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Some Morse Theoretic Aspects of Holomorphic Vector Bundles

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§ Introduction

In this paper we shall consider certain theory of stationary points or loci arising from families of holomorphic sections of holomorphic vector bundles. The notion of Morse functions will be generalized to families of holomorphic sections, called quasilinear sections or holomorphic sections in quasilinear position. One of our results shows that this kind of sections exist generically in certain cases. We shall define a particular subset of Schubert cycles, called stationary loci, associated to quasilinear holomorphic sections. We are also concerned with a relation between these loci and characteristic classes. As Morse functions give us some information of topology of differentiable manifolds, it will turn out that our loci tell us some complex analytic structure of complex manifolds.

Let M be a compact complex manifold and let $E \to M$ be a holomorphic vector bundle of rank q. We denote by $\oplus^r \Gamma(M, \mathcal{O}(E))$ the set of all the families of holomorphic r-sections $\{\sigma_1, \cdots, \sigma_r\}$ of $E \to M$, $r \leq q$. Topology of the set $\oplus^r \Gamma(M, \mathcal{O}(E))$ is naturally defined taking into consideration higher order differentials. For the definition of quasilinear sections, see Section 1. Our generic existence theorem is stated as follows.

Generic existence Theorem. *Let M be a compact complex manifold and let $E \to M$ be a holomorphic vector bundle of rank q such that each fibre is generated by global holomorphic sections. Then, for any integer $r \leq q$, the set of all families of holomorphic sections $\{\sigma_1, \cdots, \sigma_r\}$ in quasilinear position forms an open and dense subset in $\oplus^r \Gamma(M, \mathcal{O}(E))$.*

Let $\sigma_1, \cdots, \sigma_r$ be holomorphic sections of $E \to M$. The Schubert cycle denoted by $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$ is defined to be the subset of M consisting of points where $\sigma_1, \cdots, \sigma_r$ fail to be linearly independent (see § 2). If $\sigma_1, \cdots, \sigma_r$ are in quasilinear position, then it follows that the Schubert cycle $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$ has only singularities of quasilinear type (see Def. 1.1).

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Among Schubert cycles, we shall define quasilinear bordism. This bordism is certain complex analytic bordism admitting only singularities of quasilinear type. See for the precise definition, Section 4. We shall show the following bordism theorem.

Quasilinear bordism Theorem. *Let M, $E\to M$ as in the generic existence theorem. Let $\{\sigma_1, \cdots, \sigma_r\}$ and $\{s_1, \cdots, s_r\}$ be two sets of holomorphic sections of $E\to M$ in quasilinear position. Then, their associated Schubert cycles $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$ and $\mathscr{S}(s_1, \cdots, s_r)$ are quasilinearly bordant in M.*

Our third theorem is concerned with certain stationary loci associated to quasilinear holomorphic sections. Let $\sigma_1, \cdots, \sigma_r$ be quasilinear holomorphic sections of $E\to M$. The stationary locus denoted by $\Sigma(\sigma_1, \cdots, \sigma_r)$ consists of those points z in $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$ at which Plücker coordinates of planes generated by $\sigma_1(z), \cdots, \sigma_r(z)$ vary stationarily as z runs over $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$.

Stationary loci represent certain invariants of the structure of the base manifold and the bundle $E\to M$. We shall exhibit a complete calculation of their associated cohomology classes by polynomials in terms of Chern classes of M and $E\to M$ in case that the rank of the bundle is equal to the dimension of M. The result is as follows. For the definitions of cohomology class $\{\{\Sigma(\sigma_1, \cdots, \sigma_r)\}\}$ associated to the stationary locus $\Sigma(\sigma_1, \cdots, \sigma_r)$ and polynomials $R_r(M, E)$, see Section 7 and Section 9 respectively.

Stationary locus Theorem. *Let M be a compact complex manifold of dimension n and let $E\to M$ be a holomorphic vector bundle of rank n such that each fibre is generated by global holomorphic sections. Let $\{\sigma_1, \cdots, \sigma_r\}$ $(r\leq n)$ be holomorphic sections of $E\to M$ in quasilinear position. Then, the associated cohomology class of the stationary locus $\Sigma(\sigma_1, \cdots, \sigma_r)$ coincides with $R_r(M, E)$;*

$$\{\{\Sigma(\sigma_1, \cdots, \sigma_r)\}\}=R_r(M, E) \qquad \textit{in } H^{2(n-r+2)}(M, \boldsymbol{Z})$$

provided that the dimension of $\Sigma(\sigma_1, \cdots, \sigma_r)$ is strictly lower than that of $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$.

In particular, in case that M is a complex analytic surface, we see that the polynomials $R_r(M, E)$ will turn out to be Euler characteristic, arithmetic genus and so on, as follows;

$$R_2(M, TM)=2\chi(M)$$
$$R_2(M, T^*M)=24\sum_k(-1)^k \dim (H^k(M, \mathcal{O}(M))).$$

The stationary locus formula holds also for open complex manifolds. In open cases, one can show that there exists at least one set of holomorphic sections $\{\sigma_1, \cdots, \sigma_r\}$ in quasilinear position for each r, if the bundle has global holomorphic sections generating each fibre.

Section 1 and Section 2 contain definitions and basic results about quasilinearity. Section 3 is devoted to the proof of the generic existence theorem. Section 4 contains the proof of the quasilinear bordism theorem. In Section 6 we shall give an example of quasilinear bordism invariants. Section 7 contains precise definitions of stationary loci and their associated cohomology classes. Sections 8, 10 and 11 are devoted to the proof of the stationary locus theorem.

§ 1. Definition of quasilinear structure

In this section we shall recall some notions concerning quasilinearity. Quasilinear subvarieties are characterized by the type of their singularities. The quasilinear structure of singularities is modeled on some corns in the space of complex matrices. We shall denote by $\mathfrak{M}(r, s)$ the set of all $r \times s$ complex matrices. We define

$$\mathfrak{M}_k(r, s) = \{A \in \mathfrak{M}(r, s);\ \text{corank}\ (A) \geq k\},$$

where $r \leq s$, $k = 1, \cdots, r$. Notice that $\mathfrak{M}_k(r, s) \supset \mathfrak{M}_{k+1}(r, s)$ and $\mathfrak{M}_r(r, s) = O_{r,s}$ (=zero-matrix). Consider a sequence of subvarieties of $\mathfrak{M}(r, s) \times \boldsymbol{C}^t$ for an integer t,

$$\mathfrak{M}_1(r, s) \times \boldsymbol{C}^t \supset \mathfrak{M}_2(r, s) \times \boldsymbol{C}^t \supset \cdots \supset O_{r,s} \times \boldsymbol{C}^t.$$

This sequence gives a regular stratification, and the non-singular submanifold $O_{r,s} \times \boldsymbol{C}^t$ can be regarded as a center. Our model is the structure of stratification of this sequence in the neighbourhood of this center. And the quasilinearity of subvarieties is defined as follows.

Definition 1.1. A complex analytic subvariety V of a complex manifold M is said to be quasilinear if it has a regular stratification

$$V_1 - V_2,\ V_2 - V_3,\ \cdots,\ V_N,$$

where $V_1 = V$ and V_i is the set of singular points of V_{i-1}, such that for any point z of any stratum $V_i - V_{i+1}$, there exist some integers r, s, t and a biholomorphic map φ of a neighbourhood U at z in M onto a neighbourhood W at $(O_{r,s}, O_t)$ in $\mathfrak{M}(r, s) \times \boldsymbol{C}^t$ such that

$$\varphi(U \cap V_k) = W \cap (\mathfrak{M}_k(r, s) \times \boldsymbol{C}^t), \qquad k = 1, \cdots, i.$$

If V is a quasilinear subvariety of M, we also say that V has only singularities of quasilinear type.

For the structure of singularities of $\mathfrak{M}_k(r, q)$, we can show the following.

Lemma 1.2. *For any integer k, $1 \leq k \leq r$, the subvariety $\mathfrak{M}_k(r, q)$ is quasilinear in $\mathfrak{M}(r, q)$.*

Although $\mathfrak{M}_k(r, q)$ is quasilinear in the neighbourhood of the zero-matrix, it is not trivial that all the singularities of $\mathfrak{M}_k(r, q)$ are of quasilinear type. For the proof of the above lemma, we refer to H. Morimoto [8].

§ 2. Schubert cycles and quasilinear position

Let $\{\sigma_1, \cdots, \sigma_r\}$ be holomorphic sections of a holomorphic vector bundle $E \to M$. We define the associated Schubert cycle $\mathscr{S} = \mathscr{S}(\sigma_1, \cdots, \sigma_r)$ by setting

$$\mathscr{S}(\sigma_1, \cdots, \sigma_r) = \{z \in M;\ \sigma_1(z) \wedge \cdots \wedge \sigma_r(z) = 0\}.$$

Let $\sigma_1, \cdots, \sigma_r$ be represented as

$$\sigma_i(z) = \sum_{j=1}^{q} \alpha_{ij}(z) e_j(z), \quad q = \operatorname{rank}(E),$$

for some local frame $\{e_j\}$ on a small open subset U. And let $\Phi_U(z) = (\alpha_{ij}(z))$ be the holomorphic map of U into $\mathfrak{M}(r, q)$.

Definition 2.1. Holomorphic sections $\{\sigma_1, \cdots, \sigma_r\}$ are said to be in quasilinear position if at any point $z \in M$, there is a neighbourhood U such that the map Φ_U is transversal to any stratum of $\mathfrak{M}_1(r, q)$ (i.e., $\mathfrak{M}_k(r, q) - \mathfrak{M}_{k+1}(r, q)$, $1 \leq k \leq r$) at any point of U.

Note that $\Phi_U(z) \in \mathfrak{M}_1(r, q)$ if and only if $z \in \mathscr{S}(\sigma_1, \cdots, \sigma_r)$. Recalling that $\mathfrak{M}_1(r, q)$ is quasilinear in $\mathfrak{M}(r, q)$, we have the following lemma.

