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A Differentiable Sphere Theorem for Volume-Pinched Manifolds

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Dedicated to Professor I. Mogi on his 60th birthday

§ 0. Introduction

A main problem in differential geometry is to investigate the influences of geometrical quantities of complete Riemannian manifolds on the topology. The sphere theorem due to Klingenberg states that if M is a complete simply connected manifold with the sectional curvature K_M, $1/4 < K_M \leqslant 1$, then M is a topological sphere ([7]). A stronger assumption for curvature implies that M is diffeomorphic to the standard sphere ([4], [8], [10]). In the proof of these results, an estimate of the injectivity radius $i(M)$, $i(M) \geqslant \pi$, of the exponential map on M plays an essential role. On the other hand, by pinching the diameter $\mathrm{diam}(M)$ in place of the sectional curvature Grove-Shiohama has obtained the following theorem which generalizes the Klingenberg sphere theorem.

Theorem A ([6]). *If the sectional curvature and the diameter of a complete manifold M satisfy $K_M \geqslant 1$, $\mathrm{diam}(M) > \pi/2$, then M is a topological sphere.*

Recently by pinching the volume $\mathrm{Vol}(M)$, Shiohama has proved the following sphere theorem for manifolds M of positive Ricci curvature Ric_M. We denote by S^n the unit n-sphere.

Theorem B ([9]). *For given n, $-\Lambda^2$, there exists an $\varepsilon = \varepsilon(n, \Lambda)$ such that if a complete manifold M of dimension n satisfies*

$$\mathrm{Ric}_M \geqslant 1,\ K_M \geqslant -\Lambda^2,\ \mathrm{Vol}(M) \geqslant \mathrm{Vol}(S^n) - \varepsilon,$$

then M is a topological sphere.

But in the situation of Theorem A or Theorem B, it was not known for M to be diffeomorphic to the standard sphere. The purpose of this

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paper is to give a partial affirmative answer to the problem of this type. We denote by ∇R_M the covariant derivative of the curvature tensor R_M of M. We obtain the following

Main Theorem. *For given n, $\Lambda \geqslant 1$, $\Lambda_1 > 0$, there exists a $\delta = \delta(n, \Lambda, \Lambda_1) > 0$ such that if a complete manifold M of dimension n satisfies*

$$1 \leqslant K_M \leqslant \Lambda^2, \ \|\nabla R_M\| \leqslant \Lambda_1, \ \mathrm{Vol}(M) \geqslant \mathrm{Vol}(S^n) - \delta,$$

then M is diffeomorphic to the standard sphere.

To show that M is a topological sphere, it suffices to cover M by two open cells. But in our situation, we need a tool which shows when two manifolds are diffeomorphic. For the proof of Main Theorem, we use the following general result which has been applied to finiteness theorems (See [11], also [5]).

Theorem C. *For given n, Λ, Λ_1, $R > 0$, there exist $\varepsilon_1 = \varepsilon_1(n) > 0$ and $r_1 = r_1(n, \Lambda, \Lambda_1, R) > 0$ such that if complete manifolds M and $\bar{M}$ of dimension n satisfy the following conditions, then they are diffeomorphic;*

(1) $|K_M|, |K_{\bar{M}}| \leqslant \Lambda^2, \ \|\nabla R_M\|, \|\nabla R_{\bar{M}}\| \leqslant \Lambda_1, \ i(M), i(\bar{M}) \geqslant R,$

(2) *for some $\varepsilon \leqslant \varepsilon_1$ and $r \leqslant r_1$, there exist $2^{-(n+8)}r$-dense and $2^{-(n+9)}r$-discrete subsets $\{p_i\} \subset M$, $\{q_i\} \subset \bar{M}$ such that the correspondence $p_i \to q_i$ is bijective and $(1+\varepsilon)^{-1} \leqslant d(q_i, q_j)/d(p_i, p_j) \leqslant 1+\varepsilon$ for all p_i, p_j with $d(p_i, p_j) \leqslant 20r$.*

ε_1 and r_1 can be written explicitly.

By definition a subset A of M is δ-dense (resp. δ-discrete) if any $x \in M$ has the distance $d(x, A) < \delta$(resp. if any pair of distinct points in A has the distance $\geqslant \delta$).

