CHAPTER 2

Spaces, Functions, and Groups Acting on Spaces

This chapter is divided into three sections. In Section 2.1 notions are introduced that do not involve the concept of continuity. Topology is introduced in Section 2.2 and, among other things, proper mappings are defined. The most interesting questions deal with the interaction between algebra and topology, and these are treated in Section 2.3. For simplicity of notation in this and several subsequent chapters, we shall denote spaces by symbols X, Y, etc., instead of $\mathfrak{X}$, $\mathfrak{Y}$, etc. From Chapter 8 on we shall revert to the latter notation and reserve X, Y, etc. for random variables, as in Chapter 1.

2.1. Spaces, functions, groups, and group action. Let X and Y be two arbitrary spaces and f a function $X \to Y$. Instead of "function" the names **mapping** or **map** are also used. The **range** of f is range $f = \{f(x) : x \in X\}$. If range $f = Y$, f is said to be **onto**, or **surjective**. If $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in X$, then f is called **one-to-one** (or 1-1), or **injective**. In that case f^{-1} is defined as a function on range f onto X. If f is both 1-1 and onto, it is also called **bijective**. In that case there is a 1-1 correspondence between X and Y, and f^{-1} is defined on all of Y. For arbitrary $f : X \to Y$, f^{-1} is always defined as a set function: for $B \subset Y$,

$f^{-1}(B) = \{x \in X : f(x) \in B\}$. If Z is a third space and $g : Y \to Z$, then the **composition** of f and g is the function $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$. The **identity function**, or **identity map**, on X is the function $i_X : X \to X$ given by $i_X(x) = x$ for every $x \in X$.

If X and Y are two spaces, their **product**, denoted $X \times Y$, is the set of all (x, y) with $x \in X$, $y \in Y$. The space $X \times Y$ is also called a **product space**. A **product set** $A \times B \subset X \times Y$, with $A \subset X$, $B \subset Y$, is the set of all (x, y) with $x \in A$, $y \in B$. The **projection** of $X \times Y$ onto X, denoted pr_1, is the function $\mathrm{pr}_1(x, y) = x$. Similarly, $\mathrm{pr}_2(x, y) = y$. If Z is a third space and $f_1 : Z \to X$, $f_2 : Z \to Y$ two functions, then (f_1, f_2) denotes the function $Z \to X \times Y$ defined by $(f_1, f_2)(z) = (f_1(z), f_2(z))$. In contrast, if R is the real line and $f_1 : X \to R$, $f_2 : Y \to R$, then $f_1 \otimes f_2$ denotes the function $X \times Y \to R$ defined by $(f_1 \otimes f_2)(x, y) = f_1(x) f_2(y)$. Finally, if for $i = 1, 2$, there are spaces X_i, Y_i and functions $f_i : X_i \to Y_i$, then $f_1 \times f_2$ is the function $X_1 \times X_2 \to Y_1 \times Y_2$ that maps (x_1, x_2) to $(f_1(x_1), f_2(x_2))$.

A **linear space** (or **vector space**) over the real numbers R is a space X on which is defined addition $x + y$ and multiplication cx, for $x, y \in X$, $c \in R$. These operations have to satisfy certain axioms, such as associativity and commutativity of addition, distributivity of multiplication, existence of a neutral element 0, and others, For a complete list of axioms, see, e.g., Taylor (1965, 1985), Section 3-12, or Dunford and Schwartz (1958), Section I-11. In this context the real numbers are often called **scalars**. A 1-1 correspondence between two linear spaces is called a **linear isomorphism** if it preserves the linear operations.

A **normed linear space** is a linear space X on which is defined a real valued function, called **norm** and denoted $\| \ \|$, with the following properties: $\|x\| \geq 0$ for every $x \in X$, and $= 0$ if and only if $x = 0$; $\|cx\| = |c| \|x\|$ for $c \in R$; $\|x + y\| \leq \|x\| + \|y\|$. In words, the norm is a nonnegative function on X (and actually positive on $X - \{0\}$) that is positively homogenous and satisfies the triangle inequality. A **semi-norm** or **pseudo-norm** satisfies all the conditions of a norm except that $\|x\| = 0$ need not imply $x = 0$. If $\|\ \|$ is a norm on X, then $d(x,y) = \|x - y\|$ is a distance function on $X \times X$ so that a normed linear space is a special case of a **metric space.** In R^n with points $x = (x_1, \dots, x_n)$ the **Euclidean norm** is $\|x\| = (\sum x_i^2)^{\frac{1}{2}}$. If X and Y are linear spaces, then a function $f : X \to Y$ is called **linear** if $f(c_1x_1 + c_2x_2) = c_1f(x_1) + c_2f(x_2)$ for $x_i \in X$, $c_i \in R$. If $Y = R$, then a linear function $X \to R$ is often called a **linear functional** on X.

