Geometry of Geodesics and Related Topics
Advanced Studies in Pure Mathematics 3, 1984
pp. 423–450

Topology of Complete Noncompact Manifolds

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§ 0. Introduction

A well known theorem due to Cohn-Vossen [8] states that if an oriented complete, noncompact and finitely connected Riemannian manifold M of dimension 2 admits the total curvature $c(M)=\int_M G\,dM$, where G is the Gaussian curvature and dM is the volume element of M, then $c(M)\leqq 2\pi\chi(M)$. From this fact he proved that if M has nonnegative Gaussian curvature, then M is either diffeomorphic to a plane R^2 or else isometric to a flat cylinder $S^1\times R$ or a flat open Möblius strip. This pioneering work of Cohn-Vossen has been extended by Cheeger, Gromoll and Meyer in [7], [20] and others to obtain the structure theorem for complete noncompact Riemannian manifolds of nonnegative sectional curvature. The structure theorem states that if a complete noncompact Riemannian manifold M has nonnegative sectional curvature, then there exists a compact totally geodesic submanifold S of M (which is called the soul of M and has dimension $\geqq 0$) such that M is homeomorphic (or even diffeomorphic, see [33]) to the total space of the normal bundle $\nu(S)$ over S in M. The proof is done by constructing a family of compact totally convex sets exhausting M. It turns out that this family of compact totally convex sets is nothing but the sublevels of a convex function which is obtained by Busemann functions for rays emanating from an arbitrary fixed point.

Thus Busemann functions play an essential role in the study of complete noncompact Riemannian manifolds. This function has been introduced by Busemann (see Section 22 in [2]) in order to establish a theory of parallels for straight lines on a straight G-space on which every two points are joined by a unique distance realizing geodesic.

One of the purposes of this survey note is to study fundamental theorems on complete noncompact Riemannian manifolds of nonnegative curvature which has been obtained by Cheeger, Gromoll Meyer and To-

Received January 8, 1983.
Revised April 1, 1983.
* Dedicated to Professor I. Mogi on his 60th birthday

ponogov [7], [20], [44] and others from a point of view of Busemann functions for rays. This attempt has been done by Wu in [47] for generalized Busemann functions, where the convexity, subharmonicity and plurisubharmonicity of Busemann functions under certain conditions for curvatures are discussed. We shall first of all in Section 1 study geometric and fundamental properties of Busemann functions without curvature assumptions which have been discussed by Busemann [2] and Cheeger-Gromoll [7]. Making use of the fundamental properties of Busemann functions we shall give in Section 2 a simple proof for the convexity, subharmonicity and plurisubharmonicity of Busemann functions under certain conditions for curvature. The basic idea of our proof of the above properties for Busemann functions is going along with the line settled by Wu in [47]. The simplification of our proof is acheived by showing the property of Busemann functions (which is seen in Lemma 1.2) which states that a Busemann function F_γ for a ray γ is supported by another Busemann function $F_\sigma + F_\gamma(p)$ at each point p on M, where $p=\sigma(0)$ and σ is asymptotic to γ. Thus our proof is more direct and simpler than that of Wu's.

Making use of the Grauert theorem, Greene and Wu have proved in [17] that if a complete noncompact Kaehler manifold has positive sectional curvature, then it is a Stein manifold. This is because M admits a convex exhaustion function and the convexity of a function on M implies plurisubharmonicity of it (see [14]). Such manifolds are interested from a point of view of the function theory of complex variables, and discussed in [39], but we do not discuss here.

On the other hand, the subharmonicity of a function on M however does not give any topological restrictions to M. Subharmonic functions always exists on any connected, complete noncompact Riemannian manifold. This is because every such an M admits $2m+1$ smooth harmonic functions together which form a proper embedding of M into R^{2m+1} (see [16]). It will be proved in Section 2 that if a complete noncompact M has nonnegative Ricci curvature, then every Busemann function on M is subharmonic. The only one metric consequence for such an M of nonnegative Ricci curvature is that the volume of M is unbounded. This has recently been proved by Wu in [48]. It has recently been proved by Schoen and Yau [36] that if a complete noncompact Riemannian manifold of dimension 3 has positive Ricci curvature, then M is diffeomorphic to R^3.

The convexity of a function on M gives topological restrictions to M as well as metric restrictions of course. The structure theorem for complete noncompact manifolds of nonnegative sectional curvature is a special case of a more general result of the Greene-Shiohama theorem which states that if a complete M admits a convex function f which is not constant on any open subset of M, then for a fixed value $a > \inf_M f$ there exists a homeomorphism H: $M_a^a(f) \times (\inf_M f, \infty) \to M - \{x \in M; f(x) = \inf_M f\}$ such that $f(H(y, \alpha)) = \alpha$ for all $y \in M_a^a(f)$ and for all $\alpha > \inf_M f$, where $M_a^a(f)$ is the a-level set of f which is a topological submanifold of dimension $m-1$. Other results concerning with convex functions on complete manifolds will be summarized in Section 3.

In the pioneering work of Cohn-Vossen [9], he has proved that if M is a complete Riemannian manifold which is diffeomorphic to R^2 and if the total curvature exists and if M admits a straight line, then the total curvature is non-positive. It follows from this and his previous results that if a complete, noncompact Riemannian 2-manifold of nonnegative Gaussian curvature admits a straight line, then M is isometric to either a flat cylinder $S^1 \times R$ or else to E^2. This has been extended by Toponogov [44], known as the Toponogov splitting theorem, which states that if a complete manifold of nonnegative sectional curvature admits a straight line, then it is isometric to the Riemannian product $N \times R$. Further generalization has been obtained by Cheeger and Gromoll in [6] for manifolds of nonnegative Ricci curvature. The notion of asymptotic rays will be usefull to prove the splitting theorem and to describe fundamental properties of Busemann functions as well. A flat strip theorem is the first step of the proof of splitting theorem, which shows the existence of a flat totally geodesic surface bounded by two rays asymptotic to a fixed ray under certain conditions for curvature and asymptotic rays. A Clifford translation on an Hadamard manifold also provides a flat strip between biasymptotic lines each of which is translated along itself by the Clifford translation, for detail see [39]. Now suppose that M is isometric to the Riemannian product $N \times R$, and let $\gamma(t) := (x, t)$, $X \in N$, $t \geqq 0$. Then the Busemann function for this γ takes a special from which is not only convex but affine. A natural extension of the Toponogov splitting theorem has been obtained by Innami in [25] which states that a complete M admits a nontrivial affine function if and only if M is isometric to the Riemannian product $N \times R$, where N is a level set of the function which is totally geodesic and totally convex as well. This result will be discussed in Section 4.

The above result and the Greene-Shiohama theorem for convex functions apply to obtain an elementary proof of a gap theorem established recently by Greene and Wu in [18], which states that if a complete manifold M of nonnegative sectional curvature has zero curvature outside a compact set and if M is simply connected at infinity, then M is isometric to the Euclidean m-space E^m. This will be stated in Section 5.

At the end of this survey note let us go back to the original pioneering works of Cohn-Vossen. We find that there is an interesting problem which

Cohn-Vossen did not discuss and which is left for us. The problem is to investigate geometric significance of the existence of total curvature on the Riemannian structure of a complete noncompact Riemannian 2-manifold. It has been investigated by Maeda in [30] that there is a relationship between the total curvature and the mass of rays emanating from a fixed point on a complete manifold of nonnegative Gaussian curvature diffeomorphic to R^2. Roughly speaking, the Maeda theorem states that on a complete manifold diffeomorphic to R^2 of nonnegative Gaussian curvature if the total curvature is small, then the mass of rays emanating from an arbitrary fixed point on it is large. For detailed arguments see this Proceedings [29]. We shall here discuss about the relation between the total curvature and the behavior of Busemann functions on a finitely connected complete noncompact Riemannian 2-manifolds each of which has one end.

The emphasis of this note is to show that we can prove all basic results for noncompact complete manifolds of nonnegative sectional curvature obtained by Toponogov, Cheeger, Gromoll and Meyer and others without using the Rauch comparison theorem nor the Toponogov comparison theorem. The basic tools used in our observation here are the first and second variation formulas for lengths of 1-parameter variations along geodesics and the minimization theorem for index forms and the index comparison theorem which are seen in the standard text books on Riemannian geometry such as [1], [5], [19] and [31].

§ 1. Busemann functions

The purpose of this section is to state the fundamental properties of Busemann functions on a complete noncompact Riemannian manifold M of dimension $m \geqq 2$. Unless otherwise stated geodesics on M are parametrized by arc length. It is the nature of completeness and noncompactness that through every point p on M there passes a *ray* $\gamma: [0, \infty) \to M$ with $\gamma(0) = p$, e.g., $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \geqq 0$, where d is the distance function on M induced from the Riemannian metric. A geodesic is called to be a *straight line* iff for any $t_1, t_2 \in R$, $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$. The *Busemann function* F_γ for a ray $\gamma: [0, \infty) \to M$ is defined by

$$F_\gamma(x) := \lim_{t \to \infty} [t - d(x, \gamma(t))], \qquad x \in M.$$

The $t - d(x, \gamma(t))$ is monotone increasing with t and is bounded above by $d(p, x)$, where $p = \gamma(0)$. Therefore it converges uniformly on every compact set of M. The notion of asymptote is needed to describe the fundamental properties of Busemann functions. Let $\gamma: [0, \infty) \to M$ be a ray.

A ray $\sigma: [0, \infty) \to M$ is said to be *asymptotic* to γ if there is a sequence of minimizing geodesics $\{\sigma_j\}$, each σ_j satisfying $\sigma_j(0)=q_j$ with $\lim_{j\to\infty} q_j=\sigma(0)$ and $\sigma_j(l_j)=\gamma(t_j)$ for some divergent sequence $\{t_j\}$ and they satisfy $\dot\sigma(0)=\lim_{j\to\infty} \dot\sigma_j(0)$, where $\dot\sigma(0)$ is the tangent vector to σ at 0.

For a point x on M and for an $r>0$ let $B(x, r):=\{y \in M;\ d(x, h)<r\}$ and let $S(x, r):=\{y \in M;\ d(x, y)=r\}$. For a function $f: M\to R$ a *sublevel set* and a *level set* of f are denoted by $M^a(f):=\{x \in M;\ f(x)\leqq a\}$ and $M^a_a(f):=\{x \in M; f(x)=a\}$. With these notations the fundamental properties of Busemann functions obtained in [2] and [7] are stated as follows.

Theorem 1.1. *Let $\gamma: [0, \infty)\to M$ be a ray. Then the following statements are true.*

(1) *F_γ is Lipschitz continuous with the Lipschitz constant* 1.