After preparing some basic results in Section 1, an estimate of injectivity radius is given in Section 2. Main Theorem is proved in Section 3.

§ 1. Preliminary results

Let M be an n-dimensional complete manifold with $K_M \geqslant 1$. Then Bonnet's theorem implies that $\mathrm{diam}(M) \leqslant \pi$, hence M is compact. For fixed points $x \in M$ and $p \in S^n$, let I be a linear isometry from S^n_p to M_x. We denote by M_x the tangent space to M at x, and set $\mathfrak{S}_x := \{v \in M_x;\ d(\exp_x v, x) = \|v\|\}$ and $U := \exp_p(I^{-1} \circ \mathring{\mathfrak{S}}_x)$, where $\mathring{A}$ denotes the interior of a set A. Let $C(x)$ denote the cut locus of x. Then the map $F: U \to M - C(x)$ defined by $F = \exp_x \circ I \circ \exp_p^{-1}$ is a diffeomorphism and the Rauch comparison theorem implies that $\|dF\| \leqslant 1$. For a positive number r, we denote by $v(r)$ the volume of an r-ball in S^n.

Lemma 1.1. *If a complete manifold M of dimension n satisfies* $K_M \geqslant 1$, $\mathrm{Vol}(M) \geqslant \mathrm{Vol}(S^n) - \delta$, *then* $\mathrm{Vol}(F(A)) \geqslant \mathrm{Vol}(A) - \delta$ *for every measurable subset A of U.*

Proof. Since $C(x)$ has measure zero, the Rauch comparison theorem implies that $\mathrm{Vol}(U-A) \geqslant \mathrm{Vol}(M-F(A))$. By the assumption we get

$$\begin{aligned}\mathrm{Vol}(F(A)) &= \mathrm{Vol}(M) - \mathrm{Vol}(M-F(A)) \\ &\geqslant \mathrm{Vol}(S^n) - \delta - \mathrm{Vol}(U-A) \\ &\geqslant \mathrm{Vol}(A) - \delta. \end{aligned}$$

Q.E.D.

Remark. Owing to Bishop's result (See [1]), we can use the Ricci curvature instead of the sectional curvature in the previous lemma.

The following lemma will play an important role in the proof of our Main Theorem. For the proof see Lemma 4.3 in [2].

Lemma 1.2. *Let M be a Riemannian manifold of dimension n and let* $\{x_i\}$ *be a normal coordinate system. Set* $r := (\sum x_i^2)^{1/2}$. *Then there exist continuous functions* $\Omega_1 \colon \boldsymbol{R}_+ \times \boldsymbol{R}_+ \to \boldsymbol{R}_+$, $\Omega_2 \colon \boldsymbol{N} \times \boldsymbol{R}_+ \times \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *such that if* $|K_M| \leqslant \Lambda^2$, $\|\nabla R_M\| \leqslant \Lambda_1$ *on the normal coordinate neighborhood, then*

$$\left\| \nabla_{\partial/\partial r} \frac{\partial}{\partial x_i} \right\| \leqslant \Omega_1(\Lambda, r), \qquad \left\| \nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} \right\| \leqslant \Omega_2(n, \Lambda, \Lambda_1, r).$$

For example, Ω_1 *and* Ω_2 *are taken as*

$$\begin{aligned} \Omega_1 &= 2\Lambda^2 r \cdot \exp((2\Lambda+1)r) \\ \Omega_2 &= \Omega_1 + 6n(n-1)(\Lambda_1 r \cdot \exp(2\Lambda r) + \Lambda^2(3+\exp(2\Lambda r)) \\ &\quad + 4n^{3/2}\Lambda^2 r^2 \cdot \exp(2\Lambda r)))r \cdot \exp(n^2\Lambda^2(1 + r\cdot\exp(2\Lambda r) + 2\Lambda + 1)r). \end{aligned}$$

§ 2. Estimate of injectivity radius

For a fixed point x in a complete manifold M, let y realize the minimum distance from x to its cut locus. Then either there is a minimal geodesic from x to y along which y is conjugate to x, or there are precisely two minimal geodesics from x to y which form a geodesic loop at x. In the case $K_M \geqslant 1$, we observe the influence of the distance $d(x, C(x))$ on the total volume of M. From now on, all geodesics are assumed to be parametrized by arc length. We denote by $J(t)$ the Jacobian of exponential map on S_p^n at a point of distance equal to t from the origin:

$$J(t) = \left(\frac{\sin t}{t}\right)^{n-1}.$$

Lemma 2.1. *Let M be an n dimensional complete manifold with $K_M \geqslant 1$ and let $x \in M$ and $y \in C(x)$ satisfy $d(x, y) = d(x, C(x)) =: l$. Suppose that there is a geodesic loop γ: $[0, 2l] \to M$ at x with $\gamma(l) = y$. Then*

