



CHAPTER 14

**Computation of zeros of nonlinear monotone mappings
in certain Banach spaces, by T. M. Sow, M. Ndiaye, M.
Sene and N. Djitte**

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Abstract. Let E be a uniformly convex real Banach space with uniformly Gâteaux differentiable norm and E^* its dual space and let $A : E \rightarrow E^*$ be a bounded and uniformly monotone mapping such that $A^{-1}(0) \neq \emptyset$. In this paper, we introduce an new explicit iterative algorithm that converges strongly to the unique zeros of A . The results proved here are applied to the convex optimization problem.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\|\cdot\|_H$. An operator $A : H \rightarrow H$ is called *monotone* if

$$\langle Ax - Ay, x - y \rangle_H \geq 0 \quad \forall x, y \in H,$$

Interest in monotone operators stems mainly from their usefulness in numerous applications. Consider, for example, the following: Let $f : H \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper lower semi continuous and convex function. The *subdifferential*, $\partial f : H \rightarrow 2^H$ of f at $x \in H$ is defined by

$$\partial f(x) = \{x^* \in H : f(y) - f(x) \geq \langle y - x, x^* \rangle \quad \forall y \in H\}.$$

It is easy to check that $\partial f : H \rightarrow 2^H$ is a *monotone operator* on H , and that $0 \in \partial f(x)$ if and only if x is a minimizer of f . Setting $\partial f \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is solving for a minimizer of f .

Several existence theorems have been established for the equation $Au = 0$ when A is of the monotone-type (see e.g., [Diemling \(1985\)](#), [Pascali and Sburian \(1978\)](#)).

The extension of the monotonicity definition to operators from a Banach space into its dual has been the starting point for the development of non-linear functional analysis. The monotone maps constitute the most manageable class because of the very simple structure of the monotonicity condition. The monotone mappings appear in a rather wide variety of contexts since they can be found in many functional equations. Many of them appear also in calculus of variations as subdifferential of convex functions. ([Pascali and Sburian \(1978\)](#), p. 101).

Let E be a real normed space, E^* its topological dual space. The map $J : E \rightarrow 2^{E^*}$ defined by

$$Jx := \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \cdot \|x^*\|, \|x\| = \|x^*\|\}$$

is called the *normalized duality map* on E . where, \langle, \rangle denotes the generalized duality pairing between E and E^* .

REMARK 12. Note also that a duality mapping exists in each Banach space. We recall from [Alber \(1994\)](#) some of the examples of this mapping in $l_p, L_p, W^{m,p}$ -spaces, $1 < p < \infty$.

(i) $l_p : Jx = \|x\|_{l_p}^{2-p} y \in l_q, x = (x_1, x_2, \dots, x_n, \dots)$
and $y = (x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \dots, x_n|x_n|^{p-2}, \dots),$

(ii) $L_p : Ju = \|u\|_{L_p}^{2-p} |u|^{p-2} u \in L_q,$

(iii) $W^{m,p} : Ju = \|u\|_{W^{m,p}}^{2-p} \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (|D^\alpha u|^{p-2} D^\alpha u) \in W^{-m,q},$

where $1 < q < \infty$ is such that $1/p + 1/q = 1$.

It is well known that E is smooth if and only if J is single valued. Moreover, if E is a reflexive smooth and strictly convex Banach space, then J^{-1} is single valued, one-to-one, surjective and it is the duality mapping from E^* into E . Finally, if E has uniform Gâteaux differentiable norm, then J is norm-to-weak* uniformly continuous on bounded sets.

A map $A : E \rightarrow E^*$ is called *monotone* if for each $x, y \in D(A)$, the following inequality holds:

$$\langle Ax - Ay, x - y \rangle \geq 0.$$

A is called *uniformly monotone* if there exists a continuous increasing function $\gamma(t), \gamma(0) = 0$, such that for each $x, y \in D(A)$, the following inequality holds:

$$\langle Ax - Ay, x - y \rangle \geq \gamma(\|x - y\|).$$

If here $\gamma(t) = ct^p, p \geq 2$, where $c > 0$ is a positive constant, we have

$$\langle Ax - Ay, x - y \rangle \geq c\|x - y\|^p.$$

A map $A : E \rightarrow E$ is called *accretive* if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0.$$

A is called *strongly accretive* if there exists $k \in (0, 1)$ such that for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq k\|x - y\|^2.$$

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, *monotonicity* and *accretivity* coincide. For accretive-type operator A , solutions of the equation $Au = 0$, in many cases, represent *equilibrium state* of some dynamical system (see e.g., **Chidume (2009)**, p.116).

