Advanced Studies in Pure Mathematics 2, 1983
Galois Groups and their Representations
pp. 15–36

Hodge Structures of Shimura Varieties Attached to the Unit Groups of Quaternion Algebras

Takayuki Oda

Introduction

a) Let G be a reductive algebraic group defined over an algebraic number field F. Then it is believed that for any "geometric" primitive modular form f over the adelization G_A of G, there exists a motif M_f naturally attached to f (cf. Deligne [1]). It is already one question to give a precise meaning to the term "geometric". But one might set aside this temporarily, by agreeing to the following. When F is totally real and the quotient space $(G\times_{F,\sigma}\boldsymbol{R})(\boldsymbol{R})/K$ of the real Lie group of the real points of $G\times_{F,\sigma}\boldsymbol{R}$ by a maximal compact group K of $(G\times_{F,\sigma}\boldsymbol{R})(\boldsymbol{R})$, has an invariant complex structure for any embedding: $\sigma\colon F\hookrightarrow\boldsymbol{R}$, those modular forms f which belong to the automorphic representations $\pi=\otimes_v\pi_v$ of G_A with infinite components π_σ belonging to the discrete series of $(G\times_{F,\sigma}\boldsymbol{R})(\boldsymbol{R})$, are geometric.

Then it is a problem to know how to attach motives M_f to the primitive forms f of the above type. We want to discuss this problem for modular forms f of weight 2 with respect to the Hilbert modular groups or the unit groups of quaternion algebras, i.e. when F is totally real and $G=GL(2)$ or $G=B^\times$ for some indefinite quaternion algebra B over F. This case is very special. Nevertheless, the answer is by no means trivial as we shall see soon. In this paper, we announce some results on the Hodge structures attached to f, which are generalizations of the results of the previous paper [9].

b) In order to explain the purpose of this paper, we start with some conjectures proposed in [9], which we reformulate here, employing the terminology of motives in Deligne [2].

Let F be a totally real algebraic number field of degree g over $\boldsymbol{Q}$. Let us call a modular cusp form f over the adelization $GL_2(A)$ of $GL_2(F)$ a primitive Hilbert modular cusp form, if it generates an irreducible automorphic representation $\pi_f=\otimes_v\pi_v$ of $GL_2(A)$ whose infinite component π_σ belongs to the discrete series of $GL_2(\boldsymbol{R})$ for any embedding $\sigma\colon F\hookrightarrow\boldsymbol{R}$,

Received October 4, 1982.

and moreover f is the extreme vector of π_v for any infinite place v and f is a new vector of π_v for any finite place v of F.

For any primitive Hilbert modular form f of weight 2 with trivial central character and with conductor $\mathfrak{n}$, we expect that the motif M_f is defined as follows. Let $X_{\mathfrak{n}}$ be the Hilbert modular variety attached to "$\Gamma_0(\mathfrak{n})$", which is a disjoint union of a finite number of arithmetic quotients of H^g by the congruence subgroups. Here H is the complex upper half plane. Let $H^g(X_{\mathfrak{n}})$ be the g-th motif of $X_{\mathfrak{n}}$ (cf. [2]), and let $\{W.\}$ be the weight filtration. Then $W_gH^g(X_{\mathfrak{n}})$ would be a polarized motif of weight g. If g is even, the space $W_gH^g(X_{\mathfrak{n}})$ contains some trivial elements η_Λ obtained as the cup products of the Chern classes η in $W_2H^2(X_{\mathfrak{n}})$ corresponding to the line bundles of the automorphy factors. Let us discard this part by putting $H^g_{sp}(X_{\mathfrak{n}})=W_gH^g(X_{\mathfrak{n}})$ if g is odd, and $H^g_{sp}(X_{\mathfrak{n}})=$ the orthogonal complement of η_Λ's with respect to the polarization in $W_gH(X_{\mathfrak{n}})$ if g is even.

For any pair of divisors, $\mathfrak{m}$, $\mathfrak{d}$ of $\mathfrak{n}$ satisfying, $\mathfrak{m}\mathfrak{d}\,|\,\mathfrak{n}$, we can define the "face" morphism $j_{\mathfrak{m},\mathfrak{d}}\colon X_{\mathfrak{n}}\to X_{\mathfrak{m}}$ as usual, which induces a homomorphism $j^*_{\mathfrak{m},\mathfrak{d}}\colon H^g(X_{\mathfrak{m}})\to H^g(X_{\mathfrak{n}})$ of motives. We define the new part $H^g_{sp}(X_{\mathfrak{n}})^{\mathrm{new}}$ of $H^g_{sp}(X_{\mathfrak{n}})$ as the orthogonal complement of the images of $j^*_{\mathfrak{m},\mathfrak{d}}$ for various $\mathfrak{m}$, $\mathfrak{d}$ in $H^g_{sp}(X_{\mathfrak{n}})$ with respect to the polarization.

Since any Hecke operator $T(\mathfrak{q})$ $((\mathfrak{q},\mathfrak{n})=1)$ is regarded as an algebraic correspondence of $X_{\mathfrak{n}}$, the Hecke algebra R generated by $T(\mathfrak{q})$ $((\mathfrak{q},\mathfrak{n})=1)$ acts on $H^g_{sp}(X_{\mathfrak{n}})^{\mathrm{new}}$. Thus we can consider the decomposition of the R-module $H^g_{sp}(X_{\mathfrak{n}})^{\mathrm{new}}$ into simple submodules.

Let $R^{\mathrm{new}}_{\boldsymbol{Q}}$ be the subalgebra over $\boldsymbol{Q}$ in End $(H^g_{sp}(X_{\mathfrak{n}})^{\mathrm{new}})$ generated by the image of R. Then $R^{\mathrm{new}}_{\boldsymbol{Q}}$ is expected to be a commutative semisimple algebra, hence is a direct sum of finite algebraic number fields K. For each primitive form f, we shall find a unique subfield K of $R^{\mathrm{new}}_{\boldsymbol{Q}}$, or a unique primitive idempotent e in $R^{\mathrm{new}}_{\boldsymbol{Q}}$ such that $K=eR^{\mathrm{new}}_{\boldsymbol{Q}}=R^{\mathrm{new}}_{\boldsymbol{Q}}e=eR^{\mathrm{new}}_{\boldsymbol{Q}}e \simeq K_f$. Here K_f is the field generated by the eigenvalues of f over $\boldsymbol{Q}$, or the field of moduli of the automorphic representation π_f. Then we put

$$M_f=eH^g_{sp}(X_{\mathfrak{n}})^{\mathrm{new}}$$

and call M_f the motif attached to f.

The motif M_f should have the following properties:

(i) M_f is a motif of weight g with a homomorphism

$$\theta_f\colon K_f\longrightarrow \mathrm{End}\,(M_f),$$

and M_f is a K_f-module of rank 2^g via θ_f.

(ii) M_f has a polarization $\Phi_f\colon M_f\times M_f\to\boldsymbol{Q}(-g)$ such that

$$\Phi_f(\theta_f(a)x,y)=\Phi_f(x,\theta_f(a)y)$$

for any $a \in K_f$ and any $x, y \in M_f$.

Moreover there exists the unique K_f-bilinear form $\Phi_f\colon M_f \times M_f \rightarrow K_f$ such that $\Phi_f = \mathrm{tr}_{K_f/Q}(\psi_f)$.

Thus M_f is a motif with coefficient in K_f by the terminology of Deligne [2], which we simply call a K_f-motif.

Then, the conjecture A^{split} in Chapter 0 of [9] is found to be the Hodge realization of the following conjecture.

Conjecture A^{split}. *Let f be a primitive Hilbert modular cusp form of weight* 2, *and let M_f be the motif attached to f. Then there exists an abelian variety A_f of dimension $[K_f\colon Q]$ defined over F with a homomorphism*

$$\theta_f\colon K_f \longrightarrow \mathrm{End}(A_f) \otimes_Z Q$$

such that there exists an isomorphism of K_f-motives

$$M_f = \bigotimes_{\sigma \in \Sigma_f} {}_{K_f} H^1(A_f)_\sigma.$$

Here Σ_f is the set of all the embeddings of F into the algebraic closure $\bar{F}$ of F, and $H^1(A_f)_\sigma$ is the "conjugate" of the 1-motif $H^1(A_f)$ of A_f with respect to σ in some sense. Note there that the tensor product in the right hand side is considered over K_f.

Roughly speaking, the conjecture A^{split} implies that the g-th cohomology group of any Hilbert modular variety is a direct sum of the 1-st cohomology group of certain varieties.

In addition to this conjecture, we may further expect the following generalization of the conjecture of Weil uniformization (cf. Chap. 0, iii) of [9]).

Conjecture W. *For any elliptic curve E defined over F, there exists a primitive Hilbert modular cusp form f of weight* 2 *with $K_f = Q$, such that E is isogenous to the elliptic curve A_f over F.*

c) Let us make our talk more practical and substantial by replacing all the motives in question by the corresponding Hodge structures and considering only the Hodge realizations of the conjectures A^{split} and W, because the existence of the theory of motives is not yet proved by logic.

Let us try to justify Conjecture W. As one notices soon, the first thing to be done is to show that Conjecture W is valid for any elliptic curve E over F, which is obtained as a factor of the jacobian varieties of Shimura curves defined over F. What does this mean? Let us assume that the Shimura curve in question is obtained as a union of the arithmetic quotients of H by congruence subgroups of the unit groups of a quaternion algebra B over F, which split at only one infinite place of F. In this case, generally speaking the elliptic curve E itself is an abelian variety $A_{f'}$ attached to a certain modular cusp form f' of weight 2 over $B_A^\times$. Conjecture W implies that A_f and $A_{f'}$ are isogenous over F. Then it is natural to ask what is the relation between f and f' and to suspect that f is obtained by the Eichler-Shimizu correspondence. Consequently, we are naturally led to the following two conjectures A and B.

In order to simplify our exposition, from now on we assume that the class number in the narrow sense of F is one, i.e. the class number of F is one and the group E_F^+ of totally positive units has index 2^g in the group E_F of units in O_F. Here O_F is the ring of integers in F.