Lemma 2.2. *If holomorphic sections $\{\sigma_1, \cdots, \sigma_r\}$ are in quasilinear position, then the Schubert cycle $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$ has only singularities of quasilinear type.*

§ 3. Proof of the generic existence theorem

This section is devoted to the proof of the generic existence theorem. For holomorphic sections $\{\sigma_1, \cdots, \sigma_r\}$ of the bundle $E \to M$, we define subvarieties $S_k(\sigma_1, \cdots, \sigma_r) \subset M$, $k = 1, \cdots, r$ by setting,

$$S_k(\sigma_1, \cdots, \sigma_r)=\{z \in M; \Phi_U(z) \in \mathfrak{M}_k(r, q)\},$$

where U is some neighbourhood of z and the map Φ_U: $U \to \mathfrak{M}(r, q)$ is the associated map defined in Section 2. Notice that $S_1(\sigma_1, \cdots, \sigma_r)$ coincides with the Schubert cycle $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$. We define some order of quasilinear position as follows.

Definition 3.1. Holomorphic sections $\{\sigma_1, \cdots, \sigma_r\}$ are said to be in quasilinear position of order k if for any point z_0 in M there is some neighbourhood U at z_0 such that the map $\Phi_U: U \to \mathfrak{M}(r, q)$ is transversal on U to any stratum of $\mathfrak{M}_{r-k+1}(r, q)$.

In particular, quasilinear position of order r is equivalent to the quasilinear position defined in 2.1.

From the quasilinear structure of $\mathfrak{M}_k(r, q)$, the following lemma follows easily.

Lemma 3.2. *If holomorphic sections $\{\sigma_1, \cdots, \sigma_r\}$ are in quasilinear position of order k at $z_0 \in S_{r-k+1}(\sigma_1, \cdots, \sigma_r)-S_{r-k+2}(\sigma_1, \cdots, \sigma_r)$, then $\{\sigma_1, \cdots, \sigma_r\}$ are already in quasilinear position of order r on some neighbourhood of z_0 in M.*

In general, if $V=V_1 \supset \cdots \supset V_r$ is quasilinear, then the transversality to V_l-V_{l+1} implies the transversality to V_s-V_{s+1}, for any s, $1 \leq s \leq l$ on some small neighbourhood of V. Especially, the quasilinearity of $\mathfrak{M}_k(r, q)$ gives the above lemma.

From the same reason, we have;

Lemma 3.3. *If holomorphic sections $\{\sigma_1, \cdots, \sigma_r\}$ are in quasilinear position of order k at $z_0 \in S_{r-k+1}(\sigma_1, \cdots, \sigma_r)-S_{r-k+2}(\sigma_1, \cdots, \sigma_r)$, then there exist $\delta>0$ and some neighbourhood U of z_0 in M such that if holomorphic sections $s_1, \cdots, s_r$ satisfies $\|s_i-\sigma_i\|_U<\delta$, then $\{s_1, \cdots, s_r\}$ are in quasilinear position of order r.*

From previous two lemmas, we show the stability of quasilinear position of order k.

Proposition 3.4. *Let M be a compact complex manifold and $E \to M$ a holomorphic vector bundle of rank q. Then the set of holomorphic sections $\{\sigma_1, \cdots, \sigma_r\}$ (r fixed) which are in quasilinear position of order k, $1 \leq k \leq r$ forms an open subset in the set of all holomorphic r-sections $\oplus^r \Gamma(M, \mathcal{O}(E))$.*

Proof. If $z \in S_{r-k+1}(\sigma_1, \cdots, \sigma_r)$, then z is contained in some stratum S_l-S_{l+1}, $r-k+1 \leq l \leq r$. From Lemma 4.3, there are some neighbourhood U_z and $\delta_z>0$ such that if $\|s_i-\sigma_i\|_{S_l}<\delta_z$ then $\{s_i\}$ are in quasilinear position of order s, $s \geq l$. Therefore the lemma follows from the compactness of $S_{r-k+1}(\sigma_1, \cdots, \sigma_r)$. Q.E.D.

Now we proceed to prove the density of holomorphic sections in quasilinear position. Throughout the remainder of this section, we assume that the bundle $E \to M$ has global sections $\tau_1, \cdots, \tau_N$ which generate each fibre. We begin with showing the existence of particular neighbourhoods.

Lemma 3.5. *Fix $z_0 \in M$ and an integer $k \leq r-1$. Suppose that holomorphic k-sections $\{u_1, \cdots, u_k\}$ are linearly independent at z_0. Then there exists a compact neighbourhood of z_0 in M which we shall denote by $U(z_0; u_1, \cdots, u_k)$ such that for any global holomorphic sections $\{s_1, \cdots, s_{r-k}\}$ and any $\varepsilon > 0$ there exist some global holomorphic sections $\bar{s}_1, \cdots, \bar{s}_{r-k}$ such that* (a) *$\|s_i - \bar{s}_i\|_M < \varepsilon$ and* (b) *holomorphic sections $\{u_1, \cdots, u_k, \bar{s}_1, \cdots, \bar{s}_{r-k}\}$ are in quasilinear position of order $k+1$ on $U(z_0; u_1, \cdots, u_k)$.*

Proof. From the assumption we can take $\tau_{t_1}, \cdots, \tau_{t_{q-k}}$ so that holomorphic sections $u_1, \cdots, u_k, \tau_{t_1}, \cdots, \tau_{t_{q-k}}$ form a local frame on some compact neighbourhood U of z_0. We shall show that this neighbourhood satisfies our requirements.

Let arbitrary holomorphic sections $s_1, \cdots, s_{r-k}$ be given. We deform these sections into the following form;

$$\bar{s}_i = s_i + \sum_{j=1}^{q-k} \varepsilon_j^i \tau_{t_j}, \qquad i = 1, \cdots, r-k,$$
$$\mathscr{E} = (\varepsilon_j^i), \qquad \varepsilon_j^i \in \boldsymbol{C}.$$

Since $u_1, \cdots, u_k, \tau_{t_1}, \cdots, \tau_{t_{q-k}}$ form a basis for each fibre of U, $s_1, \cdots, s_{r-k}$ can be expressed as

$$s_i = \sum_{l=1}^{k} \alpha_l^i(z) u_l + \sum_{j=1}^{q-k} \beta_j^i(z) \tau_{t_j}$$

and hence, we have

$$\bar{s}_i = \sum_{l=1}^{k} \alpha_l^i(z) u_l + \sum_{j=1}^{q-k} (\beta_j^i(z) + \varepsilon_j^i) \tau_{t_j}.$$

Therefore the associated map $\Phi_U \colon U \to \mathfrak{M}(r, q)$ (see § 2) with respect to sections $\{u_1, \cdots, u_k, \bar{s}_1, \cdots, \bar{s}_{r-k}\}$ under the local frame $u_1, \cdots, u_k, \tau_{t_1}, \cdots, \tau_{t_{q-r}}$ has the following form;

$$\left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline \alpha_l^i(z) & \beta_j^i(z) + \varepsilon_j^i \end{array}\right].$$

Take as $\mathscr{E}_0=(\varepsilon_{0j}^i)\in\mathfrak{M}(r-k, q-k)$ a sufficiently small regular value of the map which sends each $z\in U$ to $(-\beta_j^i(z))\in\mathfrak{M}(r-k, q-k)$. Then holomorphic sections $\{u_1, \cdots, u_k, \bar{s}_1, \cdots, \bar{s}_{r-k}\}$ satisfy our quasilinear position requirement of order $k+1$. Q.E.D.

The neighbourhood $U(z_0; u_1, \cdots, u_k)$ will be called approximating neighbourhood at z_0 with respect to holomorphic sections $u_1, \cdots u_k$. In case $k=0$, it follows from the proof of the above lemma that there is a neighbourhood $U(z_0)$ such that for any global holomorphic sections $s_1, \cdots, s_r$ there are holomorphic sections $\bar{s}_1, \cdots, \bar{s}_r$ satisfying (a) and (b) with $k=0$.

Lemma 3.6. *Let $\{\sigma_1, \cdots, \sigma_r\}$ be holomorphic sections of $E\to M$ and K a compact subset of M. Let k be an integer, $0\leq k\leq r-1$. Suppose that there exists a subset $\{t_1, \cdots, t_k\}\subset\{1, \cdots, r\}$ such that $\sigma_{t_1}, \cdots, \sigma_{t_k}$ are linearly independent at each point of K (in case $k=0$, we suppose nothing). Then for any $\varepsilon>0$ there exist global holomorphic sections $s_1, \cdots, s_r$ which are in quasilinear position of order $k+1$ and satisfy $\|\sigma_i-s_i\|_M<\varepsilon$, $(i=1, \cdots, r)$.*

Proof. We may assume $\{t_1, \cdots, t_k\}=\{1, \cdots, k\}$. From Lemma 3.5, there are compact neighbourhoods $U(z; \sigma_1, \cdots, \sigma_k)$ at each point $z\in K$. Hence K has a finite covering $\{U(z_i; \sigma_1, \cdots, \sigma_k)\}$, $i=1, \cdots, N$.