For approximating a solution of $Au = 0$, assuming existence, where $A : E \rightarrow E$ is of accretive-type, Browder (1967) defined an operator $T : E \rightarrow E$ by $T := I - A$, where I is the identity map on E . He called such an operator *pseudo-contractive*. It is trivial to observe that zeros of A correspond to fixed points of T . For Lipschitz strongly pseudo-contractive maps, Chidume (1987) proved the following theorem.

Theorem C1. (Chidume (1987)) Let $E = L_p$, $2 \leq p < \infty$, and $K \subset E$ be nonempty closed convex and bounded. Let $T : K \rightarrow K$ be a strongly pseudo-contractive and Lipschitz map. For arbitrary $x_0 \in K$, let a sequence $\{x_n\}$ be defined iteratively by $x_{n+1} = (1 - \lambda_n)x_n + \lambda_nTx_n$, $n \geq 0$, where $\{\lambda_n\} \subset (0, 1)$ satisfies the following conditions: (i) $\sum_{n=1}^{\infty} \lambda_n = \infty$, (ii) $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Then, $\{x_n\}$ converges strongly to the unique fixed point of T .

By setting $T := I - A$ in Theorem C1, the following theorem for approximating a solution of $Au = 0$ where A is a strongly accretive and bounded operator can be proved.

Theorem C2. Let $E = L_p$, $2 \leq p < \infty$. Let $A : E \rightarrow E$ be a strongly accretive and bounded map. Assume $A^{-1}(0) \neq \emptyset$. For arbitrary $x_0 \in K$, let a sequence $\{x_n\}$ be defined iteratively by $x_{n+1} = x_n - \lambda_nAx_n$, $n \geq 0$, where $\{\lambda_n\} \subset (0, 1)$ satisfies the following conditions: (i) $\sum_{n=1}^{\infty} \lambda_n = \infty$, (ii) $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Then, $\{x_n\}$ converges strongly to the unique solution of $Au = 0$.

The main tool used in the proof of Theorem C1 is an inequality of Bynum (1976). This theorem signaled the return to extensive research efforts on inequalities in Banach spaces and their applications to iterative methods for solutions of nonlinear equations. Consequently, Theorem C1 has been generalized and extended in various directions, leading to flourishing areas of research, for the past thirty years or so, for numerous authors (see e.g., Censor and Reich (1996), Chidume (1987), Chidume (1986), Chidume (2002), Chidume and Bashir (2007), Chidume *et al.* (2005), Chidume *et al.* (2006), Chidume and Osilike (1999), Deng (1993), Zhou and Jia (1996), Liu (1995), Oihou (1990), Reich (1977), Reich (1978), Reich (1979), Reich and Sabach (2009), Reich and Sabach (2010), Weng (1991), Xiao (1998), Xu (1992), Xu (1991), Xu (1991), Berinde *et al.* (2014), Moudafi (2003), Moudafi (2010), Moudafi and Thera (1997), Xu and Roach (1991), Zhu (1994) and a host of other authors). Recent monographs emanating from these researches include those by Berinde (2007), Chidume (2009),

Goebel and Reich (1984), and William and Shahzad (2014).

Unfortunately, the success achieved in using geometric properties developed from the mid 1980s to early 1990s in approximating zeros of *accretive-type mappings* has not carried over to approximating zeros of *monotone-type operators* in general Banach spaces. Part of the problem is that since A maps E to E^* , for $x_n \in E$, Ax_n is in E^* . Consequently, a recursion formula containing x_n and Ax_n may not be well defined.

Attempts have been made to overcome this difficulty by introducing the inverse of the normalized duality mapping in the recursion formulas for approximating zeros of monotone-type mappings.

Recently, **Diop et al. (2016)** introduced an iterative schema and proved the following strong convergence theorem for approximation of the solution of equation $Au = 0$ in 2-uniformly convex real Banach spaces. In particular they proved the following theorem.