Let B be an indefinite quaternion algebra over F with discriminant $\mathfrak{d}_B$, and let $N(B)$ (resp. $R(B)$) be the set of infinite places of F at which B is not ramified (resp. is ramified). We denote by n the cardinality of the set $N(B)$. Since we assume that the class number of F is one, all the maximal order in B are isomorphic by the strong approximation. Choose a maximal order O_B in B. Then we denote by $O_{B+}^\times$ the subgroup of the unit group $O_B^\times$ of O_B, consisting of those elements α of O_B whose reduced norms $\nu(\alpha)$ are totally positive. For any place v in $N(B)$, the completion B_v of B at v is isomorphic to $M_2(\boldsymbol{R})$. Therefore via v, we can define a homomorphism $O_{B+}^\times \to GL_2^+(\boldsymbol{R})$. The product of all these homomorphism for $v \in N(B)$ defines a homomorphism $O_{B+}^\times \to GL_2^+(\boldsymbol{R})^n$. By means of this homomorphism, we can define the action of $O_{B+}^\times$ on H^n as usual.

Let v be a finite place of F which does not divide $\mathfrak{d}_B$. Then the completion $O_{B,v}$ of O_B at v is isomorphic to $M_2(O_{F,v})$, where $O_{F,v}$ is the completion of O_F at v. Fix an isomorphism $i_v\colon O_{B,v} \xrightarrow{\sim} M_2(O_{F,v})$ for each v not dividing $\mathfrak{d}_B$. Let $\mathfrak{m}$ be an ideal of O_F coprime to $\mathfrak{d}_B$. Then we define a congruence subgroup $\Gamma_0(\mathfrak{m}; B)$ of $O_{B+}^\times$ by

$$\Gamma_0(\mathfrak{m}; B) = \left\{ \alpha \in O_{B+}^\times \mid i_v(\alpha) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } c \in \mathfrak{m} O_{F,v} \right\}.$$

Let f' be a holomorphic primitive modular cusp form on H^n of weight 2 with respect to $\Gamma_0(\mathfrak{m}; B)$. If $B = M_2(F)$, f' is a primitive Hilbert modular cusp form and we can attach a Hodge structure $H^g(M_f, \boldsymbol{Q})$ of weight g (cf. § 1 and 2), which is the Hodge realization of the motif M_f. If $B \not\cong M_2(F)$, then similarly as for Hilbert modular forms, we can attach a Hodge structure $H^n(M'_f, \boldsymbol{Q})$ of weight n to f', which is a substructure of the Hodge structure $H^n(\Gamma_0(\mathfrak{m}; B) \backslash H^n, \boldsymbol{Q})$ of the complex projective variety $\Gamma_0(\mathfrak{m}; B) \backslash H^n$.

As we see later (cf. § 1 and 2), the Hodge structure $H^n(M_{f'}, \boldsymbol{Q})$ has the following properties:

(i) $H^n(M_{f'}, \boldsymbol{Q})$ is a rational Hodge structure of weight n with a homomorphism $\theta_{f'}: K_{f'} \to \mathrm{End}\,(H^n(M_{f'}, \boldsymbol{Q}))$, and $H^n(M_{f'}, \boldsymbol{Q})$ is of rank 2^n over $K_{f'}$ via $\theta_{f'}$.

(ii) There is a polarization $\Phi_{f'}: H^n(M_{f'}, \boldsymbol{Q}) \times H^n(M_{f'}, \boldsymbol{Q}) \to \boldsymbol{Q}(-n)$ satisfying

$$\Phi_{f'}(\theta_{f'}(a)x, y) = \Phi_{f'}(x, \theta_{f'}(a)y)$$

for any $a \in K_{f'}$ and any $x, y \in H^n(M_{f'}, \boldsymbol{Q})$, where $K_{f'}$ is the subfield of $\boldsymbol{C}$ generated by the eigenvalues of f' with respect to $T(\mathfrak{q})$ $((\mathfrak{q}, \mathfrak{d}_B\mathfrak{m})=1)$. Moreover there exists a $K_{f'}$-bilinear form $\psi_{f'}$ on $H^n(M_{f'}, \boldsymbol{Q})$ with value in K_f such that $\Phi_{f'} = \mathrm{tr}_{K_{f'}/\boldsymbol{Q}}(\psi_{f'})$.

Then as a generalization of Conjecture $\mathrm{A}^{\mathrm{split}}$, we propose the following.

Conjecture A (Hodge realization version). *Let f' be a holomorphic modular cusp form on H^n of weight 2 with respect to $\Gamma_0(\mathfrak{m}; B)$ which is a common eigenform of all Hecke operators $T(\mathfrak{q})$ with $(\mathfrak{q}, \mathfrak{d}_B\mathfrak{m})=1$. Then for the Hodge structure $H^n(M_{f'}, \boldsymbol{Q})$ there exists an abelian variety $A_{f'}$ defined over F with a homomorphism*

$$\theta_{f'}: K_{f'} \longrightarrow \mathrm{End}\,(A_{f'}) \otimes_{\boldsymbol{Z}} \boldsymbol{Q},$$

such that there is an isomorphism of $K_{f'}$-Hodge structures

$$H^n(M_{f'}, \boldsymbol{Q}) = \bigotimes_{v \in N(B)} {}_{K_{f'}} H^1(A_{f'}^{(v)}, \boldsymbol{Q}).$$

Here $A_{f'}^{(v)}$ is the abelian variety defined over $\boldsymbol{C}$ obtained from $A_{f'}$ by means of the scalar extension with respect to the embedding $F \hookrightarrow \boldsymbol{C}$ corresponding to v.

Remark. If $n=1$, the above conjecture says nothing except that $A_{f'}$ is defined over F, which is shown by the theory of canonical models of Shimura [15].

Let us formulate another conjecture related with the Eichler-Shimizu correspondence. Suppose that two indefinite quaternion algebras B_1 and B_2 over F are given, and consider a holomorphic modular cusp form f_1 (resp. f_2) on H^{n_1} (resp. on H^{n_2}) of weight 2 with respect to $\Gamma_0(\mathfrak{m}_1; B_1)$ (resp. $\Gamma_0(\mathfrak{m}_2; B_2)$) for some ideal $\mathfrak{m}_1$ (resp. $\mathfrak{m}_2$) of O_F, which is a common eigenform of all the Hecke operators $T(\mathfrak{n})$ with $(\mathfrak{n}, \mathfrak{d}_{B_1}\mathfrak{m}_1)=1$ (resp. with

$(\mathfrak{n}, \mathfrak{d}_{B_2}\mathfrak{m}_2)=1$). Then we say that two modular forms f_1 and f_2 correspond in the sense of Eichler-Shimizu, if the eigenvalues of f_1 and f_2 with respect to $T(\mathfrak{n})$ coincide for any Hecke operator $T(\mathfrak{n})$ with $(\mathfrak{n}, \mathfrak{m}_1\mathfrak{m}_2\mathfrak{d}_{B_1}\mathfrak{d}_{B_2})=1$ (cf. [4], [12], [13], [14], and § 16 of [7]).

Conjecture B. *Let f_1 (resp. f_2) be a holomorphic modular form of weight 2 on H^{n_1} (resp. on H^{n_2}) with respect to $\Gamma_0(\mathfrak{m}_1; B_1)$ (resp. $\Gamma_0(\mathfrak{m}_2; B_2)$) which is a common eigenform of all Hecke operators. Suppose that Conjecture* A *is valid for f_1 and f_2, and let A_{f_1} and A_{f_2} be the abelian varieties over F obtained in Conjecture* A *for f_1 and f_2, respectively. Then, if f_1 and f_2 correspond in the sense of Eichler-Shimizu, there exists an isogeny*

$$A_{f_1}^{(v)} \sim A_{f_2}^{(v)}$$

over $v(F)$ ($\hookrightarrow C$), compatible with the actions of $K_{f_1}=K_{f_2}$, for any $v \in N(B_1)\cap N(B_2)$.

d) Though we cannot yet prove Conjectures A and B completely because of some limitation of our present method, we have some results stated below. Before formulating our main results, we need some preparation, because we have to impose some condition on the modular form f so that our method works.

Let $P_\infty(F)$ be the set of all infinite places of F, and let Sgn (F) be the set of mappings of the set $P_\infty(F)$ to the set $\{+1, -1\}$. In other words, Sgn (F) is the set of vectors of length g with components equal to $+1$ or to -1. Let χ be a quadratic Dirichlet character of O_F with conductor $\mathfrak{k}$, and let $\tilde{\chi}$ be the corresponding character of the idele group I_F. Let α be the element of $I_F/(F^\times I_F^2)$ corresponding to $\tilde{\chi}$ by the reciprocity law of the class field theory. Then for any χ, we define Sgn (χ) as the vector in Sgn (F) such that its v-th component is $+1$ or -1, according as the v-component of the idele class α is positive or negative for each $v \in P_\infty(F)$.

Definition. Let B be an indefinite quaternion algebra over F, and suppose that f is a holomorphic modular form of weight 2 on H^n with respect to $\Gamma_0(\mathfrak{m}; B)$ for some ideal $\mathfrak{m}$ of O_F, which is a common eigenform of all Hecke operators $T(\mathfrak{n})$ ($(\mathfrak{n}, \mathfrak{d}_B\mathfrak{m})=1$). Then f is called *strongly admissible*, if for any given element S of Sgn (F), there exists a quadratic Dirichlet character χ of O_F with signature Sgn $(\chi)=S$ and with conductor coprime to $\mathfrak{m}\mathfrak{d}_B$ such that $L(1, f, \chi)\neq 0$.