From the property of $U(z_1; \sigma_1, \cdots, \sigma_k)$ in Lemma 3.5, we can deform $\{\sigma_1, \cdots, \sigma_k, \sigma_{k+1}, \cdots, \sigma_r\}$ into the form $\{\sigma_1, \cdots, \sigma_k, s_1^1, \cdots, s_{r-k}^1\}$ such that

$$\|\sigma_{k+j}-s_j^1\|_M<\frac{\varepsilon}{N}$$

and $\{\sigma_1, \cdots, \sigma_k, s_1^1, \cdots, s_{r-k}^1\}$ are in quasilinear position of order $k+1$. Notice that under the deformation, sections $\sigma_1, \cdots, \sigma_k$ remain unchanged. Therefore we can apply Lemma 3.5 and use the neighbourhood $U(z_2; \sigma_1, \cdots, \sigma_k)$ in order to deform $\{\sigma_1, \cdots, \sigma_k, s_1^1, \cdots, s_{r-k}^1\}$. By induction on i, we obtain holomorphic sections $s_1^i, \cdots, s_{r-k}^i$, $i=1, \cdots, N$ such that sections $\{\sigma_1, \cdots, \sigma_k, s_1^i, \cdots, s_{r-k}^i\}$ are in quasilinear position of order $k+1$ on $U(z_i; \sigma_1, \cdots, \sigma_k)$ for each i and satisfy

$$\|s_j^i-s_j^{i+1}\|_M<\frac{\varepsilon}{N}, \quad 1\leq i\leq N, \quad 1\leq j\leq r-k.$$

From the stability of quasilinear position, we can take $s_1^i, \cdots, s_{r-k}^i$ so that $\{\sigma_1, \cdots, \sigma_k, s_1^i, \cdots, s_{r-k}^i\}$ are in quasilinear position of order $k+1$ on $U(z_l; \sigma_1, \cdots, \sigma_k)$, $l=1, \cdots, i$. Set $\{s_1, \cdots, s_r\}=\{\sigma_1, \cdots, \sigma_k,$

$s_1^N, \cdots, s_{r-k}^N\}$. Then holomorphic sections $\{s_1, \cdots, s_r\}$ are in quasilinear position of order $k+1$ and satisfy $\|\sigma_i - s_i\|_M < \varepsilon$ $(i=1, \cdots, r)$. Q.E.D.

Lemma 3.7. *Let $\{\sigma_1, \cdots, \sigma_r\}$ be holomorphic sections of $E \to M$ which are in quasilinear position of order k on M. Then for any $\varepsilon > 0$ there exist holomorphic sections $\{s_1, \cdots, s_r\}$ which are in quasilinear position of order $k+1$ and satisfy $\|\sigma_i - s_i\|_M < \varepsilon$, $i=1, \cdots, r$.*

Proof. From Lemma 3.2, it follows that $\{\sigma_1, \cdots, \sigma_r\}$ are already in quasilinear position of order $k+1$ on some neighbourhood $\overline{W}$ of $S_{r-k+1}(\sigma_1, \cdots, \sigma_r)$ in M. From the stability of quasilinear position, there is $\bar{\delta} > 0$ such that if $\|\sigma_i - S_i\|_{\overline{W}} < \bar{\delta}$, $i=1, \cdots, r$, then $s_1, \cdots, s_r$ are also in quasilinear position of order $k+1$ on $\overline{W}$.

By the definition of $S_{r-k+1}(\sigma_1, \cdots, \sigma_r)$, for each point $z \in M - S_{r-k+1}$, there is a subset $\{t_1, \cdots, t_k\} \subset \{1, \cdots, r\}$ such that $\sigma_{t_1}, \cdots, \sigma_{t_k}$ are linearly independent at z. Therefore $M - \overline{W}$ has the following covering consisting of ${}_nC_k$ of compact subsets of $M - \overline{W}$;

$$M - \overline{W} = \bigcup W(t_1, \cdots, t_k),$$

where for each point $z \in W(t_1, \cdots, t_k)$, $\sigma_{t_1}, \cdots, \sigma_{t_k}$ are linearly independent at z. We shall arrange this covering in some order and denote by

$$M - \overline{W} = \bigcup_{i=1}^{N} W_i, \quad W_i = W(t_1(i), \cdots, t_k(i)), \quad i=1, \cdots, N(={}_rC_k).$$

Notice that there are $\delta_i > 0$ such that if $\|s_j - \sigma_j\|_{W_i} < \delta_i$, $j=1, \cdots, r$ then $s_{t_1(i)}, \cdots, s_{t_k(i)}$ are linearly independent at any point in W_i. Denote by δ the minimum of $\{\delta_i\}$, $i=1, \cdots, N$.

We shall prove the lemma by induction on i. From Lemma 3.6, it follows that there are global holomorphic sections $s_1^1, \cdots, s_r^1$ in quasilinear position of order $k+1$ on W_1 which satisfy

$$\|s_j^1 - \sigma_j\|_M < \frac{1}{N} \min\{\varepsilon, \bar{\delta}, \delta\}.$$

From the stability of quasilinear position, there is $\delta^{(1)} > 0$ such that if $\|s_i - s_i^1\|_{W_1} < \delta^{(1)}$ then $\{s_i\}$ are also in quasilinear position of order $k+1$ on W_1.

Since $\|s_j^1 - \sigma_j\|_{W_2} < \delta$, holomorphic sections $s_{t_1(2)}^1, \cdots, s_{t_k(2)}^1$ are linearly independent at any point in W_2. Therefore again by Lemma 3.6, we have holomorphic sections $s_1^2, \cdots, s_r^2$ in quasilinear position of order $k+1$ on W_2 which satisfy

$$\|s_j^2 - s_j^1\|_M < \frac{1}{N} \min\{\varepsilon, \bar{\delta}, \delta, \delta^{(1)}\}.$$

Moreover there is $\delta^{(2)}>0$ such that if $\|s_j-s_j^2\|_{W_2}<\delta^{(2)}$ then $\{s_i\}$ are also in quasilinear position of order $k+1$ on W_2.

By induction on i, we have holomorphic sections $s_1^N, \cdots, s_r^N$ in quasilinear position of order $k+1$ on W_N which satisfy

$$\|s_j^N-\sigma_j\|_M<\varepsilon,$$
$$\|s_j^N-s_j^i\|_{W_i}<\delta^{(i)}, \qquad \text{for any } i,$$
$$\|s_j^N-\sigma_j\|_{\overline{W}}<\bar{\delta}.$$

Consequently, $\{s_1^N, \cdots, s_r^N\}$ are in quasilinear position of order $k+1$ on $M-\overline{W}$ and on $\overline{W}$. This completes the proof of the lemma. Q.E.D.

Lemma 3.7 yields by induction on k, the following density of holomorphic sections in quasilinear position.

Proposition 3.8. *Let M be a compact complex manifold and $E\to M$ a holomorphic vector bundle of rank q which has global holomorphic sections generating each fibre. Then the set of holomorphic sections $\{\sigma_1, \cdots, \sigma_r\}$ (r fixed) in quasilinear position on M forms a dense subset in $\oplus^r \Gamma(M, \mathcal{O}(E))$, the set of all holomorphic r-sections of $E\to M$.*

The generic existence theorem follows from Proposition 3.4 and Proposition 3.8. This completes the proof of the generic existence theorem.

§ 4. The quasilinear bordism theorem

In this section we define an equivalent relation, called quasilinear bordism, for the set of quasilinear subvarieties in M, and we state our quasilinear bordism theorem, which asserts the existence of such bordism between Schubert cycles of holomorphic sections in quasilinear position. The proof of the theorem will be given in the next section.

Definition 4.1. Quasilinear subvarieties X_1 and X_2 of the same dimension k in M are said to be quasilinearly bordant in the strong sense, if there exists a quasilinear subvariety W in $M\times \boldsymbol{C}$ of dimension $k+1$ such that

$$X_1=W\cap M\times\{0\}, \qquad X_2=W\cap M\times\{1\},$$

and such that $M\times\{0\}$ and $M\times\{1\}$ cut transversally each stratum of W.

Definition 4.2. Quasilinear subvarieties V_1 and V_2 in M are said to be quasilinearly bordant if there exists a sequence of quasilinear subvarieties $X_1, \cdots, X_N$ in M such that for any $1\leq i\leq N$ the subvariety X_i is quasilinearly bordant to X_{i+1} in the strong sense and such that V_1 and V_2 are quasilinearly bordant to X_1 and X_N in the strong sense respectively.

§ 5. Proof of the quasilinear bordism theorem

This section is devoted to the proof of the quasilinear bordism theorem.

Let us consider the disjoint union of copies of $M\times \boldsymbol{C}$;

$$\bigcup_{i=1}^{N} X_i, \qquad X_i=M\times \boldsymbol{C}.$$

We denote by χ the quotient space (with singularities) of this union under the following identification;

$$\underset{M\times\{1\}}{\overset{X_i}{\bigcup}} \cong \underset{M\times\{0\}}{\overset{X_{i+1}}{\bigcup}}, \qquad i=1, \cdots, N-1.$$

Vector bundles and their sections on χ are said to be holomorphic if they are holomorphic when they are restricted to each X_i.

Let $E\to M$ be a holomorphic vector bundle. We consider copies $\{\bar{E}_i\}$ of holomorphic vector bundle $\bar{E}$ which is induced by the canonical projection $X_i=M\times \boldsymbol{C}\to M$ from the bundle E on M. This gives rise a holomorphic vector bundle denoted by $\boldsymbol{E}$ on the space χ such that $\boldsymbol{E}$ is the quotient of $\{\bar{E}_i\}$, $i=1, \cdots, N$.

Let us define two fundamental linear operations on $\Gamma(M\times \boldsymbol{C}, \bar{E})$. For each holomorphic section $\sigma(x, z)$ of $\bar{E}\to M\times \boldsymbol{C}$, $x\in M$, $z\in \boldsymbol{C}$, we set

$$\mu(\sigma)(x, z)=z\sigma(x, z),$$
$$\nu(\sigma)(x, z)=(1-z)\sigma(x, z).$$

Notice that

$$\mu(\sigma)=0, \quad \nu(\sigma)=\sigma, \qquad \text{on } M\times\{0\}$$
$$\mu(\sigma)=\sigma, \quad \nu(\sigma)=0, \qquad \text{on } M\times\{1\}.$$

We begin with the following lemma.

Lemma 5.1. *Fix an integer k, $1\leq k\leq N$. For any holomorphic section σ of the bundle $\bar{E}_k\to X_k$, there exists a holomorphic section $\bar{\sigma}$ of the bundle $\boldsymbol{E}\to\chi$ satisfying the following;*

i) $\bar{\sigma}_k=\sigma$, *on* X_k.

ii) $\bar{\sigma}_{k-1}=\sigma$, *on* $M\times\{1\}$ *and* $\bar{\sigma}_{k-1}=0$, *on* $M\times\{0\}$,

iii) $\bar{\sigma}_{k+1}=\sigma$, *on* $M\times\{0\}$ *and* $\bar{\sigma}_{k+1}=0$, *on* $M\times\{1\}$.

v) $\bar{\sigma}_i=0$, *for any* $i\neq k-1, k, k+1$,

where $\bar{\sigma}_j$ *denotes the restriction of* $\bar{\sigma}$ *to each* X_j, $j=1, \cdots, N$.