(1) *if $l > \pi/2$, then $C(x)$ consists of the single point y. In particular, M is homeomorphic to S^n and* $\mathrm{Vol}(M) \leqslant v(l)$.

(2) *if $l \leqslant \pi/2$, then $\mathfrak{S}_x \subset \bar{B}(0, \pi - l)$. Furthermore* $\mathrm{Vol}(M) \leqslant v(\pi/2)$.

Proof. (1) For any $z \in C(x)$ let γ_0 be a minimal geodesic from y to z. Set $l_1 := d(x, z)$, $l_2 := d(y, z)$. Since γ is a geodesic loop, we may assume that $\langle \dot{\gamma}_0(0), -\dot{\gamma}(l) \rangle \geqslant 0$. Then the Toponogov comparison theorem and the fact $l_1 \geqslant l$ imply

$$(*) \qquad \cos l \geqslant \cos l_1 \geqslant \cos l_2 \cdot \cos l.$$

It follows that if $l > \pi/2$, then $l_2 = 0$ since $\cos l \cdot (1 - \cos l_2) \geqslant 0$.

(2) Suppose that $l \leqslant \pi/2$ and $d(x, z) = l_1 > \pi - l$. Then it turns out

$$-\cos l = \cos(\pi - l) > \cos l_1 \geqslant \cos l \cdot \cos l_2,$$

and hence $\cos l \cdot (1 + \cos l_2) < 0$. This is a contradiction. Therefore $\mathfrak{S}_x \subset \bar{B}(0, \pi - l)$. We now prove the second part of (2). For a unit tangent vector v at x, let t_v denote the distance from x to the cut point along the geodesic with direction v. We show that $t_v + t_{-v} \leqslant \pi$. Set $\alpha := \measuredangle(v, \dot{\gamma}(0))$, $l_2 := d(\exp t_v v, y)$. Then the Toponogov comparison theorem implies

$$\cos l_2 \geqslant \cos l \cdot \cos t_v + \sin l \cdot \sin t_v \cdot \cos \alpha.$$

Together with $(*)$ this yields

$$\cos t_v \geqslant \cos l (\cos l \cdot \cos t_v + \sin l \cdot \sin t_v \cdot \cos \alpha),$$

and hence $\cot t_v \geqslant \cot l \cdot \cos \alpha$. Similarly $\cot t_{-v} \geqslant -\cot l \cdot \cos \alpha$. Therefore we have

$$\sin(t_v + t_{-v}) = \sin t_v \cdot \sin t_{-v} (\cot t_v + \cot t_{-v}) \geqslant 0,$$

and $t_v + t_{-v} \leqslant \pi$. For a fixed v, we may assume that $t_v < \pi/2$ and $t_{-v} > \pi/2$. Then we get that for any $t \in [t_v, \pi/2]$, $J(t) \geqslant J(\pi - t)$, and hence

$$\int_{t_v}^{\pi/2} J(t)dt \geqslant \int_{t_v}^{\pi/2} J(\pi - t)dt = \int_{\pi/2}^{\pi - t_v} J(t)dt \geqslant \int_{\pi/2}^{t_{-v}} J(t)dt.$$

It follows that

$$\mathrm{Vol}(M) \leqslant \int_{\mathfrak{S}_x} J(\|v\|)dv \leqslant \int_{\bar{B}(0,\pi/2)} J(\|v\|)dv = v(\pi/2).$$

Q.E.D.

Since by the previous lemma $\mathrm{Vol}(M) > 1/2\,\mathrm{Vol}(S^n)$ implies that y is conjugate to x, we are forced to treat the conjugate point case. To do this, we need upper bounds on $|K_M|$ and $\|\nabla R_M\|$.