Here $L(s, f)$ is the Dirichlet series defined by the Euler product

$$L(s, f)=\prod_{\mathfrak{q}\nmid\mathfrak{m}\mathfrak{d}_B}(1-a_\mathfrak{q}N\mathfrak{q}^{-s}+N\mathfrak{q}^{1-2s})^{-1}$$

with $a_\mathfrak{q}$ the eigenvalue of f with respect to $T(\mathfrak{q})$, and $N\mathfrak{q}$ the absolute norm of the prime ideal $\mathfrak{q}$ in K_f, and $L(s, f, \chi)$ is the twist of $L(s, f)$ with respect to the quadratic character, i.e.

$$L(s, f, \chi) = \prod_{\mathfrak{q} \nmid \mathfrak{d}_B \mathfrak{m}} (1 - \chi(\mathfrak{q}) a_{\mathfrak{q}} N\mathfrak{q}^{-s} + N\mathfrak{q}^{1-2s})^{-1}.$$

As a partial answer to Conjecture A, we have the following:

Theorem A. *Let B be an indefinite quaternion algebra over F, and let f be a holomorphic modular form of weight 2 on H^n with respect to $\Gamma_0(\mathfrak{m}; B)$, which is a common eigenform of all Hecke operators. Assume that f is strongly admissible. Then for each infinite place v of F in $N(B)$, we can find an abelian variety A_f^v of dimension $[K_f : \boldsymbol{Q}]$ defined over $\boldsymbol{C}$ with a homomorphism*

$$\theta_f^v : K_f \longrightarrow \mathrm{End}\,(A_f^v) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$$

such that there is an isomorphism of K_f-Hodge structures

$$H^n(M_f, \boldsymbol{Q}) = \bigotimes_{\substack{K_f \\ v \in N(B)}} H^1(A_f^v, \boldsymbol{Q}).$$

Moreover, if n is odd, the abelian variety A_f^v is defined over the algebraic number field $v(F)$ for each $v \in N(B)$.

Remark 1. If n is even, we do not know in general whether or not the abelian variety A_f^v is defined over some algebraic number field. Assume that n is odd. Then, if $F/\boldsymbol{Q}$ is a galois extension, the abelian variety $A_f^{v'}$ for $v' \in N(B)$ is K_f-isogenous (i.e. isogenous compatible with the actions of K_f) to the conjugate of A_f^v which is obtained from A_f^v by means of the scalar extension $v(F) \simeq v(F)' \hookrightarrow \boldsymbol{C}$, *over* $\boldsymbol{C}$. If $F/\boldsymbol{Q}$ is not a galois extension, we know nothing about the conjugacy between A_f^v ($v \in N(B)$) in general.

We have the following partial answer to Conjecture B.

Theorem B. *Let B_1 and B_2 be two indefinite quaternion algebras over F, and suppose that f_1 (resp. f_2) is a holomorphic modular forms of weight 2 on H^{n_1} (resp. on H^{n_2}) with respect to $\Gamma_0(\mathfrak{m}_1; B_1)$ (resp. $\Gamma_0(\mathfrak{m}_2; B_2)$), which is an eigenform of all Hecke operators. Assume that both f_1 and f_2 are strongly admissible. Then, if f_1 and f_2 correspond in the sense of Eichler-Shimizu, the abelian varieties $A_{f_1}^v$ and $A_{f_2}^v$ constructed by Theorem A for each $v \in N(B_1) \cap N(B_2)$, are mutually $K_{f_1} = K_{f_2}$-isogenous over $\boldsymbol{C}$.*

Remark 2. Let f be a modular form considered in Theorem A. As we have noted in Remark 1, we do not know in general whether or not

A_f^v is defined over algebraic number field, if n is even. However it may happen that there exists another modular form f' of weight 2 over another quaternion algebra B', which is the Eichler-Shimizu correspondence of f, such that $n' = \sharp(N(B'))$ is odd. Then by Theorems B and A, A_f^v is isogenous to an abelian variety defined over an algebraic number fields, hence A_f^v itself is defined over a number field.

But unfortunately this never happens for (say) Hilbert modular form f of conductor 1 with even degree g, because of the sum theorem of Hasse invariant.

Remark 3. Theorem B is related with the recent results of Ribet [10] and Shimura [18]. We discuss it in a later section.

§ 1. Hodge structures of Shimura varieties

In this section, we verify some basic facts on the Hodge structures of Shimura varieties. Similarly as in Introduction, we assume that the class number of F in the narrow sense is one.

Let B be an indefinte quaternion algebra over F, and let us choose a maximal order O_B. We denote by $O_{B+}^\times$ the subgroup of the unit group $O_B^\times$ of $O_B^\times$ consisting of elements with totally positive reduced norms. Let $\mathfrak{n}$ be an ideal of O_F coprime to the conductor $\mathfrak{d}_B$ of B, and let us define the congruence subgroup $\Gamma_0(\mathfrak{n}; B)$ of $O_{B+}^\times$ as in Introduction, which acts on the product H^n of the complex upper half plane H. Here n is the cardinality of the set $N(B)$ of the infinite places of F, at which B splits. The quotient analytic variety $\Gamma_0(\mathfrak{n}; B)\backslash H^n$ is a projective algebraic variety if B is not isomorphic to $M_2(F)$, or a quasi-projective algebraic variety if $B \cong M_2(F)$. We put $X_\mathfrak{n} = \Gamma_0(\mathfrak{n}; B)\backslash H^n$. Then $X_\mathfrak{n}$ has only finite number of isolated quotient singularities, hence $X_\mathfrak{n}$ is a rational homology manifold.

Let $H^n(X_\mathfrak{n}, \boldsymbol{Q})$ be the n-th cohomology group with rational coefficients of X. Then by Deligne [3], $H^n(X_\mathfrak{n}, \boldsymbol{Q})$ has a mixed Hodge structure. Let $\{W_\cdot\}$ be the weight filtration of $H^n(X_\mathfrak{n}, \boldsymbol{Q})$. Then $W_{n-1}H^n(X_\mathfrak{n}, \boldsymbol{Q}) = \{0\}$, because $X_\mathfrak{n}$ is a rational homology manifold (cf. Theorem (8.2.4) of III of [3]). Moreover, if $B \not\cong M_2(F)$, $W_n H^n(X_\mathfrak{n}, \boldsymbol{Q}) = H^n(X_\mathfrak{n}, \boldsymbol{Q})$, because $X_\mathfrak{n}$ is compact in this case. If $B \cong M_2(B)$, we have the following.

Proposition 1.1. *Put $\Gamma_0(\mathfrak{n}) = \Gamma_0(\mathfrak{n}; B)$ for $B = M_2(F)$. Then if s is the number of $\Gamma_0(\mathfrak{n})$-equivalence classes of cusps with respect to $\Gamma_0(\mathfrak{n})$. Then* $\operatorname{rank}_{\boldsymbol{Q}} \{H^n(X_\mathfrak{n}, \boldsymbol{Q}) / W_{n-1}H^n(X_\mathfrak{n}, \boldsymbol{Q})\} = s$, *and*

$$W_{i+1}H^n(X_\mathfrak{n}, \boldsymbol{Q}) = W_i H^n(X_\mathfrak{n}, \boldsymbol{Q}) \qquad \text{for } n \leqq i \leqq 2n-2,$$

and

$$W_{2n}H^n(X_{\mathfrak{n}}, \boldsymbol{Q})/W_{2n-1}H^n(X_{\mathfrak{n}}, \boldsymbol{Q}) = \overset{s}{\oplus}\, \boldsymbol{Q}(-n),$$

if $g=n>1$ (*note that* $g=n$ *in this case*).

Remark 1. The elements of $H^g(X_{\mathfrak{n}}, \boldsymbol{Q})/W_gH^g(X_{\mathfrak{n}}, \boldsymbol{Q})$ are represented by certain holomorphic g-forms of the "third" kind obtained from the Eisenstein series.

Remark 2. If $g=n=1$, i.e. $F=\boldsymbol{Q}$, the above proposition is also valid by putting $s=$ the number of $\Gamma_0(\mathfrak{n})$-equivalence classes of cusps minus 1.

We can prove the above proposition by means of the Poincaré residue mapping and the toroidal compactifications of $X_{\mathfrak{n}}$. Details are omitted.

Let $z=(z_1, \cdots, z_n) \in H^n$ be the coordinates of the point z in H^n with $z_i=x_i+\sqrt{-1}\,y_i$ ($x_i, y_i \in \boldsymbol{R}, y_i>0$) for each i ($1 \leqq i \leqq n$). Then we put

$$\eta_i=(2\pi\sqrt{-1})^{-1}y_i^{-2}dz_i\wedge d\bar{z}_i \qquad \text{for each } i \ (1\leqq i\leqq n).$$

It is easy to see that each η_i defines a $GL_2^+(\boldsymbol{R})$-invariant real (1, 1) type 2-form on H^n. Hence η_i defines *a* 2-form on X, and an element of $H^2_{DR}(X_{\mathfrak{n}}, \boldsymbol{C})$, which we denote by the same symbol η_i.

Proposition 1.2. *For each* i $(1\leqq i\leqq n)$, η_i *defines an element of* $W_2H^2(X_{\mathfrak{n}}, \boldsymbol{Q})$.

The proof of this proposition is similar to that of Proposition 1.9 of [9].

For any subset Λ of $\{1, \cdots, n\}$ with cardinality λ, the 2λ-form $\eta_\Lambda = \bigwedge_{i\in\Lambda}\eta_i$ defines an element of $W_{2\lambda}H^{2\lambda}(X_{\mathfrak{n}}, \boldsymbol{Q})$. Especially if n is even, the element η_Λ with $\lambda=n/2$ defines an element in $W_nH^n(X_{\mathfrak{n}}, \boldsymbol{Q})$.

Definition. If n is even, we put $H^n_{sp}(X_{\mathfrak{n}}, \boldsymbol{Q})=$ the orthogonal complement of η_Λ's with $\lambda=\sharp(\Lambda)=n/2$ in $W_nH^n(X_{\mathfrak{n}}, \boldsymbol{Q})$ with respect to the intersection form.

If n is odd, we put $H^n_{sp}(X_{\mathfrak{n}}, \boldsymbol{Q})=W_nH^n(X_{\mathfrak{n}}, \boldsymbol{Q})$.