A **group** G is a set (also denoted G) together with a binary operation that assigns to each ordered pair of elements of G another element of G. The binary operation is usually written as a multiplication and is then called **group multiplication**: $g_1g_2 = g_3$, where the g_i are elements of G. This group multiplication is required to be **associative**: $g_1(g_2g_3) = (g_1g_2)g_3$; there must be an **identity element** $e \in G$ that has the property $eg = ge = g$ for very $g \in G$; and each $g \in G$ must have an **inverse** g^{-1}, for which $gg^{-1} = g^{-1}g = e$ holds. In general, g_1g_2 need not be equal to g_2g_1 for all $g_1, g_2 \in G$, but if it is, then G is called **commutative** or **abelian**. A **subgroup**, say G_0, of G is a subset of G that is also a group if it inherits the group multiplication from G. If G and H are two groups, then a function $\phi : G \to H$ is called a **homomorphism** if it preserves group multiplication: $\phi(g_1g_2) = \phi(g_1)\phi(g_2)$, $g_1, g_2 \in G$. This implies $\phi(e_G) = e_H$ (where e_G, e_H are the identities in G, H, respectively) and $\phi(g^{-1}) = (\phi(g))^{-1}$. Further group notions will be introduced as the need arises.

A simple example of an abelian group is the real line R with 0 removed, where group multiplication is ordinary multiplication of real numbers. Then $e = 1$ and $g^{-1} = \frac{1}{g}$ for every $g \in R - \{0\}$. This group possesses the subgroup R_+^*, the set of all positive reals under multiplication. Another abelian group that appears often in applications is R under addition; i.e., group multiplication is ordinary addition of real numbers. Here $e = 0$ and $g^{-1} = -g$. The set of all integers forms a subgroup and so does the set of all rationals. A typical example of a non-abelian group is the **general linear group**

$GL(n)$, which is the set of all real nonsingular $n \times n$ matrices ($n \geq 2$) with group multiplication = matrix multiplication. There are several interesting subgroups, also non-abelian, e.g., the group $O(n)$ of $n \times n$ orthogonal matrices encountered in Chapter 1. Also the group $UT(n)$ of $n \times n$ upper triangular matrices with positive diagonal elements; similarly the lower triangular matrices $LT(n)$. An example of an abelian subgroup of $GL(n)$ is all $n \times n$ diagonal matrices non of whose diagonal elements are 0. The identity in $GL(n)$ and in all of its subgroups is $e = I_n = \text{diag}(1, \dots, 1)$, i.e., the $n \times n$ identity matrix. An example of the kind of group that falls outside the scope of this monograph (because the group is not locally compact) is the set of all functions that map R 1-1 onto R, with group multiplication = composition of functions.

Throughout this monograph groups will usually be denoted by the symbols G, H, or K. The identity element of a group G is written e or e_G. Let G be a group and X an arbitrary space. An **action** of G on X to the **left** (or: an action of G on the left of X) is any function $a : G \times X \to X$ with the following properties: (i) for every $g \in G$, $a(g, \cdot) : X \to X$ is bijective; (ii) $a(e, x) = x$ for every $x \in X$; (iii) $a(g_2, a(g_1, x)) = a(g_2 g_1, x)$ for every $g_1, g_2 \in G$, $x \in X$. If on the right-hand side of (iii) $g_2 g_1$ is replaced by $g_1 g_2$, then we say that G acts on X to the **right**. It is customary, for simplicity of notation, to suppress the symbol for the function a, and write simply gx for $a(g, x)$ if the action is to the left. In that notation the three defining properties of left action read

(i) for every $g \in G$, the function $x \to gx$ is bijective;
(ii) $ex = x$ for every $x \in X$;
(iii) $g_2(g_1 x) = (g_2 g_1)x$ for every $g_1, g_2 \in G$, $x \in X$.

If the action is to the right one writes xg; then (iii) above changes to $(xg_1)g_2 = x(g_1 g_2)$. From (i)–(iii) it follows that the action of any $g \in G$ followed by the action of g^{-1} produces the identity transformation i_X on X. If $gx = x$ for every $g \in G$, $x \in X$, then the action of G is said to be **trivial**. At the other extreme, G is said to act **freely** if every $g \in G$ except $g = e$ moves every $x \in X$; i.e., $gx \neq x$ unless

$g = e$. In the majority of applications the action is somewhere in between, i.e., for each $x \in X$ there are some group elements g other than e that leave x fixed. It will often be convenient to state the action in the form $x \to gx$, with gx replaced by an explicit formula.

