Advanced Studies in Pure Mathematics 3, 1984
Geometry of Geodesics and Related Topics
pp. 333–386

Applications of Laplacian and Hessian Comparison Theorems

Atsushi Kasue*)

§ 0. Introduction

Rauch [41] proved a fundamental theorem on the lengths of Jacobi fields, called the Rauch comparison theorem. After him, Berger [4], Warner [48] and Heintze and Karcher [27] etc. had extended the Rauch comparison theorem. Especially, Heintze and Karcher showed a very general comparison theorem for the length and volume distortion of the normal exponential map of a submanifold. The proof of their comparison theorem in turn tells us some useful informations about the "*local*" behaviour of the Laplacian and Hessian of the distance function to a submanifold (cf. Greene and Wu [25] in the case when a submanifold is a point). On the other hand, Wu [49] has proved that, in certain situations, the Laplacian and the Hessian of a distance function in an appropriate weak sense can be "*globally*" estimated from above (cf. also Calabi [10], Cheeger and Gromoll [13, 14], Yau [51]). Moreover, making use of the method by Wu, we have shown in [32] general comparison theorems on the Laplacian and the Hessian of a distance function. The purpose of the present paper is to give several applications of our comparison theorems.

0.1. We shall first describe our Laplacian and Hessian comparison theorems. Let M be a Riemannian manifold with (possibly empty) boundary ∂M. We write M_0 for the interior of M ($M=M_0$ if $\partial M=\phi$). Let X be a smooth vector field on M. We consider the second order elliptic operator $L_X=\Delta+X$ acting on functions, where Δ denotes the Laplace operator $\left(\text{i.e., locally } \Delta=\sum \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}\left(\sqrt{G}g^{ij}\frac{\partial}{\partial x^j}\right)\right)$. For a semi-continuous function φ on a neighborhood of a point x in M, an extended real number $S_X\varphi(x)$ is defined by

$$S_X\varphi(x):=\liminf_{r\to 0}\left\{\int_{\partial B_r(x)} -\frac{\partial G_r(x,\xi)}{\partial\nu(\xi)}\varphi(\xi)d\xi-\varphi(x)\right\}\Big/\int_{B_r(x)} G_r(x,y)dy,$$

Received December 26, 1982.
*) Research supported partly by Grant-in-Aid for Scientific Research.

where $B_r(x)$ is the metric ball with radius r around x(r is sufficiently small), $G_r(y, z)$ denotes the Green function of L_X with respect to the Dirichlet problem on $B_r(x)$ (cf. e.g., [31]) and $\nu(\xi)$ is the outer unit normal vector field on the metric sphere $\partial B_r(x)$. In the case when $X\equiv 0$, we write $S\varphi(x)$ for $S_0\varphi(x)$. We remark that if φ is smooth near x, $S_X\varphi(x)=L_X\varphi(x)$. Let N be a closed subset of M. We write $\rho_N(x)$ for the distance between a point $x\in M$ and N. For a point $x\in M_0\backslash N$, a geodesic $\sigma\colon[0, a]\to M$ is called a *distance minimizing geodesic from N to x* if $\rho_N(\sigma(t))=t$ for $t\in[0, a]$, $\sigma(a)=x$ and $\sigma(t)\subset M_0$ for $t\in(0, a]$.

Theorem 0.1 (cf. [32: Theorem (2.28) and Lemma (2.5)]). *Let N be a closed subset of a Riemannian manifold M of dimension m and x a point of $M_0\backslash N$. Suppose there exists a distance minimizing geodesic $\sigma\colon[0, a]\to M$ from N to x. Let $\mathscr{R}(t)$ (resp. $\eta(t)$) be a continuous function on $[0, a]$ such that the Ricci curvature in direction $\dot\sigma(t)$ is bounded from below by $(m-1)\mathscr{R}(t)$ (resp. $\langle X, \dot\sigma(t)\rangle\leqq\eta(t)$). Then for any nonincreasing C^2-function ψ on $[0, a]$, we have*

$$S_X(\psi\circ\rho_N)(x)\geqq\{\psi''+\psi'((m-1)\log f_{\mathscr{R}})'+\psi'\eta\}(a), \tag{0.1}$$

where $f_{\mathscr{R}}$ is the solution of the classical Jacobi equation:

$$f_{\mathscr{R}}''(t)+\mathscr{R}(t)f_{\mathscr{R}}(t)=0 \qquad \text{with } f_{\mathscr{R}}(0)=0 \text{ and } f_{\mathscr{R}}'(0)=1. \tag{0.2}$$

Moreover when N is a point and σ can be extended to a distance minimizing geodesic $\tilde\sigma\colon[0, \tilde a]\to M$ $(a<\tilde a)$ from N through x, the equality holds in (0.1) *if and only if the sectional curvature of any plane tangent to $\dot\sigma(t)$ is equal to $\mathscr{R}(t)$ and $\langle X, \dot\sigma(t)\rangle=\eta(t)$ $(t\in[0, a])$.*

When N is a smooth hypersurface of M, we have a better estimate than (0.1) (cf. [32: Lemma (2.27)]). That is:

Theorem 0.2 (cf. [ibid. : Theorem (2.28) and Lemma (2.8)]). *Let N be a closed hypersurface of a Riemannian manifold M of dimension m and x a point of $M_0\backslash N$. Suppose there exists a distance minimizing geodesic $\sigma\colon[0, a]\to M$ from N to x. Let $\mathscr{R}$ and η be as in Theorem* 0.1. ; *let Λ be a real number such that the trace of $S_{\dot\sigma(0)}$ is bounded from above by $(m-1)\Lambda$, where $S_{\dot\sigma(0)}$ denotes the second fundamental form of N with respect to $\dot\sigma(0)$* (i.e., $\langle S_{\dot\sigma(0)}V, U\rangle=\langle\nabla_V\dot\sigma(0), U\rangle$). *Then for any nonincreasing C^2-function ψ on $[0, a]$, we have*

$$S_X(\psi\circ\rho_N)(x)\geqq\{\psi''+\psi'((m-1)\log h_{\mathscr{R},\Lambda})'+\psi'\eta\}(a), \tag{0.3}$$

where $h_{\mathscr{R},\Lambda}(t)$ is the solution of the classical Jacobi equation:

(0.4) $\quad h''_{\mathscr{R},\Lambda}(t)+R(t)h_{\mathscr{R},\Lambda}(t)=0 \qquad$ *with* $h_{\mathscr{R},\Lambda}(0)=1$ *and* $h'_{\mathscr{R},\Lambda}(0)=\Lambda.$

Moreover when σ *can be extended to a distance minimizing geodesic* $\tilde{\sigma}$: $[0, \tilde{a}] \to M$ *from* N *through* x, *the equality holds in* (0.3) *if and only if the sectional curvature of any plane tangent to* $\dot{\sigma}(t)$ *is equal to* $\mathscr{R}(t)$, N *is umbilic at* $\sigma(0)$ (i.e., $\langle S_{\dot{\sigma}(0)}V, U\rangle=\Lambda\langle V, U\rangle$) *and* $\langle X, \dot{\sigma}(t)\rangle=\eta(t)$ ($t \in [0, a]$).

In [32], we have actually proved the above theorems for the Laplacian Δ, using the second variational formula of arc lengths and the method developed by Wu [49]. However it is easily seen that the same arguments as in [32] are applicable to the proofs of Theorems 0.1 and 0.2. In fact, we shall apply Theorems 0.1 and 0.2 of the above forms to the study of some function theoretic properties concerning the operator L_X on a complete noncompact Riemannian manifold (cf. Section 5).

We remark here that the Hessian $\nabla^2(\psi \circ \rho_N)$ in an appropriate weak sense can be estimated from below in terms of the sectional curvature along σ and the second fundamental form of N if N is a submanifold (cf. [32: Theorem (3.31)]). But this fact will not be used in this paper.

As for a lower estimate of the Laplacian of a distance function, we have the following

Theorem 0.3 (cf. [ibid.: Lemma (2.11) and Theorem (2.49)]). *Let* N *be a closed submanifold of a Riemannian manifold* M *of dimension* m *and* x *a point of* $M_0\backslash N$. *Suppose there exists a distance minimizing geodesic* σ: $[0, a]\to M$ *from* N *through* $x=\sigma(a')(a'<a)$, (*so that* ρ_N *is smooth near* x). *Let* $\mathscr{K}$ *be a continuous function on* $[0, a']$ *such that the sectional curvature of any tangent plane containing* $\dot{\sigma}(t)$ *is bounded from above by* $\mathscr{K}(t)$ ($t \in [0, a']$); *in the case when* $\dim N>0$, *let* Γ *be a real number such that all the eigenvalues of the second fundamental form* $S_{\dot{\sigma}(0)}$ *of* N *is bounded from below by* Γ. *Let* $h_{\mathscr{K},\Gamma}$ (*resp.* $f_{\mathscr{K}}$) *be the solution of equation* (0.4) *defined by* $\mathscr{K}$ *and* Γ (*resp. the solution of equation* (0.2) *defined by* $\mathscr{K}$). *Suppose* $h_{\mathscr{K},\Gamma}$ *is positive on* $[0, a']$. *Then the Hessian* $\nabla^2\rho_N$ *of* ρ_N *has an estimate*:

$$(\nabla^2\rho_N)_x(V, V)\geqq(\log h_{\mathscr{K},\Gamma})'(a')\{\|V\|^2-\langle\dot{\sigma}(a'), V\rangle^2\}$$

for any $V \in M_x$, *and in addition, if* $V \in d(\exp_N)_{\dot{\sigma}(0)}(\Pi)$,

$$(\nabla^2\rho_N)_x(V, V)\geqq(\log f_{\mathscr{K}})'(a')\{\|V\|^2-\langle\dot{\sigma}(a'), V\rangle^2\},$$

where Π *denotes the vertical subspace in the tangent space at* $\dot{\sigma}(0)$ *of the normal bundle* $\nu(N)$ *for* N. (*We take* $d(\exp_N)_{\dot{\sigma}(0)}(\Pi)=M_x$ *if* $\dim N=0$.) *In particular*,

$$\Delta\rho_N(x)\geqq\{n(\log h_{\mathscr{K},\Gamma})'+(m-n-1)(\log f_{\mathscr{K}})'\}(a').$$

0.2. Now we shall assume M is a complete noncompact Riemannian manifold of dimension m with (possibly empty) boundary ∂M. For a geodesic ray $\gamma: [0, \infty) \to M$ (we assume $\gamma(t) \in M_0$ for $t>0$ if ∂M is not empty), we define a function $B_\gamma: M \to R$ by

$$B_\gamma(x): = \lim_{t\to\infty} \{\mathrm{dis}\,(\gamma(t), x) - t\}.$$

This function B_γ is called the *Busemann function* associated with a ray γ. (cf. e.g., [19: p. 56]). Let $B_\gamma^t := t - \mathrm{dis}\,(\gamma(t), *)$. Then $|B_\gamma^t(x)| = |\mathrm{dis}\,(\gamma(t), x) - \mathrm{dis}\,(\gamma(0), \gamma(t))| \leqq \mathrm{dis}\,(\gamma(0), x)$, by the triangle inequality, so that the family $\{B_\gamma^t\}$ is uniformly bounded on compact subsets of M. Moreover if $s<t$, then

$$\begin{aligned} B_\gamma^s(x) - B_\gamma^t(x) &= \mathrm{dis}\,(\gamma(s), x) - \mathrm{dis}\,(\gamma(t), x) + t - s \\ &= \mathrm{dis}\,(\gamma(s), x) - \mathrm{dis}\,(\gamma(t), x) + \mathrm{dis}\,(\gamma(t), \gamma(s)) \\ &\geqq 0, \end{aligned}$$

again by the triangle inequality. Thus the family $\{B_\gamma^t\}$ is also nonincreasing and hence the convergence of $\lim_{t\to\infty} B_\gamma^t$ to B_γ is uniform on compact sets. In particular, the Busemann function B_γ is continuous. We note that when t is fixed, the level sets $\{B_\gamma^t \equiv \text{constant}\}$ are precisely the metric spheres about the point $\gamma(t)$. Since $B_\gamma^t \downarrow B_\gamma$ as $t \uparrow +\infty$, we may thus think intuitively of the level sets of B_γ itself as the "metric sphere about the point $\gamma(\infty)$" and we may think of B_γ as "the distance from $\gamma(\infty)$". This is the intuitive meaning of the Busemann function. Note also that $B_\gamma(\gamma(t)) = -t$ for all $t \geqq 0$.

Now we shall state a result on SB_γ:

Theorem 0.4 (cf. [33: Lemmas (1.12) and (1.13)]). *Let M be a complete noncompact Riemannian manifold of dimension m. Suppose the Ricci curvature of M is bounded from below by some nonpositive constant $(m-1)R$. Then the Busemann function B_γ associated with any geodesic ray γ if ∂M is empty (resp. a distance minimizing geodesic ray γ from ∂M if ∂M is not empty) satisfies*

$$S(-B_\gamma) \geqq -(m-1)\sqrt{-R}$$

on M (resp. $\mathscr{H}_\gamma: = \{x \in M: B_\gamma(x) < 0\}$).

Theorem 0.4 implies that if M has nonnegative Ricci curvature and ∂M is empty, the Busemann function B_γ associated with any ray γ is superharmonic on M. (cf. the paragraph 0.5 below). This was proved by Cheeger and Gromoll [13] (cf. also [49]). (When we consider the operator L_x in Theorem 0.4, we see also that $S_X(-B_\gamma) \geqq (m-1)\sqrt{-R} - A$

on M or $\mathscr{H}_\gamma$ if the length $\|X\|$ of $X \leqq A$. But this estimate will not be used in this paper.)

We shall now describe the contents of each section. In the following, let M be a connected complete Riemannian manifold of dimension m with (possibly empty) boundary ∂M.

0.3. As we mentioned after Theorem 0.4, Cheeger and Gromoll proved in [13] that if ∂M is empty and the Ricci curvature of M is nonnegative, then the Busemann function associated with any ray is superharmonic on M. From this result, they showed that M as above is isometric to the direct product $N \times \boldsymbol{R}^k$ $(k \geqq 0)$, where N contains no lines and $\boldsymbol{R}^k$ has its standard flat metric. They also showed in [14] that if M is a convex subset with boundary ∂M in a Riemannian manifold of nonnegative sectional curvature, the distance function to ∂M is concave on M. Later, making use of this result, Burago and Zalgaller obtained in [9] a theorem on such a manifold M saying that (1) the number of components of ∂M is not greater than 2, (2) if there are exactly two components Γ_1 and Γ_2 of ∂M, M is isometric to the direct product $[0, a] \times \Gamma_1$, and (3) if ∂M is connected and compact, but M is noncompact, M is isometric to the direct product $[0, \infty) \times \partial M$.

In Section 1, using Theorem 0.1, Theorem 0.2 and Theorem 0.4, we shall prove, roughly speaking, a generalization of the above result by Burago and Zalgaller from the view point of Ricci curvature. More precisely, we shall show that, in the case when M has nonnegative Ricci curvature and ∂M is a smooth hypersurface whose mean curvature with respect to the inner normal is nonpositive, (1) if ∂M is disconnected and it has a compact component Γ, M is isometric to the direct product $[0, a] \times \Gamma$, and (2) if ∂M is connected and compact, but M is noncompact, M is isometric to the direct product $[0, \infty) \times \partial M$ (cf. Theorem 1.2 (1) and Theorem 1.4 (2)). Moreover we shall prove an analogue of Cheng's maximum diameter theorem (cf. [16] or Theorem 4.2 in Section 4).

The results of this section have been proved in the author's previous paper [33].

0.4. We assume M is a connected *compact* Riemannian manifold of dimension m with smooth boundary ∂M and consider the following eigenvalue problem:

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0 & \text{on } M, \\ \varphi = 0 & \text{on } \partial M. \end{cases}$$

We write $\lambda_1(M)$ for the first eigenvalue of the above equation. Let R and Λ be two real numbers such that the Ricci curvature of M is bounded from below by $(m-1)R$ and the trace of S_ν is bounded from above by $(m-1)\Lambda$, where S_ν denotes the second fundamental form of ∂M with respect to the unit inner normal vector field ν on ∂M. We call such a manifold M *a Riemannian manifold of class* (R, Λ) for the sake of brevity (cf. Section 1). Recently, Li and Yau [36] have given, among other things, computable lower bounds for $\lambda_1(M)$ in terms of R, Λ and the inradius $\mathscr{I}_M$ of M (i.e., $\mathscr{I}_M = \sup\{\mathrm{dis}(x, \partial M) : x \in M\}$). In particular, their estimate (cf. [ibid. : Theorem 11]) is optimum in the case when $R=0$ and $\Lambda=0$. More precisely, they have proved that if M is of class $(0, 0)$, $\lambda_1(M)$ is greater than or equal to $\pi^2/4\mathscr{I}_M^2$; the equality is attained for a flat cylinder. Their method is based on a gradient estimate of the first eigenfunction. Moreover, Gallot [23] has also showed another computable lower bound for $\lambda_1(M)$, estimating the Cheeger's isoperimetric constant in terms of R, Λ and $\mathscr{I}_M$. On the other hand, before the works mentioned above, Reilly [42] showed that if $R>0$ and $\Lambda=0$, then $\lambda_1(M)$ is not less than mR and the equality holds if and only if M is isometric to the closed hemisphere of the standard sphere of constant curvature R.

In Section 2, we shall show that for a Riemannian manifold M of class (R, Λ), $\lambda_1(M)$ has a lower bound depending on R, Λ and $\mathscr{I}_M$, and the equality holds if and only if M is isometric to a model space of class (R, Λ) (cf. Theorem 2.1). We remark that our estimate coincides with the above one due to Li and Yau when $R=\Lambda=0$, and our result contains the above theorem by Reilly as the special case: $R>0$ and $\Lambda=0$.

The results of this section have been proved in the author's previous paper [34].

0.5. We assume M is a connected *compact* Riemannian manifold of dimension m with smooth boundary ∂M and consider a Poisson equation:

$$\begin{cases} \Delta u + Q = 0 & \text{on } M, \\ u = 0 & \text{on } \partial M, \end{cases}$$

where Q is a smooth function on M. We write U_Q for the solution of the above equation. When M is a (simply connected) domain of Euclidean plane $\boldsymbol{R}^2$ and $Q\equiv 2$, U_2 and its integral $2\int_M U_2$ are, respectively, called the warping function of M and the torsional rigidity of M, and it is a classical problem to obtain geometric bounds for U_2, ∇U_2 and $2\int_M U_2$ etc. (cf. e.g., [2], [39]).

In Section 3, we shall show that if M is of class (R, Λ), U_Q is bounded from above by a continuous function of the form: $F(m, R, \Lambda, \mathscr{I}_M, Q^*)\circ\rho$, where $\rho = \mathrm{dis}(*, \partial M)$, $Q^*(t) = \max\{Q(x) : x \in M, \rho(x) = t\}$ $(0 \leqq t \leqq \mathscr{I}_M)$

and $F(m, R, \Lambda, \mathscr{I}_M, Q^*)(t)$ is a continuous function on $[0, \mathscr{I}_M]$ determined by m, R, Λ, $\mathscr{I}_M$ and Q^*, and moreover the equality holds for some $x \in M_0$ if and only if M is a model of class (R, Λ) (cf. Theorem 3.1). Moreover in the case when $Q \equiv 1$, we shall prove that if the Ricci curvature of M is greater than or equal to $(m-1)R$, U_1 is bounded from below by a continuous function of the form: $G(R) \circ \rho$, where $G(R)(t)$ is a continuous function determined by R (cf. Theorem 3.2).

0.6. In Section 4, we shall show a volume estimate for a domain in a certain Riemannian manifold (cf. Proposition 4.1) and as its application, we shall prove the following

Theorem (Theorem 4.1). *Let m be a positive integer and let $K \in (1, \infty)$ and $\mathscr{V}_0 \in (0, \omega_m)$ be given constants, where ω_m denotes the volume of the unit sphere $S^m(1)$ in Euclidean space of dimension $m+1$. Then there exists for any number $V \in (\mathscr{V}_0, \omega_m)$ a constant $d(m, K, \mathscr{V}_0; V) \in (0, \pi)$, depending on m, K, $\mathscr{V}_0$ and V, such that for a complete m-dimensional Riemannian manifold M whose boundary is empty and which satisfies*

$$(0.5)\qquad \begin{cases} \text{the Ricci curvature } \geqq (m-1) \\ |\text{the sectional curvature}| \leqq K \\ \text{the volume } \mathrm{Vol}_m(M) \geqq \mathscr{V}_0, \end{cases}$$

if the diameter $d(M) \geqq d(m, K, \mathscr{V}_0; V)$, then $\mathrm{Vol}_m(M) \geqq V$.

