let us denote by $T_0(\tau)$ the matrix $(\theta^l_{m,n}(\tau))_{\substack{m=1,2,\cdots,l+1\\ n=0,2,\cdots,2l}}$. Because of the symmetry relation (3.13) we have $\theta^l_{2m+1,4n+2}(\tau)=0$ and $\theta^l_{2m,4n}(\tau)=0$. In other words, the square matrix $T_0(\tau)$ splits into two blocks:

$$T_1(\tau)=(\theta^l_{m,n}(\tau))_{\substack{m=1,3,\cdots,2[l/2]+1\\ n=0,4,\cdots,4[l/2]}}$$

and

$$T_2(\tau)=(\theta^l_{m,n}(\tau))_{\substack{m=2,4,\cdots,2[(l+1)/2]\\ n=2,6,\cdots,4[(l+1)/2]-2}}.$$

Then Proposition 6 follows from

Proposition 7.

$$\det T_1(\tau)=\begin{cases}\eta(\tau)\eta(2\tau)^{l+1} & l\text{: even,}\\ \eta(2\tau)^{l+2}\eta(4\tau)^{-1} & l\text{: odd}\end{cases}$$

$$\det T_2(\tau)=\begin{cases}\eta(2\tau)^{l} & l\text{: even,}\\ \eta(\tau)\eta(2\tau)^{l-1}\eta(4\tau) & l\text{: odd.}\end{cases}$$

Proof. The transformation property (3.12) implies

$$\theta^{B'}_{L,\mu}\left(\frac{\tau}{-2\tau+1}\right)=\frac{2\tau-1}{16(l+1)(l+2)}\sum_{\mu'\in L^*/L}s(\mu,\mu')\theta^{B'}_{L,\mu'}(\tau)$$

with

$$s(\mu,\mu')=\sum_{\lambda\in L^*/L}e^{2\pi i(B(\mu+\mu',\lambda)+B(\lambda,\lambda))}.$$

Note that for $\lambda'=\lambda+\begin{pmatrix}0\\1/2\end{pmatrix}$ we have $B(\lambda', \lambda')=B(\lambda, \lambda)$, and for $\lambda''=\lambda+\begin{pmatrix}1/2\\0\end{pmatrix}$ we have

$$B(\lambda'', \lambda'')=\begin{cases}B(\lambda, \lambda) & l\text{: even}\\ B(\lambda, \lambda)+\frac{1}{2} & l\text{: odd.}\end{cases}$$

If we set

$$\mu=\begin{bmatrix}\dfrac{m}{2(l+2)}\\ \dfrac{n}{8(l+1)}\end{bmatrix} \quad\text{and}\quad \mu'=\begin{bmatrix}\dfrac{m'}{2(l+2)}\\ \dfrac{n'}{8(l+1)}\end{bmatrix},$$

then

$$B(\mu+\mu', \lambda'-\lambda)=-\frac{n+n'}{2} \quad\text{and}\quad B(\mu+\mu', \lambda''-\lambda)=\frac{m+m'}{2}.$$

Thus we can show that

$$s(\mu, \mu')=0 \qquad \text{if } n\not\equiv n' \bmod 2$$

and

$$s(\mu, \mu')=0 \qquad \begin{array}{ll}\text{if} & l\text{: even,} \quad m\not\equiv m' \bmod 2\\ \text{or} & l\text{: odd,} \quad m\equiv m' \bmod 2.\end{array}$$

By using (3.13) we obtain

$$T_j\left(\frac{\tau}{-2\tau+1}\right)=(2\tau-1)S_jT_{j*}(\tau)S'_j, \qquad j=1, 2, \tag{5.5}$$

with some matrices S_j and S'_j. Here $1^*=1$, $2^*=2$ (l: even) and $1^*=2$, $2^*=1$ (l: odd), respectively. By a similar argument we can show that

$$T_j\left(\frac{\tau}{-4\tau+1}\right)=(4\tau-1)\tilde{S}_jT_j(\tau)\tilde{S}'_j, \qquad j=1, 2, \tag{5.6}$$

with some matrices $\tilde{S}_j$ and $\tilde{S}'_j$.

The transformation property (3.12) implies that $\det T_0(\tau)=\det T_1(\tau)\cdot\det T_2(\tau)$ is a modular form of weight $l+1$ with some multiplier system for $\Gamma_0(2)$. We can show that for $\tau\to i\infty$

$$\theta^l_{m,n}(\tau)=O(q^{m^2/4(l+2)-(2m-1)^2/16(l+1)}), \qquad 1\leq m\leq l+1, \tag{5.7}$$

and

$$\theta^l_{m,2n}(\tau)=O(q^{m^2/4(l+2)-(m-1)^2/4(l+1)}), \qquad 1\leq m\leq l+1. \tag{5.8}$$

Using (3.13), (5.7) and (5.8) we can estimate the order of zeros at cusps 0 and ∞ for det $T_0(\tau)$, and we obtain det $T_0(\tau)=\eta(\tau)\eta(2\tau)^{2l+1}$.