(2) *For every $a \in F_\gamma(M)$, $M^a_a(F_\gamma)=\lim_{t\to\infty} S(\gamma(t), t-a)$.*

(3) *For every $a, b \in F_\gamma(M)$ with $a\leqq b$, $M^a(F_\gamma)=\{y \in M^b(F_\gamma);\ d(y, M^b_b(F_\gamma))\geqq b-a\}$, and $M^a_a(F_\gamma)=\{y \in M^b(F_\gamma);\ d(y, M^b_b(F_\gamma))=b-a\}$.*

(4) *A unit speed geodesic $\sigma: [0, \infty)\to M$ is a ray asymptotic to γ if and only if $F_\gamma\circ\sigma(t)=t+F_\gamma\circ\sigma(0)$ for all $t\geqq 0$.*

(5) *For every $x \in M$ and for every $a \in F_\gamma(M)$ with $a>F_\gamma(x)$ if $\sigma: [0, l)\to M$ is a minimizing geodesic with $\sigma(0)=x$ and $\sigma(l) \in M^a_a(F_\gamma)$ such that $l=d(x, M^a_a(F_\gamma))$, then the extension of σ to $[0, \infty)$ is a ray asymptotic to γ.*

(6) *F_γ is differentiable at x if x is an interior point of some ray asymptotic to γ.*

Proof of (1). For every $x, y \in M$, one has $|F_\gamma(x)-F_\gamma(y)|\leqq d(x, y)$ since $|d(x, \gamma(t))-d(y, \gamma(t))|\leqq d(x, y)$ for all $t\geqq 0$.

Proof of (2). It is sufficient to show that $\liminf_{t\to\infty} S(\gamma(t), t-a)=M^a_a(F_\gamma)$ and $\limsup_{t\to\infty} S(\gamma(t), t-a)=M^a_a(F_\gamma)$. Let x be a point on the lower limit. For any $\varepsilon>0$ there is a $t_\varepsilon>0$ such that if $t>t_\varepsilon$, then $B(x, \varepsilon)\cap S(\gamma(t), t-a)\neq\phi$. This means that $t-a-\varepsilon\leqq d(x, \gamma(t))\leqq t-a+\varepsilon$ holds for all $t>t_\varepsilon$. Therefore $F_\gamma(x)=a$ is verified by letting $\varepsilon\to 0$. Let y be a point on $M^a_a(F_\gamma)$. Since $t-d(y, \gamma(t))$ is monotone increasing in t and its limit is a, for any $\varepsilon>0$ there exists a $t_\varepsilon>0$ such that if $t_1>t_\varepsilon$, then there is a point y_1 on a minimizing geodesic joining y to $\gamma(t_1)$ such that $d(y_1, \gamma(t_1))=t_1-a$ and $d(y, y_1)\leqq\varepsilon$. Therefore if $t_2\geqq t_1$ then $d(y_1, \gamma(t_2))\leqq t_2-a$ and $d(y, \gamma(t_2))\geqq t_2-a$ implies that the minimizing geodesic from y to y_1 intersects $S(\gamma(t_2), t_2-a)$ for all $t_2\geqq t_1$, and hence $y \in \liminf_{t\to\infty} S(\gamma(t), t-a)$.

To prove the second relation let z be a point on the upper limit. (It is obvious that $M^a_a(F_\gamma)\subset\limsup S(\gamma(t), t-a)$.) There is a monotone divergent sequence $\{t_j\}$ and points $\{z_j\}$ such that $\lim z_j=z$ and $z_j \in S(\gamma(t_j), t_j-a)$. Then $F_\gamma(z)=\lim\,[t_j-d(z_j, \gamma(t_j))]+\lim d(z_j, z)=a$. This proves $\limsup S(\gamma(t), t-a)\subset M^a_a(F_\gamma)$.

Proof of (3). It follows from (1) that if x and y are points on $M^a(F_\gamma)$ and on $M^b_b(F_\gamma)$ respectively such that $d(x, y)=d(x, M^b_b(F_\gamma))$, then $d(x, y) \geqq |F_\gamma(x)-F_\gamma(y)| \geqq b-a$. Conversely let $x \in M^b(F_\gamma)$ satisfy $d(x, M^b_b(F_\gamma)) \geqq b-a$ and let y be a point on $M^b_b(F_\gamma)$ such that $d(x, y)=d(x, M^b_b(F_\gamma))$. For a sufficiently large $t>0$ let q_t be the point of intersection of $M^b_b(F_\gamma)$ with a minimizing geodesic joining x to $\gamma(t)$. From triangle inequality it follows that $d(x, q_t) \leqq d(x, y)+d(y, \gamma(t))-d(q_t, \gamma(t))$. Since y and q_t belong to $\lim_{t\to\infty} S(\gamma(t), t-b)$, $\lim_{t\to\infty} [d(y, \gamma(t))-d(q_t, \gamma(t))]=0$, and hence $d(q_t, x)$ is bounded above for all large t. Thus there is a monotone divergent sequence $\{t_j\}$ such that if $q_j := q_{t_j}$, then $\{q_j\}$ converges to a point q on $M^b_b(F_\gamma)$. Hence $F_\gamma(x)=\lim_{j\to\infty} [t_j-d(q_j, \gamma(t_j))]=\lim_{j\to\infty} [t_j-d(q_j, \gamma(t_j))] -\lim_{j\to\infty} d(x, q_j)=b-d(x, q) \leqq b-d(x, M^b_b(F_\gamma))=a$.

The above argument shows that if x is a point on $M^a_a(F_\gamma)$ and if q_j is the point of intersection of $M^b_b(F_\gamma)$ with a minimizing geodesic joining x to $\gamma(t_j)$ for some monotone divergent sequence $\{t_j\}$ such that $\{q_j\}$ converges to a point q, then $d(x, q)=d(x, M^b_b(F_\gamma))=b-a$ and hence $M^a_a(F_\gamma) \subset \{x \in M^b(F_\gamma);\, d(x, M^b_b(F_\gamma))=b-a\}$. Conversely if $x \in M^b(F_\gamma)$ satisfies $d(x, M^b_b(F_\gamma))=b-a$ and if $\{q_j\}$ and y are taken on $M^b_b(F_\gamma)$ as in the above argument, then $d(x, q_j) \geqq b-a$ and $d(x, q_j) \leqq d(x, y)+d(y, \gamma(t_j))-d(q_j, \gamma(t_j))$. It follows from $\lim_{j\to\infty} [d(y, \gamma(t_j))-d(q_j, \gamma(t_j))]=0$ that $\lim_{j\to\infty} d(x, q_j)=b-a$, and hence $F_\gamma(x)=\lim_{j\to\infty} [t_j-d(x, q_j)-d(q_j, \gamma(t_j))] =F_\gamma(q)-(b-a)=a$. $M^a_a(F_\gamma)=\{y \in M^b(F_\gamma);\, d(y, M^b_b(F_\gamma))=b-a\}$ is now obvious.

To prove (4) let $\sigma\colon [0, \infty)\to M$ be a ray asymptotic to γ. Then there exists a sequence of points $\{p_j\}$ with $\lim_{j\to\infty} p_j=\sigma(0)$ and a sequence of minimizing geodesics $\{\sigma_j\}$ each $\sigma_j\colon [0, l_j]\to M$ satisfying $\sigma_j(0)=p_j$, $\sigma_j(l_j) =\gamma(t_j)$ for some monotone divergent sequence $\{t_j\}$ such that $\dot\sigma(0)= \lim_{j\to\infty} \dot\sigma_j(0)$. For every fixed $t \geqq 0$ if $z_j=\sigma_j(t)$, then $\{z_j\}$ converges to $\sigma(t)$ and $F_\gamma \circ \sigma(t)=\lim_{j\to\infty} [t_j-d(z_j, \gamma(t_j))]+\lim_{j\to\infty} [d(z_j, \gamma(t_j))-d(\sigma(t), \gamma(t_j))] =\lim_{j\to\infty} [t_j-(l_j-t)]=t+F_\gamma \circ \sigma(0)$. Conversely if a unit speed geodesic $\sigma\colon [0, \infty)\to M$ satisfies that $F_\gamma \circ \sigma(t)=t+F_\gamma \circ \sigma(0)$ for all $t \geqq 0$, then (1) implies that $d(\sigma(0), \sigma(t)) \geqq |F_\gamma(\sigma(0))-F_\gamma(\sigma(t))|=t$ holds for all $t \geqq 0$. This means that $\sigma|[0, t]$ is minimizing for all $t \geqq 0$, and hence it is a ray. That σ is asymptotic to γ is proved by showing that for any $\varepsilon>0$ $\sigma|[\varepsilon, \infty)$ is a unique ray emanating from $\sigma(\varepsilon)$ asymptotic to γ. Indeed, if it is established that $\sigma|[\varepsilon, \infty)$ is a unique asymptotic ray to γ, then for a sequence $\{\varepsilon_n\}$ with $\lim_{n\to\infty} \varepsilon_n=0$ there is a divergent sequence $\{t_n\}$ and minimizing geodesics $\{\tau_n\}$ each $\tau_n\colon [0, l_n]\to M$ satisfying $\tau_n(0)=\sigma(\varepsilon_n)$ and $\tau_n(l_n)=\gamma(t_n)$ and $\lim_{n\to\infty} \measuredangle(\dot\sigma(\varepsilon_n), \dot\tau_n(0))=0$. Therefore $\lim_{n\to\infty} \dot\tau_n(0)=\dot\sigma(0)$ and σ is asymptotic to γ. It follows from (3) together with the assumption $F_\gamma \circ \sigma(t)= t+F_\gamma \circ \sigma(0)$ that for any fixed $\varepsilon \in (0, 1)$ and $t \geqq 1$ one has $d(\sigma(\varepsilon), \sigma(t))= d(\sigma(\varepsilon), M^t_t(F_\gamma))=t-\varepsilon$. If there is a point $q \neq \sigma(t)$ on $M^t_t(F_\gamma)$ such that

$d(\sigma(\varepsilon), q)=d(\sigma(\varepsilon), M_t^t(F_\gamma))$, then $d(\sigma(\varepsilon'), M_t^t(F_\gamma))<(\varepsilon-\varepsilon')+(t-\varepsilon)=t-\varepsilon'$ for all ε' in $(0, \varepsilon)$, a contradiction. This fact means together with the first statement of (4) that every ray emanating from $\sigma(\varepsilon)$ and asymptotic to γ must coincide with $\sigma|[\varepsilon, \infty)$, and hence there is a unique asymptotic ray to γ emanating from $\gamma(\varepsilon)$.