Lemma 2.2. *Let M be a complete manifold of dimension n with $|K_M| \leqslant \Lambda^2$, $\|\nabla R_M\| \leqslant \Lambda_1$, and let $x \in M$ and $y \in C(x)$ satisfy $d(x, y) = d(x, C(x)) := l$. Suppose that there is a unit vector $w \in (M_x)_{lv}$ such that $d(\exp_x)(w) = 0$. Let W be the parallel vector field on M_x with $W(0) = w$. Then for an arbitrary $\varepsilon \in (0, \pi)$, there are $t_1 \in (l/2, l)$ and $\eta = \eta(n, \Lambda, \Lambda_1, l, \varepsilon)$ such that $B(t_1v, \eta) \subset \mathring{\mathfrak{S}}_x$ and $\|d(\exp_x)(W)\| \leqslant \varepsilon$ on $B(t_1v, \eta)$. η is monotone decreasing in n, Λ, Λ_1, l and monotone increasing in ε.*

Proof. Let $\{e_i\}$ be an orthonormal frame at x with $e_1 = w$ and $\{x_i\}$ the normal coordinate system on $M - C(x)$ based on $\{e_i\}$. By Lemma 1.2

$$\left\|\nabla_{\partial/\partial r}\frac{\partial}{\partial x_i}\right\| \leqslant \Omega_1(\Lambda, l), \quad \left\|\nabla_{\partial/\partial x_i}\frac{\partial}{\partial x_j}\right\| \leqslant \Omega_2(n, \Lambda, \Lambda_1, l) \quad \text{on } B(x, l).$$

Notice that $\left\|\frac{\partial}{\partial x_1}(\gamma_v(t))\right\| = \|d(\exp_x)(W(tv))\| \to 0$ as $t \uparrow l$. Since $\left|\frac{d}{dr}\left\|\frac{\partial}{\partial x_1}\right\|\right| \leqslant \left\|\nabla_{\partial/\partial r}\frac{\partial}{\partial x_1}\right\|$, setting $t_1 := l - \varepsilon/(2\Omega_1(\Lambda, l))$ we have $\left\|\frac{\partial}{\partial x_1}(\gamma_v(t_1))\right\| \leqslant \varepsilon/2$. If $\eta := \varepsilon/(2n\Omega_2(n, \Lambda, \Lambda_1, l))$, then $B(t_1v, \eta) \subset \mathring{\mathfrak{S}}_x$ and since $\left|\frac{\partial}{\partial x_i}\left\|\frac{\partial}{\partial x_1}\right\|\right| \leqslant \left\|\nabla_{\partial/\partial x_i}\frac{\partial}{\partial x_1}\right\|$, we get that for every $u \in B(t_1v, \eta)$

$$\left|\left\|\frac{\partial}{\partial x_1}(\exp_x u)\right\| - \left\|\frac{\partial}{\partial x_1}(\exp_x t_1v)\right\|\right| \leqslant n\Omega_2(n, \Lambda, \Lambda_1, l)\|u - t_1v\|$$

$$\leqslant n\Omega_2(n, \Lambda, \Lambda_1, l)\cdot\eta \leqslant \varepsilon/2,$$

hence $\|d(\exp_x)(W(u))\| = \left\|\frac{\partial}{\partial x_1}(\exp_x u)\right\| \leqslant \left\|\frac{\partial}{\partial x_1}(\exp_x t_1v)\right\| + \varepsilon/2 \leqslant \varepsilon$.

Q.E.D.

Proposition 2.3. *For given n, $\Lambda \geqslant 1$, $\Lambda_1 > 0$ and for $\varepsilon \in (0, \pi/2)$, there exists a $\delta_1 = \delta_1(n, \Lambda, \Lambda_1, \varepsilon) > 0$ such that if M is a complete manifold of dimension n such that $1 \leqslant K_M \leqslant \Lambda^2$, $\|\nabla R_M\| \leqslant \Lambda_1$, $\mathrm{Vol}(M) \geqslant \mathrm{Vol}(S^n) - \delta_1$, then $i(M) > \pi - \varepsilon$.*

Proof. Let M satisfy $\mathrm{Vol}(M) \geqslant \mathrm{Vol}(S^n) - \delta_1$ for some $\delta_1 > 0$. Suppose that $l := i(M) \leqslant \pi - \varepsilon$. Then two points $x, y \in M$ with $y \in C(x)$, $d(x, y) = l$ satisfy the following (1) or (2);