Remark. The intersection form is well-defined on $W_nH^n(X_{\mathfrak{n}}, \boldsymbol{Q})$, because the cohomology classes in $W_nH^n(X_{\mathfrak{n}}, \boldsymbol{Q})$ has compact support. Moreover, it is not difficult to check that it defines a non-degenerate bilinear form on $W_nH^n(X_{\mathfrak{n}}, \boldsymbol{Q})$ by Poincaré duality.

In order to describe the Hodge decomposition of $H^n_{sp}(X_{\mathfrak{n}}, \boldsymbol{Q})\otimes_{\boldsymbol{Q}}\boldsymbol{C}$, it is convenient to introduce here certain non-holomorphic involutive automorphisms of the Shimura variety $X_{\mathfrak{n}}$, which are investigated by Shimura [16] and Shin [11] in a different context.

By the assumption on F, the subgroup E_F^+ of totally positive units has index 2^g in the unit group E_F of O_F, and $E_F^+=E_F^2$. Let $P_\infty(F)$ be the set of infinite places of F. Then we identify any element of $P_\infty(F)$ with the corresponding embedding $F\hookrightarrow\boldsymbol{C}$. Now we define Sgn (B) as the set of all mappings of the set $N(B)$ into the set $\{+1, -1\}$, which is naturally regarded as the set of vectors of length n with components $+1$ or -1. For any given element $S=(s_\tau)_{\tau\in N(B)}$ of Sgn (B), we choose an element ε_S of E_F such that $\tau(\varepsilon)>0$ or $\tau(\varepsilon)<0$ according as $s_\tau=+1$ or $s_\tau=-1$, which is unique modulo $E_F^+=E_F^2$. Put

$$\Gamma_0^{\pm}(\mathfrak{n}; B)=\left\{\alpha\in O_B^\times \mid i_v(\alpha)=\begin{pmatrix} a & b\\ c & d\end{pmatrix},\ c\in\mathfrak{n}O_{F,v} \text{ for any finite place } v \text{ of } F\right\}.$$

Then $\Gamma_0(\mathfrak{n}; B)$ is a normal subgroup of index 2^n in $\Gamma_0^{\pm}(\mathfrak{n}; B)$. By the strong approximation theorem, we can choose an element α_S of $\Gamma_0^{\pm}(\mathfrak{n}; B)$ such that $\nu(\alpha_S)=\varepsilon_S$, where ν is the reduced norm of B over F. The class α_S modulo $\Gamma_0(\mathfrak{n}; B)$ does not depend on the choice of ε_S and α_S. For any element τ of $N(B)$, we have an isomorphism $B\otimes_{F,\tau}\boldsymbol{R}\cong M_2(\boldsymbol{R})$, which we denote by the same symbol τ. By the definition of α_S, det $\tau(\alpha_S)$ is positive or negative, according as $s_\tau=+1$ or $s_\tau=-1$.

Let us define a non-holomorphic automorphism $\tilde{F}_S$ of H^n by $\tilde{F}_S(z)=w$ with each component w_τ of $w=(w_\tau)_{\tau\in N(B)}$ given by

$$\begin{cases} w_\tau=\tau(\alpha_S)(z_\tau), & \text{if } s_\tau=+1\\ w_\tau=\tau(\alpha_S)(\bar{z}_\tau), & \text{if } s_\tau=-1\end{cases}$$

for $z=(z_\tau)_{\tau\in N(B)}\in H^n=H^{N(B)}$.

Then the automorphism $\tilde{F}_S$ of H^n belongs to the normalizer of $\Gamma_0(\mathfrak{n}; B)$. Hence, on passing to the quotient $X_{\mathfrak{n}}$, $\tilde{F}_S$ defines an automorphism F_S of $X_{\mathfrak{n}}$, which is involutive because $\tilde{F}_S^2\in\Gamma_0(\mathfrak{n}; B)$. It is easy to check that F_S does depend on the choice of ε_S and α_S. If we define the product ST of two elements S, T of Sgn (B) by the componentwise multiplication, $F_S\cdot F_T=F_{ST}=F_{TS}=F_T\cdot F_S$. If S is the special vector in Sgn (B) such that all the components are -1, F_S is denoted by F_∞. F_∞ is an anti-holomorphic automorphism of $X_{\mathfrak{n}}$.

Now let us see the Hodge decomposition of $H_{sp}^n(X_{\mathfrak{n}}, \boldsymbol{Q})\otimes_{\boldsymbol{Q}}\boldsymbol{C}$. Let $S_2(\Gamma_0(\mathfrak{n}; B))$ be the space of holomorphic modular cusp forms of weight 2 on H^n with respect to $\Gamma_0(\mathfrak{n}; B)$. For eny element $f(z)$ of $S_2(\Gamma_0(\mathfrak{n}; B))$, we define a holomorphic n-form ω_f on H^n by

$$\omega_f=(2\pi i)^n f(z)dz_1\wedge dz_2\cdots\wedge dz_n.$$

Then ω_f is $\Gamma_0(\mathfrak{n}; B)$-invariant, hence defines an element of $H^n_{DR}(X_\mathfrak{n}, \boldsymbol{C})$. For any element $S=(s_\tau)_{\tau\in N(B)}$ of $\mathrm{Sgn}(B)$, we denote by $p(S)$ the number of components $s_\tau=+1$. Then for any $S\in\mathrm{Sgn}(B)$, the pull-back $F_S^*(\omega_f)$ of ω_f with respect to F_S is a $(p(S), n-p(S))$ type n-form on $X_\mathfrak{n}$.

Theorem 1.3. *Let $H^{p,q}$ be the (p, q) part of the Hodge decomposition $H^n_{sp}(X_\mathfrak{n}, \boldsymbol{Q})\otimes_{\boldsymbol{Q}}\boldsymbol{C}=\bigotimes_{p+q=n} H^{p,q}$. Then we have a natural isomorphism*

$$H^{p,q}\cong \bigoplus_{\substack{S\in \mathrm{Sgn}(B)\\ p(S)=p}} \{F_S^*(\omega_f)\,|\, f\in S_2(\Gamma_0(\mathfrak{n}; B))\}.$$

Moreover these decomposition and isomorphisms are compatible with the action of Hecke operators.

Remark. If $B\not\cong M_2(F)$ i.e. when $X_\mathfrak{n}$ is compact, the above theorem is proved by Matsushima-Shimura [8]. If $B\cong M_2(F)$, it is shown by Harder [5]. In this case $X_\mathfrak{n}$ is noncompact. But the Kähler metric $ds^2=\sum_{i=1}^n y_i^{-2}|dz_i|^2$ on H^n defines a complete metric on $X_\mathfrak{n}$. Hence the estimates which appear in the theory of harmonic integrals are valid for square-integrable forms. The space of square-integrable cocycles in $H^n_{DR}(X_\mathfrak{n}, \boldsymbol{C})$ is naturally identified with $W_nH^n(X_\mathfrak{n}, \boldsymbol{C})=\mathrm{Image}\,(H^n_c(X_\mathfrak{n}, \boldsymbol{C})\to H^n(X_\mathfrak{n}, \boldsymbol{C}))$, where $H^n_c(X_\mathfrak{n}, \boldsymbol{C})$ is the cohomology group with compact support. A detailed exposition on this problem is found in Hida [6].

§ 2. Hodge structures attached to primitive forms of weight 2

In this section, we define the Hodge structure $H^n(M_f, \boldsymbol{Q})$ attached to a primitive form f of weight 2 in $S_2(\Gamma_0(\mathfrak{n}; B))$. The definitions and notation in the previous section is in force in this section.

Let us define the new part $H^n_{sp}(X_\mathfrak{n}, \boldsymbol{Q})^{\mathrm{new}}$ of $H^n_{sp}(X_\mathfrak{n}, \boldsymbol{Q})$ as follows. Let $\mathfrak{m}$ be a divisor of $\mathfrak{n}$, and let $\mathfrak{d}$ be a divisor of $\mathfrak{n}/\mathfrak{m}$. Let d be a totally positive generator of the ideal $\mathfrak{d}=(d)$. Then we can choose an element β of O_{B+} with $\nu(\beta)=d$ such that $\beta^{-1}\Gamma_0(\mathfrak{n}; B)\beta\hookrightarrow\Gamma_0(\mathfrak{m}; B)$. The homomorphism

$$j_{\mathfrak{m},\mathfrak{d}}\colon \gamma\in\Gamma_0(\mathfrak{n}; B)\longmapsto\beta^{-1}\gamma\beta\in\Gamma_0(\mathfrak{m}; B)$$

induces a morphism $j_{\mathfrak{m},\mathfrak{d}}\colon X_\mathfrak{n}\to X_\mathfrak{m}$, which does not depend on the choice of β. This morphism $j_{\mathfrak{m},\mathfrak{d}}$ in its turn induces an injective homomorphism of Hodge structures:

$$j^*_{\mathfrak{m},\mathfrak{d}}\colon H^n(X_\mathfrak{m}, \boldsymbol{Q})\longrightarrow H^n(X_\mathfrak{n}, \boldsymbol{Q}).$$

Then the image of $H^n_{sp}(X_\mathfrak{m}, \boldsymbol{Q})$ by $j^*_{\mathfrak{m},\mathfrak{d}}$ belongs to $H^n_{sp}(X_\mathfrak{n}, \boldsymbol{Q})$.

Now we define the new part $H^n_{sp}(X_\mathfrak{n}, \boldsymbol{Q})^{\mathrm{new}}$ of $H^n_{sp}(X_\mathfrak{n}, \boldsymbol{Q})$ as the orthogonal complement of the subspace generated by the images of $j_{\mathfrak{m},\mathfrak{d}}$ for all pairs of divisors $\mathfrak{m}$, $\mathfrak{d}$ of $\mathfrak{n}$ satisfying $\mathfrak{m}\mathfrak{d}\,|\,\mathfrak{n}$, with respect to the polarization. Similarly we define the subspace $S_2(\Gamma_0(\mathfrak{n}; B))^{\mathrm{new}}$ of new forms in $S_2(\Gamma_0(\mathfrak{n}; B))$. Let us call an element f of $S_2(\Gamma_0(\mathfrak{n}; B))$ a primitive form, if f is a new form and a common eigenform of all Hecke operators $T(\mathfrak{q})$ $((\mathfrak{q}, \mathfrak{d}_B\mathfrak{n})=1)$.