To prove (5) let σ_j: $[0, s_j]\to M$ be a minimizing geodesic with $\sigma_j(0)=\sigma(l)$ and $\sigma_j(s_j)=\gamma(t_j)$ for some monotone divergent sequence $\{t_j\}$ such that $\{\sigma_j\}$ converges to a ray σ_1. Let τ_j: $[0, u_j]\to M$ be a minimizing geodesic with $\tau_j(0)=x$, $\tau_j(u_j)=\gamma(t_j)$ and let q_j be the point of intersection $\tau_j([0, u_j])\cap M_a^a(F_\gamma)$. The intersection exists because of the assumption $F_\gamma\circ\tau_j(0)<a$ and $F_\gamma\circ\tau_j(u_j)=t_j>a$. Then $d(x, q_j)\geqq d(x, \sigma(l))=d(x, M_a^a(F_\gamma))$ and from $q_j, \sigma(l)\in\lim S(\gamma(t_j), t_j-a)$ follows $\lim [d(q_j, \gamma(t_j))-d(\sigma(l), \gamma(t_j))]=0$. Suppose that $\dot\sigma_1(0)\neq\dot\sigma(l)$. Then there is an $\eta>0$ such that $d(x, \gamma(t_j))\leqq d(x, \sigma(l))+d(\sigma(l), \gamma(t_j))-\eta$ for all large j. Letting $j\to\infty$, the above inequality yields $F_\gamma(x)=\lim [t_j-d(x, \gamma(t_j))]\geqq\lim [t_j-d(\sigma(l), \gamma(t_j))-l+\eta]=a-l+\eta$. Similarly, $F_\gamma(x)=\lim [t_j-d(x, \gamma(t_j))]=\lim [t_j-d(x, q_j)-d(q_j, \gamma(t_j))]=a-l$, a contradiction. Thus $\sigma(t)=\sigma_1(t-l)$ for all $t\geqq 0$. Since $\sigma_1|[0, \infty)$ is asymptotic to γ, (4) implies $F_\gamma\circ\sigma(t-l)=a+(t-l)$ for all $t\geqq l$. To prove that σ: $[0, \infty)\to M$ is asymptotic to γ recall that $d(\sigma(0), M_a^a(F_\gamma))=d(\sigma(0), \sigma(l))=l$. Therefore it follows that for every t in $[0, l]$, $d(\sigma(t), M_a^a(F_\gamma))=l-t$, and (3) implies $F_\gamma\circ\sigma(t)=a-t$ for all $t\in[0, l]$. In particular $F_\gamma\circ\sigma(0)=a-l$. It follows from (4) that σ is a ray which is asymptotic to γ.

To prove (6) let σ: $[0, \infty)\to M$ be a ray asymptotic to γ and let $x=\sigma(a)$ for an $a>0$. For a fixed number $c>0$ set $b=c+a$. Since σ is a ray, there exists a neighborhood V of $\sigma(b)$ such that both $S(\sigma(0), b)\cap V:=\Sigma_1$ and $S(\sigma(b+c), c)\cap V=:\Sigma_2$ are smooth hypersurfaces and they have the common unit normal $\dot\sigma(b)$ at $\sigma(b)$. It follows from (3) that in a small neighborhood U of x, $F_\gamma|U$ is expressed as $F_\gamma(y)=b+\alpha-d(y, M_{b+\alpha}^{b+\alpha}(F_\gamma))$, $y\in U$, where $\alpha:=F_\gamma\circ\sigma(0)$. If f_1, f_2: $U\to R$ are defined as $f_i(y):=d(y, \Sigma_i)$, $i=1, 2$, $y\in U$, then there exists a neighborhood $U_1\subset U$ of x such that for every $y\in U_1$ and for every $i=1, 2$, f_i is attained at a unique point in Σ_i. Thus both f_1 and f_2 are smooth on U_1.

It will be asserted that for every $y\in U_1$ $f_1(y)\leqq d(y, M_{b+\alpha}^{b+\alpha}(F_\gamma))\leqq f_2(y)$ and in particular $f_1(y)\leqq b+\alpha-F_\gamma(y)\leqq f_2(y)$ for all $y\in U_1$. It follows from (1) that for any $z\in B(\sigma(0), b)$ $|F_\gamma(z)-F_\gamma(\sigma(0))|\leqq b$, and hence $F_\gamma(z)\leqq b+F_\gamma(\sigma(0))=b+\alpha$. This proves $B(\sigma(0), b)\subset\mathrm{Int}\,(M^{b+\alpha}(F_\gamma))$. The proof of the assertion is complete if $S(\sigma(b+c), c)\subset M-\mathrm{Int}\,(M^{b+\alpha}(F_\gamma))$ is verified. In fact, these implications show that Σ_1 and Σ_2 are separated by $M_{b+\alpha}^{b+\alpha}(F_\gamma)$ in V, and hence the inequality in the assertion is proved.

To prove $S(\sigma(b+c), c)\subset M-M^{b+\alpha}(F_\gamma)$, let τ_t: $[0, l_t]\to M$ be a minimizing geodesic with $\tau_t(0)=\gamma(t)$, $\tau_t(l_t)=x$ and let q_t be the intersection of

$\tau_t([0, l_t])$ with $M^{b+\alpha}_{b+\alpha}(F_\gamma)$. As is seen in the proof of (5) there is a unique ray emanating from x and asymptotic to γ. Thus $\lim_{t\to\infty} q_t=\sigma(b)$. From $q_t \in M^{b+\alpha}_{b+\alpha}(F_\gamma)$ it follows that the point $\tau_t(t-(b+\alpha))$ lies on the subarc of τ_t with endpoints $\tau_t(0)$ and q_t. Triangle inequality implies that $B(\gamma(t), t-(b+\alpha))\supset B(\tau_t(t-(b+\alpha+c)), c)$ for all large t. Note also that $\tau_t(t-(b+\alpha))$ is the point of intersection of $\tau_t([0, l_t])$ with $S(\gamma(t), t-b-\alpha)$, and hence $\lim_{t\to\infty} \tau_t(t-b-\alpha-c)=\sigma(b+c)$. Since $S(\sigma(b+c), c)=\lim_{t\to\infty} S(\tau_t(t-b-c-\alpha), c)$ and $\lim_{t\to\infty} B(\gamma(t), t-b-\alpha)=M-\mathrm{Int}\,(M^{b+\alpha}(F_\gamma))$ the desired implication is proved.

If C is a curve fitting to a vector $v \in M_x$, then $v(f_1)\leqq\lim_{h\to 0}-[F_\gamma \circ C(h)-F_\gamma \circ C(0)]/h\leqq v(f_2)$ follows from $f_1(C(0))=f_2(C(0))=b+\alpha-F_\gamma(C(0))$ and $f_1(C(h))\leqq b+\alpha-F_\gamma(C(h))\leqq f_2(C(h))$ for $C(h) \in U_1$. Because $v(f_1)=v(f_2)=\langle v, -\dot\sigma(b)\rangle$ for every $v \in M_x$, this proves the differentiability of F_γ at x.

A ray γ: $[0, \infty)\to M$ is now fixed in the following remarks.

Remark 1. If $\{\sigma_n\}$ is a sequence of rays each of which is asymptotic to γ and such that it converges to a ray σ, then σ is again asymptotic to γ. This immediately follows from (4).

Remark 2. If there are two distinct rays emanating from a point x and each of which is asymptotic to γ, then F_γ is not differentiable at x. A ray σ: $[0, \infty)\to M$ asymptotic to γ is by definition *maximal* iff for any positive ε its extension to $[-\varepsilon, \infty)$ is not an asymptotic ray to γ. While it might happen that the extension is still a ray. Let A be the set of all starting points of maximal asymptotic rays to γ, and let B be the set of all non-differentiable points of F_γ, and let C be the set of all points which are starting points of at least two distinct rays each asymptotic to γ. Then $C\subset B\subset A$ holds and C is a dense subset of A (see [26]). However it has not been known whether A is a set of measure zero, while so is B.

As is seen in Theorem 1.1, (4), if σ is an asymptotic ray to γ, then $F_\gamma(\sigma(t))=t+F_\gamma(\sigma(0))$. The relation between F_σ and F_γ will now be discussed. This result has not been seen in literature.

Lemma 1.2. *Let γ and σ be rays such that σ is asymptotic to γ. Then*

$$F_\sigma(x)\leqq F_\gamma(x)-F_\gamma(\sigma(0)), \qquad x \in M.$$

In particular for every $l>0$, $\lim_{t\to\infty} S(\sigma(t), t-l)=M^l_l(F_\sigma)$ is contained in $M-\mathrm{Int}\,(M^{a+l}(F_\gamma))$ where $a:=F_\gamma(\sigma(0))$.

Proof. There exists a sequence of points $\{p_j\}$ on M and minimizing geodesics σ_j: $[0, s_j]\to M$ with $\sigma_j(0)=p_j$, $\sigma_j(s_j)=\gamma(t_j)$ for some divergent sequence $\{t_j\}$ such that $\lim \sigma_j(0)=\sigma(0)$. It follows from (4) that $F_\gamma(\sigma(t))$

$=t+a$. Convergence property of $\{\sigma_j\}$ implies that for any $\varepsilon>0$ and for any fixed $t>0$ there is a $j(t,\varepsilon)$ such that if $j>j(t,\varepsilon)$ then $d(\sigma(t),\sigma_j(t))<\varepsilon$. For a point x on M, $t-d(x,\sigma(t))<t-[d(x,\sigma_j(t))-\varepsilon]=t-[d(x,\gamma(t_j))-d(\gamma(t_j),\sigma_j(t))]+\varepsilon=[t_j-d(x,\gamma(t_j))]+(s_j-t_j)+\varepsilon$. Because $\lim_{j\to\infty}(s_j-t_j)=-a$ and because $\varepsilon>0$ is any, the above inequality implies that $t-d(x,\sigma(t))\leqq F_\gamma(x)-F_\gamma(\sigma(0))$ for any $t>0$.

Remark. Lemma 1.2 has the following important significance which is used in the next section. Let p be any point and σ a ray emanating from p and asymptotic to an arbitrary fixed ray γ. Then Lemma 1.2 implies that $F_\sigma+F_\gamma(\sigma(0))$ supports F_γ at p. Namely, a continuous function g is said to *support f at p* iff $f\geqq g$ in a neighborhood of p and $f(p)=g(p)$.

§ 2. Curvature and Busemann functions

The relations between curvature and Busemann functions on M are now discussed. A real valued function f on M is said to be *convex* (*affine*, respectively) iff its restriction to every geodesic on M is convex (affine respectively). If f is differentiable of class C^2 then f is convex iff its second derivative along every geodesic is nonnegative. If $-f$ is convex then f is said to be *concave*, f is by definition *subharmonic* iff for any harmonic function h defined on a connected open set U in M such that h has a continuous extension to the closure $\bar{U}$ of U such that $f=h$ on the boundary ∂U, $h\geqq f$ is satisfied on U. If f is differentiable of class C^2 then f is subharmonic iff $\Delta f\geqq 0$, where Δ is the Laplaican operator. A continuous real valued function f on a complex manifold X is said to be *plurisubharmonic* iff its restriction to every 1-dimensional complex submanifold of X is subharmonic. If f is differentiable of class C^2 and if $\alpha\colon D\to X$ is a holomorphic map of a disk D in C such that $\alpha(0)=p\in X$, then setting $v:=\alpha_*((\partial/\partial z)_0)$ one has $\partial\bar{\partial}f(v,\bar{v})=4\Delta(f\circ\alpha)(0)$. Thus a C^2 function f on X is plurisubharmoinc iff $\partial\bar{\partial}f$ is a positive semidefinite Hermitian form.