(1) there is a closed geodesic γ: $[0, 2l] \to M$ such that $\gamma(0) = \gamma(2l) = x$, $\gamma(l) = y$,

(2) y is conjugate to x along a minimal geodesic σ: $[0, l] \to M$. In the case (1), Lemma 2.1 implies that $\mathrm{Vol}(M) \leqslant v(\pi - \varepsilon)$. In the case (2), there is a unit vector $w \in (M_x)_{l\dot\sigma(0)}$ such that $d(\exp_x)(w) = 0$. Let W be the parallel vector field on M_x with $W(0) = w$. Then for $\varepsilon_1 := (1/2)J(\pi - \varepsilon)$, the constants t_1, $l/2 < t_1 < l$ and $\eta = \eta(n, \Lambda, \Lambda_1, \pi - \varepsilon, \varepsilon_1)$ as in Lemma 2.2 satisfy that $B(t_1\dot\sigma(0), \eta) \subset \mathfrak{S}_x$ and $\|d(\exp_x)(W)\| \leqslant \varepsilon_1$ on the ball $B(t_1\dot\sigma(0), \eta)$. Setting $q_1 := F^{-1}(\sigma(t_1))$, $\kappa := \eta(\sin(\pi - \varepsilon)/(\pi - \varepsilon))$, we have $\exp_x^{-1}(F(B(q_1, \kappa))) \subset B(t_1\dot\sigma(0), \eta)$, where F: $U \to M - C(x)$ is the diffeomorphism constructed in Section 1. This implies

$$\begin{aligned}\mathrm{Vol}\, F(B(q_1,)\kappa) &= \int_{B(q_1,\kappa)} \det(dF)\, dS^n \\ &< \varepsilon_1 v(\kappa)/J(t_1 + \kappa) < \varepsilon_1 v(\kappa)/J(\pi - \varepsilon) = \frac{1}{2} v(\kappa).\end{aligned}$$

On the other hand, Lemma 1.1 implies that $\mathrm{Vol}\, F(B(q_1, \kappa)) \geqslant v(\kappa) - \delta_1$. It turns out that $\delta_1 > (1/2)v(\kappa)$. Thus the required δ_1 is obtained as $\delta_1 = (1/2)v(\kappa)$ since $v(\varepsilon) > (1/2)v(\kappa)$. Q.E.D.

Remark. By the remark in Section 1, an observation similar to Lemma 2.1 yields that the previous proposition holds for the following class of manifolds M: $\mathrm{Ric}_M \geqslant 1$. $|K_M| \geqslant \Lambda^2$, $\|\nabla R_M\| \leqslant \Lambda_1$.

The proof of Proposition 2.3 suggests that it is possible to bound $\|dF\|$ from below. This is done in Lemma 2.5.

Lemma 2.4. *If a complete manifold M satisfies that $K_M \geqslant 1$ and $M - B(x, \pi - \varepsilon) \neq \phi$ for an ε, then* $\mathrm{diam}(M - B(x, \pi - \varepsilon)) \leqslant 2\varepsilon$.

Proof. For arbitrary two point y_1 and y_2 of $M - B(x, \pi - \varepsilon)$ let γ_i be a minimal geodesic from x to y_i and let l_i denote the length $L(\gamma_i)$ of γ_i, $i = 1, 2$. Let $\bar\gamma_i$ denote geodesics in S^n such that $\bar\gamma_1(0) = \bar\gamma_2(0)$, $\measuredangle(\bar\gamma_1'(0), \gamma_2'(0)) = \measuredangle(\gamma_1'(0), \gamma_2'(0))$. Then the Toponogov comparison theorem implies

$$d(y_1, y_2) = d(\gamma_1(l_1), \gamma_2(l_2)) \leqslant d(\bar\gamma_1(l_1), \bar\gamma_2(l_2)) \leqslant 2\varepsilon. \qquad \text{Q.E.D.}$$

Lemma 2.5. *For given n, $\Lambda \geqslant 1$, $\Lambda_1 > 0$ and for $\varepsilon \in (0, \pi/2)$, $\varepsilon' \in (0, 1)$, there exists a $\delta_2 = \delta_2(n, \Lambda, \Lambda_1, \varepsilon, \varepsilon') > 0$ ($\leqslant \delta_1(n, \Lambda, \Lambda_1, \varepsilon/2)$) such that if M is a complete manifold of dimension n such that $1 \leqslant K_M \leqslant \Lambda^2$, $\|\nabla R_M\| \leqslant$*

Λ_1, $\mathrm{Vol}(M) \geqslant \mathrm{Vol}(S^n) - \delta_2$, then $1 \geqslant \|dF\| \geqslant 1-\varepsilon'$ on $B(p, \pi-\varepsilon) \subset S^n$, where F: $U \to M - C(x)$ is the diffeomorphism constructed in Section 1.

