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Conformal and Killing Vector Fields on Complete Non-compact Riemannian Manifolds

Shinsuke Yorozu

0. In this note, we introduce the notion of vector fields with finite global norms, in order to discuss the vector fields on non-compact Riemannian manifolds. It should seem to be natural notion because we have some generalizations of well-known results for compact Riemannian manifolds (cf. [3], [9]). These generalizations are our main results. Our discussions are restricted to conformal and Killing vector fields. We show some examples in which the relations between the volumes of complete non-compact Riemannian manifolds and the global norms of Killing vector fields are discussed. For Killing vector fields with finite global norms, the case of complete non-compact Riemannian manifolds without boundary has stated in [11], and the case of non-compact Riemannian manifolds with boundary has stated in [12]. Our idea is based on in [1], [4], [6] and [10]. The case of affine and projective vector fields with finite global norms may be discussed similarly, but this case is not stated in this note (cf. [13]).

The discussions of different point of views appeared in [5] and [7].

We shall be in C^∞-category. The manifolds considered are connected and orientable.

1. Let M be a complete non-compact Riemannian manifold (without boundary) of dimension m. We denote the Riemannian metric (resp. the Levi-Civita connection) on M by g (resp. ∇). Let g_{ij} denote the components of g with respect to a local coordinate system $(x^1, \cdots, x^m)$, and (g^{ij}) denotes the inverse matrix of the matrix (g_{ij}). We set $\nabla_i = \nabla_{\partial/\partial x^i}$ and $\nabla^i = g^{ij}\nabla_j$.

For two $(0, s)$-tensor fields T and S on M, we denote the local scalar product (resp. the global scalar product) of T and S by $\langle T, S\rangle$ (resp. $\langle\langle T, S\rangle\rangle$), that is,

$$\langle T, S\rangle = \frac{1}{s!} T_{i_1\cdots i_s} S^{i_1\cdots i_s}$$

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$$\langle\langle T, S\rangle\rangle=\int_M \langle T, S\rangle\, dvol$$

where $T_{i_1\cdots i_s}$ and $S_{j_1\cdots j_s}$ denote the components of T and S respectively, and

$$S^{i_1\cdots i_s}=g^{i_1j_1}\cdots g^{i_sj_s}S_{j_1\cdots j_s}.$$

We set $\|T\|^2=\langle\langle T, T\rangle\rangle$ and we remark that $\|T\|^2\leqq+\infty$.

Let $T\otimes S$ denote the tensor product of two tensor fields T and S, for example,

$$(T\otimes S)_{ij}=T_iS_j$$

for two (0, 1)-tensor fields T and S.

We denote the space of all s-forms on M by $\Lambda^s(M)$, and let $\Lambda_0^s(M)$ denote the subspace of $\Lambda^s(M)$ composed of forms with compact supports. Let $L_2^s(M)$ be the completion of $\Lambda_0^s(M)$ with respect to the scalar product $\langle\langle\ ,\ \rangle\rangle$. The operator $d: \Lambda^s(M)\to\Lambda^{s+1}(M)$ denotes the exterior derivative and $\delta: \Lambda^s(M)\to\Lambda^{s-1}(M)$ is defined by

$$\delta=(-1)^{sm+m+1}*d*,$$

where $*$ denotes the star operator. Then we have

$$\langle\langle d\xi, \eta\rangle\rangle=\langle\langle\xi, \delta\eta\rangle\rangle$$

for any $\xi\in\Lambda^s(M)$ and $\eta\in\Lambda^{s+1}(M)$, one of which has compact support. The Laplacian operator Δ is defined by

$$\Delta=d\delta+\delta d.$$

For a 1-form ξ, we have

(1) $$(d\xi)_{ij}=\nabla_i\xi_j-\nabla_j\xi_i$$

(2) $$(\delta\xi)=-\nabla^i\xi_i$$

(3) $$(\Delta\xi)_i=-\nabla^j\nabla_j\xi_i+R_i^j\xi_j$$

where $R(\partial/\partial x^i, \partial/\partial x^j)\,\partial/\partial x^k=R_{kij}^h\,\partial/\partial x^h$, $R_{ki}=R_{khi}^h$, $R_i^j=g^{jk}R_{ki}$ and R_{ki} denote the components of the Ricci tensor of ∇. Here and hereafter, we use the Einstein summation convention.

Through this note, we identify the vector fields on M and its dual 1-forms with respect to g and they are represented by the same letters. For a vector field $\xi=\xi^i\,\partial/\partial x^i$ on M, we have its dual 1-form $\xi=\xi_j\,dx^j=g_{ji}\xi^i\,dx^j$.