The purpose of this section is to give a simple proof of the following theorem. The first part of Theorem 2.1 was proved by Cheeger-Gromoll in [6], [7] and then a more detailed discussion for generalized Busemann functions has been developed by Wu as Fundamental Theorems A, B and C in [47]. Making use of the fundamental properties of Busemann functions developed in the previous section, we shall give a technical simplification of Wu's proof.

Theorem 2.1 (Cheeger-Gromoll [6, 7], Wu [47]). *Let M be a complete noncompact Riemannian manifold and let F_γ be a Busemann function.*

(1) *If the sectional curvature of M is nonnegative everywhere, then F_γ is convex.*

(2) *If the Ricci curvature is nonnegative everywhere, then F_γ is subharmonic.*

(3) *If M is Kaehler and if the holomorphic bisectional curvature is nonnegative everywhere, then F_γ is plurisubharmonic.*

The idea of the proof of Theorem 2.1 which we shall use here is provided by Wu in [45] and is sketched as follows. (1), (2) and (3) are treated by the same principle. Namely, the convexity, subharmonicity and plurisubharmonicity of a continuous function f on M is guaranteed by showing that the modulus of convexity, subharmonicity and plurisubharmonicity for f is nonnegative every where. If f is differentiable of class C^2, then the modulus of convexity, subharmonicity and plurisubharmonicity of f at a point p on M is nothing but the infimum of second derivatives of f at p along all unit speed geodesics passing through p, $\Delta f(p)$, and $\inf_\alpha \partial\bar{\partial} f(v, \bar{v})$, where the infimum is taken over all holomorphic maps of a disk in C such that $\alpha(0)=p$ and $v=\alpha_*((\partial/\partial z)_0)$. In order to verify the nonnegativity of the corresponding modulus for f at a point p on M, it suffices to construct a sequence of continuous functions $\{f_n\}$ which has the following properties: (1) f_n converges uniformly to f on every compact set, (2) for each n f_n supports f at p, namely, $f \geqq f_n$ in a neighborhood of p and $f(p)=f_n(p)$, (3) for each n the corresponding modulus for f at p is bounded below by $-1/n$. Then the proof is achieved because if f_n supports f at p, then the corresponding modulus for f at p is bounded below by that of f_n at p, which converges to 0 as $n \to \infty$.

Thus the crucial step of our proof is to construct a sequence of functions with the properties mentioned above. Let F_γ be a Busemann function and let p be an arbitrary fixed point. For a sufficiently small fixed neighborhood U of p we shall construct a sequence of continuous functions $\{F_n\}$ each F_n is defined on $\bar{U}$ with the following properties: (1) F_n converges uniformly to a Busemann function (F_σ) on $\bar{U}$ which supports F_γ at p, (2) for each n F_n supports F_σ at p, (3) for each n there is an open set U_n of p such that $U_{n+1} \subset U_n \subset U$ and F_n is *smooth* on U_n, (4) if the sectional curvature is nonnegative then the modulus of convexity for F_n at p is bounded below by $-1/n$, (5) if the Ricci curvature is nonnegative then the modulus of subharmonicity for F_n at p is bounded below by $-1/n$, (6) if M is Kaehler and if the holomorphic bisectional curvature is nonnegative then the modulus of plurisubharmonicity for F_n is bounded below by $-1/n$.

Note that the Busemann function stated in (1) is not the original one but the one for an asymptotic ray to γ emanating from p, and which supports F_γ at p.

Proof of Theorem 2.1. Let $\gamma: [0, \infty) \to M$ be a ray and let p be an arbitrary fixed point. Let $\sigma\colon [0, \infty)\to M$ be a ray emanating from p and asymptotic to γ and set $a:=F_\gamma(p)$. It follows from Theorem 1.1, (4) and (3) that $F_\gamma(\sigma(t))=t+a$ for all $t\geqq 0$ and $F_\gamma(x)=l+a-d(x, M^{l+a}_{l+a}(F_\gamma))$ for any fixed $l>0$ and for any x in $M^{l+a}(F_\gamma)$. For a small neighborhood U of p we may consider $\bar U\subset \mathrm{Int}\,(M^c(F_\gamma))$ where $c=a+1$, and hence $F_\gamma|U$ is expressed as

$$F_\gamma(x)=c-d(x, M^c_c(F_\gamma)), \qquad x\in U.$$

Note that $\sigma(1)\in M^1_1(F_\sigma)$. If follows from Lemma 1.2 that $M^1_1(F_\sigma)$ is contained in $M-\mathrm{Int}\,(M^c(F_\gamma))$ and hence $S(\sigma(t), t-1)$ is contained in $M-\mathrm{Int}\,(M^c(F_\gamma))$ for all $t>1$. Therefore every minimizing geodesic from a point x in U to $S(\sigma(t), t-1)$ passes through a point on $M^c_c(F_\gamma)$. This fact implies that for every x in U and for every fixed $t>1$

$$d(x, M^c_c(F_\gamma))\leqq d(x, S(\sigma(t), t-1)).$$

For every large t $p=\sigma(0)$ is not a cut point to $\sigma(t)$ along $\sigma|[0, t]$ and hence there is a neighborhood V_t of $\sigma(1)$ and a neighborhood $U_t\subset U$ of p such that $\Sigma(\sigma, t):=V_t\cap S(\sigma(t), t-1)$ is a smooth hypersurface and such that U_t does not intersect the cut locus of $\sigma(t)$. Since all geodesics hitting orthogonally to $\Sigma(\sigma, t)$ pass through $\sigma(t)$ at length $t-1$, $\Sigma(\sigma, t)$ has no focal point in U_t. Therefore for every large t the function $F_t\colon U\to R$ defined as

$$F_t(x):=1-d(x, \Sigma(\sigma, t)), \qquad x\in U$$

is continuous on $\bar U$ and smooth on U_t. It is clear that $F_t(p)=F_\sigma(p)=0$ for all t and $F_t(x)\leqq F_\sigma(x)$ for all $x\in U$. F_t converges uniformly to F_σ as $t\to\infty$ and furthermore $F_\sigma+a$ supports F_γ at p.

We now want to show that for a suitably chosen divergent sequence $\{t_n\}$ if $F_n:=F_{t_n}$ then the corresponding modulus for F_n at p is bounded by $-1/n$ for every n.

The first step of the computation of the corresponding modulus for F_n is to show that the second fundamental form $A_{\dot\sigma(1)}$ of $\Sigma(\sigma, t)$ at $\sigma(1)$ has the following properties: (1) if the sectional curvature is nonnegative then the eigenvalues of $A_{\dot\sigma(1)}$ are not less than $-1/(t-1)$, (2) if the Ricci curvature is nonnegative then the trace of $A_{\dot\sigma(1)}$ is not less than $-(m-1)/(t-1)$, (3) if the holomorphic bisectional curvature is nonnegative then for every unit vector v at $\sigma(1)$ tangent to $\Sigma(\sigma, t)$, $\langle A_{\dot\sigma(1)}v, v\rangle+\langle A_{\dot\sigma(1)}Jv, Jv\rangle \geqq -2/(t-1)$. This follows from the second variation formula along $\sigma|[1, t]$. Let v be a unit vector at $\sigma(1)$ tangent to $\Sigma(\sigma, t)$ and let Y and $\tilde E$ be the Jacobi field and the parallel field along $\sigma|[1, t]$ such that $Y(1)=\tilde E(1)=v$ and $Y(t)=0$. Y is associated with the 1-parameter geodesic variation whose variational curves are minimizing geodesics joining $\sigma(t)$ to points on a curve in $\Sigma(\sigma, t)$ fitting v. Since all variational curves have the same length $t-1$, the second variation formula gives

$$\int_1^t (\langle Y', Y'\rangle - \langle R(Y, \dot{\sigma})\dot{\sigma}, Y\rangle)dt + \langle A_{\dot{\sigma}(1)}v, v\rangle = 0.$$

It is elementary that if a geodesic contains no conjugate pair, then the Jacobi field minimizes the index form I among all piecewise smooth vector fields along it with the same values at both endpoints. Thus $I(Y, Y) \leqq I(E, E)$, where $E(s) := (t-s)\tilde{E}(s)/(t-1)$, $1 \leqq s \leqq t$. If the sectional curvature is nonnegative, then $\langle A_{\dot{\sigma}(1)}v, v\rangle \geqq -I(E, E) \geqq -1/(t-1)$. If the Ricci curvature is nonnegative, then taking an orthonormal basis $v_1, v_2, \cdots v_m = \dot{\sigma}(1)$ at $M_{\sigma(1)}$, one has $\sum_{i=1}^{m-1} I(E_i, E_i) \geqq -(m-1)/(t-1)$, where $E_i(s) := (t-s)\tilde{E}_i(s)/(t-1)$ and $\tilde{E}_i$ is the parallel field with $\tilde{E}_i(1) = v_i$ as before. Thus trace $A_{\dot{\sigma}(1)} \geqq -(m-1)/(t-1)$. Finally if the holomorphic bisectional curvature is nonnegative and if Y and Z are Jacobi fields along $\sigma|[1, t]$ with $Y(1) = v$, $Z(1) = Jv$ and $Y(t) = Z(t) = 0$, then $\langle A_{\dot{\sigma}(1)}v, v\rangle + \langle A_{\dot{\sigma}(1)}Jv, Jv\rangle \geqq -(I(E, E) + I(JE, JE)) \geqq -2/(t-1)$.

Now let $F_n \colon U \to R$ be defined by $F_n := F_{t_n}$ for $t_n > n(m-1)+1$. Then the required inequalities for the second fundamental forms of $\Sigma(\sigma, t_n)$ are satisfied.

To calculate the modulus of convexity for F_n at p let $v \in M_p$ be an arbitrary unit vector and c a geodesic fitting v. Let P be a unit parallel field along $\sigma|[0, 1]$ such that $v = \cos\theta\, P(0) + \sin\theta\, \dot{\sigma}(0)$ for some constant $\theta \in [0, 2\pi)$. Let Y be the unique N-Jacobi field along $\sigma|[0, 1]$ with $Y(0) = v$ and $Y(1) \in T_{\sigma(1)}\Sigma(\sigma, t_n)$ namely, Y is associated to a 1-parameter geodesic variation whose variational geodesics are all distance-minimizing geodesics from points on c to $\Sigma(\sigma, t_n)$. It follows from the second variation formula together with the minimization theorem (see Theorem 4, p. 228 in [1]) that

$$-\frac{d^2}{ds^2}F_n(c(s))\Big|_{s=0} \leqq I(\cos\theta \cdot P, \cos\theta \cdot P) - \langle A_{\dot{\sigma}(1)} \cos\theta \cdot P(1), \cos\theta \cdot P(1)\rangle.$$

Since the sectional curvature is nonnegative, $\cos^2\theta \cdot I(P, P) \leqq 0$ and hence $-F_n(c(0))'' \leqq -\cos^2\theta \langle A_{\dot{\sigma}(1)}P(1), P(1)\rangle \leqq 1/n(m-1)$. This proves (1) of Theorem 2.1.