Let $\mathrm{End}\,(H^n_{sp}(X_{\mathfrak{n}}, \boldsymbol{Q})^{\mathrm{new}})$ be the endomorphism algebra of the rational Hodge structure $H^n_{sp}(X_{\mathfrak{n}}, \boldsymbol{Q})^{\mathrm{new}}$. Then we denote by $R^{\mathrm{new}}_{\boldsymbol{Q}}$ its subalgebra generated by the images of the Hecke operators $T(\mathfrak{q})$ $((\mathfrak{q}, \mathfrak{d}_B\mathfrak{n})=1)$ over $\boldsymbol{Q}$. Then we have the following.

Proposition 2.1. *$R^{\mathrm{new}}_{\boldsymbol{Q}}$ is a commutative semisimple algebra of finite rank over $\boldsymbol{Q}$.*

It suffices to show that $R^{\mathrm{new}}_{\boldsymbol{Q}}\otimes_{\boldsymbol{Q}}\boldsymbol{C}$ is semisimple, which follows from the strong multiplicity one theorem and the Eichler-Shimizu correspondence

$$S_2(\Gamma_0(\mathfrak{n}; B))^{\mathrm{new}} \xrightarrow{\sim} S_2(\Gamma_0(\mathfrak{n}\mathfrak{d}_B; M_2(F)))^{\mathrm{new}}.$$

By this proposition, $R^{\mathrm{new}}_{\boldsymbol{Q}}$ is a direct sum of simple components, each of which is a finite algebraic number field.

Let f be a primitive form of weight 2 in $S_2(\Gamma_0(\mathfrak{n}; B))^{\mathrm{new}}$. Then we can find a unique primitive idempotent e_f in $R^{\mathrm{new}}_{\boldsymbol{Q}}$ such that

$$\omega_f \in e_f H^n_{sp}(X_{\mathfrak{n}}, \boldsymbol{Q})^{\mathrm{new}}\otimes_{\boldsymbol{Q}}\boldsymbol{C}.$$

Moreover there exists an isomorphism

$$e_f R^{\mathrm{new}}_{\boldsymbol{Q}} = R^{\mathrm{new}}_{\boldsymbol{Q}} e_f = e_f R^{\mathrm{new}}_{\boldsymbol{Q}} e_f \xrightarrow{\sim} K_f.$$

Definition. For any primitive form f of weight 2 in $S_2(\Gamma_0(\mathfrak{n}; B))$, we put

$$H^n(M_f, \boldsymbol{Q}) = e_f H^n_{sp}(X_{\mathfrak{n}}, \boldsymbol{Q}).$$

The field $e_f R^{\mathrm{new}}_{\boldsymbol{Q}} e_f$ acts on $H^n(M_f, \boldsymbol{Q})$ by the restriction of the action of $R^{\mathrm{new}}_{\boldsymbol{Q}}$ on $H^n_{sp}(X_{\mathfrak{n}}, \boldsymbol{Q})^{\mathrm{new}}$. Hence via the isomorphism $e_f R^{\mathrm{new}}_{\boldsymbol{Q}} e_f \xrightarrow{\sim} K_f$, K_f also acts on $H^n(M_f, \boldsymbol{Q})$. In other words we have a homomorphism $\theta_f\colon K_f \to \mathrm{End}\,(H^n(M_f, \boldsymbol{Q}))$. It is easy to check that the polarization $\Phi_{X_{\mathfrak{n}}}$ of $H^n_{sp}(X_{\mathfrak{n}}, \boldsymbol{Q})$ defines a polarization

$$\Phi_f\colon H^n(M_f, \boldsymbol{Q}) \times H^n(M_f, \boldsymbol{Q}) \longrightarrow \boldsymbol{Q}(-n)$$

on $H^n(M_f, \boldsymbol{Q})$ by restriction. Because $\Phi_{X_{\mathfrak{n}}}$ satisfies

$$\Phi_{X_{\mathfrak{n}}}(T(\mathfrak{q})^*(x), y)=\Phi_{X_{\mathfrak{n}}}(x, T(\mathfrak{q})^*(y))$$

for any $\mathfrak{q}$ coprime to $\mathfrak{d}_B\mathfrak{n}$ and any $x, y \in W_n H^n(X_{\mathfrak{n}}, \boldsymbol{Q})$, Φ_f satisfies

$$\Phi_f(\theta_f(a)x, y)=\Phi_f(x, \theta_f(a)y)$$

for any $a \in K_f$ and any $x, y \in H^n(M_f, \boldsymbol{Q})$.

Now let us recall the non-holomorphic involutive automorphisms F_S ($S \in \mathrm{Sgn}(B)$) of $X_{\mathfrak{n}}$ defined in the previous section, which induce the actions $F_{S*}=F_S^*$ on $H^n_{sp}(X_{\mathfrak{n}}, \boldsymbol{Q})$. Since each F_S^* commutes with the action of Hecke operators, F_S^* holds the subspace $H^n(M_f, \boldsymbol{Q})$. Let us define a function $\langle\, ,\rangle$ on $\mathrm{Sgn}(B)\times\mathrm{Sgn}(B)$ with values in $\{+1, -1\}$ by

$$\langle S, T\rangle=\prod_{\tau\in N(B)}(s_\tau, t_\tau)_\infty \quad \text{for } S=(s_\tau)_{\tau\in N(B)} \quad \text{and} \quad T=(t_\tau)_{\tau\in N(B)}$$

in $\mathrm{Sgn}(B)$, where $(\, ,)$ is the local Hilbert symbol for $\boldsymbol{R}$ defined by $(a, b)_\infty=-1$, if $a<0$ and $b<0$, and $(a, b)_\infty=+1$, otherwise.

Definition. For any $S \in \mathrm{Sgn}(B)$ we define a subspace $H^n(M_f, \boldsymbol{Q})_S$ of $H^n(M_f, \boldsymbol{Q})$ by

$$H^n(M_f, \boldsymbol{Q})_S=\{\delta \in H^n(M_f, \boldsymbol{Q})\,|\,F_S^*(\delta)=\langle T, S\rangle\delta \qquad \text{for any } T \in \mathrm{Sgn}(B)\}.$$

Clearly we have a direct sum decomposition

$$H^n(M_f, \boldsymbol{Q})_S=\bigoplus_{S\in\mathrm{Sgn}(B)} H^n(M_f, \boldsymbol{Q})_S.$$

Proposition 2.2. *For each $S \in \mathrm{Sgn}(B)$, $H^n(M_f, \boldsymbol{Q})_S$ is a K_f-module of rank* 1.

The following proposition follows immediately from the above proposition.

Proposition 2.3. *There exists a unique K_f-bilinear form*

$$\psi_f\colon H^n(M_f, \boldsymbol{Q})\times H^n(M_f, \boldsymbol{Q})\longrightarrow K_f$$

such that $\Phi_f=\mathrm{tr}_{K_f/\boldsymbol{Q}}(\psi_f)$.

In terms of ψ_f, we can define the canonical basis of $H^n(M_f, \boldsymbol{Q})$ as follows.

Definition. A system $\{\delta_S\,|\,S \in \mathrm{Sgn}(B)\}$ of elements of $H^n(M_f, \boldsymbol{Q})$ is called a canonical basis of $H^n(M_f, \boldsymbol{Q})$, if for each $S \in \mathrm{Sgn}(B)$ the element δ_S generates the space $H^n(M_f, \boldsymbol{Q})_S$ over K_f, and

$$\psi_f(\delta_S, \delta_T) = \begin{cases} \varepsilon(S) & \text{if } T = -S, \\ 0, & \text{otherwise,} \end{cases}$$

where $\varepsilon(S) = \langle S, S_- \rangle$ with $S_- = (-1, -1, \cdots, -1)$.

Remark. If $\{\eta_S \mid S \in \mathrm{Sgn}(B)\}$ is another canonical basis of $H^n(M_f, \boldsymbol{Q})$, then there is a set of elements $\{a_S \mid S \in \mathrm{Sgn}(B)\}$ of K_f satisfying $a_S a_{-S} = 1$ for any $S \in \mathrm{Sgn}(B)$, such that $\eta_S = \theta_f(a_S)\delta_S$ for any $S \in \mathrm{Sgn}(B)$.

In order to define the fundamental system of periods for f with respect to a canonical basis of $H^n(M, \boldsymbol{Q})$, we investigate the $(n, 0)$ part of the Hodge structure $H^n(M_f, \boldsymbol{Q})$. For the totally real field K_f, we identify any embedding $\sigma\colon K_f \hookrightarrow \boldsymbol{R}$ with its extension $K_f \hookrightarrow \boldsymbol{C}$. Let Σ_f be the set of all embeddings of K_f into $\boldsymbol{R}$, or into $\boldsymbol{C}$. Then

$$H^n(M_f, \boldsymbol{Q}) \otimes_{\boldsymbol{Q}} \mathbf{C} = \bigoplus_{\sigma \in \Sigma_f} H^n(M_f, \boldsymbol{Q}) \otimes_{K_f, \sigma} \boldsymbol{C}.$$

It is easy to see that the $(n, 0)$ component of the Hodge structure $H^n(M_f, \boldsymbol{Q}) \otimes_{K_f, \sigma} \boldsymbol{C}$ is given by ω_{f^σ}, where f^σ is the companion of f with respect to $\sigma \in \Sigma_f$ defined as follows: Let f be a primitive form in $S_2(\Gamma_0(\mathfrak{n}; B))^{\mathrm{new}}$ with eigenvalues $a_{\mathfrak{q}}$ $(a_{\mathfrak{q}} \in K_f)$ for each $T(\mathfrak{q})$ $((\mathfrak{q}, \mathfrak{d}_B\mathfrak{n}) = 1)$: $T(\mathfrak{q})f = a_{\mathfrak{q}}f$. Then the companion f^σ of f with respect to $\sigma\colon K_f \hookrightarrow \boldsymbol{C}$ is an element of $S_2(\Gamma_0(\mathfrak{n}; B))^{\mathrm{new}}$ such that $T(\mathfrak{q})f^\sigma = \sigma(a_{\mathfrak{q}})f^\sigma$ for any $\mathfrak{q}$ coprime to $\mathfrak{d}_B\mathfrak{n}$, which is unique up to constant multiple in $S_2(\Gamma_0(\mathfrak{n}; B))^{\mathrm{new}}$.