Definition 1. A vector field ξ on M is called a vector field with *finite global norm* if its dual 1-form with respect to g belongs in $L_2^1(M)\cap\Lambda^1(M)$, i.e. $\xi\in L_2^1(M)\cap\Lambda^1(M)$.

Definition 2. A vector field ξ on M is called a *conformal vector field with characteristic function* λ if

$$\mathscr{L}_\xi g=2\lambda g \tag{4}$$

where $\mathscr{L}$ denotes the Lie derivative operator and λ is a function on M. If λ is vanishes identically, ξ is called a *Killing vector field*, that is,

$$\mathscr{L}_\xi g=0. \tag{5}$$

We have that (4) and (5) are expressed locally by

$$\nabla_i\xi_j+\nabla_j\xi_i=2\lambda g_{ij} \tag{4$'$}$$

and

$$\nabla_i\xi_j+\nabla_j\xi_i=0 \tag{5$'$}$$

respectively.

2. Let o be a point of M and fix it. For each point $p\in M$, we denote by $\rho(p)$ the geodesic distance from o to p. We set

$$B(r)=\{p\in M\,|\,\rho(p)<r\}$$

for any $r>0$. We may choose a C^∞-function μ on $\boldsymbol{R}$ satisfying

$$\begin{array}{ll} 0\leqq\mu(t)\leqq 1 & \text{for any } t\in\boldsymbol{R} \\ \mu(t)=1 & \text{for } t\leqq 1 \\ \mu(t)=0 & \text{for } t\geqq 2. \end{array}$$

For every $r>0$, we set

$$w_r(p)=\mu(\rho(p)/r)$$

for any $p\in M$, and then w_r is a Lipschitz continuous function on M. The function w_r has the following properties:

$$\begin{array}{ll} 0\leqq w_r(p)\leqq 1 & \text{for any } p\in M \\ \operatorname{supp} w_r\subset B(2r) & \\ w_r(p)=1 & \text{for any } p\in B(r) \end{array}$$

$$\lim_{r\to\infty} w_r = 1$$

$$|dw_r| \leqq \frac{C}{r} \qquad \text{almost everywhere on } M$$

where $C>0$ is a constant independent of r (cf. [1], [4], [10]). Then we have

Lemma 1 (cf. [1], [4]). *For any $\xi \in \Lambda^s(M)$, there exists a positive constant A independent of r such that*

$$\|dw_r \otimes \xi\|^2_{B(2r)} \leqq \frac{A}{r^2} \|\xi\|^2_{B(2r)}$$

$$\|dw_r \wedge \xi\|^2_{B(2r)} \leqq \frac{A}{r^2} \|\xi\|^2_{B(2r)}$$

$$\|dw_r \wedge * \xi\|^2_{B(2r)} \leqq \frac{A}{r^2} \|\xi\|^2_{B(2r)}$$

where $\|\xi\|^2_{B(2r)} = \langle\!\langle \xi, \xi \rangle\!\rangle_{B(2r)} = \int_{B(2r)} \langle \xi, \xi \rangle \, dvol.$

Now we remark that, for any $\xi \in L^s_2(M) \cap \Lambda^s(M)$, $w_r\xi$ is an s-form with compact support and $w_r\xi \to \xi$ $(r \to +\infty)$ in the strong sense. We have

(6) $\quad d(w_r^2\xi) = w_r^2 d\xi + 2w_r dw_r \wedge \xi \qquad$ almost everywhere on M

(7) $\quad \delta(w_r^2\xi) = w_r^2 \delta\xi - *(2w_r dw_r \wedge * \xi) \qquad$ almost everywhere on M

for any $\xi \in \Lambda^1(M)$.

Lemma 2. *For any $\xi \in \Lambda^1(M)$,*

$$4\langle\!\langle w_r dw_r \otimes \xi, \nabla\xi \rangle\!\rangle_{B(2r)} + \langle\!\langle w_r \nabla^2\xi, w_r\xi \rangle\!\rangle_{B(2r)} + 2\langle\!\langle w_r \nabla\xi, w_r \nabla\xi \rangle\!\rangle_{B(2r)} = 0,$$

where $(\nabla^2\xi)_i = \nabla^j \nabla_j \xi_i$ *and* $(\nabla\xi)_{ij} = \nabla_i \xi_j$.