To compute the modulus of subharmonicity for F_n at p, let $P_1, \cdots, P_m = \dot{\sigma}$ be an orthonormal parallel frame field along $\sigma|[0, 1]$. Then by the same reason as above,

$$-\Delta F_n(p) \leqq \sum_{i=1}^{m-1} [I(P_i, P_i) - \langle A_{\dot{\sigma}(1)}P_i(1), P_i(1)\rangle].$$

If the Ricci curvature is nonnegative then $\sum_{i=1}^{m-1} I(P_i, P_i) \leqq 0$ and hence $-\Delta F_n(p) \leqq -\text{trace } A_{\dot\sigma(1)} \leqq 1/n$, proving (2).

Finally let $\alpha: D \to M$ be a holomorphic map from a disk D in $\boldsymbol{C}$ into M with $\alpha(0)=p$ and let $v=\alpha_*((\partial/\partial z)_0)$. Because F_n is smooth in a neighborhood of p $\partial\bar\partial F_n(v, \bar v)=H_{F_n}(X, X)+H_{F_n}(JX, JX)$, where H is the usual Hessian form and v is expressed as $X-iJX$, $X \in M_p$. The Hessian form $H_{F_n}(X, X)$ is nothing but the second derivative of $F_n(c(s))$ at $s=0$ along the geodesic c fitting X. Therefore by the same reason as before $\partial\bar\partial F_n(v, \bar v) \geqq -2/n(m-1)$. This proves (3) of Theorem 2.1.

Remark 1. If M is complete and simply connected and if the sectional curvature of M is nonpositive everywhere, then F_γ is concave. This is because the distance function to a fixed point is convex and hence for every fixed t, $t-d(\cdot, \gamma(t))$ is concave. Thus the limit of these concave functions is again concave.

Remark 2. The argument in the proof of (1) also yields that if $C \subset M$ is a closed (locally) convex set with nonempty boundary and if the sectional curvature of M is nonnegative everywhere, then the function $d_C: C \to R$ defined by $d_C(x):=-d(x, \partial C)$, $x \in C$ is convex. In fact let $y \in \partial C$ be a point such that $d_C(x)=-d(x, y):=-l$, and let $\tau: [0, l] \to C$ be a minimizing geodesic with $\tau(0)=x$, $\tau(l)=y$. Consider a hypersurface $S(y):=\{\exp_y v;\ v \in M_y,\ \langle v, \dot\sigma(l)\rangle=0,\ \|v\| \leqq \text{convexity radius at } y\}$. Then $S(y)$ does not intersect $\operatorname{Int}(C)$ and hence if σ is a geodesic emanating from a point $x \in \operatorname{Int}(C)$, then $d(\sigma(u), S(y)) \geqq d(\sigma(u), \partial C)$ for all sufficiently small u. If $\theta:=\measuredangle(\dot\sigma(0), \dot\tau(0))$ and if E is a parallel field along τ with $\dot\sigma(0)=\cos\theta\dot\tau(0)+\sin\theta E(0)$, then by the same argument developed in the proof of Theorem 2.1, (1) the function $u \to d(\sigma(u), S(y))$ has second derivative at 0 bounded above by

$$\sin^2\theta \int_0^l -K(E, \dot\tau)ds \leqq 0,$$

where the eigenvalues of the second fundamental form of $S(y)$ at y are all 0. This proves the convexity of $d_C(\sigma)$.

Now the soul theorem due to Cheeger-Gromoll is a direct consequence of Theorem 2.1, (1) and Remark 2 stated above.

Theorem 2.2 (Cheeger-Gromoll, [7]). *Let M be a complete noncompact Riemannian manifold of nonnegative sectional curvature. Then there exists a compact totally geodesic submanifold without boundary which is a totally convex set of M.*

Proof. Let p be an arbitrary fixed point on M and let $F: M \to R$ be defined by $F(x):=\sup_\gamma F_\gamma(x)$, where the sup is taken over all rays emanating from p. F is clearly convex. Moreover for every $a \in F(M)$, $M^a(F)$ is compact. In fact if otherwise, $M^a(F)$ would be noncompact, and hence the total convexity of $M^a(F)$ implies that it contains a ray. F diverges on this ray, a contradiction. Thus F takes a minimum on M. We may consider by adding some constant if necessary to F, that the minimum value is 0. Remark 2 implies that if the minimum set $C_0:=M^0(F)$ has nonempty boundary, then there is a convex function $d_1\colon C_0\to R$, $d_1(x):=-d(x, \partial C_0)$. If the minimum set C_1 of d_1 has nonemtpy boundary, then a convex function is defined on C_1. Obviously the dimension of C_0 is less than m and the dimension of C_1 is less than that of C_0. Thus by induction we find a sequence of compact totally convex sets $C_0\supset C_1\supset C_2, \cdots, \supset C_k$ such that C_k has no boundary. This C_k is a soul.

The structure theorem is stated as follows.

Theorem 2.3 (Cheeger-Gromoll, [7]). *Let M be a complete and noncompact Riemannian manifold of nonnegative sectional curvature. Let S be a soul of M which is obtained in Theorem 2.2. Then M is homeomorphic to the total space of the normal bundle $\nu(S)$ over S in M.*

The proof given by Cheeger and Gromoll in [7] in sketched as follows. Using the same notations as in the proof of Theorem 2.2 we have a sequence $C_0\supset C_1\supset C_2, \cdots, \supset C_k=S$ of compact totally convex sets such that C_0 is the minimum set of F and C_k is a soul. It is proved that $M-C_0$ is homeomorphic to $M_t^t(F)\times(0, \infty)$ where $t>0$ is arbitrary fixed. As is stated in the next section such a homeomorphism can be constructed for general convex functions whose level sets are not necessarily compact. For each $i=0, 1, \cdots, k-1$ let $a_i=\max\{d(x, \partial C_i); x \in C_i\}$. Obviously $-a_i=\min_{C_i} d_{i+1}$. It follows from compactness of C_0 that there exist positive constants $\delta\leqq\varepsilon$ such that the following statements are valid (see Lemma 2.4 of [7]): (1) if $0\leqq a\leqq a'\leqq a_i$ and if $a'-a<\delta$ for $i=0, \cdots, k-1$, then for every $x \in C_i$, $d(x, C_i^a(d_{i+1}))<\varepsilon$, where $C_i^a(d_{i+1}):=\{y \in C_i; d_{i+1}(y)\leqq -a\}$, (2) if $0\leqq a<a'<a_0$ and if $a'-a<\delta$ then there is a homeomorphism $\partial C_0^a\times[0, 1]\to C_0^a-\operatorname{Int}(C_0^{a'})$ with $(x, 0)\to x$. In particular ∂C_0^a has a collar neighborhood in C_0, (3) if $a \in (a_0-\delta, a_0]$, then there is a strong deformation retract $C_0^a\to C_1$, (4) if $0<r\leqq a_0-a<\delta$, then there is a homeomorphism $C_0^a\to C_0\cap B(C_1, r)$ keeping C_1 fixed, (5) if $i=1, \cdots, k-1$ and if $0\leqq a\leqq a'\leqq a_i$ and $a'-a<r<\delta$, then there is a homeomorphism $C_0\cap B(C_i^a, r)\to C_0\cap B(C_i^{a'}, r)$ keeping C_{i+1} fixed.

The above argument shows that if $S=C_k$ is the soul of the minimum set C_0 of F, then there is a homeomorphism between $\nu(S)$ and M keeping S fixed.

A convex function f on M is said to be *strongly convex* iff for every compact set K on M there exists a positive constant η such that at each point on K the second difference quotient of f along every geodesic passing through that point is bounded below by η at that point. It follows from the argument developed in the proof of Theorem 2.1, (1) that if the sectional curvature of M is positive outside a compact set, then every Busemann function F_γ can be transformed to a strongly convex function $\chi \circ F_\gamma$ by a smooth strictly increasing convex function $\chi: R \to R$. Thus in this case the soul of M is a single point. It has been proved by Wu in [47] that if a complete noncompact Riemannian manifold has nonnegative sectional curvature and it is positive outside a compact set, then M is diffeomorphic to R^m. This is a slight generalization of the Gromoll-Meyer theorem [20] which states that every complete noncompact Riemannian manifold of positive sectional curvature is diffeomorphic to R^m.

It should also be noted that if H is a complete simply connected Riemannain manifold of nonpositive sectional curvature, then every Busemann function of H is concave and moreover it is differentiable of class C^2. This has been proved by Heintze and Im Hof in [21].

§ 3. Convex functions.

The existence of a nontrivial convex function on a complete Riemannian manifold imposes certain restrictions to the topology of the manifold. It is elementary that a convex function on a complete M is locally Lipschitz continuous (and hence automatically continuous). However it has not been known if a convex function defined on a G-space is continuous or not.

An obvious topological consequence of the existence of a convex function on a complete M is that M is noncompact. Conversely, every noncompact manifold admits a complete metric and a smooth function which is convex with respect to the metric. This is possible because such a convex function takes minimum and the minimum set has nonempty interior in which all important information on the topology of M is contained. Such a metric can be taken so that it is flat outside the minimum set (for detail see [11]). The only one metric consequence of the existence of a nontrivial convex function on M is that the volume of M is unbounded, see [50].

Therefore it is natural to assume that a convex function on M is not constant on any open subset of M. We shall call such a function to be *locally nonconstant*. The following results on complete Riemannian manifolds admitting locally nonconstant convex functions have been obtained by Greene and Shiohama in [11]. Let f: $M \to R$ be a locally nonconstant convex function. Then we have (1) if f has a nonconnected level set, then f takes minimum and the minimum set is a totally geodesic hypersurface without boundary and it has a trivial normal bundle. Moreover if $a > \inf_M f$, then $M_a^a(f)$ has exactly two components, (2) if f has a compact level, then all levels are compact and the diameter function $d(t) :=$ the diameter of $M_t^t(f)$ is monotone nondecreasing with $t \in f(M)$. (3) M with a locally nonconstant convex function has at most two ends. Moreover if f has a noncompact level, then M has exactly one end and if M has two ends, then f has compact levels.