The transformation property (5.5) implies, for even l, each det $T_j(\tau)$ ($j=1, 2$) is itself a modular form with some multiplier system for $\Gamma_0(2)$. Using (5.8) we can estimate the order of zeros at ∞ for them to show that

$$t_1(\tau)\underset{\text{def}}{=}\frac{\det T_1(\tau)}{\eta(\tau)\eta(2\tau)^{l+1}}=O(1)$$

and

$$t_2(\tau)\underset{\text{def}}{=}\frac{\det T_2(\tau)}{\eta(2\tau)^l}=O(1).$$

Since we know that $t_1(\tau)\cdot t_2(\tau)=1$, one of $t_j(\tau)$ ($j=1, 2$) does not have a pole at the cusp 0, and hence it is equal to 1. Hence we conclude $t_j(\tau)=1$ ($j=1, 2$).

For odd l, (5.6) implies that det T_j ($j=1, 2$) is a modular form with some multiplier system for $\Gamma_0(4)$. Among the cusps 0, 1/2 and ∞ for $\Gamma_0(4)$, we can easily estimate the order of zero at ∞ directly, and at 1/2 by using (5.5). Since we know that det $T_1(\tau)\cdot$det $T_2(\tau)=\eta(\tau)\eta(2\tau)^{2l+1}$, this estimation is enough to conclude Proposition 7 for odd l.

For $l=2$, not only for the determinants but also for the matrix elements $\theta^2_{m,n}(\tau)$ explicit representations in terms of Dedekind η function are available:

$$\theta^2_{1,0}(\tau)+\theta^2_{3,0}(\tau)=\eta(\tau)^2\eta\left(\frac{3\tau}{2}\right)^2\Big/\eta\left(\frac{\tau}{2}\right)\eta(3\tau),$$

$$\theta^2_{1,0}(\tau)-\theta^2_{3,0}(\tau)=\eta\left(\frac{\tau}{2}\right)\eta(2\tau)\eta(3\tau)^5/\eta(\tau)\eta\left(\frac{3\tau}{2}\right)^2\eta(6\tau)^2,$$

$$\theta^2_{1,4}(\tau)+\theta^2_{3,4}(\tau)=\eta(\tau)\eta(3\tau)^2/\eta\left(\frac{3\tau}{2}\right),$$

$$-\theta^2_{1,4}(\tau)+\theta^2_{3,4}(\tau)=\eta(\tau)\eta\left(\frac{3\tau}{2}\right)\eta(6\tau)/\eta(3\tau),$$

$$\theta^2_{2,2}(\tau)=\eta(2\tau)^2,$$
$$\theta^2_{1,1}(\tau)+\theta^2_{1,7}(\tau)=\eta(\tau)\eta(3\tau)^2/\eta(6\tau),$$
$$\theta^2_{1,1}(\tau)-\theta^2_{1,7}(\tau)=\eta(\tau)\eta(4\tau)^5/\eta(2\tau)^2\eta(8\tau)^2,$$

$$\theta^2_{1,9}(\tau)=\eta(\tau)^2\eta(6\tau)^2/\eta(2\tau)\eta(3\tau),$$
$$\theta^2_{1,5}(\tau)=\eta(\tau)\eta(8\tau)^2/\eta(4\tau).$$

We note that in this special case for $l=2$ Proposition 7 leads to the following equality between Dedekind η functions;

$$\eta(\tau)^2\eta(4\tau)\eta(6\tau)^9+\eta(2\tau)^3\eta(3\tau)^6\eta(12\tau)^3$$
$$=2\eta(\tau)\eta(2\tau)\eta(3\tau)^3\eta(4\tau)^3\eta(6\tau)^2\eta(12\tau)^2.$$

Acknowledgement. We are grateful to Professor N. Iwahori for showing us the preprint of Kac-Peterson [2]. Thanks are also due to Dr. T. Mitsui, Dr. K. Nakahara, Dr. T. Hagino and Dr. E. Shibayama for helpful consultations in operating DEC-20 system in RIMS.

References

[1] Kac, V. G. and Peterson, D. H., Affine Lie algebras and Hecke modular forms, Bull. Amer. Math. Soc., **3** (1980), 1057–1061.
[2] ——, Infinite-dimensional Lie algebras, theta functions and modular forms, preprint.
[3] Jimbo, M. and Miwa, T., Soliton equations and fundamental representations of $A^{(2)}_{2l}$, Lett. Math. Phys., **6** (1982), 463.
[4] Kac, V. G., Infinite-dimensional algebras, Dedekind's η-function, classical Möbius function and the very strange formula, Math. USSR-Izv., **2** (1968), 1271–1311; Adv. in Math., **30** (1978), 85–136.
[5] Moody, R. V., A new class of Lie algebras, J. Algebra, **10** (1968), 211–230.
[6] Kac, V. G., Kazhdan, D. A., Lepowsky, J. and Wilson, R. L., Realization of the basic representations of the Euclidean Lie algebras, Adv. in Math., **42** (1981), 83–112.
[7] Hecke, E., Über eine neuen Zusammenhang zwischen elliptischen Modulfunktionen und indefiniten quadratischen Formen, Mathematische Werke, Vandenhoeck und Ruprecht, Göttingen, 1959, p. 418.

Research Institute for
Mathematical Sciences
Kyoto University
Kyoto 606
Japan