Proof. We consider a 1-form η defined by

$$\eta = (\nabla_i \xi_j)\xi^j dx^i.$$

Then the form $*(w_r^2\eta)$ is an $(m-1)$-form with compact support in $B(2r)$. By the Stokes' theorem which is applicable to Lipschitz continuous forms (cf. [4], [10]), we have

$$\int_M d(*(w_r^2\eta)) = 0.$$

On the other hand, we have

$$d(*(w_r^2\eta))=-*\delta(w_r^2\eta).$$

Thus we have

$$\int_M *\delta(w_r^2\eta)=\int_{B(2r)} *\delta(w_r^2\eta)=0.$$

By (2) and (7), we have

$$\delta(w_r^2\eta)=-w_r^2(\nabla^i\nabla_i\xi_j)\xi^j-w_r^2(\nabla_i\xi_j)(\nabla^i\xi^j)-*(2w_r dw_r\wedge*\eta)$$

and

$$\begin{aligned}*(dw_r\wedge*\eta)&=(dw_r)_i\eta^i\\&=(dw_r)_i(\nabla^i\xi_j)\xi^j\\&=(dw_r)_i\xi_j(\nabla^i\xi^j)\\&=(dw_r\otimes\xi)_{ij}(\nabla^i\xi^j)\\&=2\langle dw_r\otimes\xi,\ \nabla\xi\rangle.\end{aligned}$$

Therefore we have

$$4\langle\langle w_r dw_r\otimes\xi,\ \nabla\xi\rangle\rangle_{B(2r)}+\langle\langle w_r\nabla^2\xi,\ w_r\xi\rangle\rangle_{B(2r)}+2\langle\langle w_r\nabla\xi,\ w_r\nabla\xi\rangle\rangle_{B(2r)}=0. \qquad \square$$

From (3), (6) and (7), we have

Lemma 3. *For any* $\xi\in A^1(M)$,

$$\begin{aligned}&\langle\langle w_r\mathscr{R}\xi,\ w_r\xi\rangle\rangle_{B(2r)}\\&\quad=\langle\langle w_r\nabla^2\xi,\ w_r\xi\rangle\rangle_{B(2r)}+\langle\langle w_r d\xi,\ w_r d\xi\rangle\rangle_{B(2r)}+2\langle\langle w_r d\xi,\ dw_r\wedge\xi\rangle\rangle_{B(2r)}\\&\qquad+\langle\langle w_r\delta\xi,\ w_r\delta\xi\rangle\rangle_{B(2r)}-2\langle\langle w_r\delta\xi,\ *(dw_r\wedge*\xi)\rangle\rangle_{B(2r)}\end{aligned}$$

where $\mathscr{R}$ *denotes the Ricci transformation on* $A^1(M)$ *defined by* $(\mathscr{R}\xi)_i=R_i^h\xi_h$.

Lemma 4. *For a conformal vector field* ξ *with characteristic function* λ *on* M,

$$\begin{aligned}&\|w_r d\xi\|^2_{B(2r)}=4\|w_r\nabla\xi\|^2_{B(2r)}-2m\|w_r\lambda\|^2_{B(2r)}\\&\|w_r\delta\xi\|^2_{B(2r)}=m^2\|w_r\lambda\|^2_{B(2r)}.\end{aligned}$$

Proof. We have

$$\langle d\xi,\ d\xi\rangle=\frac{1}{2}\{(\nabla_i\xi_j-\nabla_j\xi_i)(\nabla^i\xi^j-\nabla^j\xi^i)\}$$

$$=\frac{1}{2}\{4(\nabla_i\xi_j)(\nabla^i\xi^j)-4\lambda\nabla^j\xi_j\}$$
$$=4\langle\nabla\xi,\nabla\xi\rangle-2m\lambda^2$$
$$\langle\delta\xi,\delta\xi\rangle=(\nabla^i\xi_i)(\nabla^j\xi_j)$$
$$=\lambda^2m^2.$$

Thus we have the assertions. □

Let ξ be a conformal vector field on M with characteristic function λ. Then we have, by the Schwarz inequality, Lemma 1 and Lemma 4,

$$\begin{aligned}&|2\langle\langle w_r d\xi, dw_r\wedge\xi\rangle\rangle_{B(2r)}|\\ &\quad\leqq 2\|w_r d\xi\|_{B(2r)}\|dw_r\wedge\xi\|_{B(2r)}\\ &\quad\leqq\frac{1}{4}\|w_r d\xi\|^2_{B(2r)}+4\|dw_r\wedge\xi\|^2_{B(2r)}\\ &\quad\leqq\|w_r\nabla\xi\|^2_{B(2r)}-\frac{1}{2}m\|w_r\lambda\|^2_{B(2r)}+\frac{4A}{r^2}\|\xi\|^2_{B(2r)}\end{aligned}$$

and

$$\begin{aligned}|2\langle\langle w_r\delta\xi, *(dw_r\wedge *\xi)\rangle\rangle_{B(2r)}|&\leqq 2\|w_r\delta\xi\|_{B(2r)}\|dw_r\wedge *\xi\|_{B(2r)}\\ &\leqq\frac{1}{5}\|w_r\delta\xi\|^2_{B(2r)}+5\|dw_r\wedge *\xi\|^2_{B(2r)}\\ &\leqq\frac{1}{5}m^2\|w_r\lambda\|^2_{B(2r)}+\frac{5A}{r^2}\|\xi\|^2_{B(2r)}.\end{aligned}$$