THEOREM 36. *Let E be a 2-uniformly convex real Banach space with uniformly Gâteaux differentiable norm and E^* its dual space. Let $A : E \rightarrow E^*$ be a bounded and strongly monotone mapping such that $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in E$, let (x_n) be a sequence defined iteratively by*

$$(1.1) \quad x_{n+1} = J^{-1}(Jx_n - \alpha_n Ax_n), \quad n \geq 1$$

where J is the duality mapping from E to E^* and $(\alpha_n) \subset]0, 1[$ a real sequence satisfying the following conditions: (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$. Then, there exists γ_0 such that, if $\alpha_n < \gamma_0$, for all $n \geq 1$, then (x_n) converges strongly to the unique solution of the equation $Au = 0$.

It is purpose in this paper to introduce an explicit iterative algorithm that converges strongly to the solution of equation $Au = 0$ in certain Banach spaces including all Hilbert spaces and all l_p, L_p or $W^{m,p}$ -spaces, $1 < p < \infty$. Furthermore, our technique of proof is of independent interest.

2. Preliminaries

A normed linear space E is said to be strictly convex if the following holds:

$$\|x\| = \|y\| = 1, \quad x \neq y \Rightarrow \left\| \frac{x+y}{2} \right\| < 1.$$

The modulus of convexity of E is the function $\delta_E : (0, 2] \rightarrow [0, 1]$ defined by:

$$\delta_E(\epsilon) := \inf \left\{ 1 - \frac{1}{2} \|x + y\| : \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon \right\}.$$

E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. For $p > 1$, E is said to be p -uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$.

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to be smooth if the limit

$$\lim_{t \rightarrow 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S$ and E is Frechet differentiable if it is smooth and the limit is attained uniformly for $y \in S$.

LEMMA 19 (Zalinescu (1983)). Let $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a increasing function with $\lim_{t \rightarrow +\infty} \psi = \infty$. Then J_ψ^{-1} is single valued and uniformly continuous on bounded sets if and only if E is a uniformly convex Banach space.

LEMMA 20. Let E be a uniformly convex and smooth real Banach space. Then, the duality mapping $J_p^{-1} : E^* \rightarrow E$ is Lipschitz on bounded subsets of E^* ; that is, for all $R > 0$, there exists a positive constant m_2 such that

$$\|J_p^{-1}x^* - J_p^{-1}y^*\| \leq m_2\|x^* - y^*\|,$$

for all $x^*, y^* \in E^*$ with $\|x^*\| \leq R, \|y^*\| \leq R$.

PROOF. From lemma 19, J_p^{-1} is uniformly continuous on bounded subset of E^* implies that for all $R > 0$, there exists a nondecreasing function ψ_0 with $\psi_0(0) = 0$ such that:

$$\|J_p^{-1}x^* - J_p^{-1}y^*\| \leq \psi_0(\|x^* - y^*\|),$$

for all $x^*, y^* \in E^*$ with $\|x^*\| \leq R, \|y^*\| \leq R$. By taking $(\psi_0\|x^* - y^*\|) := m_2\|x^* - y^*\|$, the result follows. \square

THEOREM 37. Xu (1991) Let $p > 1$ and $r > 0$ two real numbers and E be a Banach space. The following are equivalent:

- (i) E is uniformly convex;
- (ii) There exists increasing function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $g(0) = 0$ such that

$$\|x + y\|^p \geq \|x\|^p + p\langle y, f_x \rangle + g(\|y\|), \quad \forall x, y \in B(0, r), f_x \in J_p(x).$$

(iii) There exists increasing function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $g(0) = 0$ such that $\langle x - y, f_x - f_y \rangle \geq g(\|x - y\|)$, $\forall x, y \in B(0, r)$, $f_x \in J_p(x)$, $f_y \in J_p(y)$.

DEFINITION 2. Let E be a smooth real Banach space with dual E^* .

(i) The function $\phi : E \times E \rightarrow \mathbb{R}$, defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad x, y \in E,$$

where J is the normalized duality mapping from E into E^* , (see e.g, [Alber \(1994\)](#).)