2.1.1. EXAMPLE. An example of a group action that occurs often in statistics consists of $X = R^n$, and $G = GL(n)$ or one of its subgroups. The group action (to the left of X) is defined in the obvious way: if $x \in X$, considered as an $n \times 1$ column vector, and $g = C \in G$, then with gx is meant Cx (matrix multiplication). Thus, the action of a single C is a nonsingular linear transformation of X. The validity of requirement (i) above follows from the fact that the equation $Cx = y$ has the unique solution $x = C^{-1}y$. The validity of (ii) and (iii) is immediate. If $G = \{e\}$, i.e., G consists of the trivial group that has only one element I_n, then the action of G is of course trivial. In all other cases the action is not trivial for there is always some point x that is moved to another position by some $g \in G$. On the other hand, the action is usually not free. For instance, if $G = GL(n)$ and $x = (1, 0, \dots, 0)'$, then for any $C \in G$ whose first column equals x we have $Cx = x$. □

2.1.2. EXAMPLE. Let $X = R^n$ as in Example 2.1.1 but now take $G = R$ under addition, so that $e = 0$. Choose an arbitrary fixed nonzero vector $v \in X$ and define the action of G on X by $x \to x + bv$, $b \in R$. Thus, the action of a single $g = b$ is a translation of X which moves every point unless $b = 0$, i.e., $g = e$. Therefore, this G acts freely. The example can easily be extended to $G = R^m$ $(m \leq n)$ under vector addition, with action $x \to x + \sum_1^m b_i v_i$, where all b_i are $\in R$ and the $v_i \in X$ are linearly independent. If $m = n$, we have a case where $G = X$ and the action is vector addition: $x \to x + v$, $v \in G$. □

For any arbitrary space X and group G one can always define the action to be trivial: $gx = x$ for every $x \in X$, $g \in G$. This may not seem very interesting, and by itself it is not useful. But it does find application in the context of product spaces. For instance, if $X \times Y$ is a product space and the action of G is defined on X in some natural way, then we may want to extend this action to $X \times Y$ by $(x, y) \to (gx, y)$. This corresponds to defining the action of G on Y to be trivial. For an example of this see Section 7.5.

For a given action of G on X the **orbit** of $x \in X$ is defined as $Gx = \{gx : g \in G\}$. If there is another group H acting on X, then we shall have to distinguish between G-orbits and H-orbits. For $x, y \in X$, either $Gx = Gy$ or Gx and Gy are disjoint. Thus, the orbits furnish a partitioning of X and the property of two points lying on the same orbit is an equivalence relation. If X contains only one orbit, the action is said to be **transitive**, or **G is transitive over X**. In that case, for any $x, y \in X$ there exists $g \in G$ such that $y = gx$.

2.1.3. EXAMPLE. The simplest example of a transitive group action is $G = X = R^n$ with action = translation, as in the special case $m = n$ of Example 2.1.2. On the other hand, if $G = R^n$ with $m < n$, then the action is not transitive. Take, for instance, $m = 1$ and the action $x \to x + bv$, $b \in R$, $v \in X$ fixed, nonzero. The orbit of x consists of all points $x + bv$, $-\infty < b < \infty$, i.e., the straight line through x in the direction v. □

2.1.4. EXAMPLE. Consider now the kind of group and action of Example 2.1.1 and suppose first that $G = GL(n)$. It is almost, but not quite, true that the action of G on $X = R^n$ is transitive. The origin 0 plays a special role: it is its own orbit. However, when the origin is deleted, the remaining set $R^n - \{0\}$ is invariant under G and the action of G is transitive. To see this, suppose that x and y are both arbitrary nonzero $n \times 1$ vectors. Then there exist matrices C_1 and C_2 in G with first columns x, y, respectively. Take $C = C_2 C_1^{-1}$, then $Cx = y$ so that x and y are on the same orbit. For the kind of statistical applications in this monograph it is usually permissible to remove from the sample space a set of Lebesgue measure zero without changing the distributions. In the present example we would simply redefine $X = R^n - \{0\}$, and then G is transitive over X, A similar phenomenon exists if $G = LT(n)$ or $UT(n)$. For instance, $G = LT(n)$ acts transitively on X if X is formed from R^n by removing the set of vectors x with first coordinate $x_1 = 0$. Such removal of sets of zero

Lebesgue measure to make the action of the group on the remaining space simpler will often occur in later applications. □

2.1.5. EXAMPLE. A typical example of nontransitive action, derived from Example 2.1.1, consists of $X = R^n$, or even $X = R^n - \{0\}$, and $G = O(n)$. Then the orbit of a point $x \in X$ is the set of all $y \in X$ for which $\|y\| = \|x\|$, i.e., the sphere about 0 with radius $\|x\|$. Here the partitioning of X furnished by the G-orbits consists of all spheres concentric with the origin. □