We remark here that if the diameter of a complete, m-dimensional Riemannian manifold M whose Ricci curvature $\geqq (m-1)$ is equal to π, M is isometric to $S^m(1)$. This is a theorem due to Cheng [16], who used his comparison theorem on the first eigenvalue of a metric ball to prove this theorem. In the course of the proof for our theorem as above, we shall give another proof of the Cheng's maximum diameter theorem (cf. Theorem 4.2). Moreover combining our theorem and a sphere theorem due to Shiohama [45], we see that if the diameter of a complete m-dimensional Riemannian manifold M which satisfies (0.5) is sufficiently close to π, M is homeomorphic to $S^m(1)$ (cf. Corollary 4.2).

0.7. We assume M is a complete noncompact Riemannian manifold of dimension m without boundary. Let X be a smooth vector field on M and $Q(\not\equiv 0)$ a nonnegative smooth function on M. A C^2-function φ on an open subset U of M is called *L_X-harmonic* if $L_X\varphi = 0$ on U. A lower semi-continuous function ϕ on U is called *L_X-superharmonic* if for any relatively compact domain V in U and every *L_X-harmonic* function φ on V with $\phi \geqq \varphi$ on ∂V, we have $\phi \geqq \varphi$ on V. When $-\phi$ is L_X-superharmonic, we say ϕ is L_X-*subharmonic*. (When $X\equiv 0$, we simply call ϕ (*super-*, *sub-*) *harmonic* if ϕ is (L_0-super-, L_0-sub-) L_0-harmonic.) It is easily seen that *an upper semicontinuous function* ϕ *on an open subset* U *is* L_X-*subharmonic if and only if* $S_X\phi\geqq 0$ *on* U (cf. e.g., [49] or [35: Theorem 15.2]). Let D be a compact domain of M with smooth boundary ∂D. We write $G_D(x, y)$ for the Green function of L_X with respect to the Dirichlet problem on D. Let $\{M_i\}_{i=1,2,\dots}$ be any increasing family of compact domains in M with smooth boundary ∂M_i. Then $\{G_{M_i}(x, y)\}$ is increasing with respect to i and set $G_M(x, y) := \lim_{i\to+\infty} G_{M_i}(x, y)$ $(\leqq +\infty)$ (cf. e.g., [31]). We call $G_M(x, y)$ *the Green function* associated with the elliptic differential operator L_X on M if $G_M(x, y)<+\infty$ $(x\neq y)$. It is known that *there exists a nonconstant positive* L_X-*superharmonic function on* M *if and only if* M *has the Green function* $G_M(x, y)$ *of* L_X (cf. [31]). In Section 5, we shall show geometric lower or upper estimates for $G_M(x, y)$ if it exists (cf. Theorem 5.1, Theorem 5.3). Moreover we shall consider the equation: $L_X u+Q=0$ on M and give criteria for existence or nonexistence of a positive solution of the above equation (cf. Theorem 5.2, Theorem 5.4). In the last part of Section 5, we shall consider the Dirichlet problem for L_X-harmonic functions "at infinity" of M under certain conditions (cf. Theorem 5.5). Section 5 is a continuation of the latter part of [32].

In connection with our results, we must mention certain previous investigations by several authors. For example, a theorem of Blanc-Fiala-Huber [28] tells us that if $m=2$ and the Gaussian curvature of M is nonnegative outside a compact set, M possesses no nonconstant positive superharmonic functions, and a theorem of Ahlfors states that if $m=2$, M is simply connected and the Gaussian curvature is bounded above by a negative constant, M has the Green function of the Laplace operator (cf. [38]). Moreover Aomoto [1] proved that if $m\geqq 3$, M is simply connected and the sectional curvature is nonpositive, there are nonconstant positive superharmonic functions on M (cf. also [18]). Recently, Ichihara [30, I] has given more general geometric criteria for existence or nonexistence of the Green function of the Laplace operator on M (cf. also [15]). In [32], we have considered the case when $X\equiv 0$ and show generalizations of the results by Ichihara. His method is similar to ours, but seems to be not applicable for the case when $X\not\equiv 0$.

We note that a solution of the equation: $\Delta u+1=0$ on a Riemannian manifold is called a quasi-harmonic function in [43], where the chapter 2 is devoted to the classification theory on quasi-harmonic functions.

The author would like to express his thanks to Professor T. Ochiai for his helpful advice and encouragement, and also Professor N. Ikeda for his useful suggestions.

Table of contents

§ 1. Ricci curvature, geodesics and some geometric properties of Riemannian manifolds with boundary

In this section, we shall state the main results in the author's previous paper [33] and give the sketches of their proofs for the completeness.

1.1. Let M be a connected, complete Riemannian manifold of dimension m with smooth boundary ∂M. Let R and Λ be two real numbers. We call M of *class* (R, Λ) if the Ricci curvature of $M \geqq (m-1)R$ and the trace of $S_\xi \leqq (m-1)\Lambda$ for any unit inner normal vector field ξ of ∂M, where S_ξ is the second fundamental form of ∂M with respect to ξ (i.e., $\langle S_\xi X, Y\rangle = \langle \nabla_X \xi, Y\rangle$). We write $\mathscr{I}_M$ for the inradius of M (i.e., $\mathscr{I}_M = \sup\{\mathrm{dis}(x, \partial M) : x \in M\} \leqq +\infty$). Let $h_{R,\Lambda} \in C^2[0, \infty)$ be the solution of equation (0.4) defined by $\mathscr{R} \equiv R$ and Λ. Set $C_1(R, \Lambda) = \inf\{t : t > 0, h_{R,\Lambda}(t) = 0\}$ and $C_2(R, \Lambda) = \inf\{t : t > 0, h'_{R,\Lambda}(t) = 0\}$. If $h_{R,\Lambda} > 0$ (resp. $h'_{R,\Lambda} > 0$) on $(0, \infty)$, we understand $C_1(R, \Lambda) = +\infty$ (resp. $C_2(R, \Lambda) = +\infty$). We remark here that $C_1(R, \Lambda) < +\infty$ *if and only if* $R > 0$, $R = 0$ *and* $\Lambda < 0$, *or* $R < 0$ *and* $\Lambda < -\sqrt{-R}$ and that $0 < C_2(R, \Lambda) < +\infty$ *if and only if* $R > 0$ *and* $\Lambda > 0$, *or* $R < 0$ *and* $-\sqrt{-R} < \Lambda < 0$.

(1.1) **Definition** A Riemannian manifold M of class (R, Λ) is said to be *a model space* if one of the following conditions holds:

(I) $C_1(R, \Lambda) < +\infty$ and M is isometric to the metric (closed) ball $B(R; C_1(R, \Lambda))$ with radius $C_1(R, \Lambda)$ in the simply connected space form of constant curvature R.

(II) $R = 0$ and $\Lambda = 0$, or $0 < C_2(R, \Lambda) < +\infty$. Moreover M is isometric to the warped product $[0, 2a] \times_h \Gamma$, where $h = h_{R,\Lambda}$, a is any positive number if $R = 0$ and $\Lambda = 0$, and $a = C_2(R, \Lambda)$ if $0 < C_2(R, \Lambda) < +\infty$. (In this case, ∂M is disconnected.)

(III) $R = 0$ and $\Lambda = 0$, or $0 < C_2(R, \Lambda) < +\infty$. Moreover ∂M is connected, there is an involutive isometry σ of ∂M without fixed points and M is isometric to the quotient space $[0, 2a]\times_h \partial M/G$, where a and h are the same as in (II), and G is the isometry group of $[0, 2a]\times_h \partial M$ whose elements consists of the identity and the involutive isometry $\hat{\sigma}$ defined by $\hat{\sigma}((t, x))=(2a-t, \sigma(x))$.

Now we shall state the main results in [33].

Theorem 1.1. *Let M be a connected, complete Riemannian manifold of class* (R, Λ). *Then*:

(1) $\mathscr{I}_M \leqq C_1(R, \Lambda)$.

(2) *If* $C_1(R, \Lambda)<+\infty$ *and* $\operatorname{dis}(p, \partial M)=C_1(R, \Lambda)$ *for some* $p \in M$, *M is isometric to the model space* $B(R; C_1(R, \Lambda))$ *of type* (I).

This theorem is an analogue of Cheng's maximum diameter theorem for compact manifolds of positive Ricci curvature (cf. [16] and Section 4).

Theorem 1.2. *Let M be a connected, complete Riemannian manifold of class* (R, Λ). *Suppose ∂M is disconnected and it has a compact connected component* Γ_1. *Then*:

(1) *If* $R=0$ *and* $\Lambda=0$, *M is isometric to the isometric product* $[0, b]\times\Gamma_1$ $(b>0)$, *that is M is a model space of type* (II) $(R=0, \Lambda=0)$.

(2) *If* $R>0$, *then* $\Lambda>0$ *and* $\min_{j\geqq 2}\operatorname{dis}(\Gamma_j, \Gamma_1)\leqq 2C_2(R, \Lambda)$, *where* $\{\Gamma_j\}_{j=1,2\ldots}$ *are the connected component of* ∂M. *Moreover if* $\min_{j\geqq 2}\operatorname{dis}(\Gamma_1, \Gamma_j)=2C_2(R, \Lambda)$, *$M$ is isometric to the warped product* $[0, 2C_2(R, \Lambda)]\times_h\Gamma_1$, *that is, M is a model space of type* (II) $(0<C_2(R, \Lambda)<+\infty)$, *where* $h=h_{R,\Lambda}$ *is the solution of equation* (0.4) *defined by* $\mathscr{R}\equiv R$ *and* Λ.

As we mentioned in Introduction, the first assertion of this theorem and the second assertion (2) of Theorem 1.4 below are, roughly speaking, a generalization of a results by Burago and Zalgaller (cf. [9: Theorem 5.2.1]).

Theorem 1.3. *Let M be a connected, complete Riemannian manifold of class* (R, Λ). *Suppose ∂M is connected and suppose there is a minimal immersion* $\iota\colon N\to M_0$ *from a Riemannian manifold N without boundary into the interior M_0 of M such that* $\dim N=\dim M-1$ *and the image* $\tilde{N}:=\iota(N)$ *is compact. Then*:

(1) *If* $R=0$, $\Lambda=0$ *and* $M\backslash\tilde{N}$ *is connected, M is a model space of type* (III) $(R=0, \Lambda=0)$. *In particular, $\tilde{N}$ is a totally geodesic hypersurface of M.*

(2) *If* $R>0$, *then* $\Lambda>0$ *and* $\operatorname{dis}(\partial M, \tilde{N})\leqq C_2(R, \Lambda)$. *Moreover if the equality holds, M is a model space of type* (III) $(R>0, \Lambda>0)$.

Theorem 1.4. *Let M be a connected, complete Riemannian manifold of class (R, Λ). Suppose ∂M is compact but M is noncompact. Then:*

(1) $R \leqq 0$.

(2) *If $R=0$ and $\Lambda=0$, ∂M is connected and M is the isometric product* $[0, \infty) \times \partial M$.

(3) *If $\Lambda<0$, $R<0$ and $\Lambda \geqq -\sqrt{-R}$. Moreover if $\Lambda=-\sqrt{-R}$, M is isometric to the warped product* $[0, \infty) \times_{\exp(-\sqrt{-R}t)} \partial M$.

We remark here that the first assertion of Theorem 1.2 has proved in [29] by a different method and that in this assertion, we cannot delete the assumption that ∂M has a compact component, in contrast to the theorem of Burago and Zalgaller cited above. In fact, it is well known that there is a non-parametric minimal hypersurface in Euclidean space $\boldsymbol{R}^m$ ($m \geqq 9$) with the form: $x^m = u(x^1, \cdots, x^{m-1})$ defined for all $(x^1, \cdots, x^{m-1})$, where u is not linear (cf. [6]). Set $M = \{(x^1, \cdots, x^m) \in R^m : u(x^1, \cdots, x^{m-1}) \leqq x^m \leqq u(x^1, \cdots, x^{m-1})+1\}$. Then M satisfies all the conditions of (1) in Theorem 1.2 except that ∂M has a compact component, but M is not isometric to the direct product $[0, b] \times \Gamma$.

1.2. Now we shall give the sketches of proofs for the above theorems. For details, see [33].

Proof of Theorem 1.1. The first assertion is well known (cf. e.g., [ibid: § 2]). Now suppose $\operatorname{dis}(p, \partial M) = C_1(R, \Lambda)$ for some $p \in M$. Set $\rho_{\partial M} := \operatorname{dis}(*, \partial M)$ and $\rho_p := \operatorname{dis}(*, p)$. Then it follows from Theorem 0.2 that

$$S(-\rho_{\partial M}) \geqq -(m-1)(h'_{R,\Lambda}/h_{R,\Lambda}) \circ \rho_{\partial M}$$

($m=$ dim M) and

$$S(-\rho_p) \geqq -(m-1)(f'_R/f_R) \circ \rho_p$$

on $\Omega := \{x \in M : 0 < \rho_{\partial M}(x) < C_1(R, \Lambda) \text{ and } 0 < \rho_p(x) < \rho_{\partial M}(p)\}$, where $f_R \in C^2[0, \infty)$ is the solution of the equation (0.2) defined by $\mathscr{R} \equiv R$. Therefore $\rho_{\partial M} + \rho_p$ satisfies

$$S(-(\rho_{\partial M} + \rho_p)) \geqq -(m-1)\{(h'_{R,\Lambda}/h_{R,\Lambda}) \circ \rho_{\partial M} + (f'_R/f_R) \circ \rho_p\} \tag{1.2}$$

on Ω. We note here that if $s>0$, $t>0$ and $s+t \geqq C_1(R, \Lambda)$,

$$(h'_{R,\Lambda}/h_{R,\Lambda})(t) + (f'_R/f_R)(s) \leqq 0.$$

This implies that the right hand side of (1.2) is nonnegative, that is, $\rho_{\partial M} + \rho_p$ is superharmonic on Ω, because $\rho_{\partial M} + \rho_p \geqq C_1(R, \Lambda)$. Set

$$\omega := \{x \in M : \rho_{\partial M}(x) > 0,\ \rho_p(x) > 0,\ \rho_{\partial M}(x) + \rho_p(x) = C_1(R, \Lambda)\}.$$

Then ω is a nonempty closed subset in Ω. Since $\rho_{\partial M} + \rho_p$ takes the minimum $C_1(R, \Lambda)$ on $\omega(\subset \Omega)$, it is equal to $C_1(R, \Lambda)$ everywhere on Ω and hence on M, by the minimum principle for superharmonic functions. Therefore we see that the exponential map $\exp_p$ at p restricted to the closed ball B with radius $C_1(R, \Lambda)$ in the tangent space M_p at p induces a diffeomorphism between B and M. Moreover we have

$$\Delta \rho_p = (m-1)(f_R'/f_R) \circ \rho_p$$

on $M \backslash \{p\}$. This shows by the equality discussion of Theorem 0.1 that for any distance minimizing geodesic $\sigma: [0, a] \to M$ from p, the sectional curvature of any plane tangent to $\dot{\sigma}$ is equal to R. Thus M is isometric to the closed ball with radius $C_1(R, \Lambda)$ in the simply connected space form of constant curvature R.

By the similar arguments, we can prove Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.4. It follows from the assumptions that there exists a distance minimizing geodesic ray $\gamma: [0, \infty) \to M$ from ∂M. In particular, $R \leqq 0$. Let $B_\gamma: M \to R$ be the Busemann function with respect to γ. Then $B_\gamma + \rho_{\partial M} \geqq 0$ on the half space $\mathscr{H}_\gamma := \{x \in M : B_\gamma(x) < 0\}$ by the triangle inequality and $(B_\gamma + \rho_{\partial M})(\gamma(t)) = 0$ for any $t \geqq 0$ by the definition of B_γ. Set $\omega := \{x \in M : B_\gamma(x) < 0,\ \rho_{\partial M}(x) + B_\gamma(x) = 0\}$. Then ω is a closed subset of $M_0 (= M \backslash \partial M)$ contained in $\mathscr{H}_\gamma$. Now we have by Theorem 0.2 and Theorem 0.4

$$S(-\rho_{\partial M}) \geqq -(m-1)(h_{R,\Lambda}'/h_{R,\Lambda}) \circ \rho_{\partial M}$$

on M and

$$S(-B_\gamma) \geqq -(m-1)\sqrt{-R}$$

on $\mathscr{H}_\gamma$, and hence

$$(1.3) \qquad S(-(\rho_{\partial M} + B_\gamma)) \geqq -(m-1)\{(h_{R,\Lambda}'/h_{R,\Lambda}) \circ \rho_{\partial M} + \sqrt{-R}\}$$

on $\mathscr{H}_\gamma$. Suppose $R=0$ and $\Lambda=0$. Then the right-hand side of (1.3) is equal to 0, that is, $\rho_{\partial M} + B_\gamma$ is superharmonic on $\mathscr{H}_\gamma$ and further it takes the minimum 0 on ω. Therefore it follows from the minimum principle for superharmonic functions that $\omega = \mathscr{H}_\gamma = M_0$ and $S(-\rho_{\partial M}) = 0$ on M_0. This implies that the exponential map $\exp_{\partial M}^{\perp}$ restricted to $\nu(\partial M)^+ = \{t\xi : t \geqq 0\}$, where ξ is the unit inner normal vector field on ∂M, induces a diffeomorphism between $\nu(\partial M)^+$ and M, and that for any distance minimizing geodesic $\sigma: [0, a] \to M$ from ∂M, the sectional curvature of any tangent plane containing $\dot{\sigma}$ is equal to 0 and ∂M is totally geodesic. Thus we see that the map $\Psi: [0, \infty] \times \partial M \to M$ defined by $\Psi(t, x) = \exp^{\perp}_{\partial M} t\xi(x)$ induces an isometry between the direct product $[0, \infty) \times \partial M$ and M. This completes the proof for the assertion (2) of Theorem 1.4. As for the assertaion (3), it follows from the assertion (2) and the existence of a distance minimizing geodesic ray from ∂M that $R<0$ and $\Lambda \geqq -\sqrt{-R}$. Now suppose $\Lambda = -\sqrt{-R}$. Then the right-hand side of (1.3) is equal to 0. Therefore the same arguments as in the preceding assertion (2) shows that the map $\Psi: [0, \infty) \times \partial M \to M$ as above induces an isometry from the warped product $[0, \infty) \times_{\exp(-\sqrt{-R}t)} \partial M$ onto M. This completes the proof of Theorem 1.4.

§ 2. Lower bounds for the first eignevalue of Laplace operator

Let M be a connected compact Riemannian manifold of dimension m with smooth boundary ∂M. Let us consider the following eigenvalue problem:

$$\begin{cases} \Delta u + \lambda u = 0 & \text{on } M \\ u = 0 & \text{on } \partial M. \end{cases} \tag{2.1}$$

We write $\lambda_1(M)$ for the first eigenvalue of (2.1). In this section, we shall apply Theorem 0.2 and Theorem 0.4 to obtain a geometric lower bound for $\lambda_1(M)$. The results of this section have been proved in [34].

2.1. In order to state and prove our results below (cf. Theorem 2.1 and Proposition 2.1), we need the following two lemmas. The first is a generalization of Barta's inequality (cf. [3]) and the second follows from simple calculations.

(2.2) **Lemma** (cf. [34]). *Suppose there are a continuous function ψ on M and a constant μ such that*

$$\begin{cases} \psi < 0 & \text{on } M_0 \ (:= M \backslash \partial M), \\ S\psi + \mu\psi \geqq 0 & \text{on } M. \end{cases}$$

Then we have

$$\lambda_1(M) \geqq \mu. \tag{2.3}$$

Moreover if ψ is smooth on some open dense subset V of M, the equality in (2.3) *implies that ψ is the first eigenfunction of* (2.1).

(2.4) **Lemma.** *Let q be a continuous function on* $[0, \alpha)$ $(\alpha>0)$ *and* β *be a positive number with* $\beta\leq\alpha$. *Let us consider the following one dimensional eigenvalue prblem*:

$$\begin{cases}\phi''+q(t)\phi'+\lambda\phi=0 & on\ [0,\beta]\\ \phi(0)=\phi'(\beta)=0.\end{cases} \tag{2.5}$$

Then the first eigenfunction ϕ *of* (2.5) *satisfies*

$$\phi\cdot\phi'>0 \tag{2.6}$$

on $(0, \beta)$.

