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Geodesic Flows and Geodesic Random Walks

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§ 0. Introduction

A flow on a set X is a family of bijections $\varphi_t\colon X\to X$, $t\in \boldsymbol{R}$, which obeys group property $\varphi_{t+s}=\varphi_t\circ\varphi_s$. Such an object (X, φ_t) arises in many contexts of mathematics. A typical example is the shift operation on a mapping space $\mathrm{Map}(\boldsymbol{R}, M)$ defined by $(\varphi_t c)(s)=c(t+s)$. Although it may seem that this flow has no interesting feature at first sight, various examples of flows in differential geometry appear in fact as subshifts of $(\mathrm{Map}(\boldsymbol{R}, M), \varphi_t)$. For instance, let M be a connected complete Riemannian manifold, and let X be the set of all geodesics $c\colon \boldsymbol{R}\to M$. Then the

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shift φ_t leaves the subset X invariant. The flow $(X, \varphi_t | X)$ is just what we call the geodesic flow of M, and a main object in this paper. The set of piecewise geodesic curves on M is also invariant under φ_t, which we call, in this paper, the shift on geodesic chains. The Wiener flow on M is defined as the restriction of φ_t to the space of continuous curves c: $\boldsymbol{R} \to M$.

In many cases, a flow (X, φ_t) has a natural invariant measure. For example, regarding the geodesic flow as the trajectory of motions of a free particle on a manifold, we can make use of Hamiltonian formalism to define an invariant measure for the flow. As for the Wiener flow, theory of Brownian motions on a manifold allows us to introduce an invariant measure. Geodesic random walks on M come up in defining a measure on the set of geodesic chains in much the same way as free motions and Brownian motions come up in each cases. Although, in this case, there is no natural $\{\varphi_t\}$-invariant measure, we can still equip a φ_1-invariant measure provided that we confine ourselves to a particular set of geodesic chains.

This article attempts to survey some geometric aspects of these flows. In the nature of the case, we must have concentrated on certain special topics. Most of results stated are known, so that if the reader would like to know the detail, he should realy read the original paper. On generality of geodesic flows, we refer to W. Klingenberg [36] [37].

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I. Geodesic Flows

§ I-1. Hamiltonian formalism and invariant measure

Let (X, φ_t) be the geodesic flow defined on a complete Riemannian manifold M. The subflow of (X, φ_t) consisting of geodesics with unit speed is denoted by (X_1, φ_t). The correspondence $c \mapsto \dot{c}(0)$ yields a bijection of X onto the tangent bundle TM, under which X_1 goes to the unit sphere bundle $S_1M = \{v \in TM; \|v\| = 1\}$ and the set of constant geodesics goes to the zero section $M \subset TM$. We define measures μ and μ_1 by the relationships

$$\int_{TM} F d\mu = \int_M dg(x) \int_{T_xM} F dv$$

$$\int_{S_1M} F d\mu = \int_M dg(x) \int_{S_xM} F dS_x(v),$$

where dg is the canonical density associated to the metric on M, dv is the

Lebesgue density of the Euclidean space T_xM and $dS_x(v)$ is the ordinary uniform density on the unit sphere $S_xM=\{v \in T_xM; \|v\|=1\}$. It is clear that the time-reversing operation $\tau: c(t)\mapsto c(-t)$ leaves the measures μ and μ_1 invariant.

The following is a special case of the Liouville theorem.

Lemma I-1-1. *The measures μ and μ_1 are invariant under φ_t-action.*

Since this is quite fundamental in the ergodic theory of geodesic flows, we will go into detail. We need some knowledge of symplectic geometry. (see [1])

A C^∞-manifold S with a closed non degenerate 2-form ω is called a symplectic manifold. Non degeneracy of ω means that to each tangent vector $v \in T_xS$, we can associate a 1-form ξ on T_xS by the formula $\omega(w, v) =\langle \xi, w\rangle$. We denote by I the isomorphism I: $T_x^*S\to T_xS$ constructed in this way. Using I, we can associate a vector field X_f for each C^∞-function f on S by putting $X_f=I(df)$, which is called the hamilton vector field associated to f. We denote by φ_t the local 1-parameter group of diffeomorphisms generated by X_f. If the relations of the exterior differentiations and Lie derivatives are used, it is easily shown that $\varphi_t^*\omega=\omega$, or equivalently $\mathscr{L}_{X_f}\omega=0$, where $\mathscr{L}$ denotes the Lie derivative. In particular, the volume element $\omega^n=\omega\wedge\cdots\wedge\omega$ is preserved by φ_t.

The cotangent bundle T^*M has a canonical symplectic structure with a form $\omega=-d\theta$, where θ is a 1-form on T^*M defined by $\theta(v)= \langle p, d\pi(v)\rangle$, $p \in T^*M$, $v \in T_pT^*M$, π: $T^*M\to M$ being the bundle projection.

We assume hereafter that M is a complete manifold, and identify T^*M with TM. It is easily seen that the measure μ coincides with ω^n (up to a constant multiple). We set: $h(p)=1/2\|p\|^2$, $p \in T^*M$. Then the flow associated to X_h is just the geodesic flow. Indeed, the equation $d\varphi_t/dt=X_h(\varphi_t)$ is written as, in terms of local coordinates

$$\begin{cases} \dfrac{dp_i}{dt}=-\dfrac{1}{2}\sum\dfrac{\partial g^{jk}}{\partial x_i}p_jp_k \\ \dfrac{dx_i}{dt}=\sum g^{ij}p_j, \end{cases} \qquad (p=\sum p_i dx_i)$$

from which the equation of geodesics comes out. Thus the invariance of the measure μ was established.

We now turn to the case of μ_1.

Lemma I-1-2. *Let (S, φ_t) be a smooth flow with a smooth invariant measure μ. Let f: $S\to N$ be a C^∞-submersion onto a C^∞-manifold N with* *a C^∞-density $d\nu$. Suppose $f\circ\varphi_t=f$. Then there exists a unique density $d\nu_n$ on each fiber $f^{-1}(n)$ such that*

$$\int_N d\nu \int_{f^{-1}(n)} F d\nu_n = \int_S F d\mu.$$

Moreover, $d\nu_n$ is invariant under φ_t.

We apply this lemma to the case $S=TM\backslash M$, $N=\boldsymbol{R}_+$, $f(p)=\|p\|$. Noting that $f^{-1}(r)=rS_1M$, we find

$$\int_0^\infty dr \int_M dg(x) \int_{rS_xM} F d(rS_x) = \int_M dg(x) \int_{T_xM} F dv = \int_S F d\mu.$$

Where $d(rS_x)$ denotes the canonical measure on the sphere rS_xM with radius r. This implies that μ_1 is an invariant measure on S_1M.

We denote by $L^2(S_1M)$ the Hilbert space of square integrable functions on S_1M with respect to the measure μ_1. The invariance of μ_1 means that the induced mapping $\varphi_t^*: L^2(S_1M)\to L^2(S_1M)$ forms a one-parameter group of unitary transformations.

Application I-1-3. Define the spherical mean operator $L_r: C_0(M)\to C_0(M)$ by

$$(L_r f)(x) = \frac{1}{\omega_{n-1}} \int_{S_xM} f(\exp rv) dS_x(v),$$

where ω_{n-1} is the volume of the $(n-1)$-dimensional unit sphere. If we denote by $\pi_*: L^2(S_1M)\to L^2(M)$ the adjoint of the pull back $\pi^*: L^2(M)\to L^2(S_1M)$, then

$$\pi_* F(x) = \frac{1}{\omega_{n-1}} \int_{S_xM} F dS_x(v), \quad \text{and} \quad L_r = \pi_* \varphi_r^* \pi^*.$$

Thus, L_r is extended to a self adjoint bounded operator $L_r: L^2(M)\to L^2(M)$ with $\|L_r\|\le 1$ (see [66]). This operator is considered as a transition operator of certain geodesic random walks on M (see Chap. III).

Application I-1-4. Let E be a Riemannian vector bundle on M. The parallel translation in E along a piecewise smooth curve $c: [a, b]\to M$ will be denoted by $P_c: E_{c(a)}\to E_{c(b)}$. We define, for $\tau>0$, the Gaussian mean value operator $K_\tau: C^\infty(E)\to C^\infty(E)$ by

$$(K_\tau s)(x) = (2\pi\tau)^{-n/2} \int_{T_xM} e^{-\|v\|^2/2\tau} P_{c_v}^{-1}(s(\exp v)) dv,$$

where $n = \dim M$, $c_v(s) = \exp sv$, $0 \le s \le 1$. This operator is intimately related to stochastic parallel displacement ([32]). We will show that K_τ is extended to a bounded operator: $L^2(E) \to L^2(E)$ with $\|K_\tau\| \le 1$, and that K_τ is a self adjoint operator. For this, we first note that

$$|(K_\tau s)(x)| \le (2\pi\tau)^{-n/2} \int_{T_xM} e^{-\|v\|^2/2\tau} |s(\exp v)|\, dv,$$

$$(2\pi\tau)^{-n/2} \int_{T_xM} e^{-\|v\|^2/2\tau} dv = 1.$$

Applying the Schwarz inequality, we get

$$\int_M |(K_\tau s)(x)|^2 dg(x) \le (2\pi\tau)^{-n/2} \int_{TM} e^{-\|v\|^2/2\tau} |s(\exp v)|^2 d\mu,$$

so that, noting $|s(\exp v)| = |s(\pi\varphi_1 v)|$ and $\|\varphi_1 v\|^2 = \|v\|^2$, we have

$$\begin{aligned}\|K_\tau s\|^2 &\le \int_M dg(x)(2\pi\tau)^{-n/2} \int_{T_xM} e^{-\|v\|^2/2\tau} |s(\pi v)|^2 dv \\ &= \int_M |s(x)|^2 dg(x) = \|s\|^2,\end{aligned}$$

which proves the first part of the assertion. We now take $s_1, s_2 \in C^0(E)$. Then

$$\begin{aligned}(K_\tau s_1, s_2)_E &= (2\pi\tau)^{-n/2} \int_M dg(x) \int_{T_xM} e^{-\|v\|^2/2\tau} \langle P_{c_v}^{-1} s_1, s_2\rangle_x dv \\ &= (2\pi\tau)^{-n/2} \int_{TM} e^{-\|v\|^2/2\tau} \langle P_{c_v}^{-1} s_1, s_2\rangle_{\pi(v)} d\mu(v).\end{aligned}$$

The function $f_{(s_1,s_2)}(v) = \langle P_{c_v}^{-1} s_1, s_2\rangle_{\pi(v)}$ satisfies the relation $f_{(s_1,s_2)}(v) = f_{(s_2,s_1)}(\varphi_{-1}(-v))$. Since the transformation $v \mapsto \varphi_{-1}(-v)$ preserves the density $d\mu$, we finally observe that the above integral equals

$$(2\pi\tau)^{-n/2} \int_{TM} e^{-\|v\|^2/2\tau} \langle P_{c_v}^{-1} s_2, s_1\rangle_{\pi(v)} d\mu(v) = (s_1, K_\tau s_2)_E.$$

The following is another consequence of the existence of an invariant measure.