(ii) The function $\phi_p : E \times E \rightarrow \mathbb{R}$, defined by

$$\phi_p(x, y) = p\left(q^{-1}\|x\|^q - \langle x, Jy \rangle + p^{-1}\|y\|^p\right), \quad \forall x, y \in E, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

(ii) Let $V_p : E \times E^* \rightarrow \mathbb{R}$ be the functional defined by:

$$V_p(x, x^*) = p\left(q^{-1}\|x\|^q - \langle x, x^* \rangle + p^{-1}\|x^*\|^p\right), \quad \forall x \in E, x^* \in E^*, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

REMARK 13. These remarks follows from Definition 2

(i) If $E = H$, a real Hilbert space, then equation (2.1) reduces to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \quad \forall x, y \in E.$$

(ii) For $p = 2$, $\phi = \phi_p$. Also, it is obvious from the definition of the function ϕ_p that

$$(2.1) \quad (\|x\| - \|y\|)^p \leq \phi_p(x, y) \leq (\|x\| + \|y\|)^p, \quad \forall x, y \in E.$$

(iii) It is obvious

$$V_p(x, x^*) = \phi_p(x, J_p^{-1}x^*), \quad \forall x \in E, x^* \in E^*.$$

LEMMA 21. [Abinu et al. \(2016\)](#) Let E be a reflexive strictly convex and smooth real Banach space with E^* as its dual. Then,

$$V_p(x, x^*) + p\langle J^{-1}x^* - x, y^* \rangle \leq V_p(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

LEMMA 22. For $p > 1$, let E be a uniformly convex real Banach space. For $r > 0$, let $B_r(0) := \{x \in E : \|x\| \leq r\}$. Then for arbitrary $x, y \in B_r(0)$,

$$\|x - y\|^p \geq \phi_p(x, y) + g(\|x\|) - \frac{p}{q}\|x\|^q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Where g is a function appearing in Theorem [37](#)

PROOF. Since E is uniformly convex real Banach space, by condition *ii* of Theorem 37, for any $x, y \in B_r(0)$, we have that

$$\|x + y\|^p \geq \|x\|^p + p\langle y, j_x \rangle + g(\|y\|).$$

Replacing y by $-y$ gives

$$\|x - y\|^p \geq \|x\|^p - p\langle y, j_x \rangle + g(\|y\|).$$

Interchanging x and y and simplifying by p , we get

$$\begin{aligned} p^{-1}\|x - y\|^p &\geq p^{-1}\|y\|^p - \langle x, j_y \rangle + p^{-1}g(\|x\|) \\ &= q^{-1}\|x\|^q - \langle x, j_y \rangle + p^{-1}\|y\|^p - q^{-1}\|x\|^q + p^{-1}g(\|x\|) \\ &= p^{-1}\phi_p(x, y) - q^{-1}\|x\|^q + p^{-1}g(\|x\|), \end{aligned}$$

so that

$$\phi_p(x, y) \leq p\left(p^{-1}\|x - y\|^p + q^{-1}\|x\|^q - p^{-1}g(\|x\|)\right),$$

which is equivalent to

$$\|x - y\|^p \geq \phi_p(x, y) + g(\|x\|) - \frac{p}{q}\|x\|^q,$$

establishing the lemma. □

LEMMA 23 (Kamimura and Takahashi (2002)). Let E be a real smooth Banach space and uniformly convex space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E . If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$, then $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.

LEMMA 24 (Tan and Xu (1993)). Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following relation:

$$a_{n+1} \leq a_n + \sigma_n \quad \forall n \geq 0.$$

Assume that $\sum_{n=0}^{\infty} \sigma_n < \infty$. Then $\lim_{n \rightarrow \infty} a_n$ exists.

3. Main results

We now prove the following results.

THEOREM 38. Let E be a uniformly convex real Banach space with uniformly Gâteaux differentiable norm and E^* its dual space. Let $A : E \rightarrow E^*$ be a bounded and uniformly monotone mapping such that $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in E$, let (x_n) be a sequence defined iteratively by

$$(3.1) \quad x_{n+1} = J_p^{-1}(J_p x_n - \alpha_n A x_n), \quad n \geq 1, \quad p \geq 2$$

where J_p is the duality mapping from E to E^* and $(\alpha_n) \subset]0, 1[$ a real sequence satisfying the following conditions: (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$. Then, there exists γ_0 such that, if $\alpha_n < \gamma_0$, for all $n \geq 1$, then (x_n) converges strongly to the unique solution of the equation $Au = 0$.