Proof. We shall first construct $\bar{\sigma}_{k+1}$. Consider the restriction $\sigma|_{M\times\{1\}}$ of σ to $M\times\{1\}\subset X_k$. Denote by s_{k+1} a holomorphic section on X_{k+1} which is induced from $\sigma|_{M\times\{1\}}$ by the canonical projection $X_{k+1}=M\times \boldsymbol{C}\rightarrow M\cong M\times\{1\}\subset X_k$. If we put $\bar{\sigma}_{k+1}=\nu(s_{k+1})$, then the section $\bar{\sigma}_{k+1}$ satisfies iii).

The construction of $\bar{\sigma}_{k-1}$ is similar to that of the section $\bar{\sigma}_{k+1}$. It is sufficient to use the operator μ instead of ν.

If we define $\bar{\sigma}_i=0$ for $i\neq k-1, k, k+1$ and $\bar{\sigma}_k=\sigma$, then this completes the proof. Q.E.D.

The holomorphic section $\bar{\sigma}$ constructed in the above lemma will be called the canonical extension of σ.

Throughout the remainder of this section, we suppose that each fibre of the bundle $E\rightarrow M$ is generated by holomorphic global sections. We consider the case of three copies $\chi=\bigcup X_i$, $\boldsymbol{E}=\bigcup \bar{E}_i$, $i=1, 2, 3$.

Let $\bigcup_{k=1}^{\infty} K_k$ be a compact covering of the complex plane $\boldsymbol{C}$ such that K_1 is a compact neighbourhood of $\{0, 1\}$. For any k, we shall denote by L_k the quotient of three copies of $M\times K_k\subset X_i$, $i=1, 2, 3$ under the canonical identification. This gives us a compact covering of the space χ, $\chi=\bigcup_{k=1}^{\infty} L_k$.

Holomorphic sections $\{h_1, \cdots, h_r\}$ of the bundle $\boldsymbol{E}\rightarrow\chi$ is said to be in quasilinear position of order k if each restriction to the bundles $\bar{E}_i\rightarrow X_i$, $i=1, 2, 3$ is in quasilinear position of order k.

Lemma 5.2. *Let* $\{\sigma_1, \cdots, \sigma_r\}$ *and* $\{s_1, \cdots, s_r\}$ *be holomorphic sections of* $E\rightarrow M$ *which are in quasilinear position on* M. *Then there exist holomorphic sections* $\{\eta_1, \cdots, \eta_r\}$ *of* $\boldsymbol{E}\rightarrow\chi$ *in quasilinear position on* L_1 *such that the restriction of* $\{\eta_1, \cdots, \eta_r\}$ *to* $M\times\{0\}\subset X_1$ *and to* $M\times\{1\}\subset X_3$ *coincide to* $\{\sigma_1, \cdots, \sigma_r\}$ *and* $\{s_1, \cdots, s_r\}$ *respectively and such that the restriction of* $\{\eta_1, \cdots, \eta_r\}$ *to* $M\times\{0\}$ *and* $M\times\{1\}\subset X_2$ *are in quasilinear position as sections of*

$$\bar{E}_2|_{M\times\{i\}}\longrightarrow X_2, \qquad i=1, 2.$$

Proof. From the assumption, each fibre of $\bar{E}_2$ is generated by some global holomorphic sections $\{v_1, \cdots, v_N\}$ on X_2. Let us denote their canonical extensions by $\bar{v}_1, \cdots, \bar{v}_N$. From the construction of the canonical extensions, it follows that

(a) $\bar{v}_i=0$ on $M\times\{0\}\subset X_1$ and on $M\times\{1\}\subset X_3$.

(b) $\bar{v}_1, \cdots, \bar{v}_N$ generate each fibre of E on $(X_1-M\times 0)\cup X_2\cup(X_3-M\times 1)$.

Extend holomorphic sections $\sigma_1, \cdots, \sigma_r$ and $s_1, \cdots, s_r$ to X_1 and X_3 respectively by the projection $M\times C\to M$, and denote them by $\sigma'_1, \cdots, \sigma'_r$ and $s'_1, \cdots, s'_r$.

Define holomorphic sections $h_1, \cdots, h_r$ of E by

$$h_i=\bar{\sigma}'_i+\bar{s}'_i, \qquad i=1, \cdots, r,$$

where $\bar{\sigma}'_i$ and $\bar{s}'_i$ are the canonical extensions of σ'_i and s'_i. By the property of canonical extensions, we have

$$\begin{aligned} &h_i=\sigma_i,\ i=1, \cdots, r \quad \text{on } M\times\{0\}\subset X_1 \\ &h_i=s_i,\ i=1, \cdots, r \quad \text{on } M\times\{1\}\subset X_3. \end{aligned}$$

And moreover $\{h_i\}$ are in quasilinear position on X_1 and X_3.

From the stability of quasilinear position, there is $\delta>0$ such that if

$$\|\xi_i-h_i\|_{L_1}<\delta,$$

then holomorphic sections $\{\xi_1, \cdots, \xi_r\}$ on χ are in quasilinear position on X_1 and on X_3.

Because of the property (b), it follows from the proof of the density of quasilinear position that we can deform $\{h_1, \cdots, h_r\}$ into $\{\eta_1, \cdots, \eta_r\}$ with

$$\eta_i=h_i+\sum_{j=1}^{N}\varepsilon^i_j\bar{v}_j, \qquad \varepsilon^i_j\in C$$

so that $\{\eta_i\}$ are in quasilinear position on $M\times K_1\subset X_2$ and satisfy

$$\|\eta_i-h_i\|_{L_1}<\delta \qquad (i=1, \cdots, r).$$

By the property of δ, sections $\{\eta_i\}$ are in quasilinear position on $M\times K_1\subset X_1$ and on $M\times K_1\subset X_3$. Therefore $\{\eta_i\}$ are in quasilinear position on $L_1\subset\chi$. From the property (a), it follows that $\{h_i\}$ remain unchanged on $M\times\{0\}\subset X_1$ and on $M\times\{1\}\subset X_3$ under the above deformation. This completes the proof. Q.E.D.

Lemma 5.3. *Let $\{h_1, \cdots, h_r\}$ be holomorphic sections of $E\to\chi$ in quasilinear position on L_k for some fixed k. Then for any $\varepsilon>0$ there exist holomorphic sections $\{\eta_1, \cdots, \eta_r\}$ in quasilinear position on L_{k+1} and on each $M\times\{j\}\subset X_i$, $i=1, 2, 3$, $j=0, 1$ which satisfy the following*:

i) $\|h_i-\eta_i\|_{L_k}<\varepsilon$ $(i=1, \cdots, r)$.

ii) $\eta_i=h_i$ $(i=1, \cdots, r)$ *on* $M\times\{0\}\subset X_1$ *and on* $M\times\{1\}\subset X_3$.

Proof. Let $\delta>0$ be the number such that if $\|s_i-h_i\|_{L_k}<\delta$ then $\{s_i\}$ are in quasilinear position on L_k. By the property (b) in the previous lemma, we know that on $L_{k+1}-L_k$ each fibre of $E\to X$ is generated by $\bar{v}_1, \cdots, \bar{v}_N$. Therefore in the same argument as the proof of Lemma 5.2, there are $\varepsilon_j^i \in C$ such that holomorphic sections

$$\eta_i=h_i+\sum_{j=1}^{N}\varepsilon_j^i\bar{v}_j, \qquad (i=1, \cdots, r)$$

are in quasilinear position on L_{k+1} and satisfy

$$\|\eta_i-h_i\|_{L_k}<\min\{\varepsilon, \delta\}.$$

The remainder part of the proof is also quite similar to that of Lemma 5.2.

Q.E.D.

Now we are in a position to prove our quasilinear bordism theorem. From Lemma 5.2 and Lemma 5.3, we have by induction on k, holomorphic sections $\{h_1^k, \cdots, h_r^k\}$ and positive number $\delta_k>0$, $k=1, 2, \cdots$, such that

i) $\|h_i^k-h_i^{k-1}\|_{L_k}<\min\left\{\dfrac{1}{2^k}, \dfrac{\delta_1}{2^k}, \dfrac{\delta_2}{2^{k-1}}, \cdots, \dfrac{\delta_{k-1}}{2^2}\right\}$.

ii) $\{h_1^k, \cdots, h_r^k\}$ are in quasilinear position on L_k and on each $M\times\{j\}\subset X_i$, $i=1, 2, 3$, $j=0, 1$.

iii) If $\|\eta_i-h_i^k\|_{L_k}<\delta_k$, then $\{\eta_i\}$ are in quasilinear position on L_k.

iv) $h_i^k=\sigma_i$ on $M\times\{0\}\subset X_1$, $h_i^k=s_i$ on $M\times\{1\}\subset X_3$.

Taking the limit of $\{h_1^k, \cdots, h_r^k\}$ as k tends to the infinity, we obtain holomorphic sections $\{h_1, \cdots, h_r\}$ of the bundle $E\to X$ in quasilinear position on X and on $M\times\{j\}\subset X_i$, $i=1, 2, 3$, $j=0, 1$, which coincides with $\{\sigma_1, \cdots, \sigma_r\}$ and $\{s_1, \cdots, s_r\}$ on $M\times\{0\}\subset X_1$ and on $M\times\{1\}\subset X_3$ respectively.

Let us denote by W_i, $i=1, 2, 3$, the Schubert cycles associated to $\{h_1, \cdots, h_r\}$ restricted to each X_i. Then W_i gives rise a strong quasilinear bordism between $W_i\cap M\times\{0\}$ and $W_i\cap M\times\{1\}$, where $M\times\{0\}$ and $M\times\{1\}$ are considered as those in X_i. Hence we have a strongly quasilinearly bordant sequence of quasilinear subvarieties of M:

$$\mathscr{S}(\sigma_1, \cdots, \sigma_r)=W_1\cap M\times\{0\}, \qquad W_1\cap M\times\{1\}=W_2\cap M\times\{0\},$$

$$W_2 \cap M \times \{1\} = W_3 \cap M \times \{0\}, \quad \text{and} \quad W_3 \cap M \times \{1\} = \mathscr{S}(s_1, \cdots, s_r).$$

This completes the proof of the quasilinear bordism theorem.