Lemma I-1-5 (A special case of Poincaré's recurrence theorem). *If M has finite volume, then the set of vectors $v \in S_1M$ such that there exists a sequence $t_1 < t_2 < \cdots \uparrow \infty$ with $\varphi_{t_i} v \to v$ is dense in S_1M.*

Application I-1-6. A smooth vector field X on a complete Riemannian manifold M is called dissipative if, for any geodesic $c: \boldsymbol{R} \to M$, the function $t \mapsto \langle X(c(t)), \dot{c}(t) \rangle$ is non-increasing. A C^∞-convex function f on M yields a dissipative vector field $X = -\operatorname{grad} f$. Killing vector fields are also dissipative because in this case $t \mapsto \langle X(c(t)), \dot{c}(t) \rangle$ is constant (see § I-2). We will show that, if M has finite volume, then any dissipative field must be Killing. Take $v \in S_1M$ such that $\varphi_{t_i}v \to v$, $t_i \uparrow \infty$. Since $\langle X(\pi(v)), v \rangle \geq \langle X(\pi\varphi_t v), \varphi_t v \rangle \geq \lim_{i\to\infty} \langle X(\pi\varphi_{t_i}v), \varphi_{t_i}v \rangle = \langle X(\pi(v)), v \rangle$, the function $t \mapsto \langle X(\pi(\varphi_t v)), \varphi_t v \rangle$ is constant. From the above lemma, it follows that for any geodesic $c: \boldsymbol{R} \to M$, $t \mapsto \langle X(c(t)), \dot{c}(t) \rangle$ is constant, so that X is a Killing vector field. Similar argument was taken up by S. T. Yau [74] in showing non-existence of convex functions on a manifold with finite volume. In fact, the above statement is considered as a partial generalization. As a corollary, one has:

Corollary I-1-7. *Let X be a conformal vector field on a complete manifold (M, g) with finite volume such that $\mathscr{L}_X g = \lambda g$ where λ is a non-positive function or non-negative function on M. Then X is a Killing vector field.*

This is a consequence of the identity: $(\mathscr{L}_X g)(u, v) = (\nabla_u X, v) + (u, \nabla_v X)$, which indeed implies that the vector field X or $-X$ is dissipative. S. Yorozu [76] has proved the corollary under the assumption that X has finite L^2-norm.

§ I-2. First integrals

Let (S, ω) be a symplectic manifold, and let f, h be C^∞-functions on S. The Poisson bracket $\{f, h\}$ is a function on S defined by

$$\{f, h\} = \frac{d}{dt}\bigg|_{t=0} f(\varphi_t x),$$

where $\{\varphi_t\}$ is the (local) 1-parameter group of diffeomorphisms generated by X_h. It is easily shown that $\{f, h\} = df(X_h) = X_h f = \omega(X_h, X_f)$, and $[X_f, X_h] = X_{\{f,h\}}$. The set of C^∞-functions on S forms a Lie algebra with respect to the bracket $\{\ ,\ \}$, and the correspondence $f \mapsto X_f$ is a Lie algebra homomorphism into the Lie algebra of C^∞-vector fields on S.

A C^∞-function f is called a first integral of h (or, of the flow associated to h) if $\{f, h\} = 0$, or equivalently if f is constant on every orbits of the flow $\{\varphi_t\}$. The function h itself is a first integral of h, so that φ_t leaves the hypersurface $\{h = \text{constant}\}$ invariant. The set of all first integrals of h forms not only a Lie subalgebra of $C^\infty(S)$, but also a ring with the ordinary multiplication. Moreover, substitution of first integrals in smooth functions yields new first integrals. First integrals of the form $F(h)$ are said to be trivial.

Return to the case of geodesic flows. We consider the function $h(p)=1/2\|p\|^2$ on the manifold $S=T^*M\backslash M\,(=TM\backslash M)$. Since $\varphi_t(p/\|p\|) =\varphi_{t/\|p\|}(p)/\|p\|$, a non constant φ_t-invariant function F on S_1M yields a non trivial first integral $f(p)=F(p/\|p\|)$, vice versa.

By definition, a function f on S is called a homogeneous polynomial of degree k along the fiber if the restriction $f|(T_x^*M\backslash(0))$ is a homogeneous polynomial of degree k for any $x\in M$. Thus the function h is a homogeneous polynomial of degree two along the fiber. Given a C^∞-function f on T^*M, we let $f\sim\sum_{k=0}^\infty f_k$ be the Taylor expansion along the fiber about zero section, where f_k is a homogeneous polynomial along the fiber of degree k. If $f|S$ is a first integral, then so is f_k.

The set of homogeneous polynomials of degree k along the fiber is canonically identified with $C^\infty(\mathscr{S}^kTM)$, the space of smooth k-symmetric tensor fields. We denote by $P_k\subset C^\infty(\mathscr{S}^kTM)$ the subspace consisting of first integrals.

Proposition I-2-1. *P_k is the solution space of a certain total differential equation of order $k+1$. In particular*, $\dim P_k<\infty$.

In fact, the equation $\{f,h\}=0, f\in C^\infty(\mathscr{S}^kTM)$ is a differential equation of first order. Differentiating this up to $k+1$-th order and solving a linear algebraic equation lead to the assertion.

The case $k=1$ is classical. Let $f(p)=\langle X,p\rangle\in P_1$, where X is a vector field on M. In terms of local coordinates, the equation $\{f,h\}=0$ reduces to

$$\sum_{ijk}\left(\xi_k\frac{\partial g^{ij}}{\partial x_k}-2\frac{\partial\xi_i}{\partial x_k}g^{jk}\right)p_ip_j=0,\qquad X=\sum\xi^i\frac{\partial}{\partial x_i}.$$

so that X must satisfy

$$\sum_k\left(\xi_k\frac{\partial g^{ij}}{\partial x_k}-\frac{\partial\xi_i}{\partial x_k}g^{jk}-\frac{\partial\xi_j}{\partial x_k}g^{ik}\right)=0.$$

This is just the equation for Killing vector fields. Thus, f is a first integral if and only if X is a Killing vector field, and $\dim P_1\leq n(n+1)/2$.

It is known that Killing vector fields are characterized by the commutativity with the Laplacian. We will see that a similar implication holds in more general situation. Let $P\colon C^\infty(M)\to C^\infty(M)$ be a differential operator of k-th order. The symbol $\sigma_k(P)$ is a homogeneous polynomial of degree k along the fiber, which is defined by

$$\sigma_k(P)(p)=\frac{1}{k!}P(\rho^k)(x),$$

where ρ is a C^∞-function on M with $\rho(x)=0$, $d_x\rho=p$. If P_1 is another differential operator of $k_1^{\text{-th}}$ order, then

$$\sigma_{k+k_1}(P\cdot P_1)=\sigma_k(P)\cdot\sigma_{k_1}(P_1)$$
$$\sigma_{k+k_1-1}([P, P_1])=\{\sigma_k(P), \sigma_{k_1}(P_1)\}.$$

Note that $\sigma_2(\Delta)(p)=\|p\|^2$. Therefore, if P commutes with Δ, then the symbol $\sigma_k(P)$ belongs to P_k.

We denote by $\mathfrak{g}$ the Lie algebra of Killing vector fields, which is, as is well-known, identified with the Lie algebra of the isometry group of M. We define a mapping P: $T^*M\to\mathfrak{g}^*$ by $\langle P(p), X\rangle=\langle X, p\rangle$. P is just what is called the Momentum mapping (see for detail [1]). Composition $h\circ P$, h: $\mathfrak{g}^*\to\boldsymbol{R}$ a C^∞-function, yields a first integral.

There is another way to construct first integrals. Suppose there are a smooth mapping Φ of a symplectic manifold S into the space of $N\times N$-hermitian matrices and a mapping B of S into the space of skew hermitian matrices such that $\Phi(\varphi_t x)=\exp tB(x)\cdot\Phi(x)\cdot\exp -tB(x)$, or equivalently $(X_h\Phi)(x)=[B(x), \Phi(x)]$. This being the case, the coefficients of the characteristic polynomial $\det(zI_N-\Phi(x))$ are first integrals of the flow φ_t.

Example I-2-2 (Geodesic flows on ellipsoids). Let E be a positive definite symmetric $n+1$-by-$n+1$ matrix. We set $M=\{x\in\boldsymbol{R}^{n+1}, \langle Ex, x\rangle=1\}$.

$$\frac{dx}{dt}=y, \quad \frac{dy}{dt}=-Ex\cdot\langle Ey, y\rangle/\|Ex\|^2, \quad x\in M, \quad \langle y, Ex\rangle=0.$$

(Generally, the equation of geodesics on a hypersurface $M=\{x_1, \cdots, x_{n+1}); f(x)=0\}$ is $dx/dt=y$, $dy/dt=-\operatorname{grad} f\cdot\operatorname{Hess}(y, y)/\|\operatorname{grad} f\|^2$. In fact, $\langle y, dy/dt\rangle=\langle y, \operatorname{grad} f\rangle=0$, hence $dy/dt=C\cdot\operatorname{grad} f$. On the other hand, $0=(d^2/dt^2)f(x(t))=\langle\operatorname{grad} f, dy/dt\rangle+\operatorname{Hess} f(y, y)$, so $C=-\operatorname{Hess} f(y, y)/\|\operatorname{grad} f\|^2$.)

We set $\Phi(x, y)=P_y(E^{-1}-x\otimes x)P_y$, where $x\otimes x$ is a linear mapping defined by $(x\otimes x)z=\langle z, x\rangle x$, and P_y is the orthogonal projection onto the orthogonal complement of the vector y. If we define a linear mapping $B(x, y)$ by $B(x, y)z=\langle Ex, z\rangle Ey-\langle Ey, z\rangle Ex$, then we find that $X_h\Phi=[B, \Phi]$ (see [51] for detail).

§ 1-3. Geodesic flows with many first integrals

Existence of many first integrals means that every orbit of the flow remains on a surface with high codimension for all moments of time. An extreme case is:

Proposition 1-3-1. *Suppose that the geodesic flow on a compact manifold M has many first integrals in the sense that the set of first integrals separates the orbits. Then all geodesics in M are periodic. The converse is also true.*

Proof. The assumption implies that every orbits are closed sets, so that these must be periodic. The opposite implication is somewhat difficult. We define a Riemannian metric on S_1M such that the orbits of the geodesic flow are geodesic curves. Indeed, this metric is given as the induced one from the so-called Sasaki metric on TM (see S. Sasaki [58] or A. L. Besse [8]). Thus, we are in position to apply the Wadsley theorem (see [8]) which asserts that, if every orbits of a flow $\{\varphi_t\}$ on a Riemannian manifold are non constant periodic geodesics, then there is an S^1-action with the same orbits as $\{\varphi_t\}$. Let $S^1p_1 \neq S^1p_2$ be two orbits, and let f' be a smooth function on S_1M such that, on each S^1p_1, S^1p_2, f' is constant and takes different values. We set $f(p)=\int_{S^1} f'(s\cdot p)ds$. Clearly f is constant on every orbits, and $f(S^1p_1)=f'(S^1p_1)\neq f'(S^1p_2)=f(S^1p_2)$. Q.E.D.

Milder than the above, but still enough to describe the shape of orbits is the case of complete integrable flows, which has a model in the geodesic flow on a flat torus.

Let $L\subset \boldsymbol{R}^n$ be a discrete subgroup of maximal rank $(=n)$, and let $M=\boldsymbol{R}^n/L$ be a flat torus with canonical metric. Then, using the identification $T^*M\simeq M\times \boldsymbol{R}^n$, we find that $\varphi_t(x, v)=(x+tv, v)$. If f: $T^*M\to\boldsymbol{R}$ denotes the function defined by $f_i(x, v)=v_i$, the *i-th* coordinate of v, then $\{f, h_i\}=0$, $\{f_i, f_j\}=0$, and $df_1\wedge\cdots\wedge df_n\neq 0$. We should note here that the mapping giving the identification $T^*M\simeq M\times\boldsymbol{R}^n$ is a symplectic diffeomorphism, where the symplectic structure is given by the form $\sum dx_i \wedge dv_i$.

Generally, the geodesic flow on M is called completely integrable if there exist n first integrals $f_1, \cdots, f_n$ such that $\{f_i, f_j\}=0$, $i, j=1, \cdots, n$, and $df_1\wedge\cdots\wedge df_n\neq 0$ on some open dense subset in $T^*M\backslash M$. This being the case, we denote by $F=f_1\times\cdots\times f_n$: $T^*M\backslash M\to\boldsymbol{R}^n$, and let $U\subset\boldsymbol{R}^n$ be an open set consisting of non-critical values of F. If $F|F^{-1}(U)$: $F^{-1}(U)\to U$ is a proper mapping, then $F|F^{-1}(v)$ is a locally trivial fibration and the fibers are a disjoint union of manifolds diffeomorphic to the torus T^n. In some circumstances, one can prove that a component of $F^{-1}(U)$ is symplectically diffeomorphic to $T^n\times U$ (see [6]).