We remark here that the first eigenfunction Φ of (2.1) for a model space $M(R, \Lambda)$ of class (R, Λ) can be written in the form:

$$\Phi=\phi\circ\rho_{\partial M}$$

where $\rho_{\partial M}=\text{dis}(M, *)$ and $\phi\in C^2[0, \mathscr{I}_{M(R,\Lambda)}]$. Since

$$\Delta\rho_{\partial M}=(m-1)(\log h_{R,\Lambda})'\circ\rho_{\partial M} \quad \text{on } M(R, \Lambda),$$

$\phi(t)$ is the first eigenfunction of (2.5) with $q=(m-1)(\log h_{R,\Lambda})'$ and $\beta=\mathscr{I}_{M(R,\Lambda)}$, where $h_{R,\Lambda}$ is the solution of equation (0.4) defined by $\mathscr{R}\equiv R$ and Λ, and $\mathscr{I}_{M(R,\Lambda)}=\max\{\rho_{\partial M}(x): x\in M(R, \Lambda)\}$.

Theorem 2.1. *Let M be an m-dimensional compact Riemannian manifold with smooth boundary ∂M. Suppose M is of class (R, Λ). Then*

$$\lambda_1(M)\geqq\lambda(R, \Lambda, \mathscr{I}_M), \tag{2.7}$$

where $\mathscr{I}_M=\max\{\text{dis}(x, \partial M): x\in M\}$, *and* $\lambda(R, \Lambda, \mathscr{I}_M)$ *is equal to the first eigenvalue of* (2.5) *with* $q=(m-1)(\log h_{R,\Lambda})'$ *and* $\beta=\mathscr{I}_M$ *if* $\mathscr{I}_M<C_1(R, \Lambda)$ (*cf. Section* 1 *for the definitions of* $h_{R,\Lambda}$ *and* $C_1(R, \Lambda)$) *and equal to the first eigenvalue of the model space* $M(R, \Lambda)$ *of class* (R, Λ) *if* $\mathscr{I}_M=C_1(R, \Lambda)$ (*cf. Theorem* 1.1). *Moreover the equality holds in* (2.7) *if and only if M is isometric to a model space* $M(R, \Lambda)$ *of class* (R, Λ).

Proof. We shall first show the theorem in the case when $\mathscr{I}_M<C_1(R, \Lambda)$. Put $q_{R,\Lambda}:=(m-1)(\log h_{R,\Lambda})'$. Then $q_{R,\Lambda}$ is a smooth function on $[0, \mathscr{I}_M]$, since $h_{R,\Lambda}$ is positive on $[0, C_1(R, \Lambda))$. Let ϕ be the first eigenfunction of (2.5) with $q=q_{R,\Lambda}$ and $\beta=\mathscr{I}_M$. We may assume that ϕ and ϕ' are both negative on $(0, \mathscr{I}_M)$ by Lemma (2.4). Applying Theorem 0.2 to $\phi\circ\rho$ ($\rho=\text{dis}(*, \partial M)$), we have

$$S(\phi \circ \rho)+\lambda(R, \Lambda, \mathscr{I}_M)\phi \circ \rho \geqq (\phi''+q_{R,\Lambda}\phi'+\lambda(R, \Lambda, \mathscr{I}_M)) \circ \rho \equiv 0. \tag{2.8}$$

Therefore it follows from Lemma (2.2) that $\lambda_1(M) \geqq \lambda(R, \Lambda, \mathscr{I}_M)$. We shall now assume that $\lambda_1(M)=\lambda(R, \Lambda, \mathscr{I}_M)$. Since ρ is smooth on the open dense subset $M \backslash \mathscr{C}(\partial M)$, where $\mathscr{C}(\partial M)$ is the cut locus of ∂M, we see by Lemma (2.2) again that $\phi \circ \rho$ is the first eigenfunction of M, that is, $\phi \circ \rho$ is smooth on M and satisfies

$$\Delta(\phi \circ \rho)+\lambda(R, \Lambda, \mathscr{I}_M)\phi \circ \rho=0 \tag{2.9}$$

on M. Hence by (2.8) and (2.9), we have

$$\Delta \rho = q_{R,\Lambda} \circ \rho$$

on $M \backslash \mathscr{C}(\partial M)$. This shows that for any distance minimizing geodesic σ: $[0, a] \to M$ from ∂M, the sectional curvature of every plane containing the tangent vector $\dot{\sigma}(t)$ is equal to R and ∂M is umbilic at $\sigma(0)$ (i.e., $S_{\sigma(0)}X, Y\rangle = \Lambda\langle X, Y\rangle$) (cf. the equality discussion of Theorem 0.2)). Then combining this fact with the smoothness of $\phi \circ \rho$ and the negativity of ϕ' on $(0, \mathscr{I}_M)$, we see that $\mathscr{C}(\partial M)=\{x \in M: \rho(x)=\mathscr{I}_M\}$. Now it is not hard to see that M is isometric to a model space of class (R, Λ), which is different from the metric ball $B(R; C_1(R, \Lambda))$ with radius $C_1(R, \Lambda)$ in the simply connected space form of constant curvature R. Now we assume $\mathscr{I}_M = C_1(R, \Lambda)$. Then it follows from Theorem 1.1 that M is isometric to $B(R; C_1(R, \Lambda))$. This completes the proof of Theorem 2.1.

We shall now give some computable lower bounds for $\lambda(R, \Lambda, \mathscr{I}_M)$ in the above theorem, because, in general, it would be very difficult to obtain the exact value of $\lambda(R, \Lambda, \mathscr{I}_M)$. (For the proofs of the results below, see [34].)

Corollary. *Let M be as in Theorem* 2.1. *Then we have*

$$\lambda_1(M) \geqq \lambda(R, \Lambda, \mathscr{I}_M) > \left[4 \max_{0 \leqq t \leqq \mathscr{I}_M} \left(\int_t^{\mathscr{I}_M} h_{R,\Lambda}^{m-1}(u)\,du \int_0^t 1/h_{R,\Lambda}^{m-1}(u)\,du\right)\right]^{-1}$$

Corollary. *Let M be as in Theorem* 2.1. *Suppose* $R=0$ *and* $\Lambda=0$. *Then we have*

$$\lambda_1(M) \geqq \lambda(0, 0, \mathscr{I}_M) = \frac{\pi^2}{4\mathscr{I}_M^2}. \tag{2.10}$$

Moreover the equality holds if and only if M is a model space of class (0, 0) (e.g, *a section of a flat cylinder*).

Corollary (Reilly [42]). *Let M be as in Theorem* 2.1. *Suppose* $R>0$ *and* $\Lambda=0$. *Then we have*

$$\lambda_1(M)\geqq mR.$$

Moreover the equality holds if and only if M is isometric to the hemisphere in the standard sphere of constant curvature R.

We remark here that estimate (2.10) was proved by Li and Yau [36] with a different method from ours and that in the case when $R>0$, we can obtain other computable estimates for $\lambda_1(M)$, making use of a result by Friedland and Hayman [22].

2.2. Before concluding this section, we shall give another estimate for $\lambda_1(M)$ in the case when M is a compact domain in a certain complete, *noncompact* Riemannian manifold N without boundary. Because of the noncompactness of N, there is a geodesic ray $\gamma: [0, \infty)\to M$. Let B_γ be the Busemann function on N with respect to γ. Then $S(-B_\gamma)$ is bounded from below by $-(m-1)\sqrt{-R}$ ($m=\dim N$) if the Ricci curvature of N is bounded from below by some nonpositive constant $(m-1)R$ (cf. Theorem 0.4). Then making use of B_γ instead of the distance function ρ to ∂M in the proof of Theorem 2.1, we can obtain the following

Proposition 2.1. *Let M be a compact domain with smooth boundary in a complete, connected and noncompact Riemannian manifold N without boundary. Suppose the Ricci curvature of N is bounded from below by some nonpositive constant* $(m-1)R$ ($m=\dim N$). *Then we have*

$$\lambda_1(M)>\begin{cases} \dfrac{\pi^2}{4d(M)^2} & (R=0), \\[2ex] \dfrac{-(m-1)^2R\exp 2(m-1)\sqrt{-R}\,d(M)}{(\exp\{(m-1)\sqrt{-R}\,d(M)\}-1+4/\pi^2)(\exp\{(m-1)\sqrt{-R}\,d(M)\}-1)^2} & (R<0), \end{cases}$$

where $d(M)$ denotes the diameter of M.

For the proof of the above proposition, see [34].

§ 3. Bounds for solutions of Poisson equations

Let M be a connected *compact* Riemannian manifold of dimension m with smooth boundary ∂M. Let us consider the following Poisson equation:

$$\text{(3.1)}\qquad \begin{cases} \Delta u+Q=0 & \text{on } M_0\ (:=M\backslash\partial M) \\ u=0 & \text{on } \partial M, \end{cases}$$

where Q is a smooth function on M. We write U_Q for the solution of (3.1), that is, $U_Q(x)=\int_M Q(y)G_M(x, y)dy$, where $G_M(x, y)$ is the Green function of the Laplacian Δ with respect to the Dirichlet problem on M. In this section, we shall show some geometric estimates for U_Q, $\max_M U_Q$ and ∇U_Q.

3.1. We shall first prove the following

Theorem 3.1. *Let M be an m-dimensional compact Riemannian manifold of class* (R, Λ) (cf. *Section* 1) *and Q a nonnegative smooth function on M. Then the solution U_Q of equation* (3.1) *has an estimate*:

$$\text{(3.2)}\qquad U_Q(x)\leqq\int_0^{\rho(x)}\left\{\int_t^{\mathscr{I}_M}(Q^*h_{R,\Lambda}^{m-1})(s)\,ds/h_{R,\Lambda}^{m-1}(t)\right\}dt$$

for $x\in M$, where $\rho(x)=\operatorname{dis}(x, \partial M)$, $\mathscr{I}_M=\max\{\rho(x)\colon x\in M\}$, $h_{R,\Lambda}$ is the solution of equation (0.4) *defined by $\mathscr{R}\equiv R$ and Λ, and*

$$Q^*(t):=\max\{Q(x)\colon x\in M,\ \rho(x)=t\}.$$

Moreover the following three conditions are equivalent to each other:

(1) *The equality holds in* (3.2) *for some $x\in M_0$,*

(2) *The equality holds in* (3.2) *everywhere on M.*

(3) *M is isometric to a model space of class* (R, Λ) (cf. *Section* 1) *and $Q=Q^*\circ\rho$.*

Proof. For the sake of brevity, we write $\varphi\circ\rho$ for the right hand side of (3.2). Then it follows from Theorem 0.2 that $\varphi\circ\rho$ satisfies

$$\text{(3.3)}\qquad S(-\varphi\circ\rho)\geqq Q^*\circ\rho$$

on M_0. Therefore we have

$$\text{(3.4)}\qquad S(U_Q-\varphi\circ\rho)\geqq -Q+Q^*\circ\rho\geqq 0$$

on M_0. This implies that $U_Q-\varphi\circ\rho$ is subharmonic on M_0, and hence inequality (3.2) holds on M, since $U_Q-\varphi\circ\rho=0$ on ∂M. Now we shall show the latter part of the theorem. It is clear that (1) and (2) are equivalent each other by the above argument. Moreover by the definition of a model space of class (R, Λ), we see that (3) implies (2). Finally let us show the converse. Suppose $U_Q\equiv\varphi\circ\rho$ on M. Then we have by (3.4)

$Q \equiv Q^* \circ \rho$ on M. In the case when $\mathcal{I}_M = C_1(R, \Lambda)$, we have shown that M is the model space of type (I) (cf. Theorem 1.1), so that we assume $\mathcal{I}_M < C_1(R, \Lambda)$ ($\leqq +\infty$). Since the equality holds in (3.3), we see that $\Delta\rho = (m-1)(\log h_{R,\Lambda})' \circ \rho$ on $M \backslash \mathcal{C}(\partial M)$, where $\mathcal{C}(\partial M)$ is the cut locus of ∂M. Therefore the latter part of Theorem 0.2 says that for any distance minimizing geodesic σ: $[0, a] \to M$ from ∂M, the sectional curvature of any plane tangent to $\dot\sigma$ is equal to R and ∂M is umbilic at $\sigma(0)$ (i.e., $\langle S_{\dot\sigma(0)}X, Y\rangle = \Lambda\langle X, Y\rangle$). Moreover we see that $\mathcal{C}(\partial M) = \{x \in M: \rho(x) = \mathcal{I}_M\}$, since φ' is positive on $[0, \mathcal{I}_M)$. These show that M is isometric to a model space of type (II) or type (III). This completes the proof of Theorem 3.1.

As an immediate consequence of Theorem 3.1, we have the following

Corollary 3.1. *Under the same notations as in Theorem* 3.1, *the following two inequalities hold*:

(3.5) $$\max_M U_Q \leqq \int_0^{\mathcal{I}_M} \left\{\int_t^{\mathcal{I}_M} (Q^* h_{R,\Lambda}^{m-1})(s)\,ds / h_{R,\Lambda}^{m-1}(t)\right\} dt,$$

and

(3.6) $$\max_{\partial M} \|\nabla U_Q\| \leqq \int_0^{\mathcal{I}_M} (Q^* h_{R,\Lambda}^{m-1})(s)\,ds.$$

Moreover the equality holds in (3.5) *or* (3.6) *if and only if the condition* (3) *in Theorem* 3.1 *holds.*

Remark. (1) When $m=2$, $Q \equiv 1$ and $R>0$, we have by (3.5) and (3.6)

$$\max_M U_1 \leqq \frac{1}{R} \log\left[\frac{1}{R}(\sigma + \Lambda\sqrt{\sigma})\right] \quad (\sigma := R + \Lambda^2),$$

$$\max_{\partial M} \|\nabla U_1\| \leqq \sqrt{\frac{\Lambda^2}{R^2} + \frac{1}{R}} + \frac{\Lambda}{R},$$

since $\mathcal{I}_M \leqq C_1(R, \Lambda)$. These bounds for $\max_M U_1$ and $\max_{\partial M} \|\nabla U_1\|$ have been proved by Sperb [47].

(2) When $Q \equiv$ const. and $R \geqq 0$, $\|\nabla U_Q\|^2$ is subharmonic on M, so that $\max_{\partial M} \|\nabla U_Q\| = \max_M \|\nabla U_Q\|$, because

$$\Delta \|\nabla U_Q\|^2 = 2\|\nabla^2 U_Q\|^2 + 2\,\mathrm{Ric}(\nabla U_Q, \nabla U_Q) \geqq 0.$$

3.2. Now we consider the case when M is a domain in a noncompact complete Riemannian manifold. Let N be a noncompact complete

Riemannian manifold without boundary and M a compact domain with smooth boundary ∂M in N. Let $\gamma: [0, \infty) \to N$ be a geodesic ray and $B_\gamma: N \to R$ the Busemann function with respect to γ. Suppose the Ricci curvature of N is bounded from below by some nonpositive constant $(m-1)R$ ($m = \dim N$). Put

$$\begin{aligned}(3.7)\qquad &\psi(R, \underline{\delta}_M, \bar{\delta}_M)(t) \\ &:= \int_{\underline{\delta}_M}^{t} \left\{ \int_{s}^{\bar{\delta}_M} \exp\,(m-1)\sqrt{-R}\,u\,du \Big/ \exp\,(m-1)\sqrt{-R}\,s \right\} ds,\end{aligned}$$

where $\underline{\delta}_M = \min\{B_\gamma(x) : x \in M\}$ and $\bar{\delta}_M = \max\{B_\gamma(x) : x \in M\}$. Then it follows from Theorem 0.4 that $\psi \circ B_\gamma$ satisfies

$$(3.8)\qquad S(-\psi(R, \underline{\delta}_M, \bar{\delta}_M) \circ B_\gamma) \geqq 1$$

on the interior M_0 of M. Therefore by the same argument as in the proof of Theorem 3.1, we see that the solution U_1 of equation (3.1) ($Q \equiv 1$) has an estimate:

$$U_1 \leqq \psi(R, \underline{\delta}_M, \bar{\delta}_M) \circ B_\gamma$$

on M_0. This shows that

$$\begin{aligned}\max_M U_1 &\leqq \max\{\psi(t) : \underline{\delta}_M \leqq t \leqq \bar{\delta}_M\} \\ &= \begin{cases} \dfrac{-1}{(m-1)^2 R}\left[\exp\{(m-1)\sqrt{-R}(\bar{\delta}_M - \underline{\delta}_M)\} - 1 - (m-1)\sqrt{-R}(\bar{\delta}_M - \underline{\delta}_M)\right] & (R<0), \\ \dfrac{(\bar{\delta}_M - \underline{\delta}_M)^2}{2} & (R=0). \end{cases}\end{aligned}$$

Since $(\bar{\delta}_M - \underline{\delta}_M)$ is less than the diameter $d(M)$ of M (cf. [49: Lemma 3.2]), we have the following

Proposition 3.1. *Let M be a compact domain in a complete noncompact Riemannian manifold N whose Ricci curvature is bounded from below by some nonpositive constant $(m-1)R$ ($m = \dim N$). Then the solution U_1 of equation* (3.1) ($Q \equiv 1$) *has an estimate*:

$$\max_M U_1 < \begin{cases} \dfrac{d(M)^2}{2} & (R=0) \\ \dfrac{-1}{(m-1)R}\left[\exp\{(m-1)\sqrt{-R}\,d(M)\} - 1 - (m-1)\sqrt{-R}\,d(M)\right] & (R<0). \end{cases}$$

3.3. Now we shall show a lower estimate for U_1.

Theorem 3.2. *Let M be a compact Riemannian manifold of dimension m with smooth boundary ∂M. Suppose the Ricci curvature of M is bounded from below by $(m-1)R$ ($R \in \boldsymbol{R}$). Then the solution U_1 of equation* (3.1) *with $Q \equiv 1$ satisfies*

$$U_1(x) \geqq \int_0^{\rho(x)} \frac{\int_0^t f_R^{m-1}(s)ds}{f_R^{m-1}} dt, \tag{3.9}$$

where $\rho(x) = \mathrm{dis}\,(x, \partial M)$ and f_R is the solution of equation (0.2) *with $\mathscr{R} \equiv R$. Moreover the equality holds for some $x \in M$ if and only if M is isometric to the metric ball in the simply connected space form of constant curvature R and x is the center of M.*

Proof. Let x be an interior point of M. We write $B_r(x)$ for the metric ball around x with radius r ($\leqq \rho(x)$). Let $G_r(y, z)$ be the Green function of $B_r(x)$ ($r \leqq \rho(x)$) (i.e., $G_r(y, z) = \lim_{i\to\infty} G_{D_i}(z, y)$, where $\{D_i\}_{i=1,2,\dots}$ is an increasing family of compact domains $D_i \subset B_r(x)$ with smooth boundary ∂D_i and $G_{D_i}(y, z)$ is the Green function of the Laplacian with respect to the Dirichlet problem on D_i). Set

$$U_r(y) := \int_{B_r(x)} G_r(y, z)dz.$$

Then we have

$$U_Q(x) \geqq U_r(x),$$

since $G_M(z, y) \geqq G_r(z, y)$ for any $y, z \in B_r(x)$. Therefore Theorem 3.2 follows from the following

(3.10) **Lemma.** *Let N be a complete Riemannian manifold of dimension m and $B_r(x)$ the metric ball around $x \in N$ with radius r ($r \leqq \mathrm{dis}\,(x, \partial N)$ if $\partial N \neq \phi$). Suppose the Ricci curvature of N is bounded from below by $(m-1)R$ ($R \in \boldsymbol{R}$). Then we have*

$$U_r(y) \geqq \int_{\mathrm{dis}(x,y)}^{r} \frac{\int_0^t f_R^{m-1}(s)ds}{f_R^{m-1}(t)} dt \tag{3.11}$$

where U_r and f_R are as above. Moreover the equality holds if and only if $B_r(x)$ is isometric to the metric ball in the simply connected space form of constant curvature R.

Proof. We write $V(y)$ for the right-hand side of (3.11). Then it follows from Theorem 0.1 that $SV+1\geqq 0$ on Int $(B_r(x))$ and $V=0$ on $\partial B_r(x)$. Therefore $S(V-U_r)\geqq 0$ on $\mathrm{Int}(B_r(x))$ and $V-U_r=0$ "almost everywhere" on $\partial B_r(x)$. Thus we have inequality (3.11) by the maximum principle for subharmonic functions. The latter part of the lemma follows from the same argument as in Theorem 3.1. This completes the proof of Lemma (3.10).