§ 6. An example of quasilinear bordism invariants

In this section we shall give an example of quasilinear bordism invariants which associate to each quasilinear subvarieties in M certain cohomology class in $H^*(M, Z)$. Throughout this section, cohomology is considered with coefficients Z.

Let V be a quasilinear subvariety of codimension k in M. Let V_s denote the set of all singular points of V. In general, we have the isomorphism;

$$H^q(M, M-V) = H^q(M-V_s, M-V)$$

for $q \leq 2k$. This follows easily from the following lemma.

Lemma 6.1. *Let M be a complex manifold and W a subvariety of complex codimension k in M. Then,*

$$H^q(M, M-W, Z) \cong 0, \qquad \text{for } q < 2k.$$

In case that V is quasilinear, we know more than the above. Suppose that V is quasilinear. Then, it follows that the complex codimension of V_s in V is not smaller than 2. Therefore we obtain by similar argument the isomorphism:

$$H^{2+2k}(M, M-V) = H^{2+2k}(M-V_s, M-V).$$

Let U be a neighbourhood of $V-V_s$ in $M-V_s$. From the Thom isomorphism and the excision property, we have

$$\begin{aligned} H^2(V-V_s) &= H^{2+2k}(U, \partial U) \\ &= H^{2+2k}(M-V_s, M-V) \\ &= H^{2+2k}(M, M-V). \end{aligned}$$

Combining with the pull back $H^{2+2k}(M, M-V) \to H^{2+2k}(M)$, we have a homomorphism:

$$\Phi_V^*: H^2(V-V_s) \to H^{2+2k}(M, Z).$$

Let K_{V-V_s} denote the canonical line bundle of $V-V_s$. Now we define the operator $\mathscr{K}$ as follows:

$$\mathscr{K}(V) = \Phi_V^*(c_1(K_{V-V_s})) \in H^{2+2k}(M, Z).$$

Theorem 6.2. *Let V_1 and V_2 be quasilinear subvarieties of M. If V_1 and V_2 are quasilinearly bordant in M, then*

$$\mathscr{K}(V_1)=\mathscr{K}(V_2).$$

Proof. It suffices to prove the result for V_1 and V_2 which are quasilinearly bordant in the strong sense. Let $W\subset M\times \boldsymbol{C}$ be a strong quasilinear bordism between V_1 and V_2. Since V_1 and V_2 are quasilinear, we may assume, by the definitions of $\mathscr{K}$, that V_1, V_2 and W are non-singular.

We identify V_1 and V_2 with $W\cap M\times\{0\}$ and $W\subset M\times\{1\}$ in $M\times \boldsymbol{C}$ respectively. We denote by $[V_i]_W$, $i=1, 2$, the normal bundles of V_i in W. Let us define a line bundle $\mathscr{L}$ on W by

$$\mathscr{L}=K_W\otimes[V_1]_W\otimes[V_2]_W,$$

where K_W denotes the canonical line bundle of W.

From the adjunction formula, it follows that the restrictions of $\mathscr{L}$ to V_1 and V_2 satisfy

$$\mathscr{L}|_{V_i}=K_{V_i}, \qquad i=1, 2.$$

Therefore cohomology classes $c_1(K_{V_1})$ and $c_1(K_{V_2})$ are extended to the cohomology class $c_1(\mathscr{L})$ of W. It is easy to prove that $c_1(K_{V_1})-c_1(K_{V_2})$ is homologous to zero. Q.E.D.

§ 7. Stationary loci and associated cohomology classes.

In this section we shall define stationary loci associated to sets of holomorphic sections as subcycles in Schubert cycles and we also define their associated cohomology classes as elements of the cohomology group $H^*(M, \boldsymbol{Z})$.

Let $\{\sigma_1, \cdots, \sigma_r\}$ be a set of holomorphic sections of a holomorphic vector bundle $E\to M$ of rank q over a compact complex manifold M. We denote by $\mathscr{S}^0(\sigma_1, \cdots, \sigma_r)$ the set of all regular points of the Schubert cycle $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$. Taking a local frame of the bundle on some small open subset U about a point $z_0\in\mathscr{S}^0(\sigma_1, \cdots, \sigma_r)$, we let

$$\Phi_U\colon U\to\mathfrak{M}(r, q), \qquad \Phi_U(z)=(\alpha_{ij}(z))$$

be a holomorphic map defined in Section 2. We denote the restriction of Φ_U to $\mathscr{S}^0(\sigma_1, \cdots, \sigma_r)$ by Φ_U^0. For each point z of $\mathscr{S}^0(\sigma_1, \cdots, \sigma_r)\cap U$ the matrix $\alpha_{ij}(z)$ defines a linear map from $\boldsymbol{C}^q$ into $\boldsymbol{C}^r$. Notice that the image of this linear map forms a hyperplane in $\boldsymbol{C}^r$, because $\Phi_U^0(z)$ is included in $\mathfrak{M}_1(r, s)$. Consequently, there is associated to each point z of

$\mathscr{S}^0(\sigma_1, \cdots, \sigma_r) \cap U$ a point of the complex projective space $\boldsymbol{C}P^{r-1}$. We thus obtain a holomorphic map:

$$\bar{\Phi}_U^0 : \mathscr{S}^0(\sigma_1, \cdots, \sigma_r) \cap U \to \boldsymbol{C}P^{r-1}.$$

Definition 7.1. Let $\{\sigma_1, \cdots, \sigma_r\}$ be a set of holomorphic sections in quasilinear position of the bundle $E \to M$. The stationary locus $\Sigma(\sigma_1, \cdots, \sigma_r)$ associated to holomorphic sections $\{\sigma_1, \cdots, \sigma_r\}$ is defined as the subset of $\mathscr{S}^0(\sigma_1, \cdots, \sigma_r)$ at which the rank of the differential $d\bar{\Phi}_U^0$ is degenerate, i.e.,

$$\operatorname{rank}(d\bar{\Phi}_U^0) < \min\{r-1, n-q+r-1\},$$

where n is the dimension of the base manifold M.

We also denote by $\Sigma(\sigma_1, \cdots, \sigma_r)$, when the stationary locus $\Sigma(\sigma_1, \cdots, \sigma_r)$ is considered as a subcycle of $\mathscr{S}^0(\sigma_1, \cdots, \sigma_r)$, the coefficients of which are naturally defined as the order of the map $\bar{\Phi}_U^0$ at $\Sigma(\sigma_1, \cdots, \sigma_r)$.

We now define associated cohomology classes of stationary loci. In case that the Schubert cycle $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$ is non singular and hence $\Sigma(\sigma_1, \cdots, \sigma_r)$ is a closed cycle, then the associated cohomology class of the stationary locus coincides with the Poincaré dual cohomology class of the cycle $\Sigma(\sigma_1, \cdots, \sigma_r)$ in $H^*(M, \boldsymbol{Z})$. For the simplicity, we suppose hereafter $q=n$. Similar argument applies to the case $2q-n+1>r$ with no assumption to q and n. If $q=n$, the integer r may vary from 1 to n.

Let $X_{\mathscr{S}}$ and X_Σ denote the sets of singular points of $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$ and $\Sigma(\sigma_1, \cdots, \sigma_r)$ respectively, and let N denote the complex codimension of the stationary locus considered as a subvariety of M. Since the case $\Sigma(\sigma_1, \cdots, \sigma_r) = \mathscr{S}(\sigma_1, \cdots, \sigma_r)$ is trivial, we suppose the dimension of $\Sigma(\sigma_1, \cdots, \sigma_r)$ is strictly smaller than that of $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$. In the latter case, we have $N=n-r+2$.

Considering a long exact sequence of cohomology classes associated to

$$M \supset M - \Sigma \supset M - (X_{\mathscr{S}} \cup \Sigma),$$

we obtain the isomorphism:

$$H^{2N}(M, M-\Sigma) \cong H^{2N}(M, M-(X_{\mathscr{S}} \cup \Sigma)),$$

because $H^*(M-\Sigma, M-(X_{\mathscr{S}} \cup \Sigma)) \cong 0$, for $* = 2N, 2N-1$, from Lemma 6.1.

Since the Schubert cycle $\mathscr{S}$ is quasilinear, we see that

$$2N,\ 2N+1 < 2(\text{complex codimension of } X_{\mathscr{S}}).$$

Therefore, it follows from Lemma 6.1,

$$H^*(M, M-(X_{\mathscr{S}} \cup X_\Sigma))=0, \qquad \text{for } * = 2N, 2N+1.$$

This yields the isomorphism:

$$H^{2N}(M, M-(X_{\mathscr{S}} \cup \Sigma)) \cong H^{2N}(M-(X_{\mathscr{S}} \cup X_\Sigma), M-(X_{\mathscr{S}} \cup \Sigma)).$$

Because $H^0(\Sigma) \cong H^0(\Sigma - X_\Sigma) \cong H^{2N}(M-(X_{\mathscr{S}} \cup X_\Sigma), M-(X_{\mathscr{S}} \cup \Sigma))$, we finally have

$$H^0(\Sigma, Z) \cong H^{2N}(M, M-\Sigma, Z).$$

Notice that stationary locus Σ is not necessarily closed in M. It is only closed in $M-X_{\mathscr{S}}$. Therefore, if the Schubert cycle $\mathscr{S}$ is not quasilinear, then the above isomorphism does not necessarily hold.

Through the last isomorphism, there is associated to the canonical class of the cycle Σ in $H^0(\Sigma, Z)$ an unique cohomology class of $H^{2N}(M, M-\Sigma, Z)$. Pulling back to $H^{2N}(M, Z)$ by the injection, this defines an unique cohomology class, which will be denoted by $\{\{\Sigma(\sigma_1, \cdots, \sigma_r)\}\}$ and will be called the associated cohomology class or the fundamental cohomology class of the stationary locus $\Sigma(\sigma_1, \cdots, \sigma_r)$.