PROOF. The proof is in two steps.

Step 1: We prove that x_n is bounded. Since $A^{-1}(0) \neq \emptyset$, let $\bar{u} \in A^{-1}(0)$. there exists $r > 0$ such that:

$$(3.2) \quad r \geq \max \left\{ \frac{4p}{q} \|\bar{u}\|^q, \phi_p(x_1, \bar{u}) \right\}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

We show that $\phi_p(\bar{u}, x_n) \leq r$ for all $n \geq 1$. The proof is by induction. By construction, $\phi_p(x_1, \bar{u}) \leq r$. Assume that $\phi_p(\bar{u}, x_n) \leq r$ for $n \geq 1$, we show that $\phi_p(\bar{u}, x_{n+1}) \leq r$. Let

$$B := \left\{ J_p x - \theta Ax, 0 \leq \theta \leq 1, \phi_p(x, \bar{u}) \leq r \right\}.$$

Since A is bounded, B is bounded, using Lemma 20. We have there exists $L \geq 0$ such that:

$$\|J_p^{-1}u^* - J_p^{-1}v^*\| \leq L\|u^* - v^*\|,$$

for all $u^*, v^* \in B$. Since A is bounded, we have

$$M_0 := L \sup \{ \|Ax\|^2, \phi_p(x, \bar{u}) \leq r \} \leq +1 < \infty,$$

Where L is a Lipschitz constant of J^{-1} . Define:

$$(3.3) \quad \gamma_0 = \min \left\{ 1, \frac{kr}{4M_0} \right\}$$

Using the definition of x_{n+1} , we compute as follows:

$$\begin{aligned} \phi_p(\bar{u}, x_{n+1}) &= \phi_p(\bar{u}, J_p^{-1}(J_p x_n - \alpha_n Ax_n)) \\ &= V_p(\bar{u}, J_p x_n - \alpha_n Ax_n). \end{aligned}$$

Using lemma 2, with $y^* = \alpha_n Ax_n$, we have:

$$\begin{aligned} \phi_p(\bar{u}, x_{n+1}) &= V(\bar{u}, J_p x_n - \alpha_n Ax_n) \\ &\leq V_p(\bar{u}, J_p x_n) - p\alpha_n \langle J_p^{-1}(J_p x_n - \alpha_n Ax_n) - \bar{u}, Ax_n - A\bar{u} \rangle \\ &= \phi_p(\bar{u}, x_n) - p\alpha_n \langle x_n - \bar{u}, Ax_n - A\bar{u} \rangle - p\alpha_n \langle J_p^{-1}(J_p x_n - \alpha_n Ax_n) - x_n, Ax_n \rangle \\ &= \phi_p(\bar{u}, x_n) - p\alpha_n \langle x_n - \bar{u}, Ax_n - A\bar{u} \rangle \\ &\quad - p\alpha_n \langle J_p^{-1}(J_p x_n - \alpha_n Ax_n) - J_p^{-1}(J_p x_n), Ax_n \rangle. \end{aligned}$$

Using the uniformly monotonicity of A , Schwartz inequality and the Lipschitz property of J_p^{-1} , we obtain

$$\begin{aligned}\phi_p(\bar{u}, x_{n+1}) &\leq \phi_p(\bar{u}, x_n) - p\alpha_n k \|x_n - \bar{u}\|^p + p\alpha_n \|J_p^{-1}(J_p x_n - \alpha_n A x_n) - J_p^{-1}(J_p x_n)\| \|A x_n\| \\ &\leq \phi_p(\bar{u}, x_n) - p\alpha_n k \|x_n - \bar{u}\|^p + p\alpha_n^2 L \|A x_n\|^2.\end{aligned}$$

Using Lemma 22

$$(3.4) \phi_p(\bar{u}, x_{n+1}) \leq \phi_p(\bar{u}, x_n) - p\alpha_n k \left(\phi_p(\bar{u}, x_n) - \frac{p}{q} \|\bar{u}\|^q + g(\|\bar{u}\|) \right) + p\alpha_n^2 M_0.$$