Examples. i) If M is a surface with a non-zero Killing vector field X, then, $f_1=h, f_2(p)=\langle X, p\rangle$ give two first integrals with $\{f_1, f_2\}=0$. In particular, a surface of revolution has completely integrable geodesic flow, and the equation of geodesics reduces to Clairaut's equations (see Besse [8]).

ii) A surface all of whose geodesics are periodic and has a common period, especially the so-called Zoll surface has completely integrable geodesic flow.

iii) If E is positive definite symmetric matrix with distinct eigenvalues, then the geodesic flow on the ellipsoid $M=\{x; \langle Ex, x\rangle=1\}$ is completely integrable. In fact, the coefficients of the characteristic polynomial $\det(zI-\Phi)$ yield an involutive system $\{f_1,\cdots,f_n\}$. Two dimensional case was first treated by Jacobi, who solved the equation of geodesics by using elliptic functions (see for modern accounts J. Moser [51] or W. Klingenberg [37]).

iv) Recently, A. S. Mishchenko proved that the geodesic flow on semisimple symmetric space is completely integrable (see [50]). Some cases have been already treated by A. Thimm [70] and K. Ii and S. Watanabe [31] independently. Certain reduction procedure by momentum mappings plays important role.

Fine structures of completely integrable Hamiltonian systems have been recently investigated by many mathematicians. Some cases are closely connected with algebraic geometric nature of the first integrals (see [78] for instanse).

§ I-4. Geodesic flows with few first integrals

Consider the property of metrics on a compact manifold M that an orbit of the geodesic flow is dense in S_1M. For a metric with this property, there exists no first integral of the geodesic flow, except for trivial one. We are interested in how big the set of metrics with this property is in the totality of metrics. The theorem due to Kolmogorov-Arnold-Moser (see [4]) says that the property is not generic (at least in the two dimensional sphere). In fact, Hamiltonian flow which is sufficiently near to a "non degenerate" completely integrable system has "many" invariant tori, which implies that in the two dimensional case the orbits are shutted in a small region between two invariant tori, so that the orbits can not be dense.

If the geodesic flow has no nontrivial first integral, then $P_{2k+1}=0$, $P_{2k}=\boldsymbol{R}h^k$. This being the case, a differential operator P with $P\Delta=\Delta P$ is a polynomial in Δ. It is likely that metrics for which $P_{2k+1}=0$, $\dim P_{2k}$

$=1$ are generic. Somewhat related with these observation is the following.

Proposition I-4-1. *For a generic metric on M, any differential operator commuting with Δ is written as $P=f(\Delta)$ as L^2-operators, where f is a continuous function.*

Proof. According to K. Uhlenbeck [72], the set of metrics that all the eigenvalues of Δ are simple is generic. Hence, for such a metric, there exists a function $C(\lambda)$ of eigenvalues λ such that $P\varphi_\lambda=C(\lambda)\varphi_\lambda$, $\Delta\varphi_\lambda=\lambda\varphi_\lambda$. Interpolating $C(\lambda)$ by a continuous function $f(\lambda)$ on $\boldsymbol{R}$ leads to the assertion.

Problem. Can one choose a polynomial f with $P=f(\Delta)$ in the above?

So far we have treated only smooth first integrals. What happens if we loosen the assumption of smoothness? The following shows that we must impose some degree of regularity on first integrals to get meaningful results.

Lemma I-4-2. *There exists always a nontrivial generalized function on S_1M which is φ_t-invariant.*

In fact, this follows from existence of closed geodesics. Let $C\subset S_1M$ be a closed orbit. Define a generalized function $\mathscr{F}$ by $\langle\mathscr{F},f\rangle=\int_C f ds$, where ds is the translation invariant density on C. Then

$$\langle\varphi_t^*\mathscr{F},f\rangle=\langle\mathscr{F},\varphi_t^*f\rangle=\int_C f\circ\varphi_t ds=\int_C f ds=\langle\mathscr{F},f\rangle.$$

The most convenient class of invariant functions turns out to be the class of measurable functions.

Theorem I-4-3 (Ergodic theorem of Birkhoff). *Let (X,φ_t) be a flow on a measure space X with a finite invariant measure μ, and let $f\in L^1(X)$. The following limits exist almost everywhere:*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\varphi_t x)=\bar{f}_+(x)$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\varphi_{-t} x)=\bar{f}_-(x),$$

and $\bar{f}_+=\bar{f}_-=\bar{f}$ almost everywhere on X. Further, $\int_X f d\mu=\int_X \bar{f} d\mu$.

We shall say that a flow (X, φ_t) in the above theorem is ergodic if either one of the following conditions is satisfied.

(1) Any $\{\varphi_t\}$-invariant measurable set has measure zero, or its complement does.

(2) Any $\{\varphi_t\}$-invariant measurable function is constant almost everywhere.

(3) For any $f \in L^1(X)$, $\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\varphi_t x)dt = \left(\int_X f d\mu\right)\mu(X)^{-1}$.

The three conditions are actually equivalent. If the geodesic flow (S_1M, φ_t) is ergodic, then orbits of almost all $v \in S_1M$ are dense in S_1M.

One of geometric conditions which guarantee ergodicity is the so-called Anosov's condition:

Theorem I-4-4. (Anosov [3]). *Let (X, φ_t) be a C^∞-flow on a compact Riemannian manifold such that the tangent bundle can be written as the Whitney sum of three φ_t-invariant continuous subbundles, $TM = E^s \oplus E^u \oplus E^o$ where on E^s φ_t is contracting, on E^u φ_t is expanding and E^o is one-dimensional and tangent to the flow. That is, there exist constants $c>0$ and $\lambda>0$ so that $\|d\varphi_t v\| \le ce^{-\lambda t}\|v\|$ when $v \in E^s$, $t>0$, and $\|d\varphi_{-t}v\| \le ce^{-\lambda t}\|v\|$ when $v \in E^u$, $t>0$. If, in addition, φ_t has an invariant smooth measure μ, then (X, φ_t) is ergodic with respect to μ.*

A flow (X, φ_t) satisfying the above condition is called of Anosov type. We will give an outline of proof. For each $x \in X$, one can construct submanifolds

$$W^s(x) = \{y;\ d(\varphi_t x, \varphi_t y) \to 0 \text{ as } t \to +\infty\} \quad \text{(stable manifold)}$$

$$W^u(x) = \{y;\ d(\varphi_{-t} x, \varphi_{-t} y) \to 0 \text{ as } t \to +\infty\} \quad \text{(unstable manifold)}$$

such that $T_x W^s(x) = E^s x$, $T_x W^u(x) = E^u$, $W^s(x)$ and $W^u(x)$ vary continuously with x in a neighbourhood of x, and $\varphi_t(W^s(x)) = W^s(\varphi_t x)$, $\varphi_t(W^u(x)) = W^u(\varphi_t x)$. In order to prove the theorem, it is enough to show that for a continuous function on X, $\bar{f}_- =$ constant almost everywhere. Let $x_0 \in X$, and U be an open neighbourhood of x_0. If, for $x \in U$, $\bar{f}_+(x)$ exists and $y \in U \cap W^s(x)$, then $\bar{f}_+(x) = \bar{f}_+(y)$ since the points $\varphi_t x$ and $\varphi_t y$ approach each other with exponential speed. Similarly, if $\bar{f}_-(x)$ exists and $y \in U \cap W^u(x)$, then $\bar{f}_-(x) = \bar{f}_-(y)$. We then choose a point $x \in U$ such that, on the submanifold $\bigcup_{|t|<\varepsilon} \varphi_t(W^s(x) \cap U)$, the point z with $\bar{f}_+(z) = \bar{f}_-(z)$ forms a set of full measure, say A. We set: $B = \bigcup_{a \in A}(W^u(a)) \cap U$, which is a subset of full measure in U. We take any two points $x_1, x_2 \in B$. Suppose $x_1 \in W^u(a_1)$, $x_2 \in W^u(a_2)$. We then have $\bar{f}_-(x_1) = \bar{f}_-(a_1) = \bar{f}_+(a_1) = \bar{f}_+(a_2) = \bar{f}_-(a_2) = \bar{f}_-(x_2)$, hence $\bar{f}_-$ is constant almost everywhere in U. The theorem is proved.

Let M be a compact Riemannian manifold with negative sectional curvature. We denote by $\tilde{M}$ the universal covering of M. Let $v \in S_x\tilde{M}$, and set $\gamma(t)=\exp tv$. We consider the limit of the geodesic sphere of radius t with the center $\gamma(t)$ as $t\uparrow\infty$:

$$H(v)=\lim_{t\to\infty}\{y \in \tilde{M};\, d(y, \gamma(t))=t\}.$$

It is proved that $H(v)$ is a C^1-hypersurface in $\tilde{M}$ through the point x, which we usually call the horosphere determined by v. We set

$$\begin{aligned}
\tilde{W}^s(v)&=\{u \in S_y\tilde{M};\, y \in H(v),\, u \in (T_yH(v))^\perp, \\
&\qquad\qquad\qquad\qquad \text{lying in the same side as } v\} \\
\tilde{W}^u(v)&=\{u \in S_y\tilde{M};\, y \in H(-v),\, u \in (T_yH(-v))^\perp, \\
&\qquad\qquad\qquad\qquad \text{lying in the same side as } v\}. \\
\tilde{E}^s_v&=T_v\tilde{W}^s(v), \quad \tilde{E}^u_v=T_v\tilde{W}^u(v).
\end{aligned}$$

Since, from the construction, the vector bundles $\tilde{E}^s$, $\tilde{E}^u$ on $S_1\tilde{M}$ are equivariant under the deck transformation group $\pi_1(M)$, we obtain vector bundles E^s and E^u on S_1M in a natural way. Estimations of the solutions of the Jacobi equations guarantee that these vector bundles satisfy the Anosov's condition. Thus we have

Theorem I-4-5 (see [6]). *If the sectional curvature of M is negative everywhere, then the geodesic flow (S_1M, φ_t) is of Anosov type, hence it is ergodic.*

Ergodicity of the geodesic flow on a surface of constant negative curvature has been long years ago proved by Hopf and Hedlund ([29], [26]).

Conversely, a manifold with geodesic flow of Anosov type has some remarkable properties (see W. Klingenberg [35]):

(a) M has no conjugate point,

(b) Every closed geodesic has index zero,

(c) The fundamental group $\pi_1(M)$ has exponential growth, and

(d) Every non-trivial abelian subgroup of Γ is infinitely cyclic, which are well-known characters of negatively curved manifolds (J. Milnor [49] Lawson-Yau [43]).

P. Eberlein has proved in [16] that the geodesic flow on a manifold without conjugate point is of Anosov type if and only if there exists no nonzero perpendicular Jacobi vector field J on a unit speed geodesic c of M such that $\|J(t)\|$ is bounded above for all $t \in \boldsymbol{R}$ (see also [17], [18], [79], [95]). Generalizations of geodesic flows are given in [96], [99].

II. Periodic Orbits

§ II-1. Class field theory for periodic orbits

Let (X, φ_t) be a flow on a compact C^∞-manifold X. We denote by P the set of periodic orbits of (X, φ_t). One basic question on periodic orbits is: How the set {period of $\mathfrak{p}$; $\mathfrak{p} \in P$} is distributed in $\boldsymbol{R}$. Generally, the set P seems to have quite complicated aspects. But one can still expect that the function $\nu(x)$, the number of periodic orbits with period at most x, has simple asymptotic property when x goes to infinity. A prototype of $\nu(x)$ is in number theory, in which the number of prime ideals is concerned. In fact, the number $\pi(x)$ of prime ideals $\mathfrak{p}$ in a number field k whose norm $N(\mathfrak{p})$ is at most x behaves like the function $x/\log x$ as x tends to infinity (prime ideal theorem). The classical way in proving this is to consider the Dedekind zeta function

$$\zeta_k(s) = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1}.$$

The singularity of $\zeta_k(s)$ on the line Re $s=1$ has influence on the asymptotic of $\pi(x)$. In this view, it is natural to consider the following zeta function

$$\zeta_X(s) = \prod_{\mathfrak{p} \in P} (1 - N_a(\mathfrak{p})^{-s})^{-1},$$

where $N_a(\mathfrak{p}) = a^{(\text{period of } \mathfrak{p})}$, $a>1$. The reason why we take up the function $N_a(\mathfrak{p})$ is that $N_a(\mathfrak{p})$ has multiplicative nature, so that N_a is regarded as an analogue of norm function of ideals.