Theorem 3.1 (Greene-Shiohama, [11]). *Let f be a locally nonconstant convex function on M. For any fixed c with $c > \inf_M f$ there exists a homeomorphism $H: M_c^c(f) \times (\inf_M f, \infty) \to M - \{x \in M; f(x) = \inf_M f\}$ such that $f(H(y, \alpha)) = \alpha$ for all $y \in M_c^c(f)$ and for all $\alpha > \inf_M f$. Moreover if the minimum set is nonempty, then H has a continuous extension $\bar{H}: M_c^c(f) \times [\inf_M f, \infty) \to M$ which is proper and surjective.*

Note that every strictly convex function is automatically locally nonconstant, since the minimum set of a strictly convex function is a single point. It has been proved by Greene and Wu in [17] that a strongly convex function on M can be approximated by smooth strongly convex functions, and hence if M admits a strongly convex exhaustion function then M is diffeomorphic to R^m. Morse-theoretic analysis is employed to construct the above diffeomorphism between M and R^m. However it has not been known whether or not a convex function can be approximated by smooth convex functions. In spite of the possible absence of smooth convex approximations, a complete Morse-theoretic analysis can be obtained even though the function might not have C^2 regularity. And the homeomorphism obtained in the above theorem can be replaced by a diffeomorphism as follows (see [12]): Let f be a locally nonconstant convex function on M. Then there exists a smooth complete hypersurface N without boundary of M which is homeomorphic to $M_c^c(f)$ and there is a diffeomorphism $D: N \times (0, \infty) \to M - \{x \in M; f(x) = \inf_M f\}$. Moreover if the minimum set is nonempty, then D has a continuous extension $\bar{D}: N \times [0, \infty) \to M$ which is proper and surjective.

As consequences of the above results it has been proved that (1) if f is strictly convex, then M is diffeomorphic to either R^m or else $N \times R$, (2) if f has a nonconnected level, then M is diffeomorphic to $N \times R$, where N is the minimum set of f, (3) if the minimum set of f has no boundary, then M is diffeomorphic to the total space of the normal bundle over the minimum set in M.

The existence of a locally convex function on M also gives certain restrictions to the Riemannian structure of M. In the case where M has positive sectional curvature, it has been proved in [20] that: (1) the isometry group $I(M)$ of M is compact, (2) the exponential map at every point on M is proper. In the case where H is a complete simply connected Riemannian manifold of nonpositive sectional curvature the Cartan theorem states that every compact subgroup of the isometry group $I(H)$ of H has a common fixed point.

The above stated results have been extended to manifolds having locally nonconstant convex functions as follows. It has been proved by Yamaguchi in [49] that (1) if M admits a strictly convex function which takes minimum, then every compact subgroup of $I(M)$ has a common fixed point, (2) if M admits a strictly convex function which has compact levels and which has no minimum, then $I(M)$ is compact. If M admits a strictly convex exhaustion function then the exponential map at every point on M is proper. In the case where a convex function has compact levels and has no minimum, M is isometric to the Riemannian product $N \times R$ if $I(M)$ is noncompact. This has been proved in [13].

For each point p on M let $C(p)$ be the cut locus and let $i(p):= d(p, C(p))$. This i is known to be continuous and plays an important role in the study of topology of compact manifolds, see Sakai [32]. It is elementary that for every p on M if $q \in C(p)$ satisfies $d(p, q)=i(p)$, then either there exists a simply closed geodesic loop at p with length $2i(p)$ passing through q or otherwise q is conjugate to p along any minimizing geodesic, see [19].

It has been proved in [45] and [28] that if M is a complete noncompact manifold whose sectional curvature K_M satisfies $0<K_M\leqq 1$, then $i(M):=\inf_M i \geqq \pi$. In fact, as is stated at the end of Section 2 there is a strongly convex exhaustion function $\tilde{F}$ on M. If there is a point q on M at which $i(q)<\pi$, then $i|M^{\tilde{F}(q)}(\tilde{F})$ takes minimum at some point p. Then $i(p)$ and the total convexity of $M^{\tilde{F}(q)}(\tilde{F})$ implies that there is a closed geodesic of length $2i(p)$ passing through p whose image is in the set. This contradicts the fact that $\tilde{F}$ is strongly convex.

The above result has been extended by Sarafutdinov in [38] as follows. If M is homeomorphic to R^m and if the sectional curvature of M satisfies $0\leqq K_M \leqq 1$, then $i(M)\geqq \pi$. A little more generally, if M admits a strictly convex exhaustion function and if the sectional curvature K_M of M satisfies $K_M \leqq 1$, then $i(M) \geqq \pi$, (see [40]).

The properness of the exponential map at every point on a complete noncompact M with positive sectional curvature has been proved by Gromoll-Meyer in [20]. This has been extended to manifolds admitting strongly convex exhaustion functions by Greene-Wu in [17]. A slight generalization of the above result will state that if M admits a strictly convex exhaustion function, then the exponential map at every point on M is proper (see [11]). Here the assumption that the function is exhaustion is inevitable. As is seen later in the example of a strictly convex nonexhaustion function with compact levels and without minimum on a surface of revolution whose profile curve is given by $y=\log x$, the exponential map at any point on such a surface is not proper. This is a direct consequence of the classical Clairaut theorem.

Finally it should be noted that the class of all complete Riemannian manifolds admitting strictly convex functions with compact levels contains properly the class of all complete Riemannian manifolds of positive sectional curvature and all complete simply connected Riemannian manifolds of nonpositive sectional curvature. In fact it is possible to construct a surface of revolution in E^3 on which there is a smooth strongly convex function with compact levels and on which the Gaussian curvature changes sign, see [40]. Such an example is constructed as follows. Let $0<a<b$ and let $h: [0, \infty)\to[0, 1]$ be a smooth function such that (1) $h(v)=0$ for $0\leqq v\leqq a$ and $h(v)=1$ for $v\geqq b$, (2) if $g(v)=v^2+h(v)$, then $g'(v)>0$ for all $v>0$ and $g''(v_0)<0$ for some $v_0 \in (a, b)$. The desired surface of revolution will be defined by $(u, v)\to(v \cos u,\ v \sin u,\ g(v))$, $u \in [0, 2\pi]$, $v\geqq 0$. Then the Gaussian curvature is negative in a neighborhood of (u, v_0) and is positive on (u, v) for $v\geqq b$ and $v\leqq a$. Then for sufficiently large n the function $f(u, v):=g^n(v)$ is a smooth strongly convex exhaustion function. A surface of revolution defined by $(u, v)\to(v \cos u,\ v \sin u,\ \log u)$ $0\leqq u\leqq 2\pi$, $v>0$ has negative Gaussian curvature and the function $f(u, v):=u$ is strongly convex and has compact levels and no minimum. Note also that the exponential map on this surface is not proper at any point on it. A small perturbation is made to the profile curve $\log u$ of this surface such that the Gaussian curvature on the new surface of revolution changes sign and there is a smooth strongly convex function with compact levels and without minimum. These examples justify our generalizations.

§ 4. Splitting theorems

The Cohn-Vossen theorem states that if a complete noncompact Riemannian 2-manifold of nonnegative Gaussian curvature admits a straight line, then it is isometric to either a plane E^2 or a flat cylinder $S^1\times R$. We shall see how this theorem has been generalized to higher dimensions. If M is isometric to the Riemannian product $N\times R$, then for every x on N $\gamma_x(t):=(x, t)$ is a straight line and γ_x and γ_y are asymptotic to each other.

Theorem 4.1. *Let $\gamma, \sigma: [0, \infty)\to M$ be rays emanating from $p=\gamma(0)$, $q=\sigma(0)$ such that σ is asymptotic to γ. Let $a=F_\gamma(q)$. Then we have:*

(1) *If M is simply connected and if the sectional curvature of M is nonpositive, then $\phi(t):=d(\sigma(t), \gamma(t+a))$, ($t\geqq 0$ if $a>0$, and $t\geqq -a$ if $a\leqq 0$) is a monotone nonincreasing convex function.* (2) *If the sectional curvature of M is nonnegative, then $\phi(t):=d(\sigma(t), \gamma(t+a))$, ($t\geqq 0$ if $a>0$, and $t\geqq -a$ if $a\leqq 0$) is monotone nondecreasing.* (3) *If the ϕ in the above* (1) *and* (2) *is constant on some interval $[c, d]$ with $c<d$, then there exists a flat strip of totally geodesic which is bounded by $\sigma([c, \infty))$, $\gamma([c+a, \infty))$ and a minimizing geodesic joining $\sigma(c)$ and $\gamma(c+a)$.*

Proof of (1). It follows from the second variation formula for a 1-parameter family of geodesic variation along geodesics joining points $\alpha(t)$ and $\beta(t)$ for any fixed two geodesics $\alpha, \beta\colon R\to M$, that the function $t\to d^2(\alpha(t), \beta(t))$ is smooth and has nonnegative second derivative for all $t\in R$. Therefore, for a divergent sequence $\{t_j\}$ and for each j if $\sigma_j\colon [0, l_j]\to M$ is a geodesic joining $q=\sigma_j(0)$ and $\gamma(t_j)=\sigma_j(l_j)$, then each $\phi_j(t):=d(\sigma_j(t), \gamma(t+a))$ is monotone nonincreasing convex on $[0, l_j]$ for $a\geq 0$ and on $[-a, l_j]$ for $a<0$, and $\lim \phi_j(t)=\phi(t)$ for all $t\geqq 0$ if $a\geq 0$, and for all $t\geqq -a$ if $a<0$. This proves (1).

Proof of (2). For every $t>0$ (if $a\geq 0$) or for every $t>-a$ (if $a<0$) note that F_γ is differentiable at $\sigma(t)$ and at $\gamma(t+a)$, and the gradient vectors to F_γ at those points are $\dot\sigma(t)$ and $\dot\gamma(t+a)$. Convexity of F_γ implies that if $\tau_t\colon [0, l]\to M$ is a minimizing geodesic with $\tau_t(l)=\sigma(t)$ and $\tau_t(0)=\gamma(t+a)$, then $\tau_t([0, l])$ is contained in $M^t(F_\gamma)$. Thus $\langle\dot\gamma(t+a), \dot\tau_t(0)\rangle\leqq 0$ and $\langle\dot\sigma(t), -\dot\tau_t(l)\rangle\leqq 0$. Since ϕ is Lipschitz continuous with the Lipschitz constant 2, it is differentiable almost everywhere and the above inequality implies that the first derivative of ϕ is nonnegative wherever it exists. This proves (2).

To prove (3) in the case of (1), note that the convexity property of the function $t\to d^2(\alpha(t), \beta(t))$ implies that through each point x on M there passes a unique ray asymptotic to γ. This fact yields that if two rays are asymptotic to γ, then they are asymptotic to each others.