Now let us choose a companion f^σ of f for each $\sigma \in \Sigma_f$. Let δ_S^σ be the image of $\delta_S \in H^n(M_f, \boldsymbol{Q})$ with respect to the natural homomorphism

$$H^n(M_f, \boldsymbol{Q}) \longrightarrow H^n(M_f, \boldsymbol{Q}) \otimes_{K_f, \sigma} \boldsymbol{C}.$$

Then $\{\delta_S^\sigma \mid S \in \mathrm{Sgn}(B)\}$ generates $H^n(M_f, \boldsymbol{Q}) \otimes_{K_f, \sigma} \boldsymbol{C}$. Since ω_{f^σ} is an element of $H^n(M_f, \boldsymbol{Q}) \otimes_{K_f, \sigma} \boldsymbol{C}$, it is represented by a linear combination of $\{\delta_S^\sigma \mid S \in \mathrm{Sgn}(B)\}$ with complex coefficients.

Definition. Choose a canonical basis $\{\delta_S \mid S \in \mathrm{Sgn}(B)\}$ of $H^n(M_f, \boldsymbol{Q})$ and a set $\{f^\sigma \mid \sigma \in \Sigma_f\}$ of companions of the given primitive form f. Then the fundamental system of periods of $H^n(M_f, \boldsymbol{Q})$ with respect to these basis is the set of complex numbers $\{W_S(f^\sigma) \mid S \in \mathrm{Sgn}(B),\ \sigma \in \Sigma_f\}$ such that

$$\omega_{f^\sigma} = \sum_{S \in \mathrm{Sgn}(B)} W_S(f^\sigma)\delta_S^\sigma \qquad \text{for each } \sigma \in \Sigma_f.$$

Theorem 2.4. (*Period relation of Riemann-Hodge*). *Under the same notation and definitions as above, we have*

$$W_S(f^\sigma)W_{-S}(f^\sigma) = 2^{-n} \int_{X_{\mathfrak{n}}} \omega_{f^\sigma} \wedge F_\infty^*(\omega_{f^\sigma})$$

for each $\sigma \in \Sigma_f$ *and any* $S \in \mathrm{Sgn}(B)$.

This theorem is only a paraphrase of the classical Riemann-Hodge period relation for the Hodge structure $H^n(M_f, \boldsymbol{Q})$. But it plays an important role in the proof of Theorem A. In fact, if $n=2$, Theorem A follows immediately from this theorem (cf. § 4–7 of [9], especially Theorem 7.2). If $n>2$, this theorem is insufficient to show Theorem A. We need more period relations, as we see in the next section.

§ 3. Eichler-Shimizu correspondence and period relations

In this section, we give an outline of the proofs of Theorems A and B, both of which are proved by a similar method.

Before starting the explanation, it is convenient to introduce here the notion of K_f-equivalence.

Definition. Let K be a totally real algbraic number field of degree d over $\boldsymbol{Q}$, and let Σ be the set of all embeddings of K into $\boldsymbol{R}$ (or into $\boldsymbol{C}$). Then two elements $\xi=(\xi_\sigma)_{\sigma\in\Sigma}$ and $\eta=(\eta_\sigma)_{\sigma\in\Sigma}$ of $K\otimes_{\boldsymbol{Q}}\boldsymbol{C}\cong\boldsymbol{C}^\Sigma\cong\boldsymbol{C}^d$ is said to be *K-equivalent*, if there exists a non-zero element a of K such that $\xi_\sigma=\sigma(a)\eta_\sigma$ for all $\sigma\in\Sigma$, and we denote this fact by $\xi \underset{K}{\approx} \eta$.

Let f be a primitive form of weight 2 in $S_2(\Gamma_0(\mathfrak{n}; B))^{\mathrm{new}}$, and let $\{W_S(f^\sigma) \mid S\in\mathrm{Sgn}(B),\ \sigma\in\Sigma_f\}$ be the fundamental system of periods of $H^n(M_f, \boldsymbol{Q})$ with respect to a canonical basis and a basis $\{f^\sigma \mid \sigma\in\Sigma_f\}$ of companions.

Let $N(B)=N_1\cup N_2$ be a partition of $N(B)$ into two disjoint subsets N_1 and N_2 of $N(B)$, and put $\mathrm{Sgn}(N_i)=\{+1, -1\}^{N_i}$: the set of all mappings of the set N_i to the set $\{+1, -1\}$ for each i $(1\leqq i\leqq 2)$, respectively. Then any element S of $\mathrm{Sgn}(B)$ is written as *a list* $S=(S_1, S_2)$ with elements S_i in $\mathrm{Sgn}(N_i)$, if we rearrange the order of the elements in $N(B)$ adequately.

Now, in the first place, we note that Theorem A follows immediately from the following period relation.

Theorem 3.1. *Let* $\{W_S(f^\sigma) \mid S\in\mathrm{Sgn}(B),\ \sigma\in\Sigma_f\}$ *be a fundamental system of periods of a primitive form f given as above. Assume that f is strongly admissible. Then for any partition* $N(B)=N_1\cup N_2$ *of the set* $N(B)$ *considered as above, we have a* K_f*-equivalence*

$$(W_{(T_1,S_2)}(f^\sigma)/W_{(S_1,S_2)}(f^\sigma))_{\sigma\in\Sigma_f} \underset{K_f}{\approx} (W_{(T_1,T_2)}(f^\sigma)/W_{(S_1,T_2)}(f^\sigma))_{\sigma\in\Sigma_f}$$

for any $S_1, T_1\in\mathrm{Sgn}(N_1)$ *and any* $S_2, T_2\in\mathrm{Sgn}(N_2)$.

Let us explain briefly how to define abelian varieties A_f^v of dimension $[K_f : \boldsymbol{Q}]$ over $\boldsymbol{C}$ with homomorphism $\theta_f^v : K_f \to \mathrm{End}\,(A_f^v) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. In general, if A is a Hilbert-Blumenthal abelian variety of dimension $[K_f : \boldsymbol{Q}]$ over $\boldsymbol{C}$ with homomorphism $\theta : K_f \to \mathrm{End}\,(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ of a totally real number field K_f, we have a decomposition

$$H^1(A, \boldsymbol{Q}) \otimes_{\boldsymbol{Q}} \boldsymbol{C} = \bigoplus_{\sigma \in \Sigma f} H^1(A, \boldsymbol{Q}) \otimes_{K, \sigma} \boldsymbol{C},$$

and a polarization

$$\Phi : H^1(A, \boldsymbol{Q}) \times H^1(A, \boldsymbol{Q}) \longrightarrow \boldsymbol{Q}(-1),$$

which is written as $\Phi = \mathrm{tr}_{K_f/\boldsymbol{Q}}(\psi)$ by a unique non-degenerate skew-symmetric K_f-bilinear form ψ on $H^1(A, \boldsymbol{Q})$ with value in K_f. Since $H^1(A, \boldsymbol{Q})$ is a K_f-module of rank 2, we can choose a basis $\{\delta_+, \delta_-\}$ of $H^1(A, \boldsymbol{Q})$ over K_f satisfying $\psi(\delta_+, \delta_+) = \psi(\delta_-, \delta_-) = 0$ and $\psi(\delta_+, \delta_-) \neq 0$.

Let $\Gamma(A, \Omega^1_{A/C})$ be the space of holomorphic 1-forms on A. Then we can diagonalize the action of K_f on $\Gamma(A, \Omega^1_{A/C})$ by choosing a basis $\{\omega_\sigma \mid \sigma \in \Sigma_f\}$ such that for each $\sigma \in \Sigma_f$ $\theta^*(a)(\omega_\sigma) = \sigma(a)\omega_\sigma$ for any $a \in K_f$. Let δ_+^σ, δ_-^σ be the images of δ_+, δ_- with respect to the canonical homomorphism

$$H^1(A, \boldsymbol{Q}) \longrightarrow H^1(A, \boldsymbol{Q}) \otimes_{K_f, \sigma} \boldsymbol{C},$$

then we can find complex numbers $w_+(\omega_\sigma)$ and $w_-(\omega_\sigma)$ such that

$$\omega_\sigma = w_+(\omega_\sigma)\delta_+^\sigma + w_-(\omega_\sigma)\delta_-^\sigma.$$

Then $(w_-(\omega_\sigma)/w_+(\omega_\sigma))_{\sigma \in \Sigma_f}$ defines an element of $K_f \otimes_{\boldsymbol{Q}} \boldsymbol{C}$ so that no components $w_-(\omega_\sigma)/w_+(\omega_\sigma)$ are real numbers. We call this the *period modulus* of A with respect to $\{\delta_+, \delta_-\}$.

The group $GL_2(K_f)$ acts on $K_f \otimes_{\boldsymbol{Q}} \boldsymbol{C}$ by

$$g(\xi) = \eta;\ \xi = (\xi_\sigma)_{\sigma \in \Sigma f} \quad \text{and} \quad \eta = (\eta_\sigma)_{\sigma \in \Sigma f} \in K_f \otimes_{\boldsymbol{Q}} \boldsymbol{C},$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K_f), \quad \text{and} \quad \eta_\sigma = (\sigma(a)\xi_\sigma + \sigma(b))/(\sigma(c)\xi_\sigma + \sigma(d))$$

for each $\sigma \in \Sigma_f$.

Then it is easy to see that the K_f-isogeny class of A defines a unique point in $K_f \otimes_{\boldsymbol{Q}} \boldsymbol{C}$ modulo $GL_2(K_f)$, and distinct K_f-isogeny classes define distinct $GL_2(K_f)$-equivalent classes of points in $K_f \otimes_{\boldsymbol{Q}} \boldsymbol{C}$.