Before paying attention to the convergence of $\zeta_X(s)$, we will observe that there is a close resemblance between periodic orbits and prime ideals, which gives another justification to consider the zeta function $\zeta_X(s)$. In fact, we can construct an analogue of class field theory in the framework of theory of flows.

Suppose an n-fold covering map $\tilde{\omega}: X \to X_0$ satisfies $\tilde{\omega} \circ \varphi_t = \varphi_t \circ \tilde{\omega}$ where (X, φ_t) and (X_0, φ_t) are flows. For a periodic orbit $\mathfrak{P}$ of (X, φ_t) the image $\tilde{\omega}(\mathfrak{P})$ is also a periodic orbit of (X_0, φ_t). Generally

$$(\text{period of } \mathfrak{P})/(\text{period of } \tilde{\omega}(\mathfrak{P})) \quad (\geq 1)$$

is an integer. We call this ratio the degree of $\mathfrak{P}$ with respect to $\tilde{\omega}$. For a periodic orbit $\mathfrak{p}$ of (X_0, φ_t), a lift of $\mathfrak{p}$ is a periodic orbit $\mathfrak{P}$ of (X, φ_t) such that $\tilde{\omega}(\mathfrak{P}) = \mathfrak{p}$. The number of lifts of $\mathfrak{p}$ is finite. In fact, if $\mathfrak{P}_1, \cdots, \mathfrak{P}_g$ are all the lifts of $\mathfrak{p}$, then

$$\sum_{i=1}^{g} \text{degree } \mathfrak{P}_i = n.$$

The covering transformation group G acts in a natural way on P_X: $\sigma\colon \mathfrak{P} \mapsto \sigma\mathfrak{P}$. If $\mathfrak{P}$ is a lift of $\mathfrak{p}$, then so is $\sigma\mathfrak{P}$. If in addition we suppose that $\tilde{\omega}\colon X \to X_0$ is a Galois covering, then G acts transitively on the set $\{\mathfrak{P}_1, \cdots, \mathfrak{P}_g\}$. These aspects of periodic orbits are reminiscent of the decomposition law for prime ideals in number fields. Much more interesting feature is that one can define an analogue of Frobenius automorphisms. For this, let $\tilde{\omega}\colon (X, \varphi_t) \to (X_0, \varphi_t)$ be a Galois covering with covering transformation group G. We choose a sequence $0 = t_0 < t_1 < \cdots < t_k =$ period of $\mathfrak{P}$ ($k =$ degree $\mathfrak{P}$) such that $\{\varphi_{t_i} x\}_{i=1,\dots,k}$ is just the intersection of $\mathfrak{P}$ and a fiber of $\tilde{\omega}$, where $x \in \mathfrak{P}$. There exists a unique σ in G such that $\sigma x = \varphi_{t_1} x$. It is easily checked that σ depends only on $\mathfrak{P}$, so we introduce the notation $(\mathfrak{P} \mid \tilde{\omega})$ representing σ, which we call the Frobenius transformation associated to $\mathfrak{P}$. The following properties are obvious.

$$(\mathfrak{P} \mid \tilde{\omega})\mathfrak{P} = \mathfrak{P}$$

$$(\mu\mathfrak{P} \mid \tilde{\omega}) = \mu(\mathfrak{P} \mid \tilde{\omega})\mu^{-1}, \quad \mu \in G$$

$$\{\sigma \in G; \sigma\mathfrak{P} = \mathfrak{P}\} = \langle(\mathfrak{P} \mid \tilde{\omega})\rangle.$$

A covering is said to be abelian if it is a Galois covering and the covering transformation group is abelian. For an abelian covering $\tilde{\omega}$, the Frobenius transformation $(\mathfrak{P} \mid \tilde{\omega})$ depends only upon the image $\tilde{\omega}(\mathfrak{P}) = \mathfrak{p}$, hence we may write as $(\mathfrak{P} \mid \tilde{\omega}) = (\mathfrak{p} \mid \tilde{\omega})$.

We now consider the free abelian group I_X generated by periodic orbits, which is regarded as a counterpart of ideal group in number theory. We make use of multiplicative notation, so that I_X consists of formal product

$$\mathfrak{a} = \mathfrak{p}_1^{a_1} \cdots \mathfrak{p}_k^{a_k}, \quad a_k \in \boldsymbol{Z}.$$

Since these elements are considered as 1-dimensional cycles on X, the homology class of $\mathfrak{a}$ can be defined. We will call $\mathfrak{a}$ principal if $\mathfrak{a}$ is homologous to zero. The set of principal cycles constitutes a subgroup of I_X, which we denote by I_X^0. The quotient group I_X/I_X^0 is a subgroup of the homology group $H_1(X, \boldsymbol{Z})$.

For a covering $\tilde{\omega}\colon (X, \varphi_t) \to (X_0, \varphi_t)$, we define a homomorphism $N\colon I_X \to I_{X_0}$ by setting $N(\mathfrak{P}) = \mathfrak{p}^f$ where $\mathfrak{p} = \tilde{\omega}(\mathfrak{P})$ and f is the degree of $\mathfrak{P}$ with respect to $\tilde{\omega}$. From very definition, the following diagram is commutative.

$$\begin{array}{ccc} I_X & \xrightarrow{N} & I_{X_0} \\ \downarrow & & \downarrow \\ H_1(X, Z) & \xrightarrow[\tilde{\omega}^*]{} & H_1(X_0, Z). \end{array}$$

We are now in position to state an analogue of the fundamental theorem in class field theory.

Proposition II-1-1. *Let (X_0, φ_t) be a flow such that for any covering $\tilde{\omega}: (X, \varphi_t) \to (X_0, \varphi_t)$, the mapping $I_X \to H_1(X, Z)$ is surjective. Then:*

i) *The index $[I_{X_0}, I^0_{X_0} \cdot N(I_X)] \leq n$ provided that $\tilde{\omega}: X \to X_0$ is an n-fold covering. The equality holds if and only if $\tilde{\omega}$ is abelian.*

ii) *Suppose $\tilde{\omega}$ is abelian. Then the correspondence $\mathfrak{p} \to (\mathfrak{p} \mid \tilde{\omega})$ yields an isomorphism of $I_{X_0}/I^0_{X_0} \cdot N(I_X)$ onto the covering transformation group G.*

iii) *For any subgroup H in I_{X_0} with finite index, containing $I^0_{X_0}$, there exists an abelian covering $\tilde{\omega}: X \to X_0$ such that $H = I^0_{X_0} \cdot N(I_X)$.*

Proof. We first observe that $I_{X_0}/I^0_{X_0} \cdot N(I_X)$ is canonically isomorphic to $H_1(X_0, Z)/\tilde{\omega}_*(H_1(X, Z))$. Note also that there exists a commutative diagram

$$\begin{array}{ccc} \pi_1(X) & \longrightarrow & \pi_1(X_0) \\ \downarrow & & \downarrow \\ H_1(X, Z) & \longrightarrow & H_1(X_0, Z), \end{array}$$

the vertical arrows being the Hurewicz homomorphisms, so that we may define a mapping Φ of the coset space $\pi_1(X_0)/\pi_1(X)$ into $H_1(X_0, Z)/\mathrm{Im}\, \tilde{\omega}_*$. But Φ is surjective, and $\sharp(\pi_1(X_0)/\pi_1(X)) = n$. This implies the inequality in i). If the equality holds, then Φ must be bijective, $\pi_1(X)$ is normal in $\pi_1(X_0)$, and Φ is a group isomorphism.

To prove ii), it suffices to check the following diagram being commutative.

$$\begin{array}{ccc} I_{X_0} & \longrightarrow & G \\ & \searrow \quad \swarrow & \\ & I_{X_0}/I^0_{X_0} \cdot N(I_X), & \end{array}$$

where the horizontal arrow is a homomorphism given by the correspondence $\mathfrak{p} \to (\mathfrak{p} \mid \tilde{\omega})$. Indeed, the commutativity comes from the definition of Frobenius transformations.

As for iii), it should be first noted that there are one-to-one correspondences between the set of subgroups H of I_{X_0} with finite index containing $I^0_{X_0}$ and the set of subgroups H' in $H_1(X_0, Z)$ with finite index, and the set of subgroups Γ in $\pi_1(X_0)$ with finite index containing the commutator subgroup $[\pi_1, \pi_1]$. Choosing Γ which corresponds to H, we get a covering $\tilde{\omega}=X=\tilde{X}_0/\Gamma\to X_0$ satisfying $H=I^0_{X_0}\cdot N(I_X)$. Here $\tilde{X}_0$ denotes the universal covering of X_0.

Remark. An example of flows (X_0, φ_t) satisfying the assumption in the above proposition is the geodesic flow (S_1M, φ_t) where $\dim M\geq 3$. In fact, note that $\pi_1(S_1M)\cong\pi_1(M)$ and $H_1(S_1M)\cong H_1(M)$. The fact that any free homotopy class $\in[\pi_1(M)]$ can be realized by a closed geodesic implies that $I_{S_1(M)}\to H_1(S_1M, Z)$ is surjective. See [68] for a bit different formulation.

§ II-2. Zeta functions and entropy

We first examine the exponent α of convergence of the zeta function $\zeta_X(s)$ introduced in the previous section, that is, a constant α with the property that $\zeta_X(s)$ converges absolutely if $\mathrm{Re}\, s>\alpha$ and diverges if $\mathrm{Re}\, s<\alpha$. For this, we set: $\pi_a(y)=\#\{p: N_a(\mathfrak{p})<y\}$. We suppose that $\pi_a(y)$ is finite for any $y>0$ and $\pi_a(y)\uparrow\infty$ as $y\uparrow\infty$. This, in fact, is the case of Anosov flows:

Theorem II-2-1. (Anosov [3]). *Let (X, φ_t) be a flow of Anosov type with a smooth invariant measure. Then there exist infinitely many periodic orbits, and the periodic orbits are dense in X.*

The following is classical in the theory of Dirichlet series.

Proposition II-2-2. $\alpha=\varlimsup\limits_{y\to\infty}\dfrac{\log\pi_a(y)}{\log y}=\dfrac{1}{\log a}\varlimsup\limits_{x\to\infty}\dfrac{\log\nu(x)}{x}.$

Proof. (Cf. [13]). The absolutely convergence of $\zeta_X(s)\Leftrightarrow$ that of $\prod_{\mathfrak{p}}(1-N_a(\mathfrak{p})^{-s})\Leftrightarrow$ that of $\sum_{\mathfrak{p}}N_a(\mathfrak{p})^{-s}\Leftrightarrow$ that of $\sum_{\mathfrak{p}}N(\mathfrak{p})^{-\mathrm{Re}\, s}$. We set $\sigma=\mathrm{Re}\, s$. Note that for $t>0$,

$$\sum_{\substack{\mathfrak{p}\\ N_a(\mathfrak{p})<t}} N_a(\mathfrak{p})^{-\sigma}=\int_\varepsilon^t\frac{d\pi_a(u)}{u^\sigma}=\frac{\pi_a(t)}{t^\sigma}+\sigma\int_\varepsilon^t\frac{\pi_a(u)}{u^{\sigma+1}}du,\quad(\pi_a(\varepsilon)=0)$$

from which the assertion follows easily.