Proof. Take a δ_2 with $\delta_2 \leqslant \delta_1(n, \Lambda, \Lambda_1, \varepsilon/2)$ and let M satisfy the conditions. Suppose that there are $q_0 \in B(p, \pi-\varepsilon)$ and a unit vector $w \in S^n_{q_0}$ such that $\|dF(w)\| < 1-\varepsilon'$, where $q_0 =: \exp_p t_0 u$, $\|u\|=1$, $u \perp w$. Set $\tilde{w} := d(\exp_p^{-1})(w)$ and let $\{e_i\}$ be an orthonormal frame at x such that $e_1 = I(\tilde{w})/\|I(\tilde{w})\|$, $e_n = I(u)$. Let $\{x_i\}$ denote the normal coordinate system based on $\{e_i\}$. Then the comparison argument yields

$$\left\|\frac{\partial}{\partial x_1}\right\| \leqslant (1-\varepsilon')\frac{\sin t_0}{t_0}, \quad \left\|\frac{\partial}{\partial x_i}\right\| \leqslant \frac{\sin t_0}{t_0}, \ (2 \leqslant i \leqslant n-1)$$

at $F(q_0)$, hence,

$$\left\|\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}\right\| \leqslant (1-\varepsilon') J(t_0).$$

Let $\tau = \tau(n, \varepsilon, \varepsilon')$ be the solution of

$$(1-\varepsilon')\left(\frac{\sin(\pi-\varepsilon)}{\pi-\varepsilon} + \tau\right)^{n-1} = (1-\varepsilon'/2) J(\pi-\varepsilon),$$

and set $\eta := (1-\varepsilon')\tau/(n\Omega_2(n, \Lambda, \Lambda_1, \pi-\varepsilon/2))$. Then for every $w \in B(t_0 I(u), \eta)$

$$\left| \left\|\frac{\partial}{\partial x_i}(\exp_x w)\right\| - \left\|\frac{\partial}{\partial x_i}(F(q_0))\right\| \right| \leqslant n\Omega_2(n, \Lambda, \Lambda_1, \pi-\varepsilon/2)\|w - t_0 I(u)\|$$
$$\leqslant (1-\varepsilon')\tau,$$

hence,

$$\left\|\frac{\partial}{\partial x_1}(\exp_x w)\right\| \leqslant (1-\varepsilon')\left(\frac{\sin t_0}{t_0} + \tau\right),$$
$$\left\|\frac{\partial}{\partial x_i}(\exp_x w)\right\| \leqslant \frac{\sin t_0}{t_0} + \tau, \ (2 \leqslant i \leqslant n-1).$$

Since $\left\|\dfrac{\partial}{\partial x_n}\right\| \leqslant 1$, this implies that on $\exp_x(B(t_0 I(u), \eta))$

$$\left\|\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}\right\| < (1-\varepsilon')\left(\frac{\sin t_0}{t_0} + \tau\right)^{n-1} \leqslant (1-\varepsilon'/2) J(t_0).$$

Setting $\kappa_1 := \eta \dfrac{\sin(\pi-\varepsilon/2)}{\pi-\varepsilon/2}$, we have $\exp_x^{-1} F(B(q_0, \kappa_1)) \subset B(t_0 I(u), \eta)$. Choose a $\kappa_2 = \kappa_2(n, \varepsilon, \varepsilon')$ which satisfies the following inequality;