As an immediate consequence of (3.2) and (3.9), we have the following

Corollary. *Let M be an m-dimensional compact Riemannian manifold of class (R, Λ). Set $P_M:=\{x \in M: U_1(x)=\max_M U_1\}$, where U_1 is the solution of equation* (3.1) $(Q\equiv 1)$. *Then we have*

$$V(m, R, \Lambda, \mathscr{I}_M)\leqq \mathrm{dis}\,(P_M, \partial M)\ (\leqq \mathscr{I}_M), \tag{3.12}$$

where $V(m, R, \Lambda, \mathscr{I}_M)$ is the positive real number defined by

$$\int_0^{V(m,R,\Lambda,\mathscr{I}_M)}\left\{\int_t^{\mathscr{I}_M} h_{R,\Lambda}^{m-1}(s)\,ds/h_{R,\Lambda}^{m-1}(t)\right\}dt=\int_0^{\mathscr{I}_M}\left\{\int_0^t f_R^{m-1}(s)\,ds/f_R^{m-1}(t)\right\}dt.$$

Moreover the equality holds in (3.12) *if and only if M is the metric ball in the simply connected space form of constant curvature R.*

Corollary 3.2. *Let M be a complete noncompact Riemannian manifold of dimension m. Suppose the Ricci curvature of M is bounded from below by some nonpositive constant $(m-1)R$.*

(1) *If ∂M is nonempty and there exists a positive solution U_1 of equation* (3.1), *then*

$$U_1>\int_0^{\rho}\left\{\int_0^t f_R^{m-1}(s)\,ds/f_R^{m-1}(t)\right\}dt \qquad (\rho=\mathrm{dis}\,(*, \partial M)).$$

(2) *If ∂M is empty, there are no positive solutions of the equation: $\Delta u+1=0$ on M.*

Proof. The first assertion follows from Theorem 3.2. As for the second assertion, let $\{M_i\}_{i=1,2\dots}$ be an increasing family of compact domains M_i of M with smooth boundary ∂M_i and U_i the solution of equation (3.1) $(Q\equiv 1)$ on M_i. Then by Theorem 3.2, we see that for each i,

$$U_i>\int_0^{\rho_i}\left\{\int_0^t f_R^{m-1}(s)\,ds/f_R^{m-1}(t)\right\}dt$$

on M_i $(\rho_i:=\mathrm{dis}\,(*, \partial M_i))$ and hence

$$\lim_{i\to\infty} U_i \geqq \lim_{i\to\infty} \int_0^{\rho_i} f_R^{m-1}(s)\, ds / f_R^{m-1}(t) \Big\} dt = +\infty.$$

This shows the second assertion.

In Section 5.2, we shall give a generalization of the second assertion of Corollary 3.2.

§ 4. A volume estimate for a domain in a Riemannian manifold and its application

Let M be a connected, complete Riemannian manifold of dimension m. Suppose M is a compact Riemannian manifold of class (R, Λ). Let U_1 be the solution of equation (3.1) with $Q \equiv 1$. Then by Stokes' theorem, we have

$$\begin{aligned} \mathrm{Vol}_m(M) &= \int_M -\Delta U_1 \\ &= \int_{\partial M} -\frac{\partial U_1}{\partial \nu} \\ &\leqq \max \|\nabla U_1\| \cdot \mathrm{Vol}_{m-1}(\partial M). \end{aligned}$$

Therefore it follows from Corollary 3.1 that

$$\mathrm{Vol}_m(M) \leqq \mathrm{Vol}_{m-1}(\partial M) \cdot \int_0^{\mathscr{I}_M} h_{R,\Lambda}^{m-1}(t) dt, \tag{4.1}$$

where $\mathscr{I}_M = \max \{\mathrm{dis}(x, \partial M) : x \in M\}$ and $h_{R,\Lambda}$ is the solution of equation (0.4) defined by R and Λ. Inequality (4.1) was proved by Heintze and Karcher [27]. In this section, we shall show a generalization of inequality (4.1) for a domain in M and give its application (cf. Theorem 4.1). Moreover in the last paragraph of this section, we shall prove a volume estimate for a domain in a noncompact complete Riemannian manifold.

4.1. We shall first prove the following

Proposition 4.1. *Let M be an m-dimensional Riemannian manifold of class (R, Λ) and D a compact domain in M with smooth boundary ∂D. Then*

$$\mathrm{Vol}_m(D) \leqq \mathrm{Vol}_{m-1}(\partial D) \cdot \max_{\delta_1(D) \leqq t \leqq \delta_2(D)} \int_t^{\delta_2(D)} h_{R,\Lambda}^{m-1}(s)\, ds / h_{R,\Lambda}^{m-1}(t), \tag{4.2}$$

where $h_{R,\Lambda}$ is the solution of equation (0.4) *defined by R and Λ, $\delta_1(D) =$* $\min \{\mathrm{dis}(x, \partial M) : x \in D\}$ *and* $\delta_2(D) = \max \{\mathrm{dis}(x, \partial M) : x \in D\}$. *Moreover* *the equality holds in* (4.2) *if and only if* $D=\{x \in M: \operatorname{dis}(x, \partial M) \geqq \delta_1(D)\}$ *and* M *is a model space of class* (R, Λ).

Remark. Simple computations show that the right-hand side of (4.2) is equal to

$$\operatorname{Vol}_{m-1}(\partial D) \int_{\delta_1(D)}^{\delta_2(D)} h_{R,\Lambda}^{m-1}(t)\, dt / h_{R,\Lambda}^{m-1}(\delta_1(D))$$

in the case when $R \geqq 0$, $R<0$ and $\Lambda \geqq 0$, or $R<0$ and $\Lambda=-\sqrt{-R}$.

Proof of Proposition 4.1. We first consider the case when $\delta_2(D)< C_1(R, \Lambda)$ $(:=\inf\{t>0: h_{R,\Lambda}(t) \leqq 0\} \leqq +\infty$ (cf. Sec. 1)). For a sufficiently small positive constant ε such that $\delta_2(D)+\varepsilon<C_1(R, \Lambda)$, we put

$$\xi_\varepsilon=-\int_{\delta_1(D)}^{\rho}\left\{\int_s^{\delta_2(D)+\varepsilon} h_{R,\Lambda}^{m-1}(u)\, du / h_{R,\Lambda}^{m-1}(S)\right\} ds,$$

where $\rho=\operatorname{dis}(*, \partial M)$. Then it follows from Theorem 0.2 in Introduction that

$$S \xi_\varepsilon \geqq 1$$

on $V_\varepsilon:=\{x \in M: \rho(x)<\delta_2(D)+\varepsilon\}$. Then by virtue of the approximation theorem by Greene and Wu (cf. [26: Lemma 1.2, Lemma 3.2 and Theorem 3.2]), we see that there exists a smooth function $\tilde{\xi}_\varepsilon$ on V_ε such that

$$\begin{cases} \Delta \tilde{\xi}_\varepsilon \geqq 1-\varepsilon, \\ \|\nabla \tilde{\xi}_\varepsilon\| \leqq \varepsilon+\displaystyle\int_\rho^{\delta_2(D)+\varepsilon} h_{R,\Lambda}^{m-1}(t)\, dt / h_{R,\Lambda}^{m-1}(\rho), \\ |\xi_\varepsilon-\tilde{\xi}_\varepsilon|<\varepsilon. \end{cases}$$

Therefore integrating by parts, we get

$$\begin{aligned} (1-\varepsilon) \operatorname{Vol}_m(D) &\leqq \int_D \Delta \tilde{\xi}_\varepsilon \\ &=\int_{\partial D} \frac{\delta \tilde{\xi}_\varepsilon}{\partial \nu} \\ &\leqq \operatorname{Vol}_{m-1}(\partial D)\left(\varepsilon+\max_{\delta_1(D) \leqq t \leqq \delta_2(D)} \int_t^{\delta_2(D)+\varepsilon} h_{R,\Lambda}^{m-1}(s)\, ds / h_{R,\Lambda}^{m-1}(t)\right), \end{aligned}$$

where ν denotes the exterior unit normal vector field on ∂D. Since ε is any small positive constant, we can obtain inequality (4.2). Now we assume that the equality in (4.2) holds. Then it is not hard to see that

$$\text{(4.3)} \qquad \mathrm{Vol}_m(D)=\mathrm{Vol}_{m-1}\partial(D)\cdot\int_{\delta_1(D)}^{\delta_2(D)} h_{R,\Lambda}^{m-1}(t)\,dt/h_{R,\Lambda}^{m-1}(\delta_1(D))$$

and

$$\text{(4.4)} \qquad D=\{x\in M\colon \mathrm{dis}\,(x,\partial M)\geqq\delta_1(D)\}.$$

Therefore it follows from (4.4) that D is a compact Riemannian manifold of class $(R, \Lambda(D))$, where $\Lambda(D)=(h'_{R,\Lambda}/h_{R,\Lambda})(\delta_1(D))$, and hence by Corollary 3.1

$$\text{(4.5)} \qquad \max_{\partial D}\|\nabla U_1\|\leqq\int_0^{\delta_2(D)-\delta_1(D)} h_{R,\Lambda(D)}^{m-1}(t)dt,$$

where U_1 is the solution of equation (3.1) ($Q\equiv 1$) on D. Noting that the right-hand side of (4.5) is equal to

$$\int_{\delta_1(D)}^{\delta_2(D)} h_{R,\Lambda}^{m-1}(t)\,dt/h_{R,\Lambda}^{m-1}(\delta_1(D)),$$

we have

$$\begin{aligned}\mathrm{Vol}_m(D)&=\int_D -\Delta U_1\\ &=\int_{\partial D}-\frac{\partial U_1}{\partial\nu}\\ &\leqq\mathrm{Vol}_{m-1}(\partial D)\int_{\delta_1(D)}^{\delta_2(D)} h_{R,\Lambda}^{m-1}(t)\,dt/h_{R,\Lambda}^{m-1}(\delta_1(D)).\\ &=\mathrm{Vol}_m\,(D) \qquad\qquad \text{by (4.3).}\end{aligned}$$

This shows that equality holds in (4.5). Therefore we see by Corollary 3.1 again that D is a model space of class $(R, \Lambda(D))$, and hence by (4.4), M is also a model space of class (R, Λ). Next we consider the case when $\delta_2(D)=C_1(R, \Lambda)$. Then it follows from Theorem 1.1 that M is isometric to the metric ball with radius $C_1(R, \Lambda)$ in the simply connected space form $M^m(R)$ of constant curvature R. Therefore, noting that $\xi_0(:=\lim_{\varepsilon\downarrow 0}\xi_\varepsilon)$ satisfies $\Delta\xi_0=1$ on D, we see by the same arguments as above that the conclusion is true. This completes the proof of Proposition 4.1.

Corollary 4.1. *Let M be a compact Riemannian manifold without boundary. Suppose the Ricci curvature is bounded from below by $(m-1)R$ ($m=\dim M$). Then for any domain D in M with smooth boundary ∂D, we have*

$$\text{(4.6)} \qquad \mathrm{Vol}_m(D) \leqq \mathrm{Vol}_{m-1}(\partial D) \int_{\mathscr{I}_{M\setminus D}}^{d(M)} f_R^{m-1}(t)\, dt / f_R^{m-1}(\mathscr{I}_{M\setminus D}),$$

where $d(M)$ *and* $\mathscr{I}_{M\setminus D}$, *respectively, denote the diameter of* M *and the in-radius of* $M\setminus D$ (i.e., $\mathscr{I}_{M\setminus D} = \max\{\mathrm{dis}(x, \partial D) \colon x \in M\setminus D\}$), *and* f_R *is the solution of equation* (0.2) *defined by* $\mathscr{R} \equiv R$. *Moreover in the case when* $R > 0$, *equality holds in* (4.6) *if and only if one of the following cases holds*:

(1) *M is isometric to the real projective space $P^m(R)$ of constant curvature R and D is the complement of the metric ball $B(R\colon r)$ of radius r* $(0 < r < \pi/2\sqrt{R})$.

(2) *M is isometric to the (standard) sphere $S^m(R)$ of constant curvature R and D is the metric ball $B(R\colon r)$* $(0 < r < \pi/\sqrt{R})$.

Proof. Choose a point p of $M\setminus D$ such that $\mathrm{dis}(p, \partial D) = \mathscr{I}_{M\setminus D}$. For any sufficiently small $\varepsilon > 0$, set $M_\varepsilon := \{x \in M \colon \mathrm{dis}(p, x) \geqq \varepsilon\}$. Then M_ε is a Riemannian manifold of class (R, Λ_ε) $(\Lambda_\varepsilon := (m-1)(\log f_R)'(\varepsilon))$, because $\Delta\rho_p \leqq (m-1)(\log f_R)'(\varepsilon)$ on $\partial M_\varepsilon (\rho_p := \mathrm{dis}(p, *))$ (cf. Theorem 0.1). Therefore we get by Proposition 4.1.

$$\begin{aligned} \mathrm{Vol}_m(D) &\leqq \mathrm{Vol}_{m-1}(\partial D) \cdot \int_{\mathscr{I}_{M\setminus D}-\varepsilon}^{\delta_2(M_\varepsilon)-\varepsilon} h_\varepsilon^{m-1}(u)\, du / h_\varepsilon^{m-1}(\mathscr{I}_{M\setminus D} - \varepsilon) \\ &\leqq \mathrm{Vol}_{m-1}(\partial D) \cdot \int_{\mathscr{I}_{M\setminus}-\varepsilon}^{d(M)-\varepsilon} h_\varepsilon^{m-1}(u)\, du / h_\varepsilon^{m-1}(\mathscr{I}_{M\setminus D} - \varepsilon), \end{aligned}$$

where h_ε is the solution of equation (0.4) defined by $\mathscr{R} \equiv R$ and $\Lambda = \Lambda_\varepsilon$. Since the right-hand side of (4.6) is equal to

$$\mathrm{Vol}_{m-1}(\partial D) \int_{\mathscr{I}_{M\setminus D}-\varepsilon}^{d(M)-\varepsilon} h_\varepsilon^{m-1}(u)\, du / h_\varepsilon^{m-1}(\mathscr{I}_{M\setminus D} - \varepsilon),$$

we have inequality (4.6). Suppose now the equality holds in (4.6). Then the latter part of Proposition 4.1 implies that each M_ε is a model space of class (R, Λ_ε) and $D = \{x \in M \colon \mathrm{dis}(x, \partial M_\varepsilon) \geqq \mathscr{I}_{M\setminus D} - \varepsilon\} = \{x \in M \colon \rho_p(x) \geqq \mathscr{I}_{M\setminus D}\}$. This shows that one of the two cases (1) and (2) as above holds. This completes the proof of Corollary 4.1.

4.2. As an application of inequality (4.6), we shall now prove the following

Theorem 4.1. *Let m be a positive integer and let $K > 1$ and $\mathscr{V}_0 \in (0, \omega_m)$ be given constants, where ω_m denotes the volume of unit sphere in Euclidean space of dimension $m+1$. Then there exists for any number $V \in (\mathscr{V}_0, \omega_m)$ a constant $d(m, K, \mathscr{V}_0; V) \in (0, \pi)$ such that for a complete m-dimensional Riemannian manifold M whose boundary is empty and which satisfies*

$$(4.7)\qquad \begin{cases} \text{the Ricci curvature} \geqq (m-1) \\ |\text{the sectional curvature}| \leqq K \\ \text{the volume } \mathrm{Vol}_m(M) \geqq \mathscr{V}_0, \end{cases}$$

if the diameter $d(M) \geqq d(m, K, \mathscr{V}_0; V)$, *then* $\mathrm{Vol}_m(M) \geqq V$.

(4.8) **Lemma.** *Let M be a compact Riemannian manifold without boundary. For any $p \in M$, we set $l_p := \max\{\mathrm{dis}(x, p) : x \in M\}$, $i_p :=$ the distance between p and the cut locus of p, and $B_r(p) := \{x \in M : \mathrm{dis}(x, p) \leqq r\}$. Suppose the Ricci curvature of M is bounded from below by $(m-1)R$ ($R \in \boldsymbol{R}$, $m = \dim M$). Then we have*

$$(4.9)\qquad \frac{\mathrm{Vol}_m(B_r(p))}{\mathrm{Vol}_m(B_{r'}(p))} \geqq \frac{\int_{l_p-r}^{d(M)} f_R^{m-1}(t)dt}{\int_{l_p-r'}^{d(M)} f_R^{m-1}(t)dt}$$

for any $r' \in (0, \min\{r, i_p\})$. Moreover if the sectional curvature of M is bounded from above by K ($K \geqq R$), we have

$$(4.10)\qquad \mathrm{Vol}_m(B_r(p)) \geqq \omega_{m-1} \int_0^{r'} f_K^{m-1}(t)dt \frac{\int_{l_p-r}^{d(M)} f_R^{m-1}(t)dt}{\int_{l_p-r'}^{d(M)} f_R^{m-1}(t)dt}.$$

Proof. For any sufficiently small positive number ε with $\varepsilon < r'/2$, the approximation theorem by Greene and Wu [26] tells us that there exists a smooth function $\rho_\varepsilon : M \to \boldsymbol{R}$ such that

$$(4.11)\qquad \begin{cases} |\rho_\varepsilon - \rho_p| < \varepsilon & \text{on } M, \\ \|\nabla \rho_\varepsilon\| < 1 + \varepsilon & \text{on } M, \\ \|\nabla \rho_\varepsilon\| > 1 - \varepsilon & \text{on } B_{i_p-\varepsilon}(p) \backslash B_\varepsilon(p), \end{cases}$$

where $\rho_p = \mathrm{dis}(p, *)$. We may assume ρ_ε is moreover a Morse function, since every smooth function on M can be approximated in the C^∞-topology by Morse functions on M (cf. [37: p.37]). For each $t \in [-\varepsilon, l_p + \varepsilon]$, set $D(\varepsilon, t) := \{x \in M : \rho_\varepsilon(x) \leqq t\}$. Then by (4.11),

$$(4.12)\qquad B_{t-\varepsilon}(p) \subset D(\varepsilon, t) \subset B_{t+\varepsilon}(p).$$

Suppose t is a regular value of ρ_ε (i.e., $d\rho_\varepsilon \neq 0$ on $\partial D(\varepsilon, t)$). Then, noting that the inradius $l_{M\backslash D(\varepsilon,t)}$ is larger than $l_p - t - \varepsilon$, we get by inequality (4.6)

$$\text{(4.13)} \qquad \mathrm{Vol}_m(D(\varepsilon, t)) \leqq \mathrm{Vol}_{m-1}(\partial D(\varepsilon, t)) \frac{\int_{l_p-t-\varepsilon}^{d(N)} f_R^{m-1}(u)du}{f_R^{m-1}(l_p-t-\varepsilon)}.$$

On the other hand, we see by (4.11) that

$$\text{(4.14)} \qquad \mathrm{Vol}_m(D(\varepsilon, t)) \geqq \frac{1}{1+\varepsilon} \int_{D(\varepsilon,t)} \|\nabla \rho_\varepsilon\| = \frac{1}{1+\varepsilon} \int_{-\varepsilon}^{t} \mathrm{Vol}_{m-1}(\partial D(\varepsilon, u))du.$$

Therefore we have by (4.13) and (4.14)

$$\text{(4.15)} \qquad \frac{f_R^{m-1}(l_p-t-\varepsilon)}{(1+\varepsilon)\int_{l_p-t-\varepsilon}^{d(M)} f_R^{m-1}(u)du} \leqq \frac{\mathrm{Vol}_m(\partial D(\varepsilon, t))}{\int_{-\varepsilon}^{t} \mathrm{Vol}_m(\partial D(\varepsilon, u))du}.$$

Integrating the both sides of (4.15) from r' to r,

$$\text{(4.16)} \qquad \left(\frac{\int_{l_p-r-\varepsilon}^{d(M)} f_R^{m-1}(t)dt}{\int_{l_p-r'-\varepsilon}^{d(M)} f_R^{m-1}(t)dt}\right)^{1/(1+\varepsilon)} \leqq \frac{\int_{-\varepsilon}^{r} \mathrm{Vol}_{m-1}(\partial D(\varepsilon, u))du}{\int_{-\varepsilon}^{r'} \mathrm{Vol}_{m-1}(\partial D(\varepsilon, u))du} = \frac{\int_{D(\varepsilon,r)} \|\nabla \rho_\varepsilon\|}{\int_{D(\varepsilon,r')} \|\nabla \rho_\varepsilon\|} \leqq \frac{(1+\varepsilon)\,\mathrm{Vol}_m(D(\varepsilon, r))}{(1+\varepsilon)\,\mathrm{Vol}_m(B_{r'-\varepsilon}(p)\backslash B_\varepsilon(p))} \quad \text{by (4.11)}.$$

Thus we obtain inequality (4.9) by taking the limit of (4.16) as $\varepsilon \downarrow 0$. Moreover if the sectional curvature of $M \leqq K$, we see by Rauch's comparison theorem that

$$\text{(4.17)} \qquad \mathrm{Vol}_m(B_{r'}(p)) \geqq \omega_{m-1} \int_0^{r'} f_K^{m-1}(t)dt$$

and hence inequality (4.10) follows from (4.9) and (4.17). This completes the proof of Lemma (4.8).