Thus, to obtain the exponent α, we have to compute $\varlimsup\log\nu(x)/x$. In general, it is rather difficult to accomplish the computation since we do not know the asymptotic of $\nu(x)$ *a-priori*. It should be noted however, that in order to get $\varlimsup\log\nu(x)/x$, much weaker information about $\nu(x)$ is enough. For instance, once one could show that for some positive constants h, C_1, C_2,

$$C_1\frac{e^{hx}}{x}\leq\nu(x)\leq C_2\frac{e^{hx}}{x},\quad x\gg 0$$

or equivalently

$$C_1'\frac{x}{\log x}\leq\pi_a(x)\leq C_2'\frac{x}{\log x},\quad a=e^h,\quad x\gg 0, \tag{1}$$

then $\lim \log \nu(x)/x=h$. In number theory, analogue of (1) is known as the Tschebyschev theorem, which can be proved in an elementary way, and is historically regarded as an important step to the complete proof of the prime number theorem. A result in this context is

Theorem II-2-3 (Bowen [10]). *Let (X, φ_t) be an Anosov flow. If we set $a=\exp h(\varphi_1)$, $h(\varphi_1)$ being the topological entropy of φ_1, then* (1) *holds, especially*

$$\alpha=\lim_{x\to\infty}\frac{\log\nu(x)}{x}=h(\varphi_1)>0.$$

The definition of topological entropy is the following: Let f: $X\to X$ be a continuous mapping of a compacg metric space X. For an open cover $A=\{A_i\}_{i\in I}$, we denote by $N_n(f, A)$ the minimum cardinality of a subcover of $A\vee f^{-1}A\vee\cdots\vee f^{-n+1}A$, where, in general, the open cover $A\vee B$ is defined as $\{A_i\cap B_j;\ A_i\in A,\ B_j\in B\}$. Then the topological entropy of f is

$$h(f)=\sup_{A}\lim_{n\to\infty}\frac{1}{n}\log N_n(f, A),$$

(see [73] for properties of $h(f)$).

Historically, the notion of entropy was introduced for measure preserving transformations by Kolmogorov, which we nowadays call measure theoretic entropy and is defined as

$$h_\mu(f)=\sup_{\mathscr{A}}\lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{i=1}^{n-1}f^{-1}\mathscr{A}\right)$$

where in this turn f: $X\to X$ is a measure-preserving transformation of the probability space (X, μ), $\mathscr{A}$ runs over finite sub σ-algebras, and in general $\mathscr{A}\vee\mathscr{B}$ means the smallest sub σ-algebra containing $\mathscr{A}$ and $\mathscr{B}$. Further

$$H(\mathscr{A})=-\sum_{i=1}^{h}\mu(A_i)\log\mu(A_i),$$

where $\{A_1\cdots A_h\}$ is the finite partition associated to $\mathscr{A}$, i.e. A_i is the non-empty set of the form $B_1\cap\cdots\cap B_h$ where $B_i=C_i$ or $X\backslash C_i$, $C_i\in\mathscr{A}$.

Relationship between topological entropy and measure-theoretic entropy is:

Theorem II-2-4. *Let X be a compact metric space and $f\colon X\to X$ be continuous. Then $h(f)=\sup_\mu h_\mu(f)$, where the supremum is taken over all f-invariant probability measures defined on the σ-algebra of Borel subsets of X.*

See [73] for the proof.

An upper bound for the entropy of a smooth mapping was given by Bowen [9] (see also Katok [39]).

Theorem II-2-5. *For a smooth mapping $f\colon X\to X$ of an n-dimensional Riemannian manifold X, we have $h(f)\leq\max(0, n\log(\sup_{x\in X}\|d_x f\|))$.*

In the case of geodesic flows on non-positively curved manifolds, there is a geometric interpretation for $h(\varphi_1)$:

Theorem II-2-6. (Manning [45] see also [82]). i) *Let M be a compact Riemannian manifold, and let $V(\tilde{x}, r)$ be the volume of the ball with center $\tilde{x}$ and radius r in the universal covering $\tilde{M}$. Then $r^{-1}\log V(\tilde{x}, r)$ converges to a limit $\lambda\geq 0$ as $r\uparrow\infty$ and λ is independent of $\tilde{x}$.*

ii) *Let $\{\varphi_t\}$ be the geodesic flow on S_1M. Then $h(\varphi_1)\geq\lambda$. Further, if M has non-positive sectional curvature, then $h(\varphi_1)=\lambda$.*

In particular, if M is a compact manifold of negative curvature, we have the following estimate of the topological entropy $h(\varphi_1)$

$$(n-1)A\leq h(\varphi_1)\leq(n-1)B,$$

where $-B^2$ and $-A^2$ are the lower and upper limit of the sectional curvature.

As for the measure theoretical entropy, Ja.Pesin [52] recently showed that for a class of manifolds containing negatively curved case

$$h_{\mu_1}(\varphi_1)=-(\mathrm{vol}\,(S_1M))^{-1}\int_{S_1M}\mathrm{tr}\,A(v)d\mu_1(v),$$

where $A(v)$ is an operator of the second fundamental form of the horosphere $H(v)$. It should be noted that generally, even in the case of negatively curved manifolds, $h_{\mu_1}(\varphi_1)$ is less than $h(\varphi_1)$. The equality holds for rank one symmetric spaces.

From now on we assume that $\alpha=1$. In [47], G. Margulis states:

Theorem II-2-7. *If M has negative sectional curvature, then there exists a constant $c>0$ such that $\pi_a(x)\sim cx/\log x$, $a=e^{h(\varphi_1)}$. Here we write $f(x)\sim g(x)$ to mean that $f(x)/g(x)\to 1$ as $x\uparrow\infty$.*

It seems, however, that no geometric interpretation for the constant c has not been given. In a special case, we can show that $c=1$ (see below).

One of most important features of the zeta functions appearing in number theory is that these are always extended to meromorphic functions defined on the whole plane $\boldsymbol{C}$. How about $\zeta_X(s)$? Does $\zeta_X(s)$ extends to $\operatorname{Re} s>1-\varepsilon$ or $\operatorname{Re} s>-\varepsilon$? The following gives a partial answer.

Theorem II-2-8 (D. Ruelle [56]). *Let (X, φ_t) be a real analytic Anosov flow whose stable and unstable manifolds form real analytic foliations. Then $\zeta_X(s)$ is meromorphically extended in $\boldsymbol{C}$.*

This applies especially to the geodesic flow of a compact manifold with constant negative curvature, in which case, connecting $\zeta_X(s)$ with the Selberg zeta function, we are able to know the location of zeros and poles (see below). But, in general cases, it seems difficult to get informations about singularities of $\zeta_X(s)$. See [21] for related questions.

Asymptotics of $\pi_a(x)$ are closely related to the behavior of $\zeta_X(s)$ on the line $\operatorname{Re} s=1$ in the following way

Proposition II-2-9. *Suppose that there exists a constant b such that $\zeta_X(s)-(b/(s-1))$ is analytically extended to* $\operatorname{Re} s\geq 1$, *and has no zero on* $\operatorname{Re} s=1$. *Then $\pi_a(x)\sim x/\log x$ as $x\uparrow\infty$.*

The proof relies heavily on the Tauberian theorem for the Dirichlet integrals (see S. Lang [42]).

We now confine ourselves to a special case. Let M be a compact locally symmetric space of negative curvature, and let $R(X, Y)Z$ be the curvature tensor on M. We denote by $\mu_1\geq\cdots\geq\mu_{n-1}>\mu_n=0$ the eigenvalues of the self adjoint operator: $X\to R(X, v)v$, where v is a unit tangent vector on M. Two point homogenity of the universal covering of M leads to the fact that $\{\mu_i\}$ does not depend upon the choice of v. Direct computation of the growth rate shows that $\lambda=\sum_{i=1}^{n-1}\mu_i^{1/2}$.

Proposition II-2-10. *Let $\zeta_X(s)$ be the zeta function associated to the geodesic flow on a negatively curved locally symmetric space M, and let*

$\Upsilon>0$ *be the first eigenvalue of the Laplace Beltrami operator on* M. *If* $\Upsilon\geq\lambda^2/4$, *then* $\zeta_X(s)$ *is non-vanishing and holomorphic in the region* Re $s>1/2$, *except for the point* $s=1$ *where* $\zeta_X(s)$ *has a simple pole. If* $\Upsilon<\lambda^2/4$, *then* $\zeta_X(s)$ *is non-vanishing and holomorphic in the region* Re $s>1/2+(1/4-\Upsilon/\lambda^2)^{1/2}$, *except for the point* $s=1$.

Thus, applying Proposition II-2-1, we obtain

Corollary II-2-11. $\pi_a(x)\sim x/\log x$ *as* $x\uparrow\infty$.

It should be noted that non-vanishing of $\zeta_X(s)$ allows us to get much stronger results about the estimates of the remainder term $\pi_a(x)-(x/\log x)$ (see D. A. Hejhal [28] for two dimensional cases).

The proof of Proposition II-2-10 is largely due to the results by Selberg [59] and Gangolli [22]. They originally introduces a bit different zeta function which takes the form

$$Z(s)=\prod_{\mathfrak{p}}\prod_{k_1,\cdots,k_{n-1}=0}^{\infty}(1-N_a(\mathfrak{p})^{-(s+k_1r_1+\cdots+k_{n-1}r_{n-1})})$$

where $r_i=\sqrt{\mu_i}/\lambda$. As was shown by Gangolli, a power of $Z(s)$ has an analytic continuation to the whole plane, and satisfies a functional identity. Further, location of zeros and poles are explicitly described (the modified "Riemann hypothesis"). Thus, making use of the relation

$$\zeta_X(s)=\frac{\prod_i Z(s+r_i)\prod_{i_1<i_2<i_3}Z(s+r_{i_1}+r_{i_2}+r_{i_3})\cdots}{Z(s)\prod_{i_1<i_2}Z(s+r_{i_1}+r_{i_2})\cdots},$$

we get the proposition.

In number theory, the prime ideal theorem is a special case of more general density theorems which are closely related to the class field theory. As was seen in the previous section, one could construct a class field theory in the framework of theory of periodic orbits. Hence, it is natural to expect that a similar implication holds in our context. This, in fact, is the case whenever locally symmetric spaces are treated.

The following is an analogue of the Tchebotarev's density theorem. As in number theory, if $\mathscr{A}$ is a set of periodic orbits, and if the limit

$$\lim_{x\to\infty}\frac{\log x}{x}\#\{\mathfrak{p}\in\mathscr{A};\,N_a(\mathfrak{p})<x\}=D(\mathscr{A})$$

exists, then we will call $D(\mathscr{A})$ the density of $\mathscr{A}$.

Proposition II-2-12 ([68]). *Let* M_0 *be a compact locally symmetric* *space of negative curvature, and let $\pi: M \to M_0$ be a finite normal Riemannian covering with covering transformation group G. We let $\tilde{\omega}: S_1M \to S_1M_0$ be the induced normal covering. Then those periodic orbits $\mathfrak{p}$ in S_1M for which there exists $\mathfrak{P}$ with $\tilde{\omega}(\mathfrak{P}) = \mathfrak{p}$ such that $(\mathfrak{P} \mid \tilde{\omega}) = \sigma (\in G)$ has density, and this density is equal to $\sharp[\sigma]/\sharp G$. Here $[\sigma]$ is the conjugacy class of σ in G.*

Applying this to abelian coverings, we get an analogue of the Dirichlet theorem for arithmetic progressions:

Corollary II-2-13. *Let H be a subgroup of $H_1(M, Z)$ of finite index, and let C be a coset in $H_1(M, Z)/H$. Then the number of periodic orbits $\mathfrak{p}$ with $N_a(\mathfrak{p}) < x$ whose homology class is in C is asymptotically equal to*

$$(\sharp(H_1(M, Z)/H))^{-1} \cdot x/\log x \qquad \text{as } x \text{ tends to infinity.}$$

Problem. Suppose that $\sharp H_1(M, Z)$ is infinite, and let $c \in H_1(M, Z)$. Then how does the function

$$\pi^c(x) = \{\mathfrak{p}: N_a(\mathfrak{p}) < x, \text{ the homology class } [\mathfrak{p}] \in c\}$$

behave asymptotically as $x \uparrow \infty$?

III. Geodesic Chains

§ III-1. Invariant measure

Let M be a connected complete Riemannian manifold. We denote by $\mathscr{C}$ the set of all continuous curves $c: \boldsymbol{R} \to M$ such that each restriction $c_i = c|[i-1, i]$, $i \in \boldsymbol{Z}$, is a geodesic curve. Elements in $\mathscr{C}$ will be called geodesic chains in M. We introduce a family of mappings $\Phi_k: c(t) \to c(t+k)$, $k \in \boldsymbol{Z}$. Since $\Phi_{h+k} = \Phi_h \circ \Phi_k$, or $\Phi_k = (\Phi_1)^k$, the family $\{\Phi_k\}_{k \in \boldsymbol{Z}}$ is regarded as a "discrete" flow on the set $\mathscr{C}$. We call Φ_k's the shifts on geodesic chains. The goal of this section is to introduce a $\{\Phi_k\}$-invariant measure on $\mathscr{C}$.