$$(1-\varepsilon'/2)J(t)/J(t+s)\leqslant 1-\varepsilon'/3$$

for every $t\in[0,\pi-\varepsilon]$ and every $s\in[0,\kappa_2]$. In fact, since the function $g(t)=\left(\frac{\sin t}{t}\right)\Big/\left(\frac{\sin(t+\kappa_2)}{t+\kappa_2}\right)$, $0\leqslant t\leqslant\pi-\kappa_2$, is monotone increasing, κ_2 is obtained as the solution of $g(\pi-\varepsilon)=\left(\frac{1-\varepsilon'/3}{1-\varepsilon'/2}\right)^{1/n-1}$. Thus if $\kappa:=\min\{\kappa_1,\kappa_2\}$, then

$$\begin{aligned}\operatorname{Vol}F(B(q_0,\kappa))&=\int_{B(q_0,\kappa)}\det(dF)\,dS^n\\&<(1-\varepsilon'/2)J(t_0)v(\kappa)/J(t_0+\kappa)\\&\leqslant(1-\varepsilon'/3)v(\kappa).\end{aligned}$$

On the other hand, Lemma 1.1 implies that $\operatorname{Vol}F(B(q_0,\kappa))\geqslant v(\kappa)-\delta_2$. Hence it appears that $\delta_2>\varepsilon' v(\kappa)/3$. Therefore the required δ_2 is given as $\delta_2=\min\{\delta_1(n,\Lambda,\Lambda_1,\varepsilon/2),\ \varepsilon' v(\kappa)/3\}$. Q.E.D.

§ 3. Proof of Main Theorem

The proof of Main Theorem is, roughly speaking, achieved as follows. In the situation of Lemma 2.5, if ε and ε' are taken sufficiently small, then $\operatorname{diam}(M-B(x,\pi-\varepsilon))$ is also small, and then F is almost isometric. As a result, for a suitable choice of $\{p_i\}\subset S^n$, $\{p_i\}$ and $\{F(p_i)\}$ will satisfy the conditions in Theorem C.

For an $\varepsilon>0$, a system of points $\{x_i\}$ in a metric space X is said to be an *ε-maximal system* if $\{x_i\}$ is maximal with respect to the property that the distance between any two of them is greater than or equal to ε. Notice that $\{x_i\}$ is an ε-maximal system if and only if it is an ε-dense and ε-discrete subset.

Proof of Main Theorem. For n, $\Lambda\geqslant 1$, Λ_1, $R=\pi/2$, let $\varepsilon_1=\varepsilon_1(n)$, $r_1=r_1(n,\Lambda,\Lambda_1,\pi/2)$ be the constants given in Theorem *C*. For $\eta:=3/4\cdot 2^{-(n+8)}r_1$, set

$$\varepsilon:=\eta\varepsilon_1/(4(1+\varepsilon_1)),\quad \varepsilon':=\varepsilon_1/(4(1+\varepsilon_1)).$$

Then the required constant δ is obtained as $\delta=\delta_2(n,\Lambda,\Lambda_1,\varepsilon,\varepsilon')$ where δ_2 is the constant given in Lemma 2.5. Let M satisfy the conditions in Main Theorem. Take an η-maximal system $\{p_i\}$ of S^n and choose a point $\bar{p}$ of S^n such that $d(\bar{p},\{p_i\})\geqslant\eta/2$. Let p be the antipodal point of $\bar{p}$, and for a fixed point $x\in M$, let $F\colon U\to M-C(x)$ be the diffeomorphism constructed in Section 1. Notice that $i(M)>\pi-\varepsilon/2$ and $1\geqslant\|dF\|\geqslant 1-\varepsilon'$ on $B(p,\pi-\varepsilon)$.

Assertion. If $y_i:=F(p_i)$, then $\{y_i\}$ is $2^{-(n+8)}r_1$-dense and $2^{-(n+9)}r_1$-discrete and satisfy $(1+\varepsilon_1)^{-1}\leqslant d(y_i, y_j)/d(p_i, p_j)\leqslant 1+\varepsilon_1$ for every $i\neq j$.

Therefore by Theorem C, M is diffeomorphic to S^n.