Before the proof of Theorem 4.1, it would be interesting to give an alternative proof of the following Cheng's theorem [16] as a corollary of Lemma (4.8).

Theorem 4.2 (Cheng). *Let M be a complete Riemannian manifold* *whose boundary is empty and whose Ricci curvature is bounded from below by $(m-1)R$. Suppose $R>0$ and $d(M)=\pi/\sqrt{R}$. Then M is isometric to the standard sphere $S^m(R)$ of constant curvature R.*

Proof. We keep the notations of Lemma (4.8). Let p be a point of M such that $l_p=d(M)$. Then inequality (4.9) $(r=d(M)=\pi/\sqrt{R})$ tells us that

$$\begin{aligned}\mathrm{Vol}_m(M)&\geqq \mathrm{Vol}_m(B_{r'}(p))\frac{\int_0^{\pi/\sqrt{R}} f_R^{m-1}(t)dt}{\int_{\pi/\sqrt{R}-r'}^{\pi/\sqrt{R}} f_R^{m-1}(t)dt}\\ &=\mathrm{Vol}_m(B_{r'}(p))\frac{\int_0^{\pi/\sqrt{R}} f_R^{m-1}(t)dt}{\int_0^{r'} f_R^{m-1}(t)dt},\end{aligned}$$

since $\int_{\pi/\sqrt{R}-r'}^{\pi/\sqrt{R}} f_R^{m-1}=\int_0^{r'} f_R^{m-1}$. Therefore taking the limit as $r'\downarrow 0$, we get $\mathrm{Vol}_m(M)\geqq\omega_{m-1}\int_0^{\pi/\sqrt{R}} f_R^{m-1}=\mathrm{Vol}_m(S^m(R))$. On the other hand, Bishop's inequality (cf. [27]) shows that $\mathrm{Vol}_m(M)\leqq\mathrm{Vol}_m(S^m(R))$. Therefore we have $\mathrm{Vol}_m(M)=\mathrm{Vol}_m(S^m(R))$. This implies that M is isometric to $S^m(R)$.

Proof of Theorem 4.1. We note first that the injectivity radius of M is larger than some positive constant $I(K,\mathscr{V}_0)$ depending on K and $\mathscr{V}_0$ (cf. [12]). Let p be a point of M such that $l_p=d(M)$, where $l_p=\max\{\mathrm{dis}(p,x): x\in M\}$. Then for any $r'\in(0, I(K,\mathscr{V}_0))$, we have by inequality (4.10) with $r=d(M)$

$$\mathrm{Vol}_m(M)\geqq\omega_{m-1}\int_0^{r'}(\sin\sqrt{K}t/\sqrt{K})^{m-1}dt\cdot\frac{\int_0^{d(M)}(\sin t)^{m-1}dt}{\int_{d(M)-r'}^{d(M)}(\sin t)^{m-1}dt}. \tag{4.17}$$

We define now a continuous function $G_V(r,d)$ $(V\in(0,\omega_m))$ on $(0, I(K,\mathscr{V}_0))\times(0,\pi]$ by

$$G_V(r,d)=\frac{\omega_{m-1}\int_0^r(\sin\sqrt{K}t/\sqrt{K})^{m-1}dt\int_0^d(\sin t)^{m-1}dt}{V\int_{d-r}^d(\sin t)^{m-1}dt}.$$

Then $G_V(r,d)$ satisfies $\lim_{r\downarrow 0} G_V(r,\pi)=\omega_m/V>1$. Therefore there exist constants $r(m, K, \mathscr{V}_0; V) \in (0, I(K, \mathscr{V}_0))$ and $d(m, K, \mathscr{V}_0; V) \in (0, \pi)$ such that

$$G_V(r(m, K, \mathscr{V}_0; V), d) \geqq 1 \tag{4.18}$$

for any $d \geqq d(m, K, \mathscr{V}_0; V)$. Hence we see by (4.17) and (4.18) that if $d(M) \geqq d(m, K, \mathscr{V}_0; V)$, $\mathrm{Vol}_m(M) \geqq G_V(r(m, K, \mathscr{V}_0; V), d(M)) \times V \geqq V$. This completes the proof of Theorem 4.1.

Combining Theorem 4.1 and a sphere theorem due to Shiohama [45], we have the following

Corollary 4.2. *Let m, K and $\mathscr{V}_0$ be as in Theorem* 4.1. *Then there exists a positive constant $d(m, K, \mathscr{V}_0) \in (0, \pi)$ such that for a complete m-dimensional Riemannian manifold M whose boundary is empty and which satisfies the conditions* (4.7), *if the diameter $d(M) \geqq d(m, K, \mathscr{V}_0)$, then M is homeomorphic to a sphere.*

Combining the above corollary and Theorem B in Croke [17], we have the following

Corollary 4.3. *Let m, K and $\mathscr{V}_0$ be as in Theorem* 4.1. *Then there exists a constant $\mu(m, K, \mathscr{V}_0) > m$ such that for a complete m-dimensional Riemannian manifold M $(\partial M = \phi)$ satisfying* (4.7), *if the first non-zero eigenvalue μ_1 of the Laplacian on M is smaller than $\mu(m, K, \mathscr{V}_0)$, M is homeomorphic to a sphere.*

4.3.

Proposition 4.2. *Let M be a complete, noncompact Riemannian manifold without boundary such that the Ricci curvature is bounded from below by some nonpositive constant $(m-1)R$ $(R \leqq 0, m = \dim M)$. Let D be a compact domain in M with smooth boundary ∂D. Then*

$$\mathrm{Vol}_m(D) < \begin{cases} d(D)\,\mathrm{Vol}_{m-1}(\partial D) & \text{if } R = 0, \\ \dfrac{\exp((m-1)\sqrt{-R}\,d(D)) - 1}{(m-1)\sqrt{-R}} \cdot \mathrm{Vol}_{m-1}(\partial D) & \text{if } R < 0, \end{cases} \tag{4.19}$$

where $d(D)$ denotes the diameter of D.

Proof. Let $\gamma: [0, \infty) \to M$ be a geodesic ray and $B_\gamma: M \to \boldsymbol{R}$ the Busemann function with respect to γ. Let $\psi(R, \underline{\delta}_D, \bar{\delta}_D)(t)$ be the function defined by (3.7), where $\underline{\delta}_D = \min\{B_\gamma(x) : x \in D\}$ and $\bar{\delta}_D = \max\{B_\gamma(x) : x \in D\}$. Then

$$S(-\psi(R, \underline{\delta}_D, \bar{\delta}_D)\circ B_\gamma)\geqq 1$$

on $D_0(=D\backslash\partial D)$ (cf. (3.8)). Therefore the same argument as in the proof of Proposition 4.1 shows an inequality:

$$(4.20)\quad \mathrm{Vol}_m(D)\leqq\begin{cases}(\bar{\delta}_D-\underline{\delta}_D)\,\mathrm{Vol}_{m-1}(\partial D) & \text{if } R=0,\\ \dfrac{\exp((m-1)\sqrt{-R}(\bar{\delta}_D-\underline{\delta}_D))-1}{(m-1)\sqrt{-R}}\cdot\mathrm{Vol}_{m-1}(\partial D) & \text{if } R<0.\end{cases}$$

Noting that for any $a\in \boldsymbol{R}$, $B_\gamma(x)=a+\mathrm{dis}\,(x, B_\gamma^{-1}(a))$ on $\{x\in M: B_\gamma(x)\geqq a\}$ (cf. [49: Lemma 3.2]), we have $d(D)\geqq\bar{\delta}_D-\underline{\delta}_D$, and hence we get inequality (4.19). This completes the proof of Proposition 4.2.

Corollary (Yau [51], Calabi [11], Wu [50]). *Let M be a noncompact complete Riemannian manifold without boundary such that the Ricci curvature is nonnegative outside a compact set. Then the volume of M is infinite.*

§ 5. Function theoretic properties of noncompact Riemannian manifolds

Let M be a connected, complete and noncompact Riemannian manifold of dimension m. (In this section, we assume M has no boundary.) Let X be a smooth vector field on M and Q $(\not\equiv 0)$ a *nonnegative* smooth function on M. We write L_X for the elliptic differential operator $\Delta+X$ acting on functions. In this section, we shall show lower or upper bounds for the Green function $G_M(x, y)$ of L_X on M if it exists. Moreover we shall consider the equation:

$$(5.1)\qquad L_Xu+Q=0$$

on M and get criteria for existence or nonexistence of a positive solution of (5.1). In the last part, we shall consider the Dirichlet problem "at infinity" of M under certain conditions.

5.1. We shall first give a lower bound for $G_M(x, y)$ if it exists. Let D be a compact domain of M with smooth boundary ∂D. We write $\rho_D(x)$ for the distance between a point x and D. Let us choose a continuous function $\mathscr{R}$ on $[0, \infty)$, a continuous function η on $[0, \infty)$ and a real constant Λ such that for any distance minimizing geodesic σ: $[0, a]\to M$ from D,

$$(5.2)\qquad \begin{cases}\text{the Ricci curvature of the direction } \dot{\sigma}(t)\geqq(m-1)\mathscr{R}(t),\\ \text{the trace of } S_{\dot{\sigma}(0)}\leqq(m-1)\Lambda,\\ \langle X, \dot{\sigma}(t)\rangle\leqq\eta(t),\end{cases}$$

where $S_{\dot\sigma(0)}$ denotes the second fundamental form of ∂D with respect to $\dot\sigma(0)$. Set $T(t):=\exp\int_0^t \eta(s)ds$ and $W_r(t):=\int_t^r 1/(Th_{\mathscr{R},\Lambda}^{m-1})(s)\, ds$, where r is a positive constant and $h_{\mathscr{R},\Lambda}$ is the solution of equation (0.4) defined by $\mathscr{R}$ and Λ. Then it follows from Theorem 0.2 that

$$S_X(W_r \circ \rho_D) \geqq 0 \tag{5.3}$$

on $M \backslash D$. Suppose there exists the Green function $G_M(x, y)$ of L_X on M. Fix any interior point x_0 of D and put $c(x_0, D):=\inf\{G(x, x_0): x \in D\}$. Then by (5.3) and the maximum principle for L_X-subharmonic functions, we see that for any $x \in M \backslash D$,

$$G_M(x, x_0) \geqq \frac{c(x_0, D)}{W_r(0)} W_r \circ \rho_D(x).$$

Thus we have shown the following

Theorem 5.1. *Let M be a connected, complete and noncompact Riemannian manifold of dimension m, X a smooth vector field on M, and $D \subset M$ a compact domain with smooth boundary ∂D. Fix any interior point x_0 of D. Suppose there exists the Green function $G_M(x, y)$ of L_X on M. Then*

$$\int_0^\infty 1\bigg/\left(h_{\mathscr{R},\Lambda}^{m-1}(t) \exp\int_0^t \eta(s)ds\right)dt < +\infty \tag{5.4}$$

and

$$G_M(x, x_0) \geqq \tilde c(x_0, D) \int_{\rho_D(x)}^\infty 1\bigg/\left(h_{\mathscr{R},\Lambda}^{m-1}(t) \exp\int_0^t \eta(s)ds\right)dt$$

for any $x \in M \backslash D$, where, $\mathscr{R}$, Λ and η are as in (5.2), *$h_{\mathscr{R},\Lambda}$ is the solution of equation* (0.4) *defined by $\mathscr{R}$ and Λ, and $\tilde c(x_0, D):=\inf\{G_M(x, x_0): x \in D\} \times$ the left-hand side of* (5.4).

Remark. Theorem 5.1 implies that if the left-hand side of (5.4) is infinite, M does not possess the Green function of L_X, That is, there are no nonconstant positive L_X-superharmonic functions on M (cf. Introduction 0.7). This has been proved in [32] when $X \equiv 0$.

5.2. We shall now show a criterion for M to have no positive solutions of equation (5.1). When M possess no nonconstant positive L_X-superharmonic functions, it is clear that equation (5.1) has no positive solutions, so that we assume in this section, *there are nonconstant L_X-*

superharmonic functions on M. Let $G_M(x, y)$ be the Green function of L_X on M. Then it is easily seen that there exist positive solutions of equation (5.1) if and only if the integral $\int_M G_M(x, y)Q(y)dy$ is finite. In the following, we shall ask whether the integral $\int_M G_M(x, y)Q(y)\,dy$ is finite or not.

Suppose the integral $\int_M G_M(x, y)Q(y)dy$ is finite. Set

$$G_M Q := \int_M G_M(x, y)Q(y)dy.$$

We fix a point x_0 of M and choose continuous functions $\mathscr{R}_0$, η_0 and q_0 on $[0, \infty)$ such that for any distance minimizing geodesic $\sigma: [0, a] \to M$ from x_0,

$$(5.5)\qquad \begin{cases} \text{the Ricci curvature in direction } \dot\sigma(t) \geqq (m-1)\mathscr{R}_0(t), \\ \langle X, \dot\sigma(t)\rangle \leqq \eta_0(t), \\ Q(\sigma(t)) \geqq q_0(t). \end{cases}$$

Let f_0 be the solution of equation (0.2) defined by $\mathscr{R}_0$ and put

$$\Phi_{0,r}(t) := \int_t^r \left\{\int_0^s (q_0 f_0^{m-1} T_0)(u)\,du/(f_0^{m-1} T_0)(s)\right\} ds,$$

where r is any positive constant and $T_0(t) := \exp \int_0^t \eta_0(u)du$. Then it follows from Theorem 0.1 that

$$S_X(\Phi_{0,r} \circ \rho_0) + Q \geqq S_X(\Phi_{0,r} \circ \rho_0) + q_0 \circ \rho_0 \geqq 0$$

($\rho_0 := \mathrm{dis}\,(x_0, *)$) on M. Since $\Phi_{0,r} \circ \rho_0$ is nonpositive on $\{x \in M: \rho_0(x) \geqq r\}$, we have by the maximum principle for L_X-subharmonic functions

$$G_M Q \geqq \Phi_{0,r} \circ \rho_0$$

on M. Thus we get

$$G_M Q \geqq \Phi_0 \circ \rho_0$$

on M, where $\Phi_0(t) := \lim_{r \to +\infty} \Phi_{0,r}(t)$, that is, *we have*

$$(5.6)\qquad G_M Q(x) \geqq \int_{\rho_0(x)}^{\infty} \left\{\int_0^s (q_0 f_0^{m-1} T_0)(u)\,du(f_0^{m-1} T_0)(s)\right\} ds$$

for any $x \in M$. Inequality (5.6) implies in turn that *if the integral*

$$(5.7) \qquad \int^{\infty} \left\{ \int_0^s (q_0 f_0^{m-1} T_0)(u)\, du / (f_0^{m-1} T_0(s)) \right\} ds$$

is infinite, $G_M Q \equiv +\infty$, i.e., *M possesses no positive solutions of equation* (5.1). This assertion, for example, tells us the following

Theorem 5.2. *Let M be a complete, connected and noncompact Riemannian manifold of dimension m, X a smooth vector field on M and* $Q(\not\equiv 0)$ *a nonnegative smooth function on M. Then M has no positive solutions of equation* (5.1) *if one of the following conditions holds*: (*In the conditions below,* ρ_0 *denote the distance function to some fixed point* $x_0 \in M$, *and* α, β *and* γ *are positive constants.*)

(1) *The Ricci curvature of* $M \geqq 0$, *the length* $\|X\|$ *of* $X \leqq \beta/\rho_0$ *and* $Q \geqq \gamma/\rho_0^2 \log \rho_0$ *outside a compact set.*

(2) *The Ricci curvature of* $M \geqq -\alpha$, $\|X\| \leqq \beta$ *and* $Q \geqq \gamma/\rho_0 \log \rho_0$ *outside a compact set.*

(3) *The Ricci curvature of* $M \geqq -\alpha\rho_0^2$, $\|X\| \leqq \beta\rho_0$ *and* $Q \geqq \gamma$ *outside a compact set.*

Remark. All the conditions of the theorem are optimum (cf. Section 5.3).

Proof of Theorem 5.2. Let us prove the third assertion and omit the proofs for the others, which will be shown by the similar arguments below. Let $\mathscr{R}_0$, η_0 and q_0 be as in (5.5). By the assumptions, we may take $\mathscr{R}_0$, η_0 and q_0, respectively, to satisfy $\mathscr{R}_0(t) = -\alpha t^2$, $\eta_0(t) = \beta t$, and $q_0 \equiv \gamma$ on $[\delta, \infty)$ for some $\delta > 0$. Then the solution f_0 of equation (0.2) defined by $\mathscr{R}_0$ satisfies $(\log f_0)'(t) \leqq \varepsilon t$ on $[\delta, \infty)$ for some $\varepsilon > 0$. In fact, set $g(t) := \exp \varepsilon t^2$. Then

$$[f_0 g' - f_0' g]_\delta^t = \int_\delta^t (f_0 g' - f_0' g)'(s) ds$$
$$= \int_\delta^t \frac{2\varepsilon + 4\varepsilon^2 s^2 - \alpha s^2}{f_0(s) g(s)} ds.$$

Therefore if $\varepsilon \geqq \max\{\alpha, f_0'(\delta)/2\delta f_0(\delta)\}$, we get $(\log f_0)'(t) \leqq (\log g)'(t) = 2\varepsilon t$ on $[\delta, \infty)$. This shows that if t is sufficiently large, we have

$$\frac{\int_0^t (q_0 f_0^{m-1} T_0)(s) ds}{(f_0^{m-1} T_0)(t)} \geqq \frac{\kappa}{t}$$

$\left(T_0(t):=\exp\int_0^t \eta_0(u)du\right)$ for some $\kappa>0$. Thus the integral (5.7) is infinite and M possesses no positive solution (5.1).

By the third assertion of Theorem 5.2 and Gauss' equation, we have the following

Corollary 5.1. *Let N be a connected Riemannian manifold whose sectional curvature is bounded from below by $-\alpha(\tilde{\rho}_0^2+1)$ for some $\alpha>0$, where $\tilde{\rho}_0$ denotes the distance to a fixed point $\tilde{x}_0$ of N. Let $\iota: M\to N$ be an isommetric immersion from a complete Riemannian manifold M without boundary into N. Suppose the length of the second fundamental form of the immersion $\iota: M\to N$ is bounded from above by $\beta(\tilde{\rho}_0\circ\iota+1)$ for some $\beta>0$. Then for any smooth vector field X on M such that $\|X\|\leqq\Upsilon(\tilde{\rho}_0\circ\iota+1)$ for some $\Upsilon>0$, there are no positive solutions of equation* (5.1) ($Q\equiv 1$) *on M.*

The several conditions as above for M to possess no positive solutions of equation (5.1) have been imposed everywhere on M. However if the integral $\int_\Omega G_M(x,y)Q(y)dy$ is infinite for an open subset Ω of M, the integral $\int_M G_M(x,y)Q(y)dy$ is also infinite. Therefore it would be desirable to find conditions on an open subset $\Omega\subset M$ under which the integral $\int_\Omega G_M(x,y)Q(y)dy$ would be infinite. In the rest of this section, we shall show a criterion for the above integral to be infinite.