Suppose that we are given a family of probability measure $\{\mu_x\}_{x \in M}$ such that i) each μ_x is assigned to the tangent space T_xM, ii) $\mu_x(-\cdot) = \mu_x(\cdot)$, and iii) the parallel translations along any geodesics preserve $\{\mu_x\}$. Further we suppose that iv) the measure μ on TM defined by the relationship

$$\int_{TM} F d\mu = \int_M dg(x) \int_{T_xM} F d\mu_x$$

is invariant under $\varphi_1: TM \to TM$, where $\{\varphi_t\}$ denotes the geodesic flow.

Associated with $\{\mu_x\}$, we define an operator L by

$$(Lf)(x)=\int_{T_xM} f(\exp_x v)d\mu_x(v).$$

We first equip a probability measure μ_x^∞ on the set $\mathscr{C}_x=\{c\in\mathscr{C}; c(0)=x\}$.

Proposition III-1-1. *There exists a measure μ_x^∞ on $\mathscr{C}_x$ satisfying*

$$\int_{\mathscr{C}_x} f(\pi_k c)d\mu_x^\infty(c)=(L^{|k|}f)(x), \quad k\in \boldsymbol{Z},$$

where π_k: $\mathscr{C}\to M$ is a mapping defined by $\pi_k(c)=c(k)$.

Proof. We identify the set $\mathscr{C}_x$ with the infinite cartesian product $\prod_{-\infty}^{\infty} T_xM=(T_xM)^{\boldsymbol{Z}}$ in the following way. Given a geodesic chain $c=(\cdots, c_{-1}, c_0, c_1, \cdots)$, we associate a sequence of tangent vectors $(\cdots, v_0^{(-1)}, v_1^{(0)}, v_2^{(1)}, \cdots)$ so that $v_i^{(i-1)}$ is the velocity of c_i at the point $c(i-1)$. Then translating these vectors $v_i^{(i-1)}$ parallelly along the piecewise smooth curve $c|[0, i-1]$ when $i\geq 1$, or $c|[i-1, 0]$ when $i\leq 0$, we get tangent vectors v_i at $c(0)=x$. As is easily checked, the correspondence $c\to(v_i)_{i\in\boldsymbol{Z}}$ yields a bijection: $\mathscr{C}_x\to\prod_{-\infty}^{\infty} T_xM$. By using this bijection, a probability measure μ_x^∞ on $\mathscr{C}_x$ is defined as the product measure $\prod_{-\infty}^{\infty}\mu_x$. In order to see that the measure μ_x^∞ satisfies the desired property, we first consider the set of finite geodesic chains, $\mathscr{C}_x(N)$, which consists of curves c: $[0, N]\to M$ such that $c|[i-1, i]$ are geodesics for $i=1, \cdots, N$. In the same way as above, we may identify $\mathscr{C}_x(N)$ with $\prod_{i=1}^{N} T_xM$, which carries the probability measure $\mu_x^N=\prod_{i=1}^{N}\mu_x$. We need the following

Lemma III-1-2.

$$\int_{\mathscr{C}_x(N_1+N_2)} Fd\mu_x^{N_1+N_2}=\int_{\mathscr{C}_x(N_1)} d\mu_x^{N_1}(c_1)\int_{\mathscr{C}_{c_1(N_1)}(N_2)} F(c_1\circ c_2)d\mu_{c_1(N_1)}^{N_2}(c_2),$$

where $c_1\circ c_2$ denotes the geodesic chain defined by

$$c_1\circ c_2(s)=\begin{cases} c_1(s), & 0\leq s\leq N_1\\ c_2(s-N_1), & N_1\leq s\leq N_1+N_2.\end{cases}$$

Proof. We shall make use of the condition iii) for $\{\mu_x\}$. The parallel translation P_{c_1}: $T_xM\to T_{c_1(N_1)}M$ along the geodesic chain c_1 induces a bijection $P_{c_1}^*$: $\mathscr{C}_x(N_2)\to\mathscr{C}_{c_1(N_1)}(N_2)$, that is to say, $P_{c_1}^*=P_{c_1}\times\cdots\times P_{c_1}$: $\mathscr{C}_x(N_2)=\prod_{i=1}^{N_2} T_xM\to\prod_{i=1}^{N_2} T_{c_1(N_1)}M=\mathscr{C}_{c_1(N_1)}(N_2)$. Since P_{c_1} preserves $\{\mu_x\}$, we find

$$\int_{\mathscr{C}_x(N_1+N_2)} F d\mu_x^{N_1+N_2} = \int_{\mathscr{C}_x(N_1)} d\mu_x^{N_1}(c_1) \int_{\mathscr{C}_x(N_2)} F(c_1 \circ P_{c_1}^*(c_2) d\mu_x^{N_2}(c_2)$$
$$= \int_{\mathscr{C}_x(N_1)} d\mu_x^{N_1}(c_1) \int_{\mathscr{C}_{c_1(N_1)}(N_2)} F(c_1 \circ c_2) d\mu_{c_1(N_1)}^{N_2}(c_2),$$

which completes the proof.

We note that for $k \geq 0$,

$$\int_{\mathscr{C}_x} f(\pi_k c) d\mu_x^\infty(c) = \int_{\mathscr{C}_x(k)} f(c(k)) d\mu_x^k(c)$$
$$\int_{\mathscr{C}_x(1)} f(c(1)) d\mu_x^1(c) = (Lf)(x).$$

In virtue of the above lemma, we find

$$\int_{\mathscr{C}_x(k)} f(c(k)) d\mu_x^k(c) = \int_{\mathscr{C}_x(1)} d\mu_x^1(c_1) \int_{\mathscr{C}_{c_1(1)}(1)} d\mu_{c_1(1)}^1(c_2)$$
$$\cdots \int_{\mathscr{C}_{c_{k-1}(1)}(1)} f(c_k(1)) d\mu_{c_{k-1}(1)}^1(c_k)$$
$$= (L^k f)(1).$$

A similar argument works for negative k (we use the condition ii), hence we are done.

Proposition III-1-3. *The measure μ^∞ on $\mathscr{C}$ defined by the relationship*

$$\int_{\mathscr{C}} F d\mu^\infty = \int_M dg(x) \int_{\mathscr{C}_x} F d\mu_x^\infty$$

is $\{\Phi_k\}$-invariant.

Proof. We shall prove that $\int_{\mathscr{C}} F(\Phi_1 c) d\mu^\infty(c) = \int_{\mathscr{C}} F(c) d\mu^\infty(c)$. A key point is that if $c \in \mathscr{C}_x$ corresponds to $\prod_{-\infty}^{\infty} v_i \in \prod_{-\infty}^{\infty} T_x M$, then $\Phi_1(c)$ corresponds to $\prod_{-\infty}^{\infty} P_{v_1}(v_{i+1})$, where P_{v_1} denotes the parallel translation along the geodesic $t \to \exp tv_1$, $0 \leq t \leq 1$. We then observe

$$\int_{\mathscr{C}} F(\Phi_1 c) d\mu^\infty(c) = \int_M dg(x) \int_{\Pi_{-\infty}^{\infty} T_x M} F\Big(\prod_{-\infty}^{\infty} P_{v_1}(v_{i+1})\Big) \prod_{-\infty}^{\infty} d\mu_x(v_i)$$
$$= \int_M dg(x) \int_{T_x M} d\mu_x(v_1) \int_{\Pi_{i\neq 1} T_x M} F\Big(\prod_{-\infty}^{\infty} P_{v_1}(v_{i+1})\Big) \prod_{i\neq 1} d\mu_x(v_i)$$
$$= \int_M dg(x) \int_{T_x M} d\mu_x(v_1) \int_{\Pi_{i\neq 1} T_y M} F\Big(\prod_{-\infty}^{\infty} u_{i+1}\Big) \prod_{i\neq 1} d\mu_y(u_i)$$

where $y=\exp v_1$, $u_1=P_{v_1}v_1=\varphi_1(v_1)$. Thus, if we put

$$f(v_1)=\int_{\Pi_{i\neq 1}T_xM} F\Big(\prod_{-\infty}^{\infty} u_{i+1}\Big)\prod_{i\neq 1} d\mu_x(u_i), \quad u_1=v_1,$$

then we have

$$\begin{aligned}\int_{\mathscr{C}} F(\Phi_1 c)d\mu^\infty(c) &= \int_{TM} f(\varphi_1 v_1)d\mu(v_1)\\ &= \int_{TM} f(v_1)d\mu(v_1)\\ &= \int_M dg(x)\int_{\Pi_{-\infty}^{\infty} T_xM} F(\prod u_i)\prod d\mu_x(u_i)\\ &= \int_{\mathscr{C}} F(c)d\mu^\infty(c).\end{aligned}$$

This completes the proof.

Remark. In view of the condition ii), the time-reversing operation $\tau: c(t)\to c(-t)$ leaves the measure μ^∞ invariant. Furthermore, the operator L turns out to be self adjoint.

Example III-1-4 ([66] [67]). The measure μ_x^r on T_xM defined by

$$d\mu_x^r=\frac{1}{r^2\omega_{n-1}}\delta(\|v\|-r)dv,$$

δ being the Dirac function, $r>0$, satisfies the conditions i)–iv). The operator L associated to this family $\{\mu_x^r\}$ is just the spherical mean operator L_r, which was already introduced in Section I-1. It is easy to see that the support of the measure μ_r^∞ is the set $\mathscr{C}_r=\{c\in\mathscr{C};\ c_i=c|[i-1,i]$ has common length $r\}$. We will call an element in $\mathscr{C}_r$ an *r-geodesic chain.* In the case $M=\boldsymbol{R}^n$, the measure μ_r^∞ is intimately related to the classical random walk problem which was originally presented by K. Pearson in 1905: A walker on the Euclidean space $\boldsymbol{R}^n$ starting from a point x walks a distance r, then changes direction at random and repeats this process N times. What is the probability density that he is at the distance R from his origin x ?, or more generally what is the probability that he drops in a subset A in $\boldsymbol{R}^n$? It should be noted that the walker's course is indicated by a polygonal path $c\in\mathscr{C}_x(N)$ in such a way that $c(k)$ corresponds to his k-th step, and that the probability that the walker is in A at the N-th step is

$$\mu_x^N\{c;\ c(N)\in A\}=(L_r^N\chi_A)(x),$$

where χ_A is the defining function of the set A. In particular, noting that the Fourier transform of $L_r f$ is

$$\Gamma\left(\frac{n}{2}\right)(\|\xi\|r/2)^{-n/2+1}J_{n/2-1}(\|\xi\|r)\hat{f}(\xi),$$

$J_\nu(x)$ being the Bessel function, we conclude that the probability density the walker finds himself at the distance R is

$$\Gamma\left(\frac{n}{2}\right)^{N-1}\int_0^\infty (\rho R/2)^{n/2}J_{n/2-1}(\rho R)(J_{n/2-1}(\rho R)(\rho r/2)^{1-(n/2)})^N d\rho.$$

(This was first obtained by Kluyver [38], see also [7] [33].)

Example III-1-5. We consider a probability density

$$d\mu_x=(2\pi\tau)^{-n/2}\exp(-\|v\|^2/2\tau)dv.$$

It is easy to see that $\{\mu_x\}$ satisfies the condition i)–iv), and the associated operator L is the Gaussian mean value operator K_τ (see § I-1). In the classical case $M=\boldsymbol{R}^n$, the measure μ_x^∞ associated with the Gaussian mean value operator is related to the Gaussian random walks in the same way as the Pearson's case (see [7]).

After these examples, the family of measures $\{\mu_x^\infty : x\in M\}$ associated with $\{\mu_x : x\in M\}$ is called geodesic random walks, which is in fact considered as a Markov process on the state space M with the transition operator L.