Thus, in order to specify the K_f-isogeny class of A_f^v, it suffices to give its period modulus. It is given by the K_f-equivalence class of ζ_S given by

$$\zeta_S = (W_{(S', \underset{\nwarrow}{-1}, S'')}(f^\sigma)/W_{(S', \underset{\uparrow}{+1}, S'')}(f^\sigma))_{\sigma \in \Sigma_f} \qquad \text{in } K_f \otimes_Q \boldsymbol{C},$$

the v-th component

where (S', S'') is an element of $\{+1, -1\}^{N(B)-\{v\}}$. By Theorem 3.1, the K_f-equivalence class ζ_S does not depend on the choice of (S', S'').

Let us discuss Theorem B in the introduction. Suppose that two primitive forms f_1 and f_2 of weight 2 with respect to $\Gamma_0(\mathfrak{n}_1; B_1)$ and $\Gamma_0(\mathfrak{n}_2; B_2)$ are given, respectively. If we put $N^+ = N(B_1) \cap N(B_2)$ and $N_i^- = N(B_i) - N^+$ for $i=1$ and $i=2$, any element S of $\mathrm{Sgn}(B_i)$ is written as a list $S = (S^+, S_i^-)$ of two vectors $S^+ \in \mathrm{Sgn}^+ = \{+1, -1\}^{N^+}$ and $S_i^- \in \mathrm{Sgn}_i^- = \{+1, -1\}^{N_i^-}$ for each i $(i=1, 2)$, respectively. Here we rearrange the order of the elements in $N(B_1)$ or $N(B_2)$ adequately.

The next theorem together with Theorem 3.1 implies Theorem B.

Theorem 3.2. *Let f_1 and f_2 be two primitive forms given as above, and assume that both f_1 and f_2 are strongly admissible. Let $\{W_S(f_1^\sigma) \mid S \in \mathrm{Sgn}(B_1), \sigma \in \Sigma_{f_1}\}$, or $\{W_T(f_2^\sigma) \mid T \in \mathrm{Sgn}(B_2), \sigma \in \Sigma_{f_2}\}$ be a fundamental system of periods of f_1 or f_2, respectively. Moreover, assume that f_1 and f_2 correspond in the sense of Eichler-Shimizu. Then $K_{f_1} = K_{f_2}$, $\Sigma_{f_1} = \Sigma_{f_2}$, and we have the period relation*:

For any $S^+, T^+ \in \mathrm{Sgn}^+$ and any $S_i^- \in \mathrm{Sgn}_i^-$ $(i=1, 2)$, the elements

$$(W_{(T^+, S_1^-)}(f_1^\sigma)/W_{(S^+, S_1^-)}(f_1^\sigma))_{\sigma \in \Sigma_{f_1}} \qquad \textit{in } K_{f_1} \otimes_Q \boldsymbol{C}$$

and

$$(W_{(T^+, S_2^-)}(f_2^\sigma)/W_{(S^+, S_2^-)}(f_2^\sigma))_{\sigma \in \Sigma_{f_2}} \qquad \textit{in } K_{f_2} \otimes_Q \boldsymbol{C}$$

are $K_{f_1} = K_{f_2}$-equivalent.

In order to explain how to prove Theorems 3.1 and 3.2, we need more notation and definitions.

Definition. Let f be a primitive form of weight 2 in $S_2(\Gamma_0(\mathfrak{n}; B))^{\mathrm{new}}$ $((\mathfrak{n}, \mathfrak{d}_B) = 1)$. Then there exists a primitive Hilbert modular form f_0 of weight 2 in $S_2(\Gamma_0(\mathfrak{n}\mathfrak{d}_B; M_2(F)))^{\mathrm{new}}$, which corresponds to f in the sense of Eichler-Shimizu. If f_0 is normalized so that the first Fourier coefficient of f_0 at ∞ is 1, f_0 is uniquely determined by f. We call f_0 *the index form of f.*

If we choose the companion f_0^σ of f_0 such that f_0^σ is normalized for each $\sigma \in \Sigma_{f_0}$, then the K_{f_0}-equivalence class of $(W_S(f_0^\sigma))_{\sigma \in \Sigma_{f_0}}$ in $K_{f_0} \otimes_Q \boldsymbol{C}$ does not depend on the choice of canonical basis of $H^g(M_{f_0}, \boldsymbol{Q})$.

When $B \not\cong M_2(F)$, the Shimura variety $X_\mathfrak{n}$ has no cusps and the modular form f has no Fourier expansions. Therefore it is a problem how to specify the normalized one in the set $C^{\times}f^{\sigma}$ of the companions of f with respect to $\sigma \in \Sigma_f$, such that its periods are comparable with those of f_0.

The following theorem is crucial in the proofs of Theorems 3.1 and 3.2.

Theorem 3.3. *Let f be a primitive form of weight 2 with respect to $\Gamma_0(\mathfrak{n}; B)$, and let f_0 be the index form of f. Assume that f is strongly admissible. Put $R(B)=P_{\infty}(F)-N(B)$.*

Then for each $\sigma \in \Sigma_f$, we can normalize the companion f^{σ} of f with respect to σ, so that for any $S \in \mathrm{Sgn}(B)$, the element

$$(W_S(f^{\sigma}))_{\sigma \in \Sigma_f} \qquad \text{in } K_f \otimes_Q C$$

is K_f-equivalent to

$$(W_{(S,R_-)}(f_0^{\sigma}))_{\sigma \in \Sigma_f} \qquad \text{in } K_{f_0} \otimes_Q C = K_f \otimes_Q C.$$

Here $R_-=(-1, -1, \cdots, -1)$ is an element of $\{+1, -1\}^{R(B)}$.

Remark. If B is totally indefinite, $R(B)=\phi$. Hence $(S, R_-)=S$ for any $S \in \mathrm{Sgn}(B)$ in this case.

Theorem 3.3 follows from the next Theorem 3.4. Let us formulate Theorem 3.4. In the first place, we construct some n-cycles on the Shimura variety $X_{\mathfrak{n}}$, which we call the *distinguished cycles.* Let W be the space of pure quaternions in B. Then we define the action of $B^{\times}$ on W by

$$w \in W \longmapsto \beta w \beta^{-1} \in W, \qquad \text{for } \beta \in B^{\times}.$$

By extension of scalars, $(B\otimes_Q R)^{\times}$ acts on $W\otimes_Q R$. Let us choose a vector $w \in W$ with $d=-\nu(w)\neq 0$ satisfying $\tau(d)>0$ if $\tau \in N(B)$ and $\tau(d)<0$ if $\tau \in R(B)$, where ν is the reduced norm map of B over F. Fix an identification $(B\otimes_Q R)^{\times}\cong GL_2(R)^n\times H^r$, and let $O_{w,R}$ be the stabilizer of $w \in W\subset W\otimes_Q R$ in $GL_2^+(R)^n\times H^r$. Then for any point $z \in X_{\mathfrak{n}}=\Gamma_0(\mathfrak{n}; B)\backslash H^n$, the $O_{w,R}$-orbit of z in $X_{\mathfrak{n}}$ is a *totally real* submanifold of $X_{\mathfrak{n}}$, and defines an element $\bar{\gamma}_w$ of $H_n(\bar{X}_{\mathfrak{n}}, Z)$, where $\bar{X}_{\mathfrak{n}}$ is the Satake compactification of $\bar{X}_{\mathfrak{n}}$ obtained from $X_{\mathfrak{n}}$ by attaching a finite number of points at infinity corresponding to the cusps of $\Gamma_0(\mathfrak{n}; B)$.

Lemma. *If $g\geqq 2$, there exists a cycle γ_w in* Image $(H_n(X_{\mathfrak{n}}, Z)\rightarrow H_n(\bar{X}_{\mathfrak{n}}, Z))$ *such that*

$$\int_{\gamma_w} \omega_h = \int_{\bar{\gamma}_w} \omega_h$$

for any element h of $S_2(\Gamma_0(\mathfrak{n}; B))$.

Remark. If $B \not\cong M_2(F)$, this lemma is trivial. When $B \cong M_2(F)$ and $g>1$, the lemma assure that the integral $\int_{\bar{\gamma}_w} \omega_h$ along the noncompact chain $\bar{\gamma}_w$ is truely a period of some proper smooth model of $X_{\mathfrak{n}}$.

Definition. We call the linear combinations of the cycles of the form γ_w with coefficient in $\boldsymbol{Q}$ in Image $(H_n(X_{\mathfrak{n}}, \boldsymbol{Q}) \rightarrow H_n(\bar{X}_{\mathfrak{n}}, \boldsymbol{Q}))$ the *distinguished cycles* on $X_{\mathfrak{n}}$.

Assume that $d=-\nu(w)$ is not a square of any element of $F^{\times}$. Then let K be the quadratic extension $K=F(\sqrt{d})$ of F, and let χ_K be quadratic Dirichlet character of O_F corresponding to the extension K/F.

Theorem 3.4. *Assume that f is strongly admissible. Then we can normalize the companions f^{σ} of f so that there exists a distinguished cycle γ independent of σ, satisfying*

$$\int_{\gamma} \omega_{f^\sigma} = \sigma(a)(2\pi)^g G(\chi_K) L(1, f^\sigma, \chi_K)$$

for some $a \in K_f$ independent of σ, where $G(\chi_K)$ is the Gaussian sum associated to χ_K.

Moreover under the same normalization, we have

$$\int_{X_{\mathfrak{n}}} \omega_{f^\sigma} \wedge F_\infty^*(\omega_{f^\sigma}) = \sigma(b) W_{(S_+, R_-)}(f_0^\sigma) W_{(S_-, R_-)}(f_0^\sigma)$$

for all $\sigma \in \Sigma_f$ with some $b \in K_f$.

Here $S_+=(+1, +1, \cdots, +1)$ and $S_-=(-1, -1, \cdots, -1)$ belong to $\operatorname{Sgn}(B)=\{+1, -1\}^{N(B)}$ *and $R_-=(-1, -1, \cdots, -1)$ to $\{+1, -1\}^{R(B)}$.*

Remark. When $d=-\nu(w)$ is a square of an element of $F^{\times}$, then $B \cong M_2(F)$ necessarily.

