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On Deformations of the C_l-Metrics on Spheres

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1. This note is a summary of our recent result concerning the existence problem of deformations of the standard metric by $C_{2\pi}$-metrics on the n-dimensional sphere S^n. By definition a riemannian metric g on a manifold M is called a C_l-metric if all of its geodesics are closed and have the common length l. Let $\{g_t\}$ be a one-parameter family of $C_{2\pi}$-metrics on S^n with g_0 being the standard one, and put

$$\frac{d}{dt} g_t|_{t=0} = h.$$

We call such a symmetric 2-form h an infinitesimal deformation. It is known that each infinitesimal deformation h satisfies the so-called zero-energy condition, i.e.,

$$\int_0^{2\pi} h(\dot{\gamma}(s), \dot{\gamma}(s)) ds = 0$$

for any geodesic $\gamma(s)$ of (S^n, g_0) parametrized by arc-length (cf. [1] p. 151). In [3] we gave another necessary condition, the second order condition, for a symmetric 2-form to be an infinitesimal deformation, and showed that there are symmetric 2-forms which satisfy the zero-energy condition, but not satisfy the second order condition in the case of S^n $(n \geqq 3)$. The present theorem is an extension of the result in [3]. We first review the second order condition, and then state the theorem.

2. Let K_2 be the vector space of symmetric 2-forms on S^n which satisfy the zero-energy condition. Let $\sharp$ be the bundle isomorphism from the cotangent bundle T^*S^n to the tangent bundle TS^n obtained by the riemannian metric g_0. Define the function $\hat{h}$ on T^*S^n for a symmetric 2-form h by

$$\hat{h}(\lambda) = h(\sharp(\lambda), \sharp(\lambda)), \ \lambda \in T^*S^n.$$

Let S^*S^n be the unit cotangent bundle with respect to the metric g_0. We

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denote by $\widetilde{H}_2$ the image of the map $\hat{}$, and by H_2 the vector space of functions on S^*S^n which are the restrictions of functions in $\widetilde{H}_2$ to S^*S^n. Put $E_0=(1/2)\hat{g}_0$. There is a homogeneous symplectic vector field $X(h)$ on T^*S^n-{0-section} such that $X(h)E_0=\hat{h}$, provided $h\in K_2$. Let G be the linear operator on $C^\infty(S^*S^n)$ defined by

$$G(f)(\lambda)=\frac{1}{2\pi}\int_0^{2\pi} f(\xi_t\lambda)dt,\ \lambda\in S^*S^n, f\in C^\infty(S^*S^n),$$

where $\{\xi_t\}$ is the geodesic flow associated with g_0. Then we can define the symmetric bilinear map F: $K_2\times K_2\to G(C^\infty(S^*S^n))$ by

$$F(f,h)=G(X(f)\hat{h}),\qquad f,h\in K_2,$$

where $X(f)\hat{h}$ is considered as a function on S^*S^n by restriction. We say $h\in K_2$ satisfies the second order condition if $F(h,h)\in G(H_2)$. It can be seen that each infinitesimal deformation satisfies the second order condition (cf. [3] Theorem 1).

3. We now assume that $n\geqq 3$. Consider S^n as the unit sphere in $\boldsymbol{R}^{n+1}$, and let ι: $S^n\to\boldsymbol{R}^{n+1}$ be the inclusion. Let $x=(x_1,\cdots,x_{n+1})$ be the canonical coordinate system on $\boldsymbol{R}^{n+1}$. Let $\boldsymbol{R}[x]_m$ be the vector space of homogeneous polynomials of degree m in the variables x, and set

$$\boldsymbol{R}[x]_{\mathrm{od}}=\sum_{k\geqq 0}\boldsymbol{R}[x]_{2k+1}.$$

It is easy to see that $(\iota^*f)g_0\in K_2$ for any $f\in\boldsymbol{R}[x]_{\mathrm{od}}$.