Remark. The associated cohomology class $\Sigma(\sigma_1, \cdots, \sigma_r)$ coincides with the natural image of the following homomorphisms through $\Phi^*_{\mathscr{S}}$ defined in the previous section:

$$H^0(\Sigma, Z) \longrightarrow H^2(\mathscr{S}-X_{\mathscr{S}}, Z) \longrightarrow H^{2N}(M, Z).$$

§ 8. Invariance of stationary loci $\Sigma(\sigma_1, \cdots, \sigma_r)$

In the previous section we have defined cohomology classes associated to stationary loci. In this section we shall see that these cohomology classes remain invariant under the change of holomorphic sections and of holomorphic vector bundles in certain equivalent classes.

Let $\{\sigma_1, \cdots, \sigma_r\}$ be a set of holomorphic sections of a holomorphic vector bundle $E \to M$ of rank q over a compact complex manifold M. We denote them as in Section 1 by

$$\sigma_i(z) = \sum_{j=1}^{q} \alpha^i_j(z) e_j(z), \qquad i=1, \cdots, r, \quad z \in U,$$

for some local frame (e_j) on a small open subset U. And we denote each column of the matrix $(\alpha^i_j(z))$ by

$$v_j(z)=\begin{bmatrix}\alpha_j^1(z)\\ \alpha_j^2(z)\\ \vdots\\ \alpha_j^r(z)\end{bmatrix}, \qquad j=1, \cdots, q.$$

We see that if $z \in \mathscr{S}^0(\sigma_1, \cdots, \sigma_r)$ then vectors $v_1(z), \cdots, v_q(z)$ span a $(r-1)$-plane in the complex Euclidean space $\boldsymbol{C}^r$. This defines a map from $\mathscr{S}^0 \cap U$ into the complex projective space $\boldsymbol{CP}^{r-1}$. Noticing that the map does not depend on the choice of a frame $\{e_j\}$ and an open subset U, we see that the map is globally well defined. We thus obtain a holomorphic map denoted by

$$f=f_{\{\sigma_1,\cdots,\sigma_r\}}: \mathscr{S}^0(\sigma_1, \cdots, \sigma_r) \longrightarrow \boldsymbol{CP}^{r-1}.$$

The map f turns out to be closely related to Grassmannian manifolds, which we shall see as follows.

Suppose that the bundle $E \to M$ has global holomorphic sections $\tau_1, \cdots, \tau_N$ which generate each fibre E_z at $z \in M$. Let $e_1, \cdots, e_q$ be a local frame as above. Then sections $\tau_1, \cdots, \tau_N$ and $\sigma_1, \cdots, \sigma_r$ can be written as

$$\tau_k(z)=\sum_{j=1}^{q} \beta_j^k(z) e_j(z), \qquad k=1, \cdots, N,$$

$$\sigma_i(z)=\sum_{j=1}^{q} \alpha_j^i(z) e_j(z), \qquad i=1, \cdots, r.$$

We consider the $(N+r)\times q$ complex matrix whose upper N rows are given by $\beta_j^k(z)$ and whose lower r rows are $\alpha_j^i(z)$. Denote its columns by

$$u_j(z)=\begin{bmatrix}\beta_j^1(z)\\ \vdots\\ \beta_j^N(z)\\ \alpha_j^1(z)\\ \vdots\\ \alpha_j^r(z)\end{bmatrix}, \qquad j=1, \cdots, q.$$

Since $\tau_1(z), \cdots, \tau_N(z)$ generate each fibre E_z, vectors $u_1(z), \cdots, u_q(z)$ span a q-plane in $\boldsymbol{C}^{N+r}$, which we shall denote by $\Psi(z)$. Therefore this gives rise a holomorphic map:

$$\Psi: M \longrightarrow G_{q,N+r-q},$$

where $G_{q,N+r-q}$ denotes the complex Grassmann manifold consisting q-

planes in C^{N+r}.

We shall see that the map f: $\mathscr{S}^0(\sigma_1, \cdots, \sigma_r) \to CP^{r-1}$ can be regarded as a projection of the map Ψ.

Theorem 8.1. *Let M be a compact complex manifold and $E \to M$ a holomorphic vector bundle of rank $q = \dim(M)$. Suppose that $E \to M$ has global holomorphic sections which generate each fibre E_z. Let $\{\sigma_1, \cdots, \sigma_r\}$, $r \leq q$ be a set of holomorphic sections of $E \to M$ in quasilinear position. Then, the cohomology class $\{\{\Sigma(\sigma_1, \cdots, \sigma_r)\}\}$ associated to the stationary locus $\Sigma(\sigma_1, \cdots, \sigma_r)$ does not depend on the choice of sections $\{\sigma_1, \cdots, \sigma_r\}$. In other words, it holds that*

$$\{\{\Sigma(\sigma_1, \cdots, \sigma_r)\}\} = \{\{\Sigma(s_1, \cdots, s_r)\}\},$$

for any other set of holomorphic sections $\{s_1, \cdots, s_r\}$ in quasilinear position.

Proof. Since $\{\sigma_1, \cdots, \sigma_r\}$ is in quasilinear position, the dimension of the Schubert cycle $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$ is equal to $r-1$. From Riemann-Hurwitz theorem, it follows that

$$[D] = K_{\mathscr{S}^0} - f^*_{\{\sigma_1, \cdots, \sigma_r\}} K_{CP^{r-1}},$$

where D denotes the ramification divisor of the map f, and $K_{\mathscr{S}^0}$ and $K_{CP^{r-1}}$ denote the canonical line bundle of $\mathscr{S}^0$ and CP^{r-1} respectively.

Because $D = \Sigma(\sigma, \cdots, \sigma_r)$ as cycles, we have

$$\{\{\Sigma(\sigma_1, \cdots, \sigma_r)\}\} = \Phi^*_{\mathscr{S}}[D],$$

where $\Phi^*_{\mathscr{S}}$ is a homomorphism of $H^2(\mathscr{S}^0, Z)$ into $H^{2+2k}(M, Z)$, $k = \operatorname{codim}(\mathscr{S})$ which decomposes into

$$H^2(\mathscr{S}) \cong H^{2+2k}(M, M-\mathscr{S}) \to H^{2+2k}(M),$$

under the Thom isomorphism and the excision. See Section 6, for the map $\Phi^*_{\mathscr{S}}$. Hence we have

$$(1) \qquad \{\{\Sigma(\sigma_1, \cdots, \sigma_r)\}\} = \Phi^*_{\mathscr{S}}(c_1(K_{\mathscr{S}^0})) - \Phi^*_{\mathscr{S}} f^*_{\{\sigma_1, \cdots, \sigma_r\}} c_1(K_{CP^{r-1}}).$$

Let us recall some results about quasilinear bordism theory in Section 5. We have shown that under the change of sections, their associated Schubert cycles remain in the same quasilinear bordism class. It has also been shown that $\mathscr{K}(\mathscr{S}) = \Phi^*_{\mathscr{S}}(c_1(K_{\mathscr{S}}))$ is a quasilinear bordism invariant. From these results we see that the first term in the right hand remains invariant under the change of holomorphic sections.

We next investigate the second term. We recall that

$$K_{CP^{r-1}} = -r[H],$$

where H is a hyperplane in CP^{r-1}.

In the Grassmann manifold $G_{q,N+r-q}$, there is defined in a canonical way the Schubert variety F with respect to the projection:

$$\pi_{r-1}\colon C^{N+r} \cong C^{N+1} \times C^{r-1} \longrightarrow C^r,$$

i.e.,

$$F = \{q\text{-plane } \tau \in G_{q,N+r-q};\ \text{codimension of } \pi_{r-1}(|\tau|) \geq 1\}.$$

Recall that the Schubert variety F corresponds to the $(q-r+2)$-th Chern class.

We may suppose that the hyperplane H is defined by the projection:

$$\pi_1\colon C^r \cong C \times C^{r-1} \longrightarrow C.$$

Considering the following commutative diagram:

$$\begin{tikzcd}
C^{N+r} = C^N \times C \times C^{r-1} \arrow[rr] \arrow[dr, "\pi_{r-1}"'] & \arrow[d, phantom, "\circlearrowright"] & C \times C^{r-1} \arrow[rr] \arrow[dl] & \arrow[d, phantom, "\circlearrowright"] & C \\
& C^{r-1} & & C^r \arrow[ul, phantom, "\wr\|"] \arrow[ur, "\pi_1"'] &
\end{tikzcd},$$

we see that $\Phi_{\mathscr{G}}^* f^*[H]$ corresponds to the pullback of the Schubert variety F, via the map Ψ. It follows that

$$-\Phi_{\mathscr{G}}^* f^*(c_1(K_{CP^{r-1}}) = (-1)^{q-r} r c_{q-r+2}(E).$$

This completes the proof of Theorem 4.1. Q.E.D.

For the later purpose, it is necessary to generalize Theorem 8.1 to the category of differentiable bundles. For this purpose, we begin with the following lemma, the proof of which is easy by an argument of algebraic topology and will be omitted.

Lemma 8.2. *Let M be a differentiable manifold and let N_1 and N_2 be submanifolds of M. We suppose that N_1 and N_2 are L-equivalent,* i.e., *there exists a differentiable manifold W in $M \times [0, 1]$ such that*

$$\partial W = N_1 - N_2, \quad W \cap (M \times \{0\}) = N_1, \quad W \cap (M \times \{1\}) = N_2.$$

Let α and β be cohomology classes of $H^*(N_1, Z)$ and $H^*(N_2, Z)$ respectively. If there is a cohomology class γ in $H^*(W, Z)$ such that

$$\alpha = i_1^* \gamma, \qquad \beta = i_2^* \gamma,$$

where $i_1: N_1 \to M$, $i_2: N_2 \to M$ are injections, then it holds that in $H^*(M, Z)$

$$\Phi^*_{N_1}(\alpha) = \Phi^*_{N_2}(\beta),$$

where $\Phi^*_{N_1}$ and $\Phi^*_{N_2}$ are canonical homomorphisms from $H^q(N_1)$ and $H^q(N_2)$ into $H^{q+k}(M)$, q=degree of α and β, k=codimension of N_1 and N_2, which were defined through the Thom isomorphism in Section 6.