The special case of the first half of the above theorem is already found in the examples of Shintani [19]. The proof of the first half is obtained by applying the results and the method of Waldspuger [20], [21]. The second half requires some trick.

Let us explain how to deduce Theorem 3.1 from Theorems 2.4 and 3.3. In the first place let us note the following fact. Let f_0 be a normalized primitive Hilbert modular cusp form of weight 2, and let $f_{0,\chi}$ be its twist with respect a quadratic character χ. Then the Hodge structures $H^g(M_{f_0}, \boldsymbol{Q})$ and $H^g(M_{f_{0,\chi}}, \boldsymbol{Q})$ are isomorphic. Moreover, if $f_{0,\chi}$ is also normalized, the element

$$(W_{SS_\chi}(f_0))_{\sigma\in\Sigma f_0} \qquad \text{in } K_{f_0}\otimes_{\boldsymbol{Q}}\boldsymbol{C}$$

and

$$(W_S(f_{0,\chi}))_{\sigma\in\Sigma f_{0,\chi}} \qquad \text{in } K_{f_{0,\chi}}\otimes_{\boldsymbol{Q}}\boldsymbol{C}=K_{f_0}\otimes_{\boldsymbol{Q}}\boldsymbol{C}$$

are $K_{f_0}=K_{f_{0,\chi}}$-equivalent. Here $S_\chi=\mathrm{Sgn}(\chi)$.

Now let f be the given primitive form. Then if $n=1$, we have nothing to prove. If $n=2$, then Theorem 2.4 implies Theorem 3.1. Therefore assume that $n\geqq 3$. Let us proceed by induction on n.

Let I be a subset of $N(B)$ with even number of elements. Then for the quaternion algebra B, we can find another quaternion B_I with the same conductor as B at the finite places of F, which splits only at the infinite places of F belonging to $N(B)-I$. Thus $N(B_I)=N(B)-I$, and the cardinality of $N(B_I)$ is $n-i$ with $i=\sharp I$. Moreover in $S_2(\Gamma_0(\mathfrak{n}; B_I))^{\mathrm{new}}$ we can find a primitive form f_I which is the Eichler-Shimizu correspondence of f. Let f_0 be the index form of f, then it is also the index form of f_I. By the assumption of induction, Theorem 3.1 is already valid for f_I. And it is also valid for any twist of f_I, and the Hodge structures attached to f_I and its twist are isomorphic. All these facts together imply Theorem 3.1 for f.

We can prove Theorem 3.2 similarly, in view of the fact that the index forms of f_1 and f_2 coincide.

§ 4. Remarks

Some portion of Theorems 3.3 and 3.4 are already obtained by Shimura [17], [18]. Assume that B is totally indefinite. Then the Shimura variety $X_{\mathfrak{n}}$ parametrizes the set of polarized abelian varieties A of dimension $2[F: \boldsymbol{Q}]$ with homomorphisms $O_B\to\mathrm{End}(A)$, with level $\mathfrak{n}$ structures. Considering a "principal congruence" subgroup Γ of $\Gamma_0(\mathfrak{n}; B)$, we have a finite covering $Y\to X_{\mathfrak{n}}$, and a universal family f: $A\to Y$ of such abelian varieties with level structures. Then $R^1f_*O_A$ is $O_F\otimes_{\boldsymbol{Q}} O_Y$-module of rank 2. Therefore, $\bigwedge^2_{O_F} R^1f_*O_A$ is a $O_F\otimes_{\boldsymbol{Q}} O_Y$-module of rank 1. Hence it is a direct sum of g invertible sheaves L_i $(1\leqq i\leqq g)$. Put $L=\otimes_{i=1}^g L_i$, then L has a descent with respect to $Y\to X_{\mathfrak{n}}$, which we denote also by L. Then the sections in $\Gamma(X_{\mathfrak{n}}, L^{\otimes -k})$ are identified with holomorphic modular forms of weight $2k$. Since $X_{\mathfrak{n}}$ and f are defined over $\bar{\boldsymbol{Q}}$, L is also defined over

$\bar{\boldsymbol{Q}}$. The sections in $\Gamma(X_n, L^{\otimes -k})$ defined over $\bar{\boldsymbol{Q}}$ correspond to arithmetic modular forms in the sense of Shimura. Then he prove that for a primitive form f of weight $2k$ defined over $\bar{\boldsymbol{Q}}$, the ratio $\langle f, f\rangle/\langle f_0, f_0\rangle$ of the Petersson metrics is an algebraic number. Here f_0 is the index form of f. Especially when $g=n=1$, in [18], he proved that $W_+(f)/W_+(f_0)$ and $W_-(f)/W_-(f_0)$ are algebraic numbers. Here we write W_+ and W_- in place of $W_{(+1)}$ and $W_{(-1)}$ in our notation.

Ribet [10] proves that the jacobian variety of Shimura curves attached to unit groups of quaternion algebras B over $\boldsymbol{Q}$, is isogenous to a factor of the jacobian variety of elliptic modular curves. This result implies our result if $F=\boldsymbol{Q}$. Conversely, Theorem B implies the existence of an isogeny *over* $\boldsymbol{C}$. If abelian varieties in question are not of CM-type, we obtain the isogeny *over* $\boldsymbol{Q}$ from the isogeny *over* $\boldsymbol{C}$, by a standard argument applying the restriction of scalars of Weil.

Let us indicate an application of Theorem B. In the previous paper, we discussed algebraic cycles of Hilbert modular surfaces ([9]). In this case $g=2$, and the Hodge structure $H^2(M_f, \boldsymbol{Q})$ attached to a primitive form f of weight 2 is of weight 2. Let $H^2(M_f, \boldsymbol{Q})_{\mathrm{alg}}$ be the subspace of algebraic cycles in $H^2(M_f, \boldsymbol{Q})$. As we have seen in Section 7 of [9], the determination of $H^2(M_f, \boldsymbol{Q})$ can be reduced to the determination of $\mathrm{Hom}_{O_f}(A_f^1, A_f^2)$ of abelian varieties A_f^i $(i=1, 2)$ such that $H^2(M_f, \boldsymbol{Q})= H^1(A_f^1, \boldsymbol{Q})\otimes_{K_f} H^1(A_f^2, \boldsymbol{Q})$. By Theorem B, these abelian varieties A_f^i are identified with factors of the Jacobian varieties of certain Shimura curves in some cases. On the other hand, if we denote by f' the conjugate form of f given by $f'(z_1, z_2)=f(z_2, z_1)$ for $(z_1, z_2)\in H^2$. Then A_f^2 is K_f-isogenous to $A_{f'}^1$. By using these facts, we can show that $H^2(M_f, \boldsymbol{Q})_{\mathrm{alg}}=0$, if f' is not equal to any twist of f.

References

[1] Deligne, P., Problems of Present Day Mathematics; (v) Non-abelian class field theory, Amer. Math. Soc. Proc. Symp. in Pure Math., **28** (Mathematical Developments arising from Hilbert Problems), 1976.

[2] ——, Valuers de fonctions L et périodes d'integrales, Amer. Math. Soc. Proc. Symp. in Pure Math., **33**, Part 2 (Automorphic forms, representations, and L-functions), (1979), 313–346.

[3] ——, Theorie de Hodge II, III, Publ. Math. Inst. Hautes Etudes Sci., **40** (1971), 5–58; **44** (1974), 5–77.

[4] Eichler, M., Quadratische Formen und Modulfunktionen, Acta Arith., **4** (1958), 217–239.

[5] Harder, G., On the cohomology of $SL(2, 0)$, Lie groups and their representations, 139–150, Adam Hilger, 1975.

[6] Hida, H., On the abelian varieties with complex multiplication as factors of the abelian variety attached to Hilbert modular forms, Japan J. Math., **5** (1979), 157–208.

[7] Jacquet, H., Langlands, R. P., Automorphic forms on $GL(2)$, Lecture Notes in Math., **114**, Springer, 1970.

[8] Matsushima, Y., Shimura, G., On the cohomology groups attached to certain vector valued differential forms on the product of the upper half planes, Ann. of Math., **78** (1963), 417–449.
[9] Oda, T., Periods of Hilbert modular surfaces, Progress in Math., **19**, Birkhäuser, 1982.
[10] Ribet, K. A., Sur les variétés abéliennes à multiplications réelles, C. R. Acad. Aco. Paris, **291**, Sér. A, (1980), 121–123.
[11] Shih, K.-Y., Anti-holomorphic automorphisms of arithmetic automorphic function fields, Ann. of Math., **103** (1976), 81–102.
[12] Shimizu, H., On the discontinuous groups operating on the product of the upper half planes, Ann. of Math., **77** (1963), 33–71.
[13] ——, On traces of Hecke operators, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **10** (1963), 1–19.
[14] ——, On zeta functions of quaternion algebras, Ann. of Math., **81** (1965), 166–193.
[15] Shimura, G., On the canonical models of arithmetic quotients of bounded symmetric domains I, II, Ann. of Math., **91** (1970), 144–222; **92** (1970), 528–549.
[16] ——, On the real points of an arithmetic quotient of a bounded symmetric domain, Math. Ann., **215** (1975), 135–164.
[17] ——, On certain zeta functions attached to two Hilbert modular forms I, II, Ann. of Math., **114** (1981), 127–164, 569–607.
[18] ——, The periods of certain automorphic forms of arithmetic type, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **28** (1982), 605–632.
[19] Shintani, T., On the construction of holomorphic cusp forms of half integral weight, Nagoya Math. J., **58** (1975), 83–126.
[20] Waldspurger, J.-L., Correspondence de Shimura, J. Math. Pures Appl., **60** (1980), 1–133.
[21] ——, Sur les coefficients de Fourier des formes modulaires de poids demientier, J. Math. Pures Appl., **61** (1981), 375–484.

Department of Mathematics
Faculty of Science
Niigata University
Igarashi, Niigata 950-21
Japan