Proof of Assertion. (1) denseness. Take an arbitrary point y in M. If $y\in \bar{B}(x, \pi-\varepsilon)$, then we can take a p_i with $d(p_i, F^{-1}(y))\leqslant\eta$. If the minimal geodesic from $F^{-1}(y)$ to p_i intersects the ball $\bar{B}(\bar{p}, \varepsilon)$, replacing the intersection by a minimal curve in the boundary $\partial\bar{B}(p, \varepsilon)$, we construct a curve from $F^{-1}(y)$ to p_i in $M-B(p, \varepsilon)$ with length$\leqslant\eta+\pi\varepsilon$. Since F is length nonincreasing, $d(y, y_i)\leqslant\eta+\pi\varepsilon<4\eta/3$. If $y\in M-\bar{B}(x, \pi-\varepsilon)$, take $y'\in\partial B(x, \pi-\varepsilon)$ and p_i with $d(y, y')\leqslant\varepsilon$, $d(p_i, F^{-1}(y'))\leqslant\eta$. Then the above argument implies

$$d(y, y_i)\leqslant d(y, y')+d(y', y_i)\leqslant\varepsilon+(\eta+\pi\varepsilon)\leqslant 4\eta/3.$$

Hence $\{y_i\}$ is $2^{-(n+8)}r_1$-dense.

(2) discreteness. For any y_i, y_j, let σ be a minimal geodesic from y_i to y_j, and let σ_1, σ_2 be the maximal geodesic segments of σ such that $\sigma_1(0)=y_i$, $\sigma_2(L(\sigma_2))=y_j$, $\mathrm{Int}\,\sigma_k\subset B(x, \pi-\varepsilon)$, $k=1, 2$ (possibly $\sigma_1=\sigma$). Notice that $L(\sigma_k)\geqslant(1-\varepsilon')L(F^{-1}\circ\sigma_k)\geqslant(1-\varepsilon')(\eta/2-\varepsilon)$ by Lemma 2.5. This yields

$$\begin{aligned} d(y_i, y_j)&\geqslant L(\sigma_1)+L(\sigma_2)\geqslant(1-\varepsilon')(L(F^{-1}\circ\sigma_1)+L(F^{-1}\circ\sigma_2))\\ &\geqslant 2(1-\varepsilon')(\eta/2-\varepsilon)>2\eta/3. \end{aligned}$$

Hence $\{y_i\}$ is $2^{-(n+9)}r_1$-discrete.

By estimating the constant $d(y_i, y_j)/d(p_i, p_j)$ we will complete the proof. Let γ be a minimal geodesic from p_i to p_j and let γ_1, γ_2 be the maximal geodesic segments of γ such that $\gamma_i(0)=p_i$, $\gamma_2(L(\gamma_2))=p_j$, $\mathrm{Int}\,\gamma_k\subset B(p, \pi-\varepsilon)$, $k=1, 2$ (possibly $\gamma_1=\gamma$). Then Lemma 2.4 implies

$$\begin{aligned} d(y_i, y_j)/d(p_i, p_j)&\leqslant(L(F\circ\gamma_1)+L(F\circ\gamma_2)+2\varepsilon)/(L(\gamma_1)+L(\gamma_2))\\ &\leqslant(L(\gamma_1)+L(\gamma_2)+2\varepsilon)/(L(\gamma_1)+L(\gamma_2))\\ &=1+2\varepsilon/(L(\gamma_1)+L(\gamma_2)), \end{aligned}$$

where $L(\gamma_k)\geqslant d(p_i, \bar{p})-\varepsilon\geqslant\eta/2-\varepsilon$, hence

$$d(y_i, y_j)/d(p_i, p_j)\leqslant 1+2\varepsilon/(\eta-2\varepsilon)<1+\varepsilon_1.$$

On the other hand, under the notations in (2), we have

$$\begin{aligned} d(y_i, y_j)/d(p_i, p_j)&\geqslant(L(\sigma_1)+L(\sigma_2))/(L(F^{-1}\circ\sigma_1)+L(F^{-1}\circ\sigma_2)+2\varepsilon)\\ &\geqslant(1-\varepsilon')(L(\sigma_1)+L(\sigma_2))/(L(\sigma_1)+L(\sigma_2)+2\varepsilon(1-\varepsilon'))\\ &=(1-\varepsilon')-2\varepsilon(1-\varepsilon')^2/(L(\sigma_1)+L(\sigma_2)+2\varepsilon(1-\varepsilon')),\\ &>1-(\varepsilon'+\varepsilon(1-\varepsilon')^2)/((1-\varepsilon)(\eta/2-\varepsilon))\\ &>(1+\varepsilon_1)^{-1}. \end{aligned}$$

Q.E.D.

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