§ III-2. Ergodicity of the shifts

Throughout this section, we suppose that M is compact, so that the measure μ^∞ on $\mathscr{C}$ constructed in the previous section is finite. We now discuss ergodicity of the dynamical system $(\mathscr{C}, \mu^\infty, \Phi_k)$. As in the case of flows (see § I-4), we shall say that $(\mathscr{C}, \mu^\infty, \Phi_k)$ is ergodic if any $\{\Phi_k\}$-invariant measurable set has measure zero, or its complement does. In our special situation there is no difficulty in showing

Lemma III-2-1. *Ergodicity of $(\mathscr{C}, \mu^\infty, \Phi_k)$ is equivalent to either one of the following conditions:*

(1) *For any Borel set A in M,*

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{-N}^{N}\chi_A(c(k))=\int_A dg\Big/\int_M dg \quad a.e.\ c.$$

(2) *For any* $f \in L^1(M)$,

$$\lim_{N\to\infty} \frac{1}{N}\left(\sum_{k=0}^{N-1} L^k f(x)\right) = \int_M f dg \Big/ \int_M dg \quad a.e.\ x.$$

(3) *If* $Lf=f$, $f \in L^1(M)$, *then* $f=\text{const.}$

(4) *For any pair of Borel sets A and B in M with positive measure, there exists a natural number k such that*

$$\int_A L^k \chi_B dg > 0.$$

Ergodicity of $(\mathscr{C}, \mu^\infty, \Phi_k)$ has a geometric consequence about geodesic chains. A sequence $\{x_i\}_{i\in Z}$ of points in M is, in general, called uniformly distributed if for any Jordan measurable set J in M, the limit

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \chi_J(x_i)$$

exists and equals $\int_J dg \Big/ \int_M dg$.

Proposition III-2-2. *If* $(\mathscr{C}, \mu^\infty, \Phi_k)$ *is ergodic, then the sequence* $\{c(i)\}_{i\in Z}$ *for almost all geodesic chains* $c \in \mathscr{C}$ *is uniformly distributed.*

We now restrict ourselves to the case of Example III-1-4. This being the case, the measure μ_x^N on $\mathscr{C}_x(N)$ has support in the product $\prod_{i=1}^N S_x^r M$, where $S_x^r M = \{v \in T_x M;\ \|v\| = r\}$, and in fact coincides with the product of the normalized canonical measure on the sphere $S_x^r M$. We define a smooth mapping $\pi_N\colon \prod^N S^r M = \bigcup_{x\in M} \prod_{i=1}^N S_x^r M \to M\times M$ by setting $\pi_N(c) = (c(0), c(N))$. We say that two points x and y are joined by an r-geodesic chain $c\colon [0, N] \to M$ if $c(0)=x$ and $c(N)=y$. Then $\bigcup_{N=1}^\infty \pi_N(\prod^N S^r M) = M\times M$ if and only if two points in M can be joined by an r-geodesic chain. On the other hand, we have, in view of Proposition III-1-1,

$$\mu_N(\pi_N^{-1}(A\times B)) = \int_A L_r^N \chi_B dg(x),$$

where μ_N is the smooth measure on $\prod^N S^r M$ defined in a natural way. Applying the Sard's theorem to the mappings π_N, we get

Proposition III-2-3 ([67]). *If any two points in M can be joined by an r-geodesic chain, then* 1 *is a simple eigenvalue of* $L_r\colon L^1(M) \to L^1(M)$, *or equivalently the dynamical system* $(\mathscr{C}_r, \mu^\infty, \Phi_1)$ *is ergodic. This, in partic-*

ular, is true if r is smaller than the injectivity radius of M.

Suppose the fundamental group $\pi_1(M)$ is infinite. Then the universal covering $\tilde{M}$ is non compact, so that there exists a geodesic ray on $\tilde{M}$ (see J. Cheeger and D. G. Ebin [12]). This guarantees that any two points of $\tilde{M}$, and hence any two points of M are joined by an r-geodesic chain. Thus we have

Proposition III-2-4. *If $\pi_1(M)$ is infinite, then 1 is a simple eigenvalue of L_r for any $r>0$.*

These results are interesting in view of generalization of mean value properties of harmonic functions on $\boldsymbol{R}^n$. It is well-known that a locally integrable function f on $\boldsymbol{R}^n$ is harmonic if and only if $L_r f=f$ for any $r>0$. This is immediately generalized to the case of Riemannian manifolds. Namely, if $L_r f=f$ for any $r>0$, then f is necessarily a harmonic function on M (the converse is in general not true). In particular, if M is compact, a function $f\in L^1(M)$ such that $L_r f=f$ for any $r>0$ must be constant. Proposition III-2-3 asserts that this is true even for a function with $L_r f=f$ for a *fixed* $r>0$.

There are several references which are concerned with mean-value properties for a different kind of mean-value operators ([24] [54] [25]).

Application III-2-5. A continuous function f on a complete Riemannian manifold M is called affine if for any geodesic $c\colon \boldsymbol{R}\to M$ the composition $f\circ c$ is a linear function on $\boldsymbol{R}$. We will give a simple proof to the following splitting theorem which has been proved by Innami [90].

Theorem. *If there is a non-constant affine function f on M, then one can find an isometry $I\colon N\times \boldsymbol{R}\to M$ with $f\circ I(x, t)=at+b$.*

Proof. We suppose first f is smooth. In virtue of the identity: $\langle \nabla_u \operatorname{grad} f, v\rangle=\operatorname{Hess} f(u, v)$, we find that the vector field $\operatorname{grad} f$ is parallel. Let $\{\varphi_t\}$ be the one parameter group of isometries generated by $\operatorname{grad} f/\|\operatorname{grad} f\|$. Since f is non-constant, $\|\operatorname{grad} f\|$ is non-zero constant, and $N=f^{-1}(0)$ is a totally geodesic submanifold. We set

$$\begin{aligned} I\colon N\times \boldsymbol{R} &\longrightarrow M \\ (x, t) &\longmapsto \varphi_t x. \end{aligned}$$

It is obvious that I is an isometry satisfying the property.

We next prove that any affine functions are smooth, in fact harmonic. Note that if f is affine, then $f(\exp rv)-f(x)=-(f(\exp -rv)-f(x))$, so that

$$\int_{S_xM} (f(\exp rv) - f(x)) dS_x(v) = 0,$$

hence $L_r f = f$. This completes the proof.

§ III-3. Wiener flows and stochastic developments

Let (Ω, φ_t) be the Wiener flow defined on a complete Riemannian manifold M, that is, Ω is the space of continuous curves $c: \boldsymbol{R} \to M$, and φ_t denotes the shift operation: $c(s) \to c(t+s)$. As in the case of shifts of geodesic chains, we may prove the same sort of existence theorem for $\{\varphi_t\}$-invariant measure on Ω.

Theorem III-3-1. *For each $x \in M$, there exists a unique finite measure μ_x on the set $\Omega_x = \{c \in \Omega;\ c(0) = x\}$ satisfying:*

i) *The measure μ_x^t on $\Omega_x(t) = \{c: [0, t] \to M;\ c(0) = x\}$ induced by the restriction mapping $\Omega_x \to \Omega_x(t)$ obeys the relationship*

$$\int_{\Omega_x(t_1+t_2)} F d\mu_x^{t_1+t_2} = \int_{\Omega_x(t_1)} d\mu_x^{t_1}(c_1) \int_{\Omega_{c_1(t_1)}(t_2)} F(c_1 \circ c_2) d\mu_{c_1(t_1)}^{t_2}(c_2).$$

ii) *The operation $c(t) \to c(-t)$ leaves μ_x invariant.*

iii) *The operator $\mathscr{H}_t$ defined by*

$$(\mathscr{H}_t f)(x) = \int_{\Omega_x(t)} f(c(t)) d\mu_x^t = \int_{\Omega_x} f(c(t)) d\mu_x, \quad t > 0,$$

which, in view of i), *forms an operator semigroup, coincides with $e^{-t\Delta/2}$, the heat semigroup. In other words, $\mathscr{H}_t f$ is the solution of the heat equation*

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) u = 0, \qquad u|_{t=0} = f.$$

iv) *The measure μ on Ω defined by the relationship*

$$\int_\Omega F d\mu = \int_M dg(x) \int_{\Omega_x} F d\mu_x$$

is $\{\varphi_t\}$-invariant.

The measure μ (resp. μ_x^t) is called the Wiener measure on Ω (resp. on $\Omega_x(t)$). The standard way for the proof of the theorem is to establish first the existence of a measure on the mapping space $M^{\boldsymbol{R}} = \mathrm{Map}(\boldsymbol{R}, M)$ which satisfies the same properties as i)–iv) (Kolmogorov's extension theorem), and next to show that the support of the measure is in Ω by using properties of the fundamental solution of the heat equation (see [20]).

Example III-3-2. In the case $M=\boldsymbol{R}^n$, the fundamental solution is $(2\pi t)^{-n/2}\exp(-\|x-y\|^2/2t)$. In view of i) and iii), the Wiener measure on $\Omega_o(t)$ should satisfy

$$\int_{\Omega o} f(c(\tau_1), \cdots, c(\tau_N))d\mu_o^t(c)=\{(2\pi\tau_1)(2\pi(\tau_2-\tau_1))\cdots(2\pi(\tau_N-\tau_{N-1}))\}^{-n/2}$$
$$\times\int_{\boldsymbol{R}^{nN}} f(x_1, \cdots, x_N)\exp\left(-\frac{1}{2t}\sum_{i=1}^N\frac{\|x_i-x_{i-1}\|^2}{\tau_i-\tau_{i-1}}\right)dx_1\cdots dx_N,$$

where $0=\tau_0<\tau_1<\cdots<\tau_N\leq t$, $x_0=\boldsymbol{o}$, and f is a continuous function on $(\boldsymbol{R}^n)^N$.

In general, we may only assert that $\mu_x(\Omega_x)\leq 1$. A sufficient condition that $\{\mu_x\}_{x\in M}$ is a family of probability measures, or equivalently that $\mathscr{K}_t 1=1$ is:

Theorem III-3-3 (S.T. Yau [75]). *If the Ricci curvature of M is bounded from below, then μ_x are probability measures.*

See K. Ichihara [30] for related topics. A manifold with the property that $\mu_x(\Omega_x)=1$ for any $x\in M$ is called stochastically complete.

A class of shifts on geodesic chains gives an approximation of the Wiener flows, which, in the classical case, is stated as the central limit theorem (see [34] [23]). Much more precise results are embodied as the stochastic developments. We briefly recall the most relevant features of subject which has been initiated by K. Ito [32] and P. Malliavin [44] (see also [20]).

In classical differential geometry, the development of a piecewise smooth curve ω: $[0, t]\to T_xM$ with $\omega(0)=\boldsymbol{o}$ (origin in T_xM) means a curve c: $[0, t]\to M$ that is uniquely determined by the equation

$$\int_0^\tau P_{c|[0,\tau]}^{-1}(\dot{c}(\tau))d\tau=\omega(\tau),$$

where $P_{c|[0,\tau]}$: $T_xM\to T_{c(\tau)}M$ denotes the parallel translation along the curve $c|[0, \tau]$. The correspondence $\omega\to c$, as is easily seen, gives a bijection of the set of all piecewise smooth curve ω in T_xM with $\omega(0)=\boldsymbol{o}$ onto the set of all piecewise smooth curve c in T_xM with $c(0)=x$. We wish to extend this correspondence to all the continuous curves ω. For this, we first define a mapping p_N: $\Omega_o(t)\to\Omega_x(t)$ in the following way. For a curve $\omega\in\Omega_o(t)$, we let ω_N be the polygonal approximation of ω, i.e.

$$\omega_N(s)=\omega\left(\frac{i-1}{N}t\right)+\frac{N}{t}\left(s-\frac{i-1}{N}t\right)\left(\omega\left(\frac{i}{N}t\right)-\omega\left(\frac{i-1}{N}t\right)\right),$$
$$\frac{i-1}{N}t\leq s\leq\frac{i}{N}t.$$

Then $p_N(\omega)$ is defined as the development of ω_N.