Using (3.4), we have:

$$(3.5) \quad \phi_p(\bar{u}, x_{n+1}) \leq \phi_p(\bar{u}, x_n) - p\alpha_n k \phi_p(\bar{u}, x_n) + p\alpha_n k \frac{p}{q} \|\bar{u}\|^q + p\alpha_n^2 M_0.$$

Finally, using the fact that $\alpha_n \leq \gamma_0$, the definition of γ_0 in (3.3) and the induction assumption, it follows

$$\begin{aligned}\phi_p(\bar{u}, x_{n+1}) &\leq (1 - p\alpha_n k) \phi_p(\bar{u}, x_n) + p\alpha_n k \frac{r}{4} + p\alpha_n k \frac{r}{4} \\ \phi_p(\bar{u}, x_{n+1}) &\leq (1 - p\alpha_n k) r + p\alpha_n k \frac{r}{4} + p\alpha_n k \frac{r}{4} \\ &\leq (1 - \frac{p\alpha_n k}{2}) r \\ &< r.\end{aligned}$$

Hence, $\phi_p(\bar{u}, x_{n+1}) \leq r$. By induction $\phi_p(\bar{u}, x_n) \leq r$ for all $n \geq 1$. Thus, from Inequality (2.1), (x_n) is bounded.

Step 2: We now prove that (x_n) converges strongly to the unique point, \bar{u} of A . Let $x^* \in A^{-1}(0)$. Following the same arguments as step 1, the boundedness of (x_n) and that A there exists a positive constant M such that :

$$(3.6) \quad \phi_p(\bar{u}, x_{n+1}) \leq \phi_p(\bar{u}, x_n) - p\alpha_n k \|x_n - \bar{u}\|^p + \alpha_n^2 M.$$

Consequently,

$$\phi_p(\bar{u}, x_{n+1}) \leq \phi_p(\bar{u}, x_n) + \alpha_n^2 M.$$

By the hypothesis that $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ and the lemma 24, we have that $\lim_{n \rightarrow \infty} \phi_p(\bar{u}, x_n)$

exists. From inequality (3.6), using the fact that, $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$, it follows that

$$\sum_{n=1}^{\infty} \alpha_n \|x_n - \bar{u}\| < \infty.$$

As $\sum_{n=0}^{\infty} \alpha_n = \infty$, then $\liminf_n \|\bar{u} - x_n\|^p = 0$. Consequently, there exists a subsequence (x_{n_k}) of (x_n) such that : $x_{n_k} \rightarrow \bar{u}$ as $k \rightarrow \infty$. Since (x_n) is bounded and J_p is norm- to-norm *weak** uniformly continuous on bounded subset of E it follows that there exists a $(\phi_p(\bar{u}, x_{n_k}))$ of $(\phi_p(\bar{u}, x_n))$ converges to 0. Therefore, $\{\phi_p(\bar{u}, x_n)\}$ converges to 0. Also, by lemma 23, $\|x_n - \bar{u}\| \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof. \square

COROLLARY 13. *Let $E := L_p, 1 < p \leq +\infty$ or $W^{m,p}$ and A be a bounded and uniformly monotone mapping such that $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in E$, let (x_n) be a sequence defined iteratively by*

$$(3.7) \quad x_{n+1} = J_p^{-1}(J_p x_n - \alpha_n A x_n), \quad n \geq 1, \quad p \geq 2$$

where J_p is the duality mapping from E to E^* and $(\alpha_n) \subset]0, 1[$ a real sequence satisfying the following conditions: (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$. Then, there exists γ_0 such that, if $\alpha_n < \gamma_0$, for all $n \geq 1$, then (x_n) converges strongly to the unique solution of the equation $Au = 0$.

PROOF. Since E is uniformly convex with uniformly Gâteaux differentiable norm, then the proof follows from Theorem 38. \square

Now, let us consider an important application.

4. Application to convex minimization problems

In this section, we study the problem of finding a minimizer of a uniformly convex function f defined from a real Banach space E to \mathbb{R} .

The following basic results are well known.