Let D be a compact domain of M with smooth boundary ∂D. Fix an interior point x_0 of D. Since the infimum of $G_M(x,x_0)$ as x ranges over M is zero, there is a connected component Ω of $M\backslash D$ such that

$$(5.8)\qquad \inf\{G_M(x,x_0): x\in\Omega\}=0.$$

Clearly Ω is noncompact. We shall now choose a continuous function $\mathscr{R}$ on $[0,\infty)$, a continuous function η on $[0,\infty)$ and a constant Λ which satisfy (5.2) for any distance minimizing geodesic $\sigma: [0,a]\to\Omega$ from D. Moreover let q be a nonnegative continuous function on $[0,\infty)$ such that $Q(x)\geqq q\circ\rho_D(x)$ for any $x\in\Omega$, where $\rho_D:=\mathrm{dis}\,(D,*)$. Let $h_{\mathscr{R},\Lambda}$ be the solution of equation (0.4) defined by $\mathscr{R}$ and Λ, and put

$$\Psi_r(t):=\int_t^r\left\{\int_0^s (qh_{\mathscr{R},\Lambda}^{m-1}T)(u)\,du/(h_{\mathscr{R},\Lambda}^{m-1}T)(s)\right\}ds$$

$\left(r>0,\ T(t):=\exp\int_0^t\eta(s)ds\right)$. Then by Theorem 0.2, we have

(5.9) $$S_X(\Psi_r \circ \rho_D)+q \circ \rho_D \geqq 0$$

on Ω. Let $\{M_i\}_{i=1,2,\dots}$ be an increasing family of compact domains $M_i \subset M$ with smooth boundary ∂M_i such that $M=\bigcup_{i=1}^{\infty} M_i$. We may assume that the interior of M_1 contains D. Set $\Omega_i := M_i \cap \Omega$ and fix any positive number r and a sufficiently large integer i so that $\{x \in \Omega : \rho_D(x) \leqq r\} \subset \Omega_i$. We write $\Theta_{i,r}$ for the solution of the equation:

$$\begin{cases} L_X \Theta_{i,r}+Q=0 & \text{on } \Omega_i, \\ \Theta_{i,r}=0 & \text{on } \partial\Omega_i \backslash D, \\ \Theta_{i,r}=\Psi_r(0) & \text{on } \partial D \cap \partial\Omega_i. \end{cases}$$

Then we have by (5.9)

$$\begin{cases} S_X(\Psi_r \circ \rho_D-\Theta_{i,r}) \geqq -q \circ \rho_D+Q \geqq 0 & \text{on } \Omega_i, \\ \Psi_r \circ \rho_D-\Theta_{i,r} \leqq 0 & \text{on } \partial\Omega_i \backslash D, \\ \Psi_r \circ \rho_D-\Theta_{i,r}=0 & \text{on } \partial D \cap \partial\Omega_i. \end{cases}$$

Therefore it follows from the maximum principle for L_X-subharmonic functions that

$$\Theta_{i,r} \geqq \Psi_r \circ \rho_D \qquad \text{on } \Omega_i,$$

and hence we have

(5.10) $$\nabla_{\nu_i}\Theta_{i,r} := \langle \nu_i, \nabla\Theta_{i,r}\rangle \leqq \Psi_r'(0)=0$$

on $\partial D \cap \partial\Omega_i$, where ν_i denotes the outer unit normal vector field on $\partial\Omega_i$. Let $G_i(x, y)$ be the Green function of L_X on M_i. Then we get by Green's formula and (5.10)

$$\begin{aligned} \int_{\Omega_i} G_i(x_0, y)Q(y)dy &= \int_{\Omega_i} -G_i(x_0, y)L_X\Theta_{i,r}(y)dy \\ &= \int_{\partial D \cap \partial\Omega_i} \Theta_{i,r}(y)\nabla_{\nu_i}G_i(x_0, y)dy \\ &\quad -\int_{\partial D \cap \partial\Omega_i} G_i(x_0, y)\nabla_{\nu_i}\Theta_{i,r}(y)dy \\ &\quad -\int_{\partial D \cap \partial\Omega_i} \Theta_{i,r}(y)G_i(x_0, y)\langle X, \nu_i\rangle dy \\ &\geqq \Psi_r(0)\int_{\partial D \cap \partial\Omega_i} \{\nabla_{\nu_i}G_i(x_0, y)-G_i(x_0, y)\langle X, \nu_i\rangle\}dy. \end{aligned}$$

Therefore we have

$$\text{(5.11)}\qquad \begin{aligned}&\int_{\Omega_i} G_i(x_0, y)Q(y)dy\\ &\qquad \geqq \Psi_r(0)\int_{\partial D\cap\partial\Omega_i}\{\nabla_{\nu_i}G_i(x_0, y)-G_i(x_0, y)\langle X, \nu_i\rangle\}dy.\end{aligned}$$

Let Σ_i be the solution of the equation:

$$\begin{cases} L_X\Sigma_i=0 & \text{on } \Omega_i, \\ \Sigma_i=0 & \text{on } \partial\Omega_i\backslash\partial D, \\ \Sigma_i=1 & \text{on } \partial D\cap\partial\Omega_i. \end{cases}$$

Then it follows from Green's formula again that

$$\int_{\partial D\cap\partial\Omega} G_i(x_0, y)\nabla_{\nu_i}\Sigma_i dy=\int_{\partial D\cap\partial\Omega_i}\{\nabla_{\nu_i}G_i(x_0, y)-G_i(x_0, y)\langle X, \nu_i\rangle\}dy.$$

This equality and (5.11) show that

$$\text{(5.12)}\qquad \int_{\Omega_i} G_i(x_0, y)Q(y)dy\geqq\Psi_r(0)\int_{\partial D\cap\partial\Omega_i} G_i(x_0, y)\nabla_{\nu_i}\Sigma_i dy.$$

Put $\Sigma:=\lim_{i\to\infty}\Sigma_i$. Then by (5.8), Σ is a L_X-harmonic function on Ω such that $\Sigma<1$. Therefore taking the limit of (5.12) as $i\uparrow\infty$, we obtain

$$\text{(5.13)}\qquad \int_{\Omega} G_M(x_0, y)Q(y)dy\geqq\Psi_r(0)\int_{\partial D\cap\partial\Omega} G_M(x_0, y)\nabla_\nu\Sigma dy$$

($\nu:=-\nabla\rho_D|_{\partial D}$). We note here that $-\nabla_\nu\Sigma>0$ on $\partial D\cap\partial\Omega$, since $\Sigma<1$ (cf. e.g., [40: Chap. 2, Theorem 7]). Thus, taking the limit of (5.13) as $r\uparrow\infty$, we have the following assertion: *if the integral*

$$\int^{\infty}\left\{\int_0^s (qh_{\mathscr{R},\Lambda}^{m-1}T)(u)\,du/(h_{\mathscr{R},\Lambda}^{m-1}T)(s)\right\}ds$$

is infinite, so is the integral $\int_\Omega G_M(x, y)Q(y)dy$, *and hence there are no positive solutions of equation* (5.1) *on* M. Therefore we see, for example, that *if one of the conditions* (1)$\sim$(3) *in Theorem* 5.2 *holds on* Ω *as above,* M *possesses no positive solutions of equation* (5.1).

5.3. We shall now give an upper bound for the Green function of L_X and moreover a sufficient condition for M to have positive solutions of equation (5.1) under certain assumptions. Let M be a connected, complete and noncompact Riemannian manifold of dimension m, X a smooth vector field on M and Q ($\not\equiv 0$) a nonnegative smooth function on M. Let D be a (*possibly noncompact*) domain with C^1 boundary ∂D. We write ρ_D (resp. ν_D) for the distance to D (resp. the outer unit normal vector field on ∂D). Set $\nu_D^+(\partial D) := \{t\nu_D(x) : x \in \partial D, t > 0\}$ and $\Omega := M \backslash D$. In general, it would be impossible to obtain a lower estimate of $S_X\rho_D$ everywhere on M, in contrast to an upper estimate of $S_X\rho_D$, so that, in order to get an upper estimate for the Green function or a criterion for M to have positive solutions of equation (5.1), we shall impose the following conditions on M, D, X and Q throughout this section:

(A.1)-the distance function ρ_D is of class C^2 on Ω.

(A.2)-there is a continuous function $\tau: [0, \infty) \to \boldsymbol{R}$ such that $\Delta\rho_D \geqq \tau \circ \rho_D$ on Ω.

(A.3)-there is a continuous function $\zeta: [0, \infty) \to \boldsymbol{R}$ such that $\langle X, \nabla\rho_D \rangle \geqq \zeta \circ \rho_D$ on Ω.

(A.4)-there exists a continuous function $q^*: [0, \infty) \to \boldsymbol{R}$ such that $q^* \geqq 0$ and $Q \leqq q^* \circ \rho_D$ on Ω.

We remark that if D is compact, we can always find ζ and q^* as above under the assumption (A.1).

(5.14) **Example.** Let M be a complete, simply connected Riemannian manifold with nonpositive sectional curvature. Then it is well known that for every point $o \in M$, the exponential map $\exp_0: M_0 \to M$ at o induces a diffeomorphism between M_0 and M. Therefore M and $D = B(o, r) := \{x \in M : \mathrm{dis}(o, x) \leqq r\}$ satisfy the assumption (A.1). Moreover let $\mathscr{K}_0$ be a nonpositive continuous function on $[0, \infty)$ such that for any $v \in M_0$ with $\|v\| = 1$, the sectional curvature for every tangent plane containing the vector $\dot\sigma_v(t)$ ($\sigma_v(t) := \exp_0 tv$) is bounded from above by $\mathscr{K}_0(t)$. Then by Theorem 0.3, we have

$$\Delta\rho_D \geqq (m-1)(\log f_0)' \circ (\rho_D + r)$$

on $M \backslash D$, where f_0 is the solution of equation (0.2) defined by $\mathscr{K}_0$.

(5.15) **Example.** Let M be a complete Riemannian manifold whose sectional curvature is bounded from above by some nonpositive constant K. Suppose M contains a totally convex closed subset C. (Recall that a closed subset C in a Riemannian manifold is said to be *totally convex* if for any geodesic $\sigma: [0, a] \to M$ whose ends are contained in C, $\sigma(t)$ belongs to C for every $t \in [0, a]$ (cf. [5]).) If C is in addition a domain with smooth boundary, it is known that M and C satisfy (A.1) (cf. [ibid.: Proposition 3.4 and Proposition 4.7). If C is a submanifold without boundary or $K<0$, we can find a totally convex domain D containing C and with smooth boundary ∂D, within any ε-neighborhood of C (cf. [32: the proof of Theorem (5.5)]). Therefore M and D satisfy (A.1). Moreover by Theorem 0.3, we have

$$\Delta\rho_D \geqq (m-1)(\log h_K)' \circ \rho_D$$

on Ω, where $h_K(t) := \cosh\sqrt{-K}t$.

(5.16) **Example.** Let M be a complete, noncompact Riemannian manifold whose sectional curvature is bounded from above by some negative constant K and bounded from below by some negative constant k ($k \leqq K < 0$). Let H be the universal covering of M and $\pi: H \to M$ the projection. Suppose the volume of M is finite. Then there is a compact domain $D \subset M$ with C^1 boundary ∂D such that M and D satisfy (A.1). Furthermore there is a family $\{\gamma_i\}_{i=1,2,\dots,n}$ of geodesic rays γ_i of H, which corresponds to the connected components $\{\Omega_i\}_{i=1,\dots,n}$ of $M\backslash D$, such that for each i, $\rho_D \circ \pi$ restricted to $\pi^{-1}(\Omega_i)$ is equal to $-B_{\gamma_i}$, where $\rho_D := \mathrm{dis}\,(D, *)$ and B_{γ_i} is the Busemann function associated with γ_i. (See [20] or [46] for these results.) Therefore by Theorem 0.4, we have

$$\Delta\rho_D \geqq -(m-1)\sqrt{-k}$$

on $M\backslash D$.

Now we shall show a criterion for M to have the Green function $G_M(x, y)$ of L_X and give an upper bound for $G_M(x, y)$. Suppose the assumptions (A.1)$\sim$(A.3) hold and further *the intergral*

$$\int^{\infty} \left\{1 \Big/ \exp\int_t^{\infty} (\tau+\zeta)(s)ds\right\}dt$$

is finite. We put

$$\Phi(t) := \int_t^{\infty} \left\{1 \Big/ \exp\int_0^{u} (\tau+\zeta)(s)ds\right\}du$$

and define a continuous function Ψ on M by $\Psi = \Phi \circ \rho_D$ on $M\backslash D$ and $\Psi \equiv \Phi(0)$ on D. Then the assumptions (A.1)$\sim$(A.3) imply that Ψ is of class C^2 on $M\backslash D$ and $L_X\Psi \leqq 0$ on $M\backslash D$. This shows that Ψ is a *positive* L_X*-superharmonic function on* W, *and hence* M *possesses the Green function* $G_M(x, y)$ *of* L_X. Moreover it follows from the maximum principle that

$$G_M(x, x_0) \leqq c\Phi \circ \rho_D(x)$$

for $x \in M\backslash D$, where x_0 is an interior point of D and $c := \max\{G_M(x, x_0)$:

$x \in M$, $\mathrm{dis}\,(x, x_0) = \mathrm{dis}\,(x_0, \partial D)\} \times (1/\Phi(0))$. We remark that *if D separates M, M has bounded nonconstant L_X-harmonic functions* (cf. Corollary (5.8) in [32]).

The above result and Examples (5.14) ~ (5.16) tell us, for instance, the following

Theorem 5.3. *Let M be a connected, complete and noncompact Riemannian manifold of dimension m and X a smooth vector field on M.*

(1) *Suppose M is simply connected and the sectional curvature is nonpositive. Fix a point o of M. Let ρ_0 and f_0 be as in Example* (5.14). *Then if, for some $\varepsilon > 0$,*

$$\|X\| \leqq (\log f_0^{m-1}(t)/t^{1+\varepsilon})' \circ \rho_0$$

outside a compact set, M possesses the Green function $G_M(x, y)$ of L_X which satisfies

$$G_M(x, x_0) \leqq \delta / \rho_0^{\varepsilon}$$

outside a compact set for some $\delta > 0$.

(2) *Suppose the sectional curvature of M is bounded from above by some negative constant K and M contains a totally convex closed set C. If, for some $\varepsilon > 0$,*

$$\|X\| \leqq (m-1)\sqrt{-K} - \varepsilon$$

outside a compact set, there exists on M the Green function $G_M(x, y)$ of L_X which satisfies

$$G_M(x, x_0) \leqq \delta / \exp \varepsilon \rho_C$$

outside a compact set for some $\delta > 0$, where x_0 is a fixed point of C and $\rho_C := \mathrm{dis}\,(C, *)$.

(3) *Suppose the volume of M is finite and the sectional curvature is bounded from above by some negative constant K and from below by some negative constant k ($k \leqq K < 0$). Let D be a compact domain as in Example* (5.16). *If, for some $\varepsilon > 0$,*

$$\langle X, \nabla \rho_D \rangle \geqq (m-1)\sqrt{-k} + \varepsilon$$

($\rho_D := \mathrm{dis}\,(D, *)$) *outside a compact set, M has the Green function $G_M(x, y)$ such that*

$$G_M(x, x_0) \leqq \delta / \exp \varepsilon \rho_D$$

outside a compact set for some $\delta > 0$, where x_0 is a fixed point of D.

Now we shall consider equation (5.1) under the assumptions (A.1) ~ (A.4). We keep the notations of those assumptions. In the following, let us prove that *if the integral* $\int_D Q(y)dy$ *is finite and the integral*

$$\int^{\infty}\left\{\int_0^t\left(q^*(s)\exp\int_0^s(\tau+\zeta)(u)du\right)ds\Big/\exp\int_0^t(\tau+\zeta)(u)du\right\}dt$$

is also finite, there are positive solutions of equation (5.1) *on* M, *and moreover if* D *is compact, there is a unique solution* U_Q *of equation* (5.1) *such that* $U_Q(x)$ *approaches to* 0 *as* $\rho_D(x)$ *tends to* ∞.

Proof. We first remark that M possesses the Green function $G_M(x, y)$ of L_X, because the integral $\int^{\infty}\left\{1\Big/\exp\int_0^t(\tau+\zeta)\right\}dt$ is finite. Set

$$\Psi:=\int_t^{\infty}\left\{\int_0^s q^*(u)\exp\int_0^u(\tau+\zeta)du\Big/\exp\int_0^s(\tau+\zeta)\right\}ds.$$

Let $\{M_i\}_{i=1,2,\dots}$ be an increasing family of compact domains $M_i\subset M$ such that for each i, the boundary ∂M_i is smooth and intersects transversally the boundary ∂D if the intersection $\partial M_i\cap\partial D$ is not empty, and $M=\bigcup_{i=1}^{\infty}M_i$. Now we fix a point x_0 of M. We may assume M_1 contains x_0. Let us consider the case when x_0 is not contained in D. We write $\{\Omega_{i,k}\}_{k=1,\dots,k(i)}$ for the connected components of $M_i\backslash D$ and Θ_i for the solution of equation:

$$\begin{cases}L_X\Theta_i+Q=0 & \text{on } M_i\backslash D,\\ \Theta_i=0 & \text{on } \partial M_i\backslash D,\\ \Theta_i=\Psi(0) & \text{on } \partial D\backslash \mathrm{Int}(M_i).\end{cases}$$

Then we have

$$\Theta_i\leqq\Psi\circ\rho_D \tag{5.17}$$

on $M_i\backslash D$. In fact, by the assumptions (A.1) ~ (A.4), we see that $\Psi\circ\rho_D$ is of class C^2 on $M\backslash D$ and satisfies $L_X\Psi\circ\rho_D+Q\leqq 0$ on $M\backslash D$. Therefore inequality (5.17) follows from the maximum principle for L_X-subharmonic functions. Moreover by (5.17), we get

$$\langle\nabla\rho_D, \nabla\Theta_i\rangle\leqq\Psi'(0)=0 \tag{5.18}$$

on $\partial D\cap\mathrm{Int}(M_i)$. Since x_0 is contained in M_i but not contained in D, we may assume x_0 is a point of $\Omega_{i,1}$. Then it follows from the Green's formula that

$$\int_{\Omega_{i,1}} Q(y)G_i(x_0, y)dy \quad (G_i(x, y):=\text{the Green function of } L_X \text{ on } M_i)$$
$$=\int_{\Omega_{i,1}} -L_X\Theta_i(y)G_i(x_0, y)dy$$
$$=\Theta_i(x_0)-\Psi(0)\int_{\partial D\cap\partial\Omega_{i,1}}\{\nabla_\nu G_i(x_0, y)-G_i(x_0, y)\langle X, \nu\rangle\}dy$$
$$+\int_{\partial D\cap\partial\Omega_{i,1}} G_i(x_0, y)\nabla_\nu\Theta_i dy,$$

where ν denotes the outer unit normal vector field on ∂D. Therefore we have by (5.18) and (5.19)

$$(5.20)\quad \begin{aligned}&\int_{\Omega_{i,1}} Q(y)G_i(x_0, y)dy\\ &\leqq \Psi\circ\rho_D(x_0)-\Psi(0)\int_{\partial D\cup\partial\Omega_{i,1}}\{\nabla_\nu G_i(x_0, y)-G_i(x_0, y)\langle X, \nu\rangle\}dy.\end{aligned}$$

Similarly for the other components $\{\Omega_{i,k}\}_{k=2,\dots,k(i)}$, we get

$$(5.21)\quad \begin{aligned}&\int_{\Omega_{i,k}} Q(y)G_i(x_0, y)dy\\ &-\Psi(0)\int_{\partial D\cap\partial\Omega_{i,k}}\{\nabla_\nu G_i(x_0, y)-G_i(x_0, y)\langle X, \nu\rangle\}dy.\end{aligned}$$

Noting that

$$-\Psi(0)\int_{\partial D\cap M_i}\{\nabla_\nu G_i(x_0, y)-G_i(x_0, y)\langle X, \nu\rangle\}dy$$
$$=\Psi(0)\int_{\partial M_i\cap D}\nabla_{\nu_i}G_i(x_0, y)dy\leqq 0,$$

where ν_i denotes the outer unit normal vector field on ∂M_i, we obtain by (5.20) and (5.21)

$$(5.22)\quad \int_{M_i} Q(y)G_i(x_0, y)dy\leqq\Psi\circ\rho_D(x_0)+\int_{D\cap M_i} Q(y)G_i(x_0, y)dy.$$

Thus, taking the limit of the both sides of (5.22) as $i\uparrow+\infty$, we have

$$(5.23)\quad \int_M Q(y)G_M(x_0, y)dy\leqq\Psi\circ\rho_D(x_0)+\int_D Q(y)G_M(x_0, y)dy.$$

In the case when x_0 is contained in D, the same calculations as above show us again inequality (5.23). By the assumption: $\int_D Q(y)dy<+\infty$, we see that the right-hand side of (5.23) is finite, and hence so is the left-hand side of (5.23). This implies that M possesses a positive solution $U_Q(x):=\int_M Q(y)G_M(x, y)dy$ of (5.1). Moreover if D is compact, it follows from (5.23) that $U_Q(x)$ tends to 0 as $\rho_D(x)\uparrow +\infty$. The uniqueness of such a solution is clear because of the maximum principle. This completes the assertion.