Thus we have, from Lemma 2 and Lemma 3,

$$\begin{aligned}&\langle\langle w_r\mathscr{R}\xi, w_r\xi\rangle\rangle_{B(2r)}\\ &\quad=-4\langle\langle w_r dw_r\otimes\xi,\nabla\xi\rangle\rangle_{B(2r)}-2\langle\langle w_r\nabla\xi, w_r\nabla\xi\rangle\rangle_{B(2r)}\\ &\qquad+\langle\langle w_r d\xi, w_r d\xi\rangle\rangle_{B(2r)}+2\langle\langle w_r d\xi, dw_r\wedge\xi\rangle\rangle_{B(2r)}\\ &\qquad+\langle\langle w_r\delta\xi, w_r\delta\xi\rangle\rangle_{B(2r)}-2\langle\langle w_r\delta\xi, *(dw_r\wedge *\xi)\rangle\rangle_{B(2r)}\\ &\quad\geqq-\frac{1}{2}\|w_r\nabla\xi\|^2_{B(2r)}-\frac{8A}{r^2}\|\xi\|^2_{B(2r)}-2\|w_r\nabla\xi\|^2_{B(2r)}\\ &\qquad+4\|w_r\nabla\xi\|^2_{B(2r)}-2m\|w_r\lambda\|^2_{B(2r)}\\ &\qquad-\|w_r\nabla\xi\|^2_{B(2r)}+\frac{1}{2}m\|w_r\lambda\|^2_{B(2r)}-\frac{4A}{r^2}\|\xi\|^2_{B(2r)}\\ &\qquad+m^2\|w_r\lambda\|^2_{B(2r)}-\frac{1}{5}m^2\|w_r\lambda\|^2_{B(2r)}-\frac{5A}{r^2}\|\xi\|^2_{B(2r)}\end{aligned}$$

$$=\frac{1}{2}\|w_r\nabla\xi\|^2_{B(2r)}+\frac{4}{5}m\left(m-\frac{15}{8}\right)\|w_r\lambda\|^2_{B(2r)}-\frac{17A}{r^2}\|\xi\|^2_{B(2r)}.$$

Thus we have

Lemma 5. *Let ξ be a conformal vector field on M with characteristic function λ and with finite global norm. If* $\limsup_{r\to+\infty}\langle\langle w_r\mathcal{R}\xi,\, w_r\xi\rangle\rangle_{B(2r)}<+\infty$, *then*

$$\limsup_{r\to+\infty}\langle\langle w_r\mathcal{R}\xi,\, w_r\xi\rangle\rangle_{B(2r)}\geqq\frac{1}{2}\|\nabla\xi\|^2+\frac{4}{5}m\left(m-\frac{15}{8}\right)\|\lambda\|^2.$$

From this lemma, we have

Theorem 1. *Suppose that a complete non-compact Riemannian manifold M has non-positive Ricci curvature. Then every conformal (or Killing) vector field on M with finite global norm is a parallel vector field. Moreover, if M has negative Ricci curvature, then there is no non-zero conformal (or Killing) vector field on M with finite global norm.*

Remark. The Killing vector field case of the above theorem was given in [11]. The above theorem is a generalization of well-known compact case (cf. [3], [9]).

Since the length of a parallel vector filed is constant, we have

Corollary 1. *Let M be a complete non-compact Riemannian manifold with non-positive Ricci curvature. If there exists a non-zero conformal (or Killing) vector field on M with finite global norm, then the volume of M is finite.*

Remark. Recently, H. Wu has proved the following theorem:

Theorem ([8]). *Let M be a complete non-compact Riemannian manifold which satisfies*

$$\textit{Ricci curvature}\geqq\frac{-\tilde{A}}{\rho^{2+\varepsilon}}$$

where ρ denotes the distance from a fixed point of M and $\tilde{A}$ and ε are positive constants. Then M has infinite volume.