REMARK 14. *It is well known that if $f : E \rightarrow \mathbb{R}$ be a real-valued differentiable convex function and $a \in E$, then the point a is a minimizer of f on E if and only if $df(a) = 0$.*

DEFINITION 3. *A function $f : E \rightarrow \mathbb{R}$ is said to be uniformly convex if there exists $\gamma(t), \gamma(0) = 0$ such that for every $x, y \in E$ the following inequality holds:*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - \gamma(\|x - y\|).$$

If here $\gamma(t) = ct^p, p \geq 2$, where $c > 0$ is a positive constant, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - \|x - y\|^p.$$

LEMMA 25. *Let E be normed linear space. let $f : E \rightarrow \mathbb{R}$ a real valued differentiable convex function. Let $df : E \rightarrow E^*$ denotes the differential map associated to f . Then the following hold. If f is bounded, then f is locally Lipschitzian, i.e., for every $x_0 \in E$ and $r > 0$, there exists $\gamma > 0$ such that f is γ -Lipschitzian on $B(x_0, r)$, i.e.,*

$$|f(x) - f(y)| \leq \gamma\|x - y\| \quad \forall x, y \in B(x_0, r).$$

LEMMA 26. *Let E be a norm linear space and $f : E \rightarrow \mathbb{R}$ a real valued convex differentiable convex function. Assume that f is bounded. Then the differentiable map, $df : E \rightarrow E^*$ is bounded.*

PROOF. For $x_0 \in E$ and $r > 0$, let $B := B(x_0, r)$. We show that $dF(B)$ is bounded. From Lemma 25, there exists $\gamma > 0$ such that

$$(4.1) \quad |f(x) - f(y)| \leq \gamma\|x - y\| \quad \forall x, y \in B.$$

Let $z^* \in df(B)$ and $x^* \in B$ such that $z^* = df(x^*)$. For $u \in E$, since B is open, then there exists $t > 0$ such that $x^* + tu \in B$. Using the fact that $z^* = df(x^*)$, convexity of F and inequality (4.1), it follows

$$\begin{aligned} \langle z^*, tu \rangle &\leq f(x^* + tu) - f(x^*) \\ &\leq t\gamma\|u\|. \end{aligned}$$

So that, $\langle z^*, u \rangle \leq \gamma\|u\| \quad \forall u \in E$. Therefore, $\|z^*\| \leq \gamma$. Hence $df(B)$ is bounded. □

LEMMA 27. Let E be normed linear space and $f : E \rightarrow \mathbb{R}$ a real-valued differentiable convex function. Assume that f is uniformly convex. Then the differential map $df : E \rightarrow E^*$ is uniformly monotone, i.e., there exists a positive constant $k > 0$ such that

$$(4.2) \quad \langle df(x) - df(y), x - y \rangle \geq k \|x - y\|^p \quad \forall x, y \in E.$$

We now prove the following theorem.

THEOREM 39. Let E be a uniformly convex real Banach space with uniformly Gâteaux differentiable norm and E^* its dual space. $f : E \rightarrow \mathbb{R}$ be a differentiable, bounded, uniformly convex real-valued function which satisfies the growth condition: $f(x) \rightarrow +\infty$ as $\|x\| \rightarrow +\infty$. For arbitrary $x_1 \in E$, let $\{x_n\}$ be the sequence defined iteratively by:

$$(4.3) \quad x_{n+1} = J_p^{-1}(J_p x_n - \alpha_n A x_n), \quad n \geq 1, \quad p \geq 2$$

where J_p is the duality mapping from E to E^* and $(\alpha_n) \subset]0, 1[$ a real sequence satisfying the following conditions: (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$. Then, f has a unique minimizer $a^* \in E$ and there exists a positive real number there exists γ_0 such that, if $\alpha_n < \gamma_0$, for all $n \geq 1$, the sequence $\{x_n\}$ converges strongly to a^* .

PROOF. Since E is reflexive, then from the growth condition, the continuity and the strict convexity of f , it follows that f has a unique minimizer a^* characterized by $df(a^*) = 0$ (Remark 14). Finally, from Lemma 25 and the fact that the differential map $df : E \rightarrow E^*$ is bounded, the proof follows from Theorem 38. \square

5. Conclusion

In this work, we proposed a new iteration scheme for the approximation of zeros of monotone mappings defined in certain Banach spaces. Our results are used to approximate minimizers of lower semi continuous and convex functions. The results obtained in this paper are important improvements of recent important results in this field.

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