Theorem. *Let $f\in\boldsymbol{R}[x]_{\mathrm{od}}$. Then $(\iota^*f)g_0$ satisfies the second order condition if and only if f has one of the following forms*:

$$\text{(i)}\qquad f\equiv h_1+h_3+\sum_{i=2}^{m}\Big(\sum_k a_kx_k\Big)^{2i}\Big(\sum_j b_{ij}x_j\Big)\qquad \mathrm{mod}\ \Big(1-\sum_i x_i^2\Big),$$

$a_k, b_{ij}\in\boldsymbol{R}$, $h_1\in\boldsymbol{R}[x]_1$, $h_3\in\boldsymbol{R}[x]_3$;

$$\text{(ii)}\qquad f\equiv h_1+h_3+cA^*h\qquad \mathrm{mod}\ \Big(1-\sum_i x_i^2\Big),$$

$h_1\in\boldsymbol{R}[x]_1$, $h_3\in\boldsymbol{R}[x]_3$, $c\in\boldsymbol{R}$, $A\in O(n+1,\boldsymbol{R})$, and

$$h=\sum_{i=2}^{10}\alpha_{2i+1}x_1^{2i+1}+\sum_{i=2}^{6}\beta_{2i+1}x_1^{2i}x_2+\sum_{i=2}^{6}\gamma_{2i+1}x_1^{2i-1}x_2^2+\delta_5x_1^2x_2^3+\varepsilon_5x_1x_2^4,$$

where the coefficients satisfy the conditions $\beta_{13}\in\boldsymbol{R}$, $\gamma_{13}\in\boldsymbol{R}-\{0\}$,

$$\alpha_5=\frac{10}{13}\gamma_{13}-\frac{5^2}{13^2\cdot 4^3\cdot 3}\frac{\beta_{13}^4}{\gamma_{13}^2}+\frac{15}{4}\frac{\beta_{13}^2}{\gamma_{13}}+45,$$

$$\alpha_7=-\frac{10}{13}\gamma_{13}-5\frac{\beta_{13}^2}{\gamma_{13}}-120,\ \alpha_9=\frac{5}{13}\gamma_{13}+\frac{15}{4}\frac{\beta_{13}^2}{\gamma_{13}}+210,$$

$$\alpha_{11}=-\frac{1}{13}\gamma_{13}-\frac{3}{2}\frac{\beta_{13}^2}{\gamma_{13}}-252,\ \alpha_{13}=\frac{1}{4}\frac{\beta_{13}^2}{\gamma_{13}}+210,$$

$$\alpha_{15}=-120,\quad \alpha_{17}=45,\quad \alpha_{19}=-10,\quad \alpha_{21}=1,$$

$$\beta_5=\frac{75}{13}\beta_{13}-\frac{5^2}{13^2\cdot 3\cdot 8}\frac{\beta_{13}^3}{\gamma_{13}},\ \beta_7=-\frac{140}{13}\beta_{13},\ \beta_9=\frac{135}{13}\beta_{13},$$

$$\beta_{11}=-\frac{66}{13}\beta_{13},\ \gamma_5=\frac{25}{13}\gamma_{13}-\frac{25}{13^2\cdot 8}\beta_{13}^2,$$

$$\gamma_7=-\frac{70}{13}\gamma_{13},\ \gamma_9=\frac{90}{13}\gamma_{13},\ \gamma_{11}=-\frac{55}{13}\gamma_{13},$$

$$\delta_5=-\frac{25}{13^2\cdot 6}\gamma_{13}\beta_{13},\quad \varepsilon_5=-\frac{25}{13^2\cdot 12}\gamma_{13}^2.$$

Remark 1. In the case of S^2, it has been proved by Guillemin [2] that all elements of K_2 are infinitesimal deformations.

Remark 2. Let u be an odd polynomial in one variable, and put $f=u(\sum_k a_k x_k)$. Then f belongs to the class (i) in Theorem. We can see that $(\iota^* f)g_0$ is an infinitesimal deformation. In fact this corresponds to a family of $C_{2\pi}$-metrics constructed by Zoll and Weinstein (cf. [1] p. 120). The details will be given in [4].

Bibliography

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