This Wu's theorem is a generalization of the result of S.T. Yau [10]. From Corollary 1, we have

Corollary 2. *Let M be a complete non-compact Riemannian manifold* *with non-positive Ricci curvature. If there exists a non-zero Killing vector field on M with finite global norm, then the group of isometries of M is compact.*

Proof. The group of isometries of a complete Riemannian manifold having finite volume is compact (cf. [2]). Thus, by this fact and Corollary 1, we have the assertion. □

We have an example:

Example 1. Let r_0 be a fixed positive number and f a function on $\boldsymbol{R}$ satisfying

$$f(r)=|r|^{-3/8} \qquad \text{for } r_0<|r|.$$

Then $\int_{-\infty}^{+\infty} f^2(r)dr=+\infty$ and $\int_{-\infty}^{+\infty} f^4(r)dr<+\infty$. Let M be a warped product Riemannian manifold $\boldsymbol{R}\times_f S^2$, that is, $ds^2=dr^2+f^2(r)\{d\theta^2+\sin^2\theta\, d\varphi^2\}$. Then

$$\text{the volume of } M=\int_{-\infty}^{+\infty}\int_0^{\pi}\int_0^{2\pi} f^2(r)\sin\theta\, dr\, d\theta\, d\varphi$$
$$=+\infty.$$

A vector field $\xi=f(r)\,\partial/\partial r$ on M is a conformal vector field. And, we have

$$\|\xi\|^2=\int_{-\infty}^{+\infty}\int_0^{\pi}\int_0^{2\pi} f^4(r)\sin\theta\, dr\, d\theta\, d\varphi$$
$$<+\infty.$$

By the method given in [6], we have

Theorem 2. *Let M be a complete non-compact Riemannian manifold having finite volume. If ξ is a conformal vector field on M with non-negative (or non-positive) characteristic function λ and with finite global norm, then ξ is a Killing vector field.*

Proof. We have, for any r,

$$\frac{1}{r}\int_{B(2r)}|\xi|\,dvol\leqq\left(\int_{B(2r)}\langle\xi,\xi\rangle\,dvol\right)^{1/2}\left(\int_{B(2r)}\left(\frac{1}{r}\right)^2 dvol\right)^{1/2}$$
$$\leqq\|\xi\|_{B(2r)}\cdot\frac{1}{r}(\mathrm{Vol}\,(M))^{1/2}$$

where $|\xi|=\sqrt{\langle\xi, \xi\rangle}$ and Vol (M) denotes the volume of M. Thus we have

$$\liminf_{r\to+\infty} \frac{1}{r}\int_{B(2r)} |\xi|\, dvol=0$$

On the other hand, we have

$$\left|\int_{B(2r)} w_r^2 \operatorname{div}\xi\, dvol\right| \leqq \frac{C}{r}\int_{B(2r)} |\xi|\, dvol$$

and

$$\operatorname{div}\xi=-m\lambda.$$

Therefore, we have

$$m\int_M \lambda\, dvol=0,$$

that is, $\lambda\equiv 0$. □

Remark. Theorem 2 holds without the finiteness of global norm of ξ. This is pointed out by Professor T. Sunada. His method differs from our method.

3. For a vector field ξ on M, we set

$$B_{ij}=\nabla_i\xi_j+\nabla_j\xi_i-\frac{2}{m}(\nabla^k\xi_k)g_{ij}$$

and

$$\hat{\eta}=B_{ij}\xi^j dx^i.$$

Then we have

Lemma 6. *It holds that*

$$\begin{aligned} &B_{ij}=B_{ji}, \qquad g^{ij}B_{ij}=0,\\ &B_{ij}\nabla^i\xi^j=\langle B, B\rangle,\\ &\nabla^k B_{kj}=\nabla^k\nabla_k\xi_j+R_j^k\xi_k+\left(1-\frac{2}{m}\right)\nabla_j\nabla^k\xi_k. \end{aligned}$$

By (2) and (7), we have

$$\begin{aligned} \delta(w_r^2\hat{\eta}) &= w_r^2\delta\hat{\eta}-*(2w_r\, dw_r\wedge *\hat{\eta})\\ &= -w_r^2(\nabla^k B_{kj})\xi^j-w_r^2 B_{kj}(\nabla^k\xi^j)-*(2w_r\, dw_r\wedge *\hat{\eta}). \end{aligned}$$

Since $\int_M *\delta(w_r^2\hat{\eta})=0$, we have

Lemma 7. *For a vector field ξ on M,*

$$\langle\langle w_r\hat{B},\ w_r\xi\rangle\rangle_{B(2r)}+\langle\langle w_rB,\ w_rB\rangle\rangle_{B(2r)}+4\langle\langle w_rdw_r\otimes\xi,\ B\rangle\rangle_{B(2r)}=0$$

where $(\hat{B})_j=\nabla^kB_{kj}$.