We are now in a position to generalize Theorem 8.1 and show the invariance of stationary loci.

Theorem 8.3. *Let M be a compact complex minifold and let $E_1 \to M$ and $E_2 \to M$ be holomorphic vector bundles of the same rank $q = \dim(M)$ such that each fibre of E_1 and E_2 are generated by global holomorphic sections. If the bundles E_1 and E_2 are equivalent as differentiable complex vector bundles, then it holds that as elements of $H^*(M, Z)$,*

$$\{\{\Sigma(\sigma_1^{(1)}, \cdots, \sigma_r^{(1)})\}\} = \{\{\Sigma(\sigma_1^{(2)}, \cdots, \sigma_r^{(2)})\}\},$$

for arbitrary holomorphic sections $\{\sigma_i^{(1)}\}$ of E_1 and $\{\sigma_i^{(2)}\}$ of E_2 which are in quasilinear position.

Proof. We put $\mathcal{S}_i = \mathcal{S}(\sigma_1^{(i)}, \cdots, \sigma_r^{(i)})$, $i=1, 2$. Since $\mathcal{S}_i$ are quasilinear, the sets of their singularities have complex codimension ≥ 2. Because we are concerned with $H^2(\mathcal{S}_i)$, we may assume that $\mathcal{S}_i$ are non-singular (cf. § 6 and 7).

Denote the normal bundle of $\mathcal{S}_i$ by $N_{\mathcal{S}_i}$, $i=1, 2$. By the proof of Theorem 8.1, it suffices to show the invariance of $\Phi_i^*(c_1(K_{\mathcal{S}_i})$.

Notice that

$$(*) \qquad c_1(K_{\mathcal{S}_i}) = c_1(N_{\mathcal{S}_i}) - c_1(T_M),$$

where T_M denotes the restriction of the tangent bundle of M to $\mathcal{S}_i$. Therefore, it is sufficient to show the invariance of $\Phi^*_{\mathcal{S}_i}(c_1(N_{\mathcal{S}_i}))$.

From the assumption, we have holomorphic mappings

$$\Psi_i: M \longrightarrow G_{q,N},$$

for sufficiently large N. Since the bundles E_1 and E_2 are equivalent as differentiable complex vector bundles, there exists a homotopy:

$$H: M \times [0, 1] \longrightarrow G_{q,N^*},$$

between Ψ_1 and Ψ_2 for sufficiently large N^*. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are both pullback of the Schubert cycle F_1 in G_{q,N^*} (modulo minus sign), there is a real subvariety W in $M\times[0, 1]$ with

$$W\cap(M\times 0)=\mathcal{S}_1, \qquad W\cap(M\times 1)=\mathcal{S}_2.$$

Because we can construct H so that W has only (real) quasilinear singularities, we may assume by the dimension argument that W is non-singular. Therefore, we see that the bundles $N_{\mathcal{S}_1}$ and $N_{\mathcal{S}_2}$ can be extended, as differentiable complex vector bundles, to the normal bundle of W. Consequently, the theorem follows easily from Lemma 8.2. Q.E.D.

§ 9. Polynomials $R_r(M, E)$

In this section, we define polynomials which arise in the stationary locus theorem. They are polynomials of Chern classes of a base manifold and a holomorphic vector bundle which represent stationary loci defined in Section 7 in case that the rank of the bundle is equal to the dimension of the base manifold.

We begin with the definition of certain symmetric polynomials with indeterminants $z_1, \cdots, z_q$. Let $\mathcal{A}=\mathcal{A}(q, r)$ be the set of all combinations consisting r elements of the set $\{1, \cdots, q\}$, i.e., the family of all the r-subsets of $\{1, \cdots, q\}$. Let $\mathcal{B}=\mathcal{B}(q, r)$ denote the set of all combinations consisting $q-r+1$ elements of the set $\mathcal{A}$. Each element β of $\mathcal{B}$ can be written as

$$\beta=\{\underbrace{\{i_1, \cdots, i_r\}, \{j_1, \cdots, j_r\}, \cdots}_{q-r+1}\}.$$

We define

$$P_{q,r}(z_1, \cdots, z_q)=\sum_{\beta\in\mathcal{B}}\Big[\sum_{\{i_1,\cdots,i_r\}\in\beta}(z_{i_1}+\cdots+z_{i_r})\prod_{\{i_1,\cdots,i_r\}\in\beta}(z_{i_1}+\cdots+z_{i_r})\Big].$$

Because the polynomial $P_{q,r}$ is a symmetric polynomial, it can be represented as a polynomial of elementary symmetric functions;

$$P_{q,r}(z_1, \cdots, z_q)=Q_{q,r}(s_1, \cdots, s_q)$$
$$s_1=z_1+\cdots+z_q,$$
$$\cdots\cdots$$
$$s_q=z_1\cdots z_q.$$

Let M be a compact complex manifold of dimension n and $E\to M$ a holomorphic vector bundle of rank n. We now define some polynomials of Chern classes of M and $E\to M$ as follows;

$$R_r(M, E)=Q_{n,r}(c_1(E), \cdots, c_n(E))-c_1(M)c_{n-r+1}(E)+rc_{n-r+2}(E).$$

Notice that the polynomial $R_r(M, E)$ defines a cohomology class in $H^{2(n-r+2)}(M, Z)$.

For small n and r, polynomials $R_r(M, E)$ have the following interresting forms. In case $n=r=2$, we have

$$R_2(M, E)=c_1^2(E)-c_1(M)c_1(E)+2c_2(E).$$

Hence, by Noether's formula, we obtain

$$R_2(M, TM)=2\chi(M), \qquad R_2(M, T^*M)=24\chi(\mathcal{O}_M).$$

§ 10. Preparatory lemmas

Before we prove the stationary locus theorem, we shall give in this section two preliminary lemmas.

We first recall the result about splitting method, refering to F. Hirzebruch [5]. Let M be a complex manifold and ξ a holomorphic $GL(q, C)$-bundle over M. We denote by L the principal bundle associated to ξ and consider the quotient

$$\tilde{M}=L/\Delta(q, C), \qquad \pi: \tilde{M}\to M,$$

where $\Delta(q, C)$ denotes the set of triangular matrices. The fibration $\pi: \tilde{M}\to M$ is a complex analytic fibre bundle with the flag manifold

$$F(q)=GL(q, C)/\Delta(q, C)$$

as fibre.

With these notations, we have the following lemma, the proof of which we refer to F. Hirzebruch [5].

Lemma 10.1. *Let $\pi: \tilde{M}\to M$ be as above. Then the structure group of the complex analytic bundle $\pi^*\xi$ over $\tilde{M}$ can be complex analytically reduced to the group $\Delta(q, C)$. Let $\xi_1, \cdots, \xi_q$ be the q diagonal complex analytic C^*-bundles. Then, $\pi^*\xi$ is differentiably equivalent to the holomorphic vector bundle $\xi_1\oplus\cdots\oplus\xi_q$.*

We continue the notations as above, and consider the homomorphism $\pi^*: H^*(M, Z)\to H^*(\tilde{M}, Z)$. For this homomorphism, we can show the following.

Lemma 10.2. *Let $\pi: \tilde{M}\to M$ as above. If $H^*(M, Z)$ contains only elements of even degree, then the homomorphism*

$$\pi^{*2k}\colon H^{2k}(M, Z)\to H^{2k}(\tilde{M}, Z)$$

is injective for any integer k.

Proof. According to Borel [1], the cohomology ring $H^*(F(n), Z)$ has the structure

$$Z[\gamma_1, \cdots, \gamma_n]/I^*(c_1, \cdots, c_n),$$

where $\gamma_1, \cdots, \gamma_n$ are regarded as indeterminates and where I^* is the ideal generated by the elementary symmetric functions $c_1, \cdots, c_n$ in the γ_i. We see that $H^*(F(n), Z)$ contains only elements of even degree and contains no torsion. Therefore,

$$H^p(M, H^q(F(n), Z))\cong H^p(M, Z)\otimes H^q(F(n), Z)\cong 0,$$

for any pair (p, q), one of which is odd.

The lemma follows easily from the calculus of the spectral sequence associated to the fibration $\pi\colon \tilde{M}\to M$. Q.E.D.

§ 11. Proof of the stationary locus theorem

This section is devoted to complete the proof of Theorem 5.1.

We have already proved in Section 8 that

$$\{\{\Sigma(\sigma_1, \cdots, \sigma_r)\}\}=\Phi^*(c_1(K_{\mathscr{S}}))+rc_{n-r+2}(\xi),$$

where $\mathscr{S}=\mathscr{S}(\sigma_1, \cdots, \sigma_r)$ is the Schubert cycle associated to holomorphic sections $\{\sigma_1, \cdots, \sigma_r\}$ and $K_{\mathscr{S}}$ denotes the canonical line bundle of $\mathscr{S}^0$. And hence, we have

$$\{\{\Sigma(\sigma_1, \cdots, \sigma_r)\}\}=\Phi^*(c_1(N_{\mathscr{S}}))-c_1(M)c_{n-r+1}(\xi)+rc_{n-r+2}(\xi),$$

where $N_{\mathscr{S}}$ is the normal bundle of the Schubert cycle $\mathscr{S}=\mathscr{S}(\sigma_1, \cdots, \sigma_r)$. In these arguments, we may suppose, without loss of generality, that $\mathscr{S}$ is non-singular, because of the quasilinearity of $\mathscr{S}$ (cf. § 7 and 8).

We shall show

$$\Phi^*(c_1(N_{\mathscr{S}}))=Q_{q,r}(c_1(\xi), \cdots, c_q(\xi)),$$

where q is not necessarily equal to n. Then this will complete the proof.