If ϕ is constant on $[c, d]$, then ϕ is constant on $[c, \infty)$. This is immediate from the convexity of ϕ. Let $l:=\phi(c)$. For each $t\leqq c$ let $\tau_t\colon [0, l]\to M$ be a geodesic with $\tau_t(0)=\gamma(t+a)$, $\tau_t(l)=\sigma(t)$. For each $s\in[0, l]$ let $\sigma_s\colon [c, \infty)\to M$ be a ray emanating from $\tau_c(s)=\sigma_s(c)$ and asymptotic to γ. It is asserted that for every $s\in[0, l]$ and for every $t\geqq c$ $\sigma_s(t)=\tau_t(s)$, and $\langle\dot\sigma_s(t), \dot\tau_t(s)\rangle=0$. In fact it follows from $\sigma(t)\in M^{t+a}_{t+a}(F_\gamma)$ for all $t\geqq c$ that $\langle\dot\sigma(t), \dot\tau_t(l)\rangle=\langle\dot\gamma(t+a), \dot\tau_t(0)\rangle=0$. Since ϕ is constant and since γ and σ are asymptotic to σ_s, (1) implies that $d(\gamma(t+a), \sigma_s(t))+d(\sigma_s(t), \sigma(t))=\phi(t)$ for all $t\geqq c$ and for all $s\in[0, l]$. This proves the assertion. This relation makes it possible to construct an isometric embedding $i: [0, l]\times[c, \infty)\to M$ by $i(s, t):=\tau_t(s)$. For the proof of i being totally geodesic it suffices to show that the second fundamental form vanishes identically. To see this let $X:=i_*(\partial/\partial s)$ and $Y:=i_*(\partial/\partial t)$. Obviously $\{X, Y\}$ forms an orthonormal frame field on the surface, and they are parallel with respect to the Riemannian connection induced through i. If ∇ is the Levi-Civita connection of M, then $\nabla_X X=\nabla_Y Y=0$ follows from the construction of i. Moreover since ϕ is constant, both X and Y are Jacobi fields along σ_s and τ_t respectively and they have the constant length. Thus they are parallel along geodesics, e.g., $\nabla_X Y=\nabla_Y X=0$. This proves the first part of (3).

To show the second part of (3), note that for every $t\in[c, d]$, $\tau_t([0, l])$ is contained entirely in $M_{t+a}^{t+a}(F_\gamma)$ and τ_t hits orthogonally to σ and γ. Let E_0 be the unit parallel field along $\gamma|[c+a, \infty)$ such that $E_0(c+a)=\dot{\tau}_c(0)$. Let $i: [0, l]\times[c, \infty)\to M$ be defined by $i(s, t):=\exp_{\gamma(t+a)} s\cdot E_0(t+a)$. Choose a small $s_0>0$ and a $t_0>c$ such that the map i is smooth on $[0, s_0]\times[c, t_0]$ and such that the length of a curve $t\to i(s, t)$, $c\leqq t\leqq t_0$ is not greater than t_0-c for all $s\in[0, s_0]$. This is guaranteed by the second variation formula. It follows from the total convexity of $M^{t_0+a}(F_\gamma)$ together with grad $F_\gamma(i(0, t_0))=\dot{\gamma}(t_0+a)$ that $i(s, t_0)\in M-\mathrm{Int}\,(M^{t_0+a}(F_\gamma))$ for all $s\in[0, s_0]$. Since $i(s, c)\in M_{a+c}^{a+c}(F_\gamma)$, Proposition 1.1, (1) implies that this curve $t\to i(s, t)$, $c\leqq t\leqq t_0$, is a minimizing geodesic segment whose length realizes the distance between $i(s, 0)=\tau_c(s)$ and $M_{t_0+a}^{t_0+a}(F_\gamma)$ which is equal to t_0-c. From Proposition 1.1, (3), its extension is a ray asymptotic to γ, say, $\sigma_s: [c, \infty)\to M$ with $\sigma_s(0)=\tau_c(s)$, $\sigma_s(t)=i(s, t)$. Moreover it follows from the second variation formula that the sectional curvature spanned by $X(s, t):=i_*(\partial/\partial s)(s, t)$ and $Y(s, t):=i_*(\partial/\partial t)(s, t)$ is 0 for any $(s, t)\in[0, s_0]\times[c, t_0]$. If E_s is the parallel unit field along $\sigma_s|[c, t_0]$ such that $E_s(c)=\dot{\tau}_c(s)$, then the same argument as above applies to find an $s_1\in(s_0, l]$ such that $i|[0, s_1]\times[c, t_0]$ is the natural smooth extension of $i|[0, s_0]\times[c, t_0]$ and keeping the same properties. By iterating this procedure one finally gets the embedding $i: [0, l]\times[c, \infty)\to M$ defined as $i(s, t)=\exp_{\gamma(t+a)} s\cdot E_0(t+a)$ such that: (1) for every $s\in[0, l]$, $t\to i(s, t)$ is a ray asymptotic to γ and emanating from $\tau_c(s)$. (Recall that there is a unique ray asymptotic to γ emanating from $\sigma(c)$, and hence the curve $t\to i(l, t)$ coincides with $\sigma(t)$), (2) for every fixed $t\geqq c$ the curve $s\to i(s, t)$ is a minimizing geodesic of length l such that $i(l, t)=\sigma(t)$ and $i(0, t)=\gamma(t+a)$, (3) the sectional curvature spanned by $X(s, t)$ and $Y(s, t)$ is zero for all $s\in[0, l]$ and for all $t\geqq c$. By the same reason as in the first case, i is totally geodesic.

On a flat totally geodesic strip as is mentioned in the above Theorem 4.1, F_γ takes a special form. A function $f: M\to R$ is said to be *affine* iff along every geodesic $\gamma: R\to M$, $f\circ\gamma: R\to R$ is expressed as $f\circ\gamma(t)=at+b$

for some constants a and b. An affine function is said to be nontrivial iff it is not constant along some geodesic. In the previous Theorem 4.1 F_γ restricted to the flat totally geodesic strip is affine. An affine function gives very strong restriction to the Riemannian structure of M as has been discussed by Innami in [25] as follows.

Theorem 4.2 (Innami [25]). *If $f\colon M\to R$ is a non-trivial affine function, then M is isometric to the Riemannian product $N\times R$, where N is a totally geodesic hypersurface in M which is a level set of f.*

Proof. The non-triviality of f implies that $\inf_M f=-\infty$, and hence M is homeomorphic to the product $M^a_a(f)\times R$. If σ is a geodesic segment joining two distinct points on a level set, then $\sigma(R)$ is contained entirely in the level, and hence every level is a complete totally geodesic hypersurface without boundary which is totally convex.

Now the basic idea of the proof of Theorem 4.2 is this. Let x be an arbitrary fixed point and let $c:=f(x)$, and let $B(x, r)$ be a strongly convex ball. Take c_1 close to c such that $d(x, M^{c_1}_{c_1}(f))=d(x, y)<r/2$ for some point y on $M^{c_1}_{c_1}(f)$. If $\sigma\colon [0, l]\to M$ is a unique minimizing geodesic with $\|\dot\sigma\|=1$, $\sigma(0)=x$, $\sigma(l)=y$, then for every $t\in[0, l]$ $d(x, M^{f:\sigma(t)}_{f\cdot\sigma(t)}(f))=t$. And in particular f is differentiable at x and $\dot\sigma(t)$ is orthogonal to the tangent hyperplane to $M^{f:\sigma(t)}_{f\cdot\sigma(t)}(f)$ at $\sigma(t)$ for all $t\in[0, l]$.

To prove this suppose that there is a point y_t on $M^{f:\sigma(t)}_{f\cdot\sigma(t)}(f)$ such that $d(x, y_t)=d(x, M^{f:\sigma(t)}_{f\cdot\sigma(t)}(f))<t$. Then the affine property of f implies that the minimizing geodesic joining x to y_t can be extended to a geodesic segment beyond y_t with length less than l and with endpoints x and a point on $M^{c_1}_{c_1}(f)$. This contradicts the distance minimizing property of σ between x and $M^{c_1}_{c_1}(f)$. The same argument applies to any point x' on the intersection of a sufficiently small neighborhood of x and $M^c_c(f)$ to verify that if y' is a point on $M^{c_1}_{c_1}(f)$ such that $d(x', M^{c_1}_{c_1}(f))=d(x', y')=l'$, then the minimizing geodesic $\sigma'\colon [0, l']\to M$ joining x' to y' hits orthogonally all levels between c and c_1. It follows from the Gauss lemma that $l=l'$, and moreover $t\to d(\sigma(t), \sigma'(t))$ is constant on $[0, l]$. It follows from Theorem A in [11] that all level sets of an affine function is connected. From connectedness of levels it follows that the local constancy of the distance $x\to d(x, M^{c_1}_{c_1}(f))$, $x\in M^c_c(f)$ extends globally to the whole level. The correspondence between $M^c_c(f)$ and $M^t_t(f)$, $c\leqq t\leqq c_1$(or $c_1\leqq t\leqq c$) via the distance minimizing geodesics is thus isometric for all t between c and c_1. Therefore $M^c_{c_1}(f)$ (or $M^{c_1}_c(f)$ if $c_1\geqq c$) is isometric to the Riemannian product $M^c_c(f)\times[0, l]$.

If $d(\sigma(t), M^c_c(f))<t$ for some $t>l$, then there is a unit speed minimizing geodesic $\tau\colon [0, t_1]\to M$ with $t_1<t$, $\sigma(t)=\tau(0)$, $\tau(t_1)\in M^c_c(f)$ and

$d(\sigma(t), M_c^c(f))=t_1$. But from the arguments developed above, $\tau(t_1-l) \in M_{c1}^{c1}(f)$ and hence $f\circ\sigma$ and $-f\circ\tau$ have the same slope, a contradiction to $t_1<t$. This fact means that for every $t>l$ and for every $t<0$, $\sigma|[0, t]$ (or $\sigma|[t, 0]$ if $t<0$) realizes the distance between $\sigma(t)$ and $M_c^c(f)$. Thus the normal exponential map Exp defined over the normal bundle of a fixed level $M_c^c(f)$ into M is an isometric embedding of $M_c^c(f)\times R$.

The Toponogov splitting theorem is a direct consequence of Theorem 4.2 which is stated as follows.

Theorem 4.3 (Toponogov [44]). *Let M be a complete noncompact Riemannian manifold of nonnegative sectional curvature. If M admits a straight line $\gamma: R\to M$* (e.g., $d(\gamma(t_1), \gamma(t_2))=|t_1-t_2|$ *for all* $t_1, t_2 \in R$), *then M is isometric to the Riemannian product $N\times R$.*

The proof technique used here is furnished by Innami, see Theorem 3 in [25].

Proof. Let F_+ and F_- be the Busemann functions for $\gamma|[0, \infty)$ and $\gamma|[0, -\infty)$ respectively. Since they are convex, the sum is convex too. For any point x on M, it follows that $F_+(x)+F_-(x)=\lim_{t\to\infty}[2t-d(x, \gamma(t))-d(x, \gamma(-t))]\leq\lim_{t\to\infty}[2t-d(\gamma(t), \gamma(-t))]=0$. Because a bounded convex function on M is constant and since $(F_+ + F_-)(\gamma(t))=0$ for every $t \in R$, the sum is identically zero. Namely, $F_+=-F_-$. This fact means that both F_+ and F_- are affine. Thus Theorem 4.2 implies the conclusion.

It should be noted that if a complete M admits a straight line γ and if the Ricci curvature of M is nonnegative, then M is isometric to the Riemannian product $N\times R$. This has been proved by Cheeger-Gromoll in [6], where they showed that the F_+ and F_- stated in the proof of Theorem 4.3 are differentiable and harmonic. And it turned out that they are affine.