Thus we have

Theorem 3. *Let M be a complete non-compact Riemannian manifold of dimension m ($\geqq 3$) and ξ a vector field on M with finite global norm. ξ is a conformal vector field if and only if ξ satisfies*

$$\nabla^k\nabla_k\xi^i+R^i_k\xi^k+\left(1-\frac{2}{m}\right)\nabla^i\nabla_k\xi^k=0. \tag{8}$$

Proof. If ξ satisfies (8), then, by Lemma 1 and Lemma 7, we have

$$\begin{aligned}\|w_rB\|^2_{B(2r)}&=-4\langle\langle w_rdw_r\otimes\xi,\ B\rangle\rangle_{B(2r)}\\&\leqq 4\|dw_r\otimes\xi\|_{B(2r)}\|w_rB\|_{B(2r)}\\&\leqq 2\left\{4\|dw_r\otimes\xi\|^2_{B(2r)}+\frac{1}{4}\|w_rB\|^2_{B(2r)}\right\}\\&\leqq\frac{8A}{r^2}\|\xi\|^2_{B(2r)}+\frac{1}{2}\|w_rB\|^2_{B(2r)}.\end{aligned}$$

Thus we have

$$\frac{1}{2}\|w_rB\|^2_{B(2r)}\leqq\frac{8A}{r^2}\|\xi\|^2_{B(2r)}.$$

Letting $r\to+\infty$, we have $\|B\|^2=0$. Therefore, we have $B=0$, that is, ξ is a conformal vector field on M. The converse is trivial. □

The following theorem is a corollary of the above theorem.

Theorem 4. *Let M be a complete non-compact Riemannian manifold and ξ a vector field on M with finite global norm. ξ is a Killing vector field if and only if ξ satisfies*

$$\nabla^k\nabla_k\xi^i+R^i_k\xi^k=0 \quad \textit{and} \quad \nabla_i\xi^i=0.$$

Example 2. In the Euclidean 3-space E^3, (8) is changed into

$$\sum_{k=1}^{3}\frac{\partial^2\xi^j}{(\partial x^k)^2}+\frac{1}{3}\frac{\partial}{\partial x^j}\left(\sum_{k=1}^{3}\frac{\partial\xi^k}{\partial x^k}\right)=0 \quad (j=1,\ 2,\ 3). \tag{8$'$}$$

Thus, we may consider a vector field ξ on E^3 defined by

$$\xi=\xi^1\,\partial/\partial x^1+\xi^2\,\partial/\partial x^2+\xi^3\,\partial/\partial x^3$$

where

$$\xi^1=(x^1)^2-\frac{2}{3}(x^2)^2-\frac{2}{3}(x^3)^2+1$$

$$\xi^2=-\frac{2}{3}(x^1)^2+(x^2)^2-\frac{2}{3}(x^3)^2+1$$

$$\xi^3=-\frac{2}{3}(x^1)^2-\frac{2}{3}(x^2)^2+(x^3)^2+1.$$

Then we have $\|\xi\|^2=+\infty$, and ξ satisfies (8)$'$, but ξ is not a conformal vector field on E^3.

Remark. Theorem 3 and Theorem 4 are generalizations of well-known results in the compact cases (cf. [9]).

4. We show some examples in which the relations between the volume of manifolds and the norms of Killing vector fields are discussed.

Let M be a warped product Riemannian manifold $\boldsymbol{R}\times_f N$ of a 1 dimensional complete non-compact Riemannian manifold $\boldsymbol{R}$ and an $m-1$ dimensional compact Riemannian manifold N, where f is a positive function on $\boldsymbol{R}$. Let $(x^1, x^2, \cdots, x^m)$ denote a local coordinate system on M such that $(x^2, \cdots, x^m)$ denotes a local coordinate system on N. The components g_{ij} of the metric tensor field g on M are expressed by

$$(g_{ij})=\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & f^2(x^1)h_{\alpha\beta} \end{array}\right) \qquad (2\leqq\alpha,\ \beta\leqq m),$$

where $h_{\alpha\beta}$ denote the components of the metric tensor field h on N. Then we have

$$\text{the volume of}\quad M=\int_M f^{m-1}(x^1)(\det(h_{\alpha\beta}))^{1/2}\,dx^1dx^2\cdots dx^m.$$