By the assertion which has just proved and Examples (5.14)~(5.16), we have the following

Theorem 5.4. *Let M be a connected, complete and noncompact Riemannian manifold of dimension m, X a smooth vector field on M and $Q(\not\equiv 0)$ a nonnegative smooth function on M.*

(1) *Suppose M is simply connected and the sectional curvature is nonpositive. We fix a point o of M. Let ρ_0 and f_0 be as in Example* (5.14). *If, for some $\varepsilon_i>0$ $(i=1, 2, 3)$,*

$$\|X\|\leqq(m-1-\varepsilon_1)(\log f_0)'\circ\rho_0,$$

and

$$Q\leqq\varepsilon_2\{(\log f_0)'(t)t^{-1-\varepsilon_3}\}\circ\rho_0$$

outside a compact set. Then there exists a unique positive solution U_Q of equation (5.1) *such that $U_Q(x)$ tends to* 0 *as $\rho_0(x)\uparrow +\infty$.*

(2) *Suppose the sectional curvature of M is bounded from above by some negative constant K and M contains a totally convex subset C. Then if, for some $\varepsilon_i>0$ $(i=1, 2, 3)$,*

$$\|X\|\leqq(m-1)\sqrt{-K}-\varepsilon_1,$$

and

$$Q\leqq\varepsilon_2/\rho_C^{1+\varepsilon_3}$$

outside a compact set and further if the integral $\int_C Q(y)dy$ is finite, M possesses positive solutions of equation (5.1). *Moreover if C is compact, there is a unique solution U_Q of equation* (5.1) *such that $U_Q(x)$ tends to* 0 *as $\rho_C(x)\uparrow +\infty$.*

(3) *Suppose the volume of M is finite and the sectional curvature of M is bounded from above by some negative constant K and from below by some negative constant k $(k\leqq K<0)$. Let D be a compact domain as in Example* (5.16). *Then if, for some $\varepsilon_i>0$ $(i=1, 2, 3)$,*

$$\langle X, \nabla\rho_D\rangle\geqq(m-1)\sqrt{-k}+\varepsilon_1,$$

and

$$Q \leqq \varepsilon_2 / \rho_D^{1+\varepsilon_3}$$

outside a compact set, or if, for some $\varepsilon_i > 0$ $(i=1, 2, 3)$,

$$\langle X, \nabla \rho_D \rangle \geqq \varepsilon_1 \rho_D^{1-\varepsilon_2}$$

and

$$Q \leqq \varepsilon_3$$

outside a compact set, there exists a unique positive solution U_Q *of equation* (5.1) *such that* $U_Q(x)$ *tends to* 0 *as* $\rho_D(x) \uparrow +\infty$.

Corollary 5.2. *Let* M *be as in the first assertion of Theorem* 5.4. *Suppose the sectional curvature of* M *is bounded from above by* $-\varepsilon_1 \rho_0^{2+\varepsilon_2}$ *and* $\|X\|$ *is bounded from above by* $\varepsilon_3 \rho_0$ *outside a compact set, where* $\varepsilon_i (i= 1, 2, 3)$ *are positive constants and* ρ_0 *denotes the distance to a fixed point* $o \in M$. *Then there is a unique solution* U_1 *of the equation*: $L_X u + 1 = 0$ *on* M *such that* $U_1(x)$ *tends* 0 *as* $\rho_0(x) \uparrow +\infty$.

Proof. Let $\mathscr{K}_0$ and f_0 be as in Example (5.14). By the assumption, we can take $\mathscr{K}_0(t) = -\varepsilon_1 t^{2+\varepsilon_2}$. Then the same calculations as in the proof of Theorem 5.2 show that $(\log f_0)'(t) \geqq \tilde{\varepsilon}_1 t^{1+\tilde{\varepsilon}_2}$ for some $\tilde{\varepsilon}_i > 0$ $(i=1, 2)$. Therefore the corollary follows from the first assertion of Theorem 5.4.

Remark. Let N be a connected compact Riemannian manifold without boundary. Let f be a smooth function on $\boldsymbol{R}$ such that $f(t) = a_1 \exp a_2 t$ for $t \leqq 0$ and $f(t) = a_3 \exp a_4 t^{2+a_5}$ for $t \geqq a_6$, where a's are all positive constants. Set $M := \boldsymbol{R} \times_f N$ (the warped product of $\boldsymbol{R}$ and N) and $D := \{(t, x) \in M : t \leqq a_6\}$. Then the assertion after Theorem 5.3 and its proof tell us that M possesses a positive solution U_1 of equation: $\Delta u + 1 = 0$ such that $U_1(t, x)$ tends to 0 as $t \uparrow +\infty$. On the other hand, since the Ricci curvature of M is bounded from below by some constnat on D, we see by Corollary 3.2 (1) that any positive solution of the above equation tends to $+\infty$ as $t \downarrow -\infty$. (See [5: pp. 26–27] for the curvature formula of warped products.)

5.4. In this section, we shall consider the Dirichlet problem "at infinity" of visibility manifolds. Let M be a complete connected Riemannian manifold of dimension m. Suppose M is simply connected and the sectional curvature is bounded from above by a negative constant K. Two geodesic rays γ_1 and γ_2 are called *equivalent* if $\operatorname{dis}(\gamma_1(t), \gamma_2(t))$ is bounded for $t \geqq 0$. The set of all equivalence classes of geodesic rays is denoted by $M(\infty)$. We assume that $\bar{M}=M\cup M(\infty)$ is equipped with the "cone topology" (i.e., a subbase for the topology is the set of open cones of geodesic rays), which makes M homeomorphic to a cell (cf. [19: Theorem 2.10]).

Let us consider the Dirichlet problem on $\bar{M}$ for the elliptic differential operator $L_X=\Delta+X$, using the Perron-Wiener-Brelot method (cf. [7: Chap. V] or [8]). The following lemma is obvious, since $\bar{M}$ is compact.

(5.24) **Lemma.** *For any L_X-superharmonic function φ, the condition*:

$$\liminf_{M\ni p\to x}\varphi(p)\geqq 0 \qquad \textit{for every } x\in M(\infty) \textit{ implies } \varphi\geqq 0.$$

Let φ be an extended real valued function on $M(\infty)$ and Σ_φ a family of lower bounded L_X-superhramonic functions ϕ such that $\liminf_{M\ni p\to x}\phi(p)\geqq\varphi(x)$ for any $x\in M(\infty)$. Then the lower envelope $\bar{D}_\varphi$ of $\Sigma_\varphi\cup\{+\infty\}$ is $+\infty$, $-\infty$ or L_X-harmonic, and $\underline{D}_\varphi\leqq\bar{D}_\varphi$, where $\underline{D}_\varphi$ is by definition $-\bar{D}_{-\varphi}$ (cf. [7: Theorem 16]). If $\bar{D}_\varphi$ is finite and $\bar{D}_\varphi=\underline{D}_\varphi$, φ is called *resolutive*. We call a point $x\in M(\infty)$ (L_X-) *regular* if for any function φ bounded above,

$$\limsup_{M\ni p\to x}\bar{D}_\varphi(p)\leqq\limsup_{M(\infty)\ni y\to x}\varphi(y)$$

(cf. [8: Sec. 18]). We see that if every point of $M(\infty)$ is regular, any continuous function φ on $M(\infty)$ is resolutive and

$$\lim_{M\ni p\to x}D_\varphi(p)=\varphi(x) \qquad (D_\varphi:=\bar{D}_\varphi)$$

for every $x\in M(\infty)$, because of Lemma (5.24) and

$$\begin{aligned}\varphi(x)=\liminf_{M(\infty)\ni y\to x}\varphi(y)&\leqq\liminf_{M\ni p\to x}\underline{D}_\varphi(p)\leqq\limsup_{M\ni p\to x}\bar{D}_\varphi(p)\\ &\geqq\limsup_{M(\infty)\ni y\to x}\varphi(y)=\varphi(x).\end{aligned}$$

In [32], we have considered the case of $L_X=\Delta$ (i.e., $X\equiv 0$) and shown that if $m=2$ or M has constant curvature outside a compact set, every point of $M(\infty)$ is regular. Let us now generalize this result.

Theorem 5.5. *Let M be a complete, simply connected Riemannian manifold of dimension m. Assume the sectional curvature is bounded from above by some negative constant K and the length $\|X\|$ of a smooth vector field X on M is bounded from above by $(m-1)\sqrt{-K}-\varepsilon$ for some positive constant $\varepsilon>0$. Suppose $m=2$, or the following conditions holds: there exist a point $o\in M$ and positive constants α, β, γ and δ such that*

$$(5.25)\quad \begin{cases} \left|\dfrac{\partial}{\partial\theta_i}\log\sqrt{G}\right| \leqq \alpha\rho^{-1-\beta}\exp 2\sqrt{-K}\rho, \\ \left|\dfrac{\partial g^{ij}}{\partial\theta_i}\right| \leqq \gamma\rho^{-1-\delta}, \end{cases}$$

where $(\rho, \theta_1, \cdots, \theta_{m-1})$ $(\rho := \mathrm{dis}\,(o, *))$ *is a polar coordinate system around* $o \in M$, $G := \det(g_{ij})$, $(g^{ij}) := (g_{ij})^{-1}$ *and* $g_{ij} := \langle \partial/\partial\theta_i, \partial/\partial\theta_j \rangle$. *Then for every point of* $M(\infty)$ *is regular, so that for any continuous function* φ *on* $M(\infty)$, *there is a unique* L_X*-harmonic function* D_φ *such that* $\lim_{M \ni p \to x} D_\varphi(p) = \varphi(x)$ *for each* $x \in M(\infty)$.

Before proving Theorem 5.5 we shall give examples of M which satisfies (5.25).

Example. Let M be a complete, symply connected Riemannian manifold whose sectional curvature is bounded from above by a negative constant K. Suppose the Riemannian metric g is *rotationally symmetric* around $o \in M$, that is, g can be written in the form:

$$g = d\rho^2 + f^2(\rho)d\Theta^2$$

in a polar coordinate system $(\rho, \theta_1, \cdots, \theta_{m-1})$ around o, where f is a smooth function on $[0, \infty)$ satisfying $f(0)=0, f'(0)=1$ and $-f''/f \leqq K$, and $d\Theta^2 := \sum_{i,j=1}^{m-1} g_{0,ij}\, d\theta_i d\theta_j$ denotes the standard metric on the unit sphere of Euclidean space $\boldsymbol{R}^m$. Then M satisfies the condition (5.25), since $g^{ij} = f^{-2}(\rho)g_0^{ij}$ and $f(t) \geqq \sinh\sqrt{-K}t/\sqrt{-K}$. Therefore another metric on M which is close enough to the above metric g in the sence of C^∞-topology satisfies all the conditions of Theorem 5.5.

Example. Let M_0 be the unit ball in $\boldsymbol{C}^n$ with Bergman metric g_0. That is, $M_0 := \{z = (z_1, \cdots, z_n) : |z| < 1\}$, $g_0 := g_{0,ij} dz_i d\bar{z}_j$ and

$$g_{0,ij} := \frac{n+1}{(1-|z|^2)}\{(1-|z|^2)\delta_{ij} + z_i \bar{z}_j\}.$$

Then it is not hard to see that M_0 satisfies (5.25). Therefore if M is a strictly pseudoconvex domain in $\boldsymbol{C}^n$ with smooth boundary which is close enough to the unit ball M_0, M with the Bergman metric satisfies all the conditions of Theorem 5.5 (cf. [24]).

Proof of Theorem 5.5. The key of the proof is to construct a "barrier" at each point $x \in M(\infty)$.

(A) Suppose the dimension of M is 2. Then every point $x \in M(\infty)$ has a fundamental neighborhood system $\mathscr{U}$ such that the complement of each $U \in \mathscr{U}$ is totally convex, because every pair of points of $M(\infty)$ can be joined by a unique geodesic line (cf. [19]) and a domain whose boundary is a geodesic line is totally convex. Moreover for each totally convex set C of M, there is a L_X-superharmonic function F_C on M such that $F_C \equiv 1$ on C and $F_C(p)$ tends to 0 as $\rho_C(p) \uparrow +\infty (\rho_C := \mathrm{dis}(C, *))$. In fact, we put $F_C \equiv 1$ on C and

$$F_C(p) := a \int_{\rho_C(p)}^{\infty} \left\{ 1 \Big/ \exp \int_0^t (\tau + \varepsilon - \sqrt{-K})(u) du \right\} dt$$

on $M \backslash C$, where $\tau(t) := \sqrt{-K} \sinh \sqrt{-K} t / \cosh \sqrt{-K} t$ and

$$a := \left(\int_0^{\infty} \left\{ 1 \Big/ \exp \int_0^t (\tau + \varepsilon - \sqrt{-K})(u) du \right\} dt \right)^{-1}.$$

Then by the assumptions, we see that F_C is L_X-superhamonic on M (cf. Example (5.15)). Therefore the theorem follows from the same arguments as in the proof of Theorem (7.3) in [32].

(B) In order to prove the theorem in the case when the metric satisfies the condition (5.25), it suffices to show that for each point $x \in M(\infty)$, there exist an open neighborhood U of $x \in M(\infty)$, and a positive L_X--superharmonic function $\mathscr{B}_x$ on $U \cap M$ such that $\mathscr{B}_x(p)$ tends to 0 as $p \to x$ and the infimum of $\mathscr{B}_x$ over the complement of any neighborhood $U' \subset U$ of x is positive (cf. [8: Theorem 15]). For the sake of brevity, we call such a function a (L_X-) *barrier at* x. In the following, let us consider the Dirichlet problem at infinity of a Riemannian manifold which satisfies more general assumptions than that of Theorem 5.5 and seek certain conditions which ensure us the existence of a barrier at each point of infinity.

(C) Let M be a connected, complete Riemannian manifold of dimension m and X a smooth vector field on M. Suppose there is a domain D with smooth boundary ∂D such that the exponential map $\exp_{\partial D}^{\perp}$ restricted to $\nu^+(\partial D) := \{t\nu_D(x) : t > 0, x \in \partial D\}$ induces a diffeomorphism between $\nu^+(\partial D)$ and $\Omega := M \backslash D$, where ν_D denotes the outer unit normal vector field on ∂D. Moreover suppose there exists a continuous function τ: $[0, \infty)$ such that the Hessian $\nabla^2 \rho$ of the distance function ρ to D satisfies

$$(5.26) \qquad (\nabla^2 \rho)_p(V, V) \geqq \tau \circ \rho(p) \| V^{\perp} \|^2$$

for any point $p \in \Omega$ and every tangent vector $V \in M_p$, where we write $V^{\perp}$ for the component of V perpendicular to $\nabla \rho$ (i.e., $V^{\perp} := V - \langle V, \nabla \rho \rangle \nabla \rho$). Let ϑ be a positive smooth function on $[0, \infty)$ such that the integral $\int_0^{\infty} 1/\vartheta(u) du$ is finite. Set $\phi(t) := \int_0^t 1/\vartheta(u) du$ ($t \in [0, \infty)$). Then a map $\mathfrak{G}$: $\Omega \to [0, \phi(\infty)) \times \partial D$ defined by $\mathfrak{G}(\exp_{\partial D}^{\perp} t\nu_D(x)) := (\phi(t), x)$ induces a diffeomorphism between Ω and $(0, \phi(\infty))\times\partial D$. We write $M(\infty)$ (resp. $\bar{M}$) for $\phi(\infty)\times\partial D$ (resp. $M\cup M(\infty)$) and assume $\bar{M}$ has the natural topology induced by $\mathfrak{G}$. Now we fix a coordinate neighborhood $\{U, \theta=(\theta_1, \cdots, \theta_{m-1})\}$ of ∂D. We may assume $\theta(U)$ contains the closed unit ball around $(0, \cdots, 0)\in \boldsymbol{R}^{m-1}$. Set $W:=\{p\in U: \sum_{i=1}^{m-1}\theta_i(p)^2\leqq 1/4\}$ and fix a point p_0 of W. Then $(s, \theta_1, \cdots, \theta_{m-1})$ $(s:=\phi(\rho))$ is a coordinate system on $\mathscr{W}:=\exp_{\partial D}^{\perp}(\{t\nu_D(x): t>0, p\in W\})$. Then the Laplace operator Δ of M can be expressed as follows:

$$\Delta=\frac{1}{\vartheta^2\circ\rho}\left[\frac{\partial^2}{\partial s^2}+\vartheta\circ\rho(\Delta\rho-(\log\vartheta)'\circ\rho)\frac{\partial}{\partial s}\right]+\Delta^{\perp},$$

where

$$\Delta^{\perp}:=\sum_{i,j=1}^{m-1}\frac{1}{\sqrt{G}}\frac{\partial}{\partial\theta_i}\left(\sqrt{G}g^{ij}\frac{\partial}{\partial\theta_j}\right),$$

$G=\det(g_{ij})$, $g_{ij}=\langle\partial/\partial\theta_i, \partial/\partial\theta_j\rangle$ and $(g^{ij})=(g_{ij})^{-1}$. For two positive constants a and b such that $a<b<\min\{1/2, \phi(\infty)\}$, we put

$$\mathscr{B}_{a,b}:=\left\{(s-\phi(\infty)-a)^2+\sum_{i=1}^{m-1}(\theta_i-\theta_i(p_0))^2\right\}^{1/2}-a$$

and

$$B_{a,b}:=\{p\in\mathscr{W}: \mathscr{B}_{a,b}(p)<a+b\}.$$

Then $\mathscr{B}_{a,b}$ is a positive smooth function on $\mathscr{W}$ such that $\mathscr{B}_{a,b}(p)$ tends to 0 as $p\in\mathscr{W}$ approaches to $p_0:=(\phi(\infty), p_0)\in M(\infty)$, and the infimum of $\mathscr{B}_{a,b}$ over the complement of any neighborhood of p_0 in $\bar{\mathscr{W}}$ is positive, where $\bar{\mathscr{W}}$ denotes the closure of $\mathscr{W}$ in $\bar{M}$. Moreover there exist positive constants ε_1 and ε_2 such that

$$\text{(5.27)}\qquad\begin{cases}\varepsilon_1<1\\ \dfrac{\partial\mathscr{B}_{a,b}}{\partial s}\leqq\varepsilon_1-1\\ \left|\dfrac{\partial^2\mathscr{B}_{a,b}}{\partial s^2}\right|\leqq\varepsilon_2\end{cases}$$

on $B_{a,b}$. Therefore if $((m-1)\tau-(\log\vartheta)'+\pi)\circ\rho\geqq 0$ on $B_{a,b}$, we have by (5.26) and (5.27)

$$\text{(5.28)}\qquad\begin{aligned}L_X\mathscr{B}_{a,b}\leqq&\frac{1}{\vartheta^2\circ\rho}[\varepsilon_2-(1-\varepsilon_1)\vartheta\circ\rho((m-1)\tau-(\log\vartheta)'+\pi)\circ\rho]\\&+\Delta^{\perp}\mathscr{B}_{a,b}+X^{\perp}\mathscr{B}_{a,b}\end{aligned}$$

on $B_{a,b}$, where π is a continuous function on $[0, \infty)$ satisfying

$$\langle X, \nabla\rho\rangle \geqq \pi \circ \rho$$

on $\mathscr{W}$. Let σ, χ and ω be continuous functions on $[0, \infty)$ which satisfy, respectively,

$$\begin{cases} \|X^{\perp}\| \leqq \sigma \circ \rho \\ \max\limits_{1\leqq i\leqq m-1} \left|\dfrac{\partial \log \sqrt{G}}{\partial \theta_i}\right| \leqq \chi \circ \rho \\ \max\limits_{1\leqq i,j\leqq m-1} \left|\dfrac{\partial g^{ij}}{\partial \theta_i}\right| \leqq \omega \circ \rho \end{cases}$$

on $\mathscr{W}$. Then there are positive constants ε_k $(k=3, 4, 5, 6)$ such that

$$|X^{\perp}\mathscr{B}_{a,b}| \leqq \varepsilon_3(\sigma T^{-1}) \circ \rho \quad \left(T(t) := \exp \int_0^t \tau(u)du\right) \tag{5.29}$$

$$\left|\sum_{i,j=1}^{m-1} g^{ij} \frac{\partial \mathscr{B}_{a,b}}{\partial \theta_i \partial \theta_j}\right| \leqq \varepsilon_4 T^{-2} \circ \rho \tag{5.30}$$

$$\left|\sum_{i,j=1}^{m-1} g^{ij} \frac{\partial \log \sqrt{G}}{\partial \theta_i} \frac{\partial \mathscr{B}_{a,b}}{\partial \theta_j}\right| \leqq \varepsilon_5(\chi T^{-2}) \circ \rho \tag{5.31}$$

$$\left|\sum_{i,j=1}^{m-1} \frac{\partial g^{ij}}{\partial \theta_i} \frac{\partial \mathscr{B}_{a,b}}{\partial \theta_i}\right| \leqq \varepsilon_6 \omega \circ \rho \tag{5.32}$$

on $B_{a,b}$. (The proof of the above inequalities (5.29) ~ (5.32) will be given at the end of the proof for Theorem 5.5). Therefore we see by (5.27) ~ ~ (5.32) that

$$L_X \mathscr{B}_{a,b} \leqq \frac{1}{\vartheta^2 \circ \rho} [\varepsilon_2 - (1-\varepsilon_1)\{\vartheta((m-1)\tau - (\log \vartheta)' + \pi)\} \circ \rho$$
$$+ \{\vartheta^2(\varepsilon_3 \sigma + \varepsilon_4 + \varepsilon_5 \chi) T^{-2}\} \circ \rho + \varepsilon_6(\vartheta^2 \omega) \circ \rho]$$

on $B_{a,b}$, and hence $L_X \mathscr{B}_{a,b} \leqq 0$ on $B_{a,b}$ for sufficiently small a and b if the following conditions hold:

$$\lim_{t\to+\infty} \Xi(t) = +\infty \quad (\Xi := \vartheta((m-1)\tau - (\log \vartheta)' + \pi)),$$

$$\limsup_{t\to+\infty} \frac{(\vartheta^2 \sigma)(t)}{(\Xi T)(t)} = 0, \tag{5.33}$$

$$\limsup_{t\to+\infty} \frac{\vartheta^2(t)}{(\Xi T^2)(t)} = 0,$$

and

$$(5.34)\qquad \begin{aligned} &\limsup_{t\to+\infty}\frac{(\vartheta^2\chi)(t)}{(\Xi T^2\omega)(t)}=0,\\ &\limsup_{t\to+\infty}\frac{(\vartheta^2\omega)(t)}{\Xi(t)}=0. \end{aligned}$$

Thus we have seen that *there is a* (L_X-) *barrier at each point* $\tilde{p}:=(\phi(\infty), p) \in M(\infty)\cap \mathscr{W}$ *under the conditions* (5.33) *and* (5.34).