Theorem III-3-4. *Suppose M is stochastically complete. Then for almost all ω (with respect to the Wiener measure on $\Omega_o(t)$), the limit $p_\infty(\omega) = \lim_{N\to\infty} p_N(\omega) \in \Omega_x(t)$ exists in probability, and*

$$\int_{\Omega o(t)} f(p_\infty(\omega)(t))d\mu_o^t(\omega) = (\mathscr{K}_t f)(x).$$

To see the connection between the theorem and shifts on geodesic chains, we consider an operator theoretic version of the theorem, which is much easier to prove. For simplicity, we assume M is compact. We define an operator $K_{N,t}$ by

$$(K_{N,t}f)(x) = \int_{\Omega o(t)} f(p_N(\omega)(t))d\mu_o^t(\omega),$$

which, in view of the equality given in Example III-3-2, is rewritten as

$$\int_{\mathscr{C}_x(N)} f(c(N))d\mu_x^N(c),$$

where μ_x^N is the measure on $\mathscr{C}_x(N)$ derived from the family $\{\mu_x\}$ of the Gaussian measure on T_xM:

$$d\mu_x = \left(2\pi\frac{t}{N}\right)^{-n/2} \exp\left(-\|v\|^2/2\frac{t}{N}\right)dv.$$

It follows from Proposition III-1-1 that $K_{N,t} = (K_{t/N})^N$. Here K_τ denotes the Gaussian mean value operator. In Section I-1, we have proved that $\|K_\tau\| \leq 1$. Easy calculations show that for any $f \in L^2(M)$. $\|K_\tau f - f\| \to 0$ as $\tau \downarrow 0$, and that for any C^∞-function f, $(K_\tau f - f)/\tau \to (1/2)\,\Delta f$ in L^2-sense. Therefore applying the Trotter-Chernoff theorem [15], we have

Proposition III-3-5 ([69]). *For any $f \in L^2(M)$, $K_{N,t}f$ converges in $L^2(M)$ as $N \uparrow \infty$, and its limit is $e^{-t\Delta/2}f$.*

Appendix 1. Small Eigenvalues of the Laplacian and Zeros of Selberg's Zeta Functions

This appendix will give a proof, as an application of the arguments in Section II-2, to a result originally established by B. Randol [100], on small eigenvalues of the Laplace operator on a compact quotient M of rank-one symmetric space of non compact type (see for related topics P. Buser [94]). We retain the notations in Section II-2.

Theorem. *Suppose $H_1(M, Z)$ is infinite. Given any integer N, one can find a finite covering $M_1 \to M$ such that the Laplacian on M_1 has at least N eigenvalues in the interval* $(0, \lambda^2/4)$, *or equivalently that the Selberg-Gangolli zeta function of M_1 has N zeros in the interval* $(1/2, 1)$.

Proof. Given a character

$$\chi\colon \pi_1(M) \longrightarrow U(1) = \{|z| = 1\},$$

we construct a flat line bundle L_χ on M in the usual way. We denote by $0 \leqq \gamma_0(\chi) \leqq \gamma_1(\chi) \leqq \cdots$ be the eigenvalues of the Laplacian Δ_χ acting on sections of L_χ. It is easy to see that for the trivial character $\mathbf{1}$, $\gamma_0(\mathbf{1}) = 0$, and that if $\chi \neq \mathbf{1}$, then $\gamma_0(\chi) > 0$. Note that the eigenvalues of Δ_χ depend continuously on the characters χ. Hence, if the trivial character $\mathbf{1}$ is not isolated in the character group of $\pi_1(M)$, then one can find a character χ near to $\mathbf{1}$ such that Δ_χ has an eigenvalue in the interval $(0, \lambda^2/4)$. The character group of $\pi_1(M)$ contains a torus whose dimension is the rank of $H_1(M, Z)$, and the rational points of the torus correspond to characters with kernels of finite index. Hence if $H_1(M, Z)$ is infinite, one can find an abelian covering $M_1 \to M$ and characters $\chi_1, \cdots, \chi_N$ factored as

$$\chi_i\colon \pi_1(M) \longrightarrow \pi_1(M)/\pi_1(M_1) \longrightarrow U(1)$$

such that Δ_{χ_i} has an eigenvalue in $(0, \lambda^2/4)$.

Given an abelian covering $M_1 \to M$ and a character χ of $\pi_1(M)/\pi_1(M_1)$, we may define an analogue of L-functions by

$$Z(s, \chi) = \prod_{\mathfrak{p}} \prod_{k_1, \cdots, k_{n-1} = 0}^{\infty} (1 - \chi(\mathfrak{p}) N_a(\mathfrak{p})^{-(s + k_1 r_1 + \cdots + k_{n-1} r_{n-1})}),$$

where we have identified the covering transformation group $\pi_1(M)/\pi_1(M_1)$ with a quotient group of the free abelian group I_M generated by prime geodesic cycles in M (see Prop. II-1-1), and have regarded χ as a character on I_M. The analogue of class field theory allows us easily to prove

Lemma. $$Z(s, M_1) = \prod_{\chi} Z(s, \chi)$$

where χ runs over all characters of the abelian group $\pi_1(M)/\pi_1(M_1)$.

As was shown by R. Gangolli [22], a power of $Z(s, \chi)$ has a meromorphic continuation whose zeros in $(1/2, 1)$ are just

$$\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\gamma_i(\chi)}{\lambda^2}} \quad \text{with } 0 < \gamma_i(\chi) < \frac{\lambda^2}{4}.$$

Combining the above argument with Lemma, we get the Theorem.

Appendix 2. Iterations of Certain Integral Operators

In connection with the operator K_τ (acting on smooth functions), we consider the operator

$$(\tilde{K}_\tau f)(x)=(2\pi\tau)^{-n/2}\int_M \exp(-d(x,y^2)/2\tau)f(y)dg(y),$$

which is closely related to DeWitt's formulation of path integral quantizations on curved spaces (B. S. DeWitt [98], K. S. Cheng [97]). The aim of this appendix is to give a rigorous proof to

Theorem. *If $\varphi\in L^2(M)$, M compact, then*

$$\tilde{K}^N_{t/N}\varphi \longrightarrow \exp t\left(\frac{1}{2}\Delta-\frac{1}{6}R\right)\cdot\varphi, \quad \textit{in } L^2\textit{-sense},$$

where R is the scalar curvature of M.

Proof. For each point x in M, there exists an open bounded cell C_x in T_xM containing the origin such that the restriction $\exp_x|C_x\colon C_x\to M$ is a diffeomorphism onto an open set in M and the complement $M\backslash\exp_x(C_x)$ has measure zero. Thus

$$\tilde{K}_\tau\varphi(x)=(2\pi\tau)^{-n/2}\int_{C_x} e^{-\|v\|^2/2\tau}f(\exp_x v)\,|\det d_v \exp_x|\,dv.$$

We note that $|\det d_v\exp_x|=1-1/6\ \mathrm{Ric}\ c(v,v)+O(\|v\|^3)$, so that we may choose a positive constant c, not depending on v, such that $|\det d_v \exp_x| \leqq 1+c\|v\|^2$ for any $v\in C_x$. Therefore,

$$\begin{aligned}\|\tilde{K}_\tau\varphi\|^2 &\leqq \int_M dg(x)\left\{(2\pi\tau)^{-n/2}\int_{C_x} e^{-\|v\|^2/2\tau}|\varphi(\exp_x v)|(1+c\|v\|^2)dv\right\}^2\\ &\leqq \int_M dg(x)\left\{(2\pi\tau)^{-n/2}\int_{T_xM} e^{-\|v\|^2/2\tau}(1+c\|v\|^2)|\varphi(\exp_x v)|dv\right\}^2\\ &=\|K_\tau|\varphi|+c\hat{K}_\tau|\varphi|\|^2,\end{aligned}$$

where we set

$$\hat{K}_\tau\varphi(x)=(2\pi\tau)^{-n/2}\int_{T_xM} e^{-\|v\|^2/2\tau}\|v\|^2\varphi(\exp_x v)dv.$$

Since $\|K_\tau\varphi\|\leqq\|\varphi\|$, $\|\hat{K}_\tau\varphi\|\leqq\tau c'\|\varphi\|$, we have

$$\|\tilde{K}_\tau\varphi\|\leqq\|K_\tau|\varphi|\|+c\|\hat{K}_\tau|\varphi|\|\leqq(1+c''\tau)\|\varphi\|.$$

Hence

$$(*)\qquad \|\tilde{K}^N_{t/N}\|\leqq\left(1+c''\frac{t}{N}\right)^N\leqq e^{c''t}.$$

To accomplish the proof, we require two lemmas.

Lemma 1. *For any positive integer k,*

$$\frac{1}{\tau^k}\int_{(1/\sqrt{\tau})C_x} e^{-\|v\|^2/2}v\cdot\varphi(x)dv \longrightarrow 0,$$

uniformly for $x\in M$, as $\tau\downarrow 0$.

Proof. It is enough to show that for any integer $k>0$,

$$\left|\int_{R\mathscr{D}}\langle u, v\rangle e^{-\|v\|^2/2}dv\right|=o(R^{-k})\cdot\|u\|$$

for $R\gg 1$, where $\mathscr{D}$ is a domain containing the unit ball $B_1(o)$, and u is a vector. In fact, noting

$$\int_{B_R(o)}\langle u, v\rangle e^{-\|v\|^2/2}dv=0,$$

we find

$$\begin{aligned}\left|\int_{R\mathscr{D}}\langle u, v\rangle e^{-\|v\|^2/2}dv\right| &= \left|\int_{R\mathscr{D}\setminus B_R(o)}\langle u, v\rangle e^{-\|v\|^2/2}dv\right| \\ &\leqq \|u\|\int_{R^n\setminus B_R(o)}\|v\|e^{-\|v\|^2/2}dv = C\|u\|\int_R^\infty s^n e^{-s^2/2}ds \\ &= o(R^{-k})\|u\|.\end{aligned}$$

Lemma 2. *For any positive integer k,*

$$\frac{1}{\tau^k}\int_{T_xM\setminus(1/\sqrt{\tau})C_x}\|v\|^2 e^{-\|v\|^2/2}dv\longrightarrow 0,$$

uniformly for $x\in M$, as $\tau\downarrow 0$.

Easy, and we omit the proof.

We now return to the proof of Theorem. Since

$$\varphi(\exp \sqrt{\tau}\, v)|\det d_{\sqrt{\tau} v} \exp_x| = \varphi(x) + \sqrt{\tau}\, v \cdot \varphi(x) + \frac{\tau}{2} \sum v^i v^j \nabla_i \nabla_j \varphi(x)$$
$$- \frac{\tau}{6} \varphi(x) \sum R_{ij} v^i v^j + O(\tau^{3/2} \|v\|^3),$$

we get

$$\tilde{K}_\tau \varphi(x) = (2\pi)^{-n/2} \int_{(1/\sqrt{\tau})C_x} e^{-\|v\|^2/2} \Big\{ \varphi(x) + \sqrt{\tau}\, v \cdot \varphi(x)$$
$$+ \tau \Big(\frac{1}{2} \sum v^i v^j \nabla_i \nabla_j \varphi(x) - \frac{1}{6} \varphi(x) \sum R_{ij} v^i v^j \Big) + O(\tau^{3/2} \|v\|^3) \Big\} dv,$$

which, in view of the above Lemmas, equals

$$\varphi(x) + \tau \Big(\frac{1}{2} \Delta \varphi(x) - \frac{1}{6} R(x) \varphi(x) \Big) + O(\tau^{3/2}).$$

On the other hand, we may easily check that $\|\tilde{K}_\tau \varphi - \varphi\| \to 0$ for any $\varphi \in C^\infty(M)$. In virtue of $(*)$, this is also true for any $\varphi \in L^2(M)$. Thus, we may apply the Trotter-Chernoff theorem, and the theorem is proved. See for related matters A. Inoue and Y. Maeda [101].

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W. Parry and M. Pollicott, An analogue of the prime number theorem for closed orbits of Axiom A flows, to appear in Ann. of Math.
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