We consider a vector field ξ on M, that is,

$$\xi=\xi^1(x^1, x^2, \cdots, x^m)\frac{\partial}{\partial x^1}+\xi^\alpha(x^1, x^2, \cdots, x^m)\frac{\partial}{\partial x^\alpha},$$

and we have

$$\text{(9)}\qquad \xi_1=\xi^1,\qquad \xi_\alpha=f^2(x^1)h_{\alpha\beta}(x^2, \cdots, x^m)\xi^\beta.$$

Lemma 8. *A vector field* $\xi=\xi^1\,\partial/\partial x^1+\xi^\alpha\,\partial/\partial x^\alpha$ *on* M *is a Killing vector field if and only if it holds that*

$$\partial\xi^1/\partial x^1=0$$
$$f^2 h_{\alpha\beta}\,\partial\xi^\beta/\partial x^1+\partial\xi^1/\partial x^\alpha=0$$
$$fh_{\alpha\gamma}(\partial\xi^\gamma/\partial x^\beta+\Gamma^\gamma_{\beta\tau}\xi^\tau)+fh_{\beta\gamma}(\partial\xi^\gamma/\partial x^\alpha+\Gamma^\gamma_{\alpha\tau}\xi^\tau)+2f'h_{\alpha\beta}\xi^1=0,$$

where $\Gamma^\gamma_{\alpha\beta}$ *denote the components of the Levi-Civita connection on* N *with respect to a local coordinate system* $(x^2, \cdots, x^m)$ *and* f' *denotes* $df(x^1)/dx^1$.

Proof. A vector field ξ on M is a Killing vector field if and only if it holds (5)′, that is,

$$\nabla_1\xi_1=0$$
$$\nabla_1\xi_\alpha+\nabla_\alpha\xi_1=0$$
$$\nabla_\alpha\xi_\beta+\nabla_\beta\xi_\alpha=0.$$

From the above facts and (9), we have the assertion. □

Example 3. Let f be the function on $\boldsymbol{R}$ defined by $f(x^1)=e^{x^1}$, and $M=\boldsymbol{R}\times_f N$. Let $\tilde{\xi}=\xi^\alpha\,\partial/\partial x^\alpha$ be a non-zero vector field on N satisfying $\tilde{\nabla}_\alpha\tilde{\xi}_\beta+\tilde{\nabla}_\beta\tilde{\xi}_\alpha=-ah_{\alpha\beta}$ where $\tilde{\nabla}$ denotes the Levi-Civita connection on N and a is a constant number. Then a vector field $\xi=a\,\partial/\partial x^1+\tilde{\xi}$ on M is a Killing vector field. We have that $\mathrm{Vol}(M)=+\infty$ and $\|\xi\|^2=+\infty$.

Example 4. Let f be the function on $\boldsymbol{R}$ defined by $f(x^1)=\exp(-(x^1)^2)$, and $M=\boldsymbol{R}\times_f N$. We take a non-zero Killing vector field $\tilde{\xi}=\xi^\alpha\,\partial/\partial x^\alpha$ on N. Then the vector field $\xi=\xi^\alpha\,\partial/\partial x^\alpha$ on M is a Killing vector field. We have that $\mathrm{Vol}(M)<+\infty$ and $\|\xi\|^2<+\infty$.

Example 5. Let r_0 be a fixed positive number, and let m_0 be a fixed positive number such that

$$\frac{1}{m+1}<m_0<\frac{1}{m-1}.$$

Let f be a function on $\boldsymbol{R}$ satisfying

$$f(x^1)=|x^1|^{-m_0}\qquad\text{for } r_0<|x^1|.$$

We remark that

$$\int_{r_0}^{+\infty}f^{m-1}(x^1)dx^1=+\infty,\qquad\int_{r_0}^{+\infty}f^{m+1}(x^1)dx^1<+\infty.$$

Let $\tilde{\xi}=\xi^{\alpha}\,\partial/\partial x^{\alpha}$ be a non-zero Killing vector field on N. Then $\xi=\xi^{\alpha}\,\partial/\partial x^{\alpha}$ is a Killing vector field on $M=\boldsymbol{R}\times_f N$, and we have that $\mathrm{Vol}\,(M)=+\infty$ and $\|\xi^2\|<+\infty$.