(D) We shall now return Θ the proof of Theorem 5.5. We keep the notations as above. At first, we put $\vartheta(t):=(t+1)^{1+\delta_1}(0<\delta_1<\min\{\beta, \delta\})$. Moreover by the assumptions of the theorem, we can take $D:=$ a metric ball around $o \in M$,

$$\tau(t):=\sqrt{-K}\sinh\sqrt{-K}\,t/\cosh\sqrt{-K}\,t,\quad \pi(t):=\varepsilon-(m-1)\sqrt{-K},$$

$$\sigma(t):=(m-1)\sqrt{-K}-\varepsilon,$$

$$\chi(t):=\alpha t^{-1-\beta}\exp 2\sqrt{-K}\,t \quad\text{and}\quad \omega(t):=\gamma t^{-1-\delta}.$$

Then the arguments of the preceding paragraph (C) show that for each $x \in M(\infty)$, there is a (L_X-) barrier at x, that is, every point of $M(\infty)$ is regular.

(E) It remains to show the inequalities (5.29)~(5.32). Inequalities (5.32) is clear because of the choice of ω. The inequalities (5.29) and (5.31) are direct concequences of the lemma below. Moreover inequality (5.30) follows from the positive semidefiniteness of the matrix $(\partial^2\mathscr{B}_{a,b}/\partial\theta_i\partial\theta_j)$ and the following lemma again.

Lemma. *Under the assumptions of the paragraph* (C), *let* Y *be a tangent vector at* $p \in \mathscr{W}$ *such that* $\langle Y, \nabla\rho\rangle=0$ *and* f *a smooth function defined near* p. *Then*:

$$(1)\qquad \|Y\|^2=\sum_{i,j=1}^{m-1} g_{ij}Y^iY^j\geqq\kappa^2\exp\int_0^{\rho(p)}2\tau(u)du\cdot\left(\sum_{i=1}^{m-1}|Y^i|^2\right),$$

$$(2)\qquad |Y\cdot f|\leqq\kappa^{-1}\exp\int_0^{\rho(\rho)}-\tau(u)du\cdot\|Y\|\left\{\sum_{i=1}^{m-1}(\partial f/\partial\theta_i)^2\right\}^{1/2},$$

where $Y=\sum_{i=1}^{m-1}Y^i(\partial/\partial\theta_i)(p)$ *and* κ *is a positive constant independent of* p, Y *and* f.

Proof. We identify $\mathscr{W}$ with $[0, \infty)\times W$ by the coordinate system $(\rho, \theta_1, \cdots, \theta_{m-1})$. Let $c: [-\varepsilon, \varepsilon]\to W_{\rho(p)}:=\rho(p)\times W$ be a smooth curve such that $c(0)=p$ and $\dot{c}(0)=Y$. Define a smooth map $\mathscr{F}: [0, \infty)\times [-\varepsilon, \varepsilon]\to\mathscr{W}$ by $\mathscr{F}(\rho, u)=(\rho, \theta_1\circ c(u), \cdots, \theta_{m-1}\circ c(u))$. Set $\tilde{Y}:=\mathscr{F}_*(\partial/\partial u)$. Then we have

$$\begin{aligned}\frac{\partial}{\partial\rho}\langle\tilde{Y},\tilde{Y}\rangle &= 2\langle\nabla_{\partial/\partial\rho}\tilde{Y},\tilde{Y}\rangle \\ &= 2\langle\nabla_{\tilde{Y}}\nabla\rho,\tilde{Y}\rangle \\ &= 2\nabla^2\rho(\tilde{Y},\tilde{Y}) \\ &\geqq 2\tau\circ\rho\cdot\|Y\|^2 \qquad \text{by (5.26).}\end{aligned}$$

Therefore we get

$$\frac{d}{d\rho}(\log\|Y\|(\rho,0))\geqq\tau\circ\rho$$

and hence, integrating the both sides, we have

$$\begin{aligned}\|Y\|^2(\rho(p),0) &\geqq \|\tilde{Y}\|^2(0,0)\exp\int_0^{\rho(p)}2\tau(u)du \\ &\geqq \kappa^2\left(\sum_{i=1}^{m-1}|Y^i|^2\right)\exp\int_0^{\rho(p)}2\tau(u)du\end{aligned}$$

for some $\kappa>0$. This proves the first assertion, from which the second assertion follows. In fact,

$$\begin{aligned}|Y\cdot f| &= \left|\sum_{i=1}^{m-1}Y^i\frac{\partial f}{\partial\theta_i}\right| \\ &\leqq \left\{\sum_{i=1}^{m-1}|Y^i|^2\right\}^{1/2}\left\{\sum_{i=1}^{m-1}\left(\frac{\partial f}{\partial\theta_i}\right)^2\right\}^{1/2} \\ &\leqq \kappa^{-1}\exp\int_0^{\rho(p)}-\tau(u)du\,\|Y\|\left\{\sum_{i=1}^{m-1}\left(\frac{\partial f}{\partial\theta_i}\right)^2\right\}^{1/2}.\end{aligned}$$

Before we state a corollary to Theorem 5.5, we recall some definitions in [19]. Let Γ be a freely acting, properly discontinuous group of isometries of a complete, simply connected Riemannian manifold M whose curvature is bounded from above by a negative constant K. We write M/Γ for the quotient manifold of M by Γ. A unit speed geodesic $\gamma(t)$ $t\geqq 0$) in M/Γ is called an *almost minimizing geodesic* if there is a positive number c such that $\mathrm{dis}(\gamma(0),\gamma(t))\geqq t-c$ for $t\geqq 0$. Two unit speed geodesics γ_1 and γ_2 in M/Γ are called *equivalent* if $\mathrm{dis}(\gamma_1(t),\gamma_2(t))$ is bounded for $t\geqq 0$. The set of all equivalence classes of almost minimizing geodesics in M/Γ is denoted by $M/\Gamma(\infty)$. Let γ be an almost minimizing geodesic in M/Γ and $\tilde{\gamma}$ a lift of γ in M. If $\tilde{\gamma}$ represents an equivalence class in $M(\infty)$-$L(\Gamma)$, where $L(\Gamma)$ is the cone limit set of Γ, γ represents, by definition, a class of $F(M/\Gamma)$. We assume that $\overline{M/\Gamma}:=M/\Gamma\cup M/\Gamma(\infty)$ is equipped with the topology induced from the cone topology and the "horocycle topology" (i.e., a subbase of the neighborhoods of a point $x \in M(\infty)$ with respect to the topology is the set of all limit balls at x) on M. Then the covering map $\pi\colon M \to M/\Gamma$ extends naturally to the covering map also denoted by π, from $M \cup O(\Gamma)$ onto $M/\Gamma \cup F(M/\Gamma)$ and the restriction map $\pi\colon O(\Gamma) \to F(M/\Gamma)$ is again a covering map, where $O(\Gamma) = M(\infty) - L(\Gamma)$. Then by the same arguments as in the proof of Theorem 5.5, we have the following

Corollary. *Let M be a Riemannian manifold which satisfies all the conditions of Theorem* 5.5. *Let Γ be a freely acting, properly discontinuous group of isometries of M. Suppose the length of a smooth vector field X on M/Γ is bounded from above by $(m-1) \times \sqrt{-K} - \varepsilon$ for some positive constant ε and $\overline{M/\Gamma}$ is compact. Then there is for any continuous function φ on $M/\Gamma(\infty)$ an L_X-harmonic function D_φ on M/Γ such that*

$$\lim_{M/\Gamma \ni p \to x} D_\varphi(p) = \varphi(x)$$

for any $x \in F(M/\Gamma)$.

We remark that $\overline{M/\Gamma}$ is compact, for example, if M/Γ is corecompact, that is, M/Γ contains a compact totally convex set, or if the dimension of M is equal to 2 and Γ is finitely generated (cf. [19]).

We shall conclude this section with the following

Remark. (1) Let M be a complete, connected and noncompact Riemannian manifold and X a smooth vector field on M. Let $(\mathscr{X}_t, \zeta, P_x, x \in M)$ be the minimal diffusion process on M with the differential generator $L_X := \Delta + X$, where ζ is the explosion time of $\mathscr{X}_t(\omega)$. If there is a positive solution U of the equation: $L_X U + 1 = 0$ on M, it follows from the Dynkin's formula that $U(x) \geqq E_x[\zeta]$ for any $x \in M$ (cf. e.g., [21]: Proposition 8B]), and hence ζ is finite almost surely, for every starting point $x \in M$. For example, if M and X are as in Corollary 5.2, we see that ζ is finite almost surely (cf. [30, II] in the case when $X \equiv 0$). On the other hand, if M and X satisfy, for instance, the condition (3) of Theorem 5.2, it turns out from the proof of the theorem and the approximation theorem due to Greene and Wu [26] (cf. the proof of Proposition 4.1 in Section 4) that there is a smooth function $\Phi\colon M \to [0, \infty)$ such that $\Phi(x) \to +\infty$ as $x \to +\infty$ in M and $L_X\, \Phi \leqq \alpha$ on M, for some constant α, and hence we see by Theorem 6A in [21] that ζ is infinite almost surely for every starting point $x \in M$ (cf. [30, II] in the case when $X \equiv 0$).

(2) Let M be a complete, simply connected Riemannian manifold of negative curvature. Recently, Sasaki [44] has proved that if the sectional curvature is "asymptotically negative constant", the Dirichlet prob-

lem for harmonic functions can be solved on $\bar{M}=M\cup M(\infty)$. However both his condition and ours in Theorem 5.5 seem to be very restrictive, and it would be wishful to solve the Dirichlet problem on $\bar{M}$ under a weaker condition. Moreover it would be interesting to describe the Martin boundary of M from a view point of geometry.

Added in proof. After the completion of this paper, the author recieved a preprint [52] from M. T. Anderson on May 7, 1983. In his paper, it is proved that a complete, simply connected Riemannian manifold whose sectional curvature is pinched by negative constants admits a wealth of global convex sets so that the Dirichlet problems for the Laplacian can be solved at infinity (cf. Theorem 5.5). The author would like to thank M. T. Anderson for sending him his preprint.

References

[1] K. Aomoto, L'analyse harmonique sur les escapes riemanniens à courbure riemannienne négative I, J. Fac. Sci. Univ. Tokyo (1966), 85–105.
[2] C. Bandle, Isoperimetric inequalities and applications, Monographs and Studies in Mathematics 7, Pitman, Boston-London-Melbourne, 1980.
[3] J. Barta, Sur la vibration fondamental d'une menbrane, Comptes Rendus de l'Acad. des Sci., Paris, **204** (1937), 472–473.
[4] M. Berger, An extension of Rauch's metric comparison theorem and some applications, Illinois J. Math., **6** (1962), 700–712.
[5] R. L. Bishop and B. O'Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc., **145** (1969), 1–49.
[6] E. Bombieri, E. de Giorgi, and E. Giusti, Minimal cones and the Bernstein problem, Invent. math., **7** (1969), 243–268.
[7] M. Brelot, Lectures on Potential Theorey, Lecture Notes No. 19, Tata Institute, 1960.
[8] M. Brelot, Le problèm de Dirichlet. Axiomatique et frontière de Martin, J. de Math., **35** (1956), 297–335.
[9] Yu. D. Burago and V. A. Zalgaller, Convex sets in Riemannian spaces of non-negative curvature, Uspekhi Mat. Nauk, **32** (1977), 3–55: Russian Math. Surveys **32** (1977), 1–57.
[10] E. Calabi, An extension of E. Hopf's maximum principle with an application to Riemannian geometry, Duke Math. J., **25** (1958), 45–56.
[11] E. Calabi, On manifolds with nonnegative Ricci curvature II, Abstract No. 720–53–6, Notices Amer. Math. Soc., A-**205**.
[12] J. Cheeger, Finiteness theorems for Riemannian manifolds, Amer. J. Math., **92** (1970), 61–74.
[13] J. Cheeger and D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature, J. Differential Geom., **6** (1971), 119–128.
[14] J. Cheeger and D. Gromoll, On the structure of complete manifolds of non-negative curvature, Ann. of Math., **96** (1972), 413–443.
[15] J. Cheeger and S. T. Yau, A lower bound for the heat kernel, Comm. Pure Appl. Math., **34** (1981), 465–480.
[16] S. Y. Cheng, Eigenvalue comparison theorem and its applications, Math. Z. **143** (1975), 289–297.
[17] C. B. Croke, An Eigenvalue Pinching theorem, Invent. math., **68** (1982), 253–256.

[18] A. Debiard, B. Gaveau and E. Mazet, Theorems de comparison en geometrie Riemannienne, Publ. RIMS. Kyoto Univ., **12** (1976), 391–425.
[19] P. Eberlein and B. O'Neill, Visibility manifolds, Pacific J. Math., **46** (1973), 45–109.
[20] P. Eberlein, Lattices in spaces of nonpositive curvature, Ann. of Math., **111** (1980), 435–476.
[21] K. D. Elworthy, Stochastic Differential Equations on Manifolds, London Math. Soc. Lecture Note Series 70, Cambridge U.P., Cambridge, 1982.
[22] S. Friedland and W. K. Hayman, Eigenvalue inequalities for the Dirichle problem on spheres and the growth of subharmonic functions, Comment Math. Helv., **51** (1976), 133–161.
[23] S. Gallot, Minorations sur le λ_1 des varietes reimannienne, Seminaire Bourbaki 33° année, 1980/81, N. 569.
[24] R. E. Greene and S. G. Krantz, Stability of the Bergman Kernel and curvature properties of bounded domains, in "Recent Developments in Several Complex Variables", Ann. of Math., Studies, **100**, Princeton Univ. Press. Princeton, N. J., 1981.
[25] R. E. Greene and H. Wu, Function Theory on Manifolds which Possess a Pole, Lecture Notes in Math., **699**, Springer-Verlarg, 1979.
[26] R. E. Greene and H. Wu, C^∞ approximations of convex, subharmonic, and plurisubharmonic functions, Ann. scient. Éc. Norm. Sup. Paris, **12** (1979), 47–84.
[27] E. Heintze and H. Karcher, A general comparison theorem with applications to volume estimates for submanifolds, Ann. Sci. École Norm. Sup. Paris. **11** (1978), 451–470.
[28] A. Huber, On subharmonic functions and differential geometry in the large, Comment. Math. Helv., **32** (1957), 13–72.
[29] R. Ichida, Riemannian manifolds with compact boundary, Yokohama Math. J., **29** (1981), 169–177.
[30] K. Ichihara, Curvature, geodesics and the Brownian motion on a Riemannian manifold I, Nagoya Math. J. **87** (1982), 101–114; II, ibid., 115–125.
[31] S. Itô, On existence of Green function and positive superharmonic functions for linear elliptic operators of second order, J. Math. Soc. Japan, **16** (1964), 299–306.
[32] A. Kasue, A Laplacian comparison theorem and function theoretic properties of a complete Riemannian manifold, Japanese J. Math., **18** (1982).
[33] A. Kasue, Ricci curvature, geodesics and some geometric properties of Riemannian manifolds with boundary, J. Math. Soc. Japan, **35** (1982).
[34] A. Kasue, On a lower bound for the first eigenvalue of the Laplace operator on a Riemannian manifold, to appear in Ann. scient. École Norm. Sup. Paris.
[35] M. Kishi, Potential Theory (in Japanese), Morikita, Tokyo, 1974.
[36] P. Li and S. T. Yau, Estimates of eigenvalue of a compact Riemannian manifold, Proc. Symp. Pure Math. Amer. Math. Soc., **36** (1980), 205–239.
[37] J. Milnor, Morse Theory, Ann. of Math. Studies, **51**, Princeton U.P. Princeton, 1969.
[38] R. Osserman, On the inequality $\Delta u \geqq f(u)$, Pacific J. Math., **7** (1957), 1641–1647.
[39] G. Pólya and G. Szegö, Isoperimetric Inequalities in Mathematical Physics, Ann. of Math. Studies, **27**, Princeton U.P., Princeton, 1951.
[40] M. H. Protter and H. F. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall, Englewood Cliffs, 1967.
[41] H. Rauch, A contribution to differential geometry in the large, Ann. of Math., **54** (1951), 38–55.
[42] R. C. Reilly, Applications of the Hessian operator in a Riemannian manifold, Indiana Univ. Math. J., **26** (1977), 459–472.

[43] L. Sario, et al., Classification Theory of Riemannian Manifolds, Lecture Notes in Math., **605**, Springer Verlag, 1977.
[44] T. Sasaki, On the Green function of a complete Riemannian or Kähler manifold with asymptotically negative constant curvature and applications, this proceeding, 387–421.
[45] K. Shiohama, A sphere theorem for manifolds of positive Ricci curvature, Trans. Amer. Math. Soc., **275** (1983), 811–819.
[46] Y. T. Siu and S. T. Yau, Compactification of negatively curved complete Kähler manifolds of finite volume, in "Seminars on Differential Geometry", Annals of Math. Studies 102, Princeton U.P. 1982.
[47] R. P. Sperb, Isoperimetric inequalities in a boundary value problem on a Riemannian manifold, J. Applied Mathematics and Physics (ZAMP), **31** (1980), 740–753.
[48] F. W. Warner, Extension of the Rauch comparison theorem to submanifolds, Trans. Amer. Math. Soc., **122** (1966), 341–356.
[49] H. Wu, An elementary method in the study of nonnegative curvature, Acta Math., **142** (1979), 57–78.
[50] H. Wu, On the volume of a noncompact manifold, Duke Math. J., **49** (1982), 71–78.
[51] S. T. Yau, Some function theoretic properties of complete Riemannian manifolds and their applications to geometry, Indiana Univ. Math. J., **25** (1976), 659–670.
[52] M. T. Anderson, The Dirichlet problem at infinity for manifolds of negative curvature, preprint.

Department of Mathematics
Osaka University
Toyonaka, 560
Japan