Recall that if $\{\sigma_1, \cdots, \sigma_r\}$ is in quasilinear position, then the map Ψ of M into the Grassmann manifold $G_{q,N}$ is transverse to the Schubert variety F(cf. § 8). Therefore, we see that $\Phi^*(c_1(N_{\mathscr{S}}))$ is expressed as the same polynomial as that of $\Phi^*(c_1(N_F))$.

Let $\pi: \tilde{G}_{q,N} \to G_{q,N}$ denote the fibration constructed in the previous section. Since the odd dimensional parts of $H^*(G_{q,N}, \boldsymbol{Z})$ are zero, it follows from Lemma 10.2 that

$$\pi^*: H^{2(q-r+2)}(G_{q,N}) \longrightarrow H^{2(q-r+2)}(\tilde{G}_{q,N})$$

is injective. Therefore, it suffices to show that the cohomology class $\Phi^*(c_1(N_{\tilde{F}}))$, $\tilde{F}=\pi^{-1}(F)$ is expressed as the polynomial $Q_{q,r}$ of $c_1(\tilde{\gamma}), \cdots, c_q(\tilde{\gamma})$, $\tilde{\gamma}=\pi^*(\gamma)$. Notice that F is also the Schubert cycle with respect to the bundle $\tilde{\gamma} \to \tilde{G}_{q,N}$, the lift of the universal bundle γ.

From Lemma 10.1, we see that the bundle $\tilde{\gamma}$ splits into $\tilde{E}_1 \oplus \cdots \oplus \tilde{E}_q$, for some holomorphic line bundles $\tilde{E}_i$ and that $\tilde{\gamma}$ is differentiably equivalent to $\tilde{E}_1 \oplus \cdots \oplus \tilde{E}_q$. From Theorem 8.3, it is sufficient to calculate $\Phi^*(c_1(N_{\mathscr{S}(s_1, \cdots, s_r)}))$ for some Schubert cycle $\mathscr{S}(s_1, \cdots, s_r)$ associated to appropriate quasilinear holomorphic sections $\{s_1, \cdots, s_r\}$ of the bundle $\tilde{E}_1 \oplus \cdots \oplus \tilde{E}_q$.

In view of all the above arguments, we see that the problem has been reduced to the following case. Let M be a complex manifold and let $\xi = E_1 \oplus \cdots \oplus E_q$ be a holomorphic vector bundle with holomorphic line bundles E_i, such that each E_i has global holomorphic sections generating each fibre, $i=1, \cdots, q$.

Let $\{\sigma_1, \cdots, \sigma_r\}$ be a set of holomorphic sections of ξ which is in quasilinear position. It suffices to prove

$$\{\{\Sigma(\sigma_1, \cdots, \sigma_r)\}\} = P_{q,r}(c_1(E_1), \cdots, c_1(E_q)).$$

The remainder part of this section is devoted to prove the above equation. And this will complete the proof of the stationary locus theorem.

We recall the notations of Section 9. Let β be an element of $\mathscr{B}$. We denote

$$\beta = \{\underbrace{\{i_1, \cdots, i_r\}, \{j_1, \cdots, j_r\}, \cdots}_{q-r+1}\}.$$

Let $\pi_{i_1, \cdots, i_r}$ and π_β denote the canonical projections:

$$\pi_{i_1, \cdots, i_r}: \bigwedge^r \xi \longrightarrow E_{i_1} \wedge \cdots \wedge E_{i_r} \quad (\cong E_{i_1} \otimes \cdots \otimes E_{i_r}),$$

$$\pi_\beta = \bigoplus_{\{i_1, \cdots, i_r\} \in \beta} \pi_{i_1, \cdots, i_r}: \bigwedge^r \xi \longrightarrow \bigoplus_{\{i_1, \cdots, i_r\} \in \beta} E_{i_1} \wedge \cdots \wedge E_{i_r}.$$

Holomorphic sections $\{\sigma_1, \cdots, \sigma_r\}$ of ξ gives rise to holomorphic sections $\pi_{i_1, \cdots, i_r}(\sigma_1 \wedge \cdots \wedge \sigma_r)$ and $\pi_\beta(\sigma_1 \wedge \cdots \wedge \sigma_r)$. We consider Schubert cycles of these sections and we set

$$\mathscr{S}_{i_1,\cdots,i_r}=\mathscr{S}(\pi_{i_1,\cdots,i_r}(\sigma_1\wedge\cdots\wedge\sigma_r)),$$
$$\mathscr{S}_\beta=\mathscr{S}(\pi_\beta(\sigma_1\wedge\cdots\wedge\sigma_r)).$$

We have proved in Section 5 that holomorphic sections in quasilinear position exist generically. Therefore, we may assume by a slight modification that sections which appear in our argument are all in quasilinear position. We suppose this hereafter without any comment.

We shall seek the structure of the normal bundle of $\mathscr{S}_\beta$. Although the variety $\mathscr{S}_\beta$ has singularities, the notion of normal bundle makes sense. This is because we are concerned with the first Chern class and because the quasilinearity of $\mathscr{S}_\beta$ yields that the set of singular points of $\mathscr{S}_\beta$ has complex codimension ≥ 2 in $\mathscr{S}_\beta$.

We set for $k=1,\cdots,q-r+1$,

$$\mathscr{S}(k)=\underbrace{\mathscr{S}_{i_1,\cdots,i_r}\cap\mathscr{S}_{j_1,\cdots,j_r}\cap\cdots}_{k}.$$

Noticing that

$$\mathscr{S}_\beta=\bigcap_{\{i_1,\cdots,i_r\}\in\beta}\mathscr{S}_{i_1,\cdots,i_r},$$

we have the following sequence of divisors:

$$M\supset\mathscr{S}(1)\supset\mathscr{S}(2)\supset\cdots\supset\mathscr{S}(q-r+1)=\mathscr{S}_\beta.$$

In the above sequence, we see that the normal bundle of $\mathscr{S}(1)$ in M is the line bundle $E_{i_1}\wedge\cdots\wedge E_{i_r}$ and that the normal bundle of $\mathscr{S}(2)$ in $\mathscr{S}(1)$ is the line bundle $E_{j_1}\wedge\cdots\wedge E_{j_r}$ and so on. Therefore, we know that the normal bundle of $\mathscr{S}_\beta$ in M is given by

$$N_{\mathscr{S}\beta}=\bigoplus_{\{i_1,\cdots,i_r\}\in\beta}(E_{i_1}\wedge\cdots\wedge E_{i_r}),$$

We obtain

$$c_1(N_{\mathscr{S}\beta})=\sum_{\{i_1,\cdots,i_r\}\in\beta}(c_1(E_{i_1})+\cdots+c_1(E_{i_r})).$$

Since $\mathscr{S}_\beta$ is the Schubert cycle of the bundle

$$\bigoplus_{\{i_1,\cdots,i_r\}\in\beta}E_{i_1}\wedge\cdots\wedge E_{i_r}$$

associated to the section $\pi_\beta(\sigma_1\wedge\cdots\wedge\sigma_r)$, it follows that $\mathscr{S}_\beta$ represents

$$\prod_{\{i_1,\cdots,i_r\}\in\beta}(c_1(E_{i_1})+\cdots+c_1(E_{i_r})).$$

Consequently, we obtain

$$\Phi^*(c_1(N_{\mathscr{S}_\beta}))=[\sum_{\{i_1,\cdots,i_r\}\in\beta}(c_1(E_{i_1})+\cdots+c_1(E_{i_r}))] \times[\prod_{\{i_1,\cdots,i_r\}\in\beta}(c_1(E_{i_1})+\cdots+c_1(E_{i_r}))].$$

Noticing that

$$\mathscr{S}(\sigma_1,\cdots,\sigma_r)=\bigcup_{\beta\in\mathscr{B}}\mathscr{S}_\beta,$$

we finally obtain

$$\Phi^*(c_1(N_{\mathscr{S}(\sigma_1,\cdots,\sigma_r)})=P_{q,r}(c_1(E_1),\cdots,c_1(E_q)).$$

This completes the proof of the stationary locus theorem. Q.E.D.

References

[1] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, Ann. of Math., **57** (1953), 115–207.
[2] M. Cornalba and P. Griffiths, Analytic cycles and vector bundles on non-compact algebraic varieties, Invent. Math., **28** (1975), 1–106.
[3] M. Eger, Sur les systèmes canoniques d'une variété algébrique, Comptes Rendus Acad. Sci. Paris, **204** (1937), 92–94.
[4] P. Griffiths, Hermitian differential geometry, chern classes, and positive vector bundles, Global Analysis, Papers in honor of K. Kodaira, Princeton Univ. Press (1969), 185–251.
[5] F. Hirzebruch, Neue topologische Methoden in der algebraischen Geometrie, Erg. d. Math., Springer, 1956.
[6] S. L. Kleiman and D. Laksov, Schubert calculus, Amer. Math. Monthly, **79** (1972), 1061–1082.
[7] J. Milnor, On the cobordism ring Ω^* and a complex analogue, Amer. J. Math., **82** (1960), 505–521.
[8] H. Morimoto, Realization of Chern classes by subvarieties with certain singularities, Nagoya Math. J., **80** (1980), 49–74.
[9] ——, On certain complex analytic cobordism between subvarieties realizing Chern classes of bundles, Nagoya Math. J., **88** (1982), 121–132.
[10] ——, On the generic existence of holomorphic sections and complex analytic bordism, Proc. Japan Acad., **58** (1982), 306–308.
[11] M. Morse, The Calculus of Variations in the Large, Amer. Math. Soc., 1934.
[12] F. Severi, Fondamenti per la geometria sulle varieta algebriche, Reindiconti del Circolo Matematico di Palermo, **33** (1909), 38–87.
[13] R. Thom, Les singularités des applications différentiables, Ann. Inst. Fourier Grenoble, **6** (1955/56), 43–87.
[14] J. A. Todd, On Giambelli's formula for incidence of linear spaces, J. Lond. Math. Soc., **6** (1931), 209–216.
[15] H. Whitney, Differentiable manifolds, Ann. of Math., **37** (1936), 645–680.

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