Example 6. Let n_0 be a fixed positive integer and ε a number such that $0<\varepsilon<2/(m-1)$. We remark that

$$\sum_{n=n_0}^{\infty} n^{-(2+\varepsilon)}(n^{1/(m-1)})^{m-1}=\sum_{n=n_0}^{\infty} n^{-(1+\varepsilon)}<+\infty$$

$$\sum_{n=n_0}^{\infty} n^{-(2+\varepsilon)}(n^{1/(m-1)})^{m+1}=\sum_{n=n_0}^{\infty} n^{-(1+\varepsilon-2/(m-1))}=+\infty.$$

We consider a function $\hat{f}$ on $\boldsymbol{R}$ such that, for each integer n ($>n_0$),

$$\begin{array}{ll} 0\leqq\hat{f}(x^1)\leqq n^{1/(m-1)} & |x^1|\in(n,\,n+1] \\ \hat{f}(x^1)=n^{1/(m-1)} & |x^1|\in[n+n^{-(2+\varepsilon)}/10,\,n+9n^{-(2+\varepsilon)}/10] \\ \hat{f}(x=0 & |x^1|\in(n+n^{-(2+\varepsilon)},\,n+1]. \end{array}$$

Then we have

$$\frac{8}{10}\times n^{-(1+\varepsilon)}\leqq\int_n^{n+1}\hat{f}^{m-1}(x^1)dx^1\leqq n^{-(1+\varepsilon)}$$

$$\frac{8}{10}\times n^{-(1+\varepsilon-(2/(m-1))}\leqq\int_n^{n+1}\hat{f}^{m+1}(x^1)dx^1\leqq n^{-(1+\varepsilon-2/(m-1))}.$$

We also consider the function $\tilde{f}$ on $\boldsymbol{R}$ defined by $\tilde{f}(x^1)=\exp\,(-(x^1)^2)$. Then we consider a function f on $\boldsymbol{R}$ such that, for each integer n ($>n_0$),

$$\begin{array}{ll} 0<f(x^1)\leqq n^{1/(m-1)} & |x^1|\in(n,\,n+1] \\ f(x^1)=n^{1/(m-1)} & |x^1|\in[n+n^{-(2+\varepsilon)}/10,\,n+9n^{-(2+\varepsilon)}/10] \\ f(x^1)=\tilde{f}(x^1) & |x^1|\in(n+n^{-(2+\varepsilon)},\,n+1]. \end{array}$$

Let $M=\boldsymbol{R}\times_f N$, and a Killing vector field $\xi=\xi^{\alpha}\,\partial/\partial x^{\alpha}$ on N induces a Killing vector field $\xi=\xi^{\alpha}\,\partial/\partial x^{\alpha}$ on M. We have that $\mathrm{Vol}\,(M)<+\infty$ and $\|\xi\|^2=+\infty$.

References

[1] A. Andreotti and E. Vesentini, Carleman estimates for the Laplace-Beltrami equation on complex manifolds, Inst. Hautes Etudes Sci. Publ. Math., **25** (1965), 313–362.

[2] G. Averous and S. Kobayashi, On automorphisms of spaces of nonpositive curvature with finite volume, Differential Geometry and Relativity (Eds. Cahen and Flato), 19–26, D. Reidel Publ. Company, Dordrecht Holland, 1976.

[3] S. Bochner, Vector fields and Ricci curvature, Bull. Amer. Math. Soc., **52** (1946), 776–797.

[4] J. Dodziuk, Vanishing theorems for square-integrable harmonic forms, Geometry and Analysis, Papers dedicated to the memory of V. K. Patodi, 21–27, Springer-Verlag, 1981.

[5] G. Gigante, Remarks on holomorphic vector fields on non-compact manifolds, Rend. Sem. Mat. Univ. Padova, **52** (1974), 211–218.

[6] L. Kaup, On Stokes' thorem for noncompact manifolds, Proc. Amer. Math. Soc., **82** (1981), 487–490.

[7] H. Wu, A remark on the Bochner technique in differential geometry, Proc. Amer. Math. Soc., **78** (1980), 403–408.

[8] ——, On the volume of a noncompact manifold, Duke Math. J., **49** (1982), 71–78.

[9] K. Yano, Integral formulas in Riemannian geometry, Marcel Dekker, INC., New York, 1970.

[10] S. T. Yau, Some function-theoretic properties of complete Riemannian manifold and their applications to geometry, Indiana Univ. Math. J. **25** (1976), 659–670.

[11] S. Yorozu, Killing vector fields on complete Riemannian manifolds, Proc. Amer. Math. Soc., **84** (1982), 115–120.

[12] ——, Killing vector fields on non-compact Riemannian manifolds with boundary, Kōdai Math. J., **5** (1982), 426–433.

[13] ——, Affine and projective vector fields on complete non-compact Riemannian manifolds, Yokohama Math. J., **31** (1983), 41–46.

Department of Mathematics
College of Liberal Arts
Kanazawa University
Kanazawa 920, Japan