Relevant results have been obtained by replacing straight lines by compact hypersurfaces with certain properties. Namely, if a complete noncompact M of nonnegative Ricci curvature admits a compact minimal hypersurface N which has no boundary and has no focal point, then M is isometric to the line bundle over N in M (see [24]). Making use of convexity argument and the second variation formulas, Burago and Zalgallaer have obtained in [4] the following. If M is a complete, connected and locally convex set in a Riemannian manifold V of nonnegative sectional curvature and if the boundary ∂M of M is nonempty, then (1) the number of components of the boundary ∂M is not greater than two, (2) if the boundary ∂M has two components, then they are isometric to each other and M is isometric to the Riemannian product $(\partial M)_1\times[0, a]$, where $(\partial M)_1$ is a component of ∂M, (3) if ∂M is connected and compact and if M is noncompact, then M is isometric to $\partial M \times [0, a]$. Making use of the minimum principle for solutions of some non-linear differential equations, Ichida has proved an extension of the above result in [22]. It has been proved in [22] that if $\overline{M}=M \cup \partial M$ is a connected manifold with a compact smooth boundary ∂M and if the Ricci curvature of $\overline{M}$ is nonnegative and if the mean curvature in the inner normal direction is nonnegative, then ∂M has at most two components. Moreover if ∂M has two components, then $\overline{M}$ is isometric to the Riemannian product $(\partial M)_1 \times [0, a]$. Further development of this type of splitting theorems has been obtained in [23].

§ 5. An elementary proof for a gap theorem

The following theorem has recently been proved by Greene and Wu in [18].

Theorem 5.1 (Greene-Wu, [18]). *Let M be a complete noncompact Riemannian manifold of nonnegative sectional curvature. If the sectional curvature is zero outside a compact set K and if M is simply connected at infinity, then M is isometric to E^m.*

M is by definition *simply connected at infinity* iff for every compact set C there is a compact set C_1 containing C such that $M-C_1$ is connected and simply connected.

Two basic facts are used in the proof of the above theorem and their proof in [18] is sketched as follows. The Sackstader theorem [35] states that if a complete Riemannian manifold N of nonnegative sectional curvature is isometrically immersed in E^{n+1} with codimension one and if the sectional curvature is positive at least one point, then it is embeded as the boundary of a smooth convex body in E^{n+1}. Second fact is the Bishop theorem for the volume of metric balls on M which states that if M has nonnegative sectional curvature, then for a fixed point x on M and for $r>0$, the function $f(r):=$[volume of $B(x, r)$]/[volume of r-ball on E^m] is monotone nonincreasing with $r \geqq 0$. In particular if $\lim_{r\to\infty} f(r)=1$, then M is isometric to E^m. In order to establish $\lim_{r\to\infty} f(r)=1$ under the assumption of Theorem 5.1, the Riemannian convolution smoothing procedure applies to a convex exhaustion function $F: M \to R$ which is constructed in the previous section by Busemann functions to provide a smooth function $\tilde{F}: M \to R$. Flatness outside a compact set implies that $\tilde{F}$ is convex outside a compact set K_1, and every sufficiently high level of $\tilde{F}$ is locally isometrically immersed into E^m with codimension one such that the second fundamental form of it is positive semidefinite everywhere. Since M is simply connected at infinity, the local isometric immersion is globally constructed and the level of $\tilde{F}$ is thus isometrically immersed into

E^m. Since the image is compact, the second fundamental form is strictly positive definite at least one point. Then the Sackstader theorem implies that the immersion is embedding, and in particular $M-K_1$ is isometric to E^m-[a convex body]. Thus $\lim_{r\to\infty} f(r)=1$ is proved.

We shall provide an elementary proof of Theorem 5.1 which is completely different from that of Greene-Wu. The splitting theorem is used for our proof in such a way that for a fixed Busemann function F_γ a sufficiently high level set is a complete totally geodesic hypersurface without boundary which is totally convex and simply connected. It will then turn out that the level set will contain $m-1$ independent lines, and hence M is isometric to $E^{m-1}\times N$ by the splitting theorem, where N is a 1-dimensional totally geodesic submanifold without boundary. Since N contains $\gamma([0, \infty)]$, N must be a straight line, and the proof is achieved.

As stated above it suffices for the proof of Theorem 5.1 to show that for a fixed Busemann function F_γ with $p=\gamma(0)$ and for a sufficiently large b, $M_b^b(F_\gamma)$ is totally convex and isometric to E^{m-1}. Let $F: M\to R$ be defined by $F:=\sup_\sigma F_\sigma$, where the sup is taken over all rays emanating from p. Choose an $a\in F(M)$ so as to satisfy $K\subset M^a(F)$. As is stated in the previous section, there is a homeomorphism $H: M_a^a(F)\times[a, \infty)\to M-\mathrm{Int}\,(M^a(F))$ such that $F(H(y, \alpha))=\alpha$ for all $y\in M_a^a(F)$ and for all $\alpha\geqq a$. It follows from simple connectedness at infinity of M that $M_a^a(F)$ is simply connected. Now let b be chosen such that $b>a$. Then $K\subset M^a(F)\subset M^b(F_\gamma)$. The following assertion is the key step for the proof.

Assertion. If M has nonnegative sectional curvature and if M is simply connected at infinity, then $M_b^b(F_\gamma)$ is simply connected.

Proof of Assertion. If levels of F_γ are all compact, then the assumption implies that $M_b^b(F_\gamma)$ is simply connected. (Under the assumption of Theorem 5.1 it is proved later that this does not occur and levels of F_γ are all noncompact.) Assume that levels of F_γ are noncompact. For an arbitrary taken closed curve $c: [0, 1]\to M_b^b(F_\gamma)$ such that $c(0)=c(1)=\gamma(b)$ there is an open ball B around $\gamma(b)$ which contains $c([0, 1])$. Through every point x on $B\cap M_b^b(F_\gamma)$ there passes a curve $t\to H(y, t)$, $t\geqq a$ with $y\in M_a^a(F)$. By means of these curves one constructs a map $\tilde{H}: B\cap M_b^b(F_\gamma)\to M_a^a(F)$ by $H(x):=y$, $x\in B\cap M_b^b(F_\gamma)$. Since $M_a^a(F)$ is simply connected there is a homotopy $h: [0, 1]\times[0, 1]\to M_a^a(F)$ such that $h(0, u)=\tilde{H}(c(u))$ and $h(1, u)=\gamma(a)$ for all $u\in[0, 1]$. The homotopy image $h([0, 1]\times[0, 1])$ is continuously deformed into the image of $\tilde{H}$ as follows: If $\dim M=3$, then $M_a^a(F)$ is homeomorphic to S^2, and hence h can be chosen so that the image of h is contained entirely in the image of $\tilde{H}$. If $\dim M\geqq 4$, then $\dim M_a^a(F)\geqq 3$. The homotopy image lying outside $\tilde{H}(B\cap M_b^b(F_\gamma))$

is covered by a finitely many convex balls $B_1, \cdots, B_k$. In each B_i the homotopy image can be swept out along a family of minimizing geodesics emanating from a fixed point outside image of the homotopy to the boundary of B_i on $M_a^a(F)$. Thus a continuous deformation of the homotopy image along $M_a^a(F)$ is obtained in such a way that the resulting homotopy on $M_a^a(F)$ between $\tilde{H} \circ c$ and a point curve $\gamma(a)$ has its image in $\tilde{H}(B \cap M_b^b(F_\gamma))$. Now this homotopy is pulled back by $\tilde{H}$ to the homotopy between c and a point curve $\gamma(b)$, in $M_b^b(F_\gamma)$. Thus the assertion is proved.

Proof of Theorem 5.1. It follows from the Assertion together with the Greene-Shiohama theorem that $M-\mathrm{Int}\,(M^b(F_\gamma))$ is simply connected, and in particular for every $t>b$ $B(\gamma(t), t-b)$ is convex. Therefore $M-\mathrm{Int}\,(M^b(F_\gamma))=\lim \bar{B}(\gamma(t), t-b)$ is convex. Since F_γ is convex, $M^b(F_\gamma)$ and $M_b^b(F_\gamma)$ are totally convex. In particular $M_b^b(F_\gamma)$ is totally geodesic hypersurface without boundary. Simple connectedness of $M_b^b(F_\gamma)$ implies that it is isometric to E^{m-1}. The proof is completed by the splitting theorem.

§ 6. Busemann functions and total curvature

As is seen in the Cohn-Vossen theorem, the total curvature of a complete noncompact Riemannian 2-manifold is not a topological invariant but it depends on the choice of the Riemannian structure. Therefore it will be natural to imagine that the total curvature should describe a certain property of the Riemannian structure which defines it. The first attempt for this line of investigation has been done by Maeda in [29], [30], where he showed that there is a relation between the total curvature and the mass of rays emanating from a fixed point on a complete Riemannian 2-manifold diffeomorphic to R^2 of nonnegative Gaussian curvature. Namely if M is such a manifold with $c(M)<2\pi$, then the measure of all unit vectors in M_p tangent to rays emanating from a fixed point p on M is bounded below by $2\pi-c(M)$. It has recently been proved by Shiga in [41] that if M is a complete finitely connected noncompact 2-manifold of nonpositive Gaussian curvature and if $c(M)>2\pi(\chi(M)-1)$, then for every point p on M the set of all unit vectors tangent to rays emanating from p has measure bounded above by $2\pi\chi(M)-c(M)$.

But here we would like to mention about the relation between the total curvature and the behavior of Busemann functions. It has been proved in [42] that for a complete noncompact 2-manifold of nonnegative Gaussian curvature, the total curvature is greater than π if and only if all Busemann functions on it are exhaustion, and the total curvature is not greater than π if and only if all Busemann functions are nonexhaustion. In this case Busemann functions are all convex and the level sets are all simultaneously compact or noncompact.

From this result one might expect a geometric significance of the total curvature in general case where manifolds are complete noncompact finitely connected whose Gaussian curvature changes sign. Namely it might be anticipated that the existence of $c(M)$ would not allow the existence of both exhaustion and nonexhaustion Busemann functions. But the behavior of Busemann functions also depends on the end structure of M. Namely it is easily seen that if M has more than one end, then every Busemann function on M has its infimum $=-\infty$, and hence Theorem 1.1, (1) implies that it is nonexhaustion. Thus the problem under consideration makes sense only when M has exactly one end. It has been proved in [43] that for a complete noncompact finitely connected Riemannian 2-manifold with one end, (1) if the total curvature is less than $(2\chi(M)-1)\pi$ then all Busemann functions are nonexhaustion, (2) if the total curvature is greater than $(2\chi(M)-1)\pi$, then all Busemann functions are exhaustion. In the case where the total curvature is equal to $(2\chi(M)-1)\pi$, there are three examples of surfaces in E^3 with total curvature π which are homeomorphic to R^2 and on which all Busemann functions are simultaneously exhaustion, non-exhaustion and on which there are both exhaustion and nonexhaustion Busemann functions. For details see [43].

An interesting problem is if the above results can be extended to 4-dimensional complete noncompact Riemannian manifolds of nonnegative sectional curvature on which the total curvature has been computed in [46].

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