2.1.6. EXAMPLE. Here is another example of nontransitive action that occurs often in applications. Let $X = R^n - \{0\}$ and $G = R^*_+$ with action $x \to cx$, $x \in X$, $c > 0$ (this may be considered a special case of Example 2.1.1 if to $c > 0$ corresponds the matrix $cI_n \in GL(n)$). The orbit of $x \in X$ is the ray emanating from 0 through x, and X is partitioned by all these rays. □

2.1.7. EXAMPLE. Another action that occurs very often in statistical application, especially multivariate analysis, consists of $X = PD(n) =$ all $n \times n$ positive definite matrices, $G = GL(n)$ or one of its subgroups, and the action defined by $S \to CSC'$ for $S \in X$, $C \in G$. The action is transitive if $G = GL(n)$ or even $LT(n)$ or $UT(n)$. However, if $G = O(n)$, then the action is not transitive and the orbit of S consists of all positive definite matrices that have the same characteristic roots as S, including multiplicities. □

The abstract space whose points are the G-orbits is called the **orbit space** (under G) and denoted X/G. The **orbit projection** $\pi : X \to X/G$ assigns to each $x \in X$ its orbit, i.e.,

$$\pi(x) = Gx, \quad x \in X, \tag{2.1.1}$$

where the right-hand side of (2.1.1) should be considered a point of X/G rather than a subset of X. If $A \subset X$ and $g \in G$, then with gA is meant the set $\{gx : x \in A\}$. This is sometimes called the **g-translate** of A. The set $GA = \{gA : g \in G\}$ is called the **saturation** of A. It may be conceived in two ways: as the union over all $g \in G$

of the g-translates of A, or as the union of all orbits that meet A. A set $A \subset X$ such that $gA = A$ for all $g \in G$ is called **invariant**. It coincides with its saturation.

Let H be a subgroup of G, then the **left coset** of $g \in G$ modulo H is the subset $gH = \{gh : h \in H\} \subset G$. These cosets are the orbits under the action of H on G to the **right** (why right rather than left action is chosen will soon become clear). The abstract space whose points are the left cosets is denoted G/H. It is known as a **homogeneous space** or **coset space**. The point of G/H corresponding to the coset gH will often also be denoted gH, or by $[g]$. The **coset projection** π is defined as in (2.1.1), but now as a function $\pi : G \to G/H$:

$$\pi(g) = gH = [g], \quad g \in G. \tag{2.1.2}$$

There is a natural transitive action of G on G/H defined by

$$g_1(gH) = (g_1 g)H, \quad g, g_1 \in G. \tag{2.1.3}$$

The **right cosets** Hg are similarly defined. The subgroup H is called **normal** if $gHg^{-1} = H$ for every $g \in G$. In that case the cosets are preserved under multiplication and G/H is a group, called the **quotient** of G and H. Furthermore, the left cosets gH coincide with the right cosets Hg.

2.1.8. EXAMPLE. Let $G = LT(2)$ and write the elements of G as $T = ((t_{ij}))$ so that $t_{12} = 0$ and $t_{11} > 0$, $t_{22} > 0$. There are several interesting subgroups H. First, let H be all $T \in G$ with $t_{11} = 1$. It is easily checked that for any $T_1 \in H$ and $T_2 \in G$ we have $T_2 T_1 T_2^{-1} \in H$ (actually, it suffices to verify this for T_2 of the form $\operatorname{diag}(a, 1)$ with $a > 0$, because this type of matrices together with H generates G). Therefore, H is normal in G. The left ($=$ right) coset of $T \in G$ consists of all $U \in G$ with $u_{11} = t_{11}$. A similar example results from taking H to be all $T \in G$ with $t_{22} = 1$. Also normal is the subgroup H consisting of all $T \in G$ with $t_{11} = t_{22} = 1$. Then each coset consists of all $T \in G$ with a fixed value of t_{21}. A subgroup that is not normal, and where consequently the left and right cosets do not coincide, is obtained by taking H as all $T \in G$ with $t_{21} = 0$, i.e., all diagonal matrices in G. It is easily verified that any left coset consists of all $T \in G$ with a fixed value of t_{21}/t_{11} (and t_{21}/t_{22} for a right coset). □

2.1.9. EXAMPLE. Another frequently occurring combination of group and subgroup in statistical applications is $G = O(n)$ and H all matrices of the form $\text{diag}(\Gamma_1, \Gamma_2)$ with $\Gamma_i \in O(n_i)$, $i = 1, 2$, $n_1 + n_2 = n$. That is, the matrices in H are block-diagonal, and each block is orthogonal. Such an H is not normal. We may further restrict Γ_i to range over a subgroup of $O(n_i)$. Of special importance is the case where $O(n_1)$ is replaced by the trivial group $\{I_{n_1}\}$. Take for instance $n_1 = 1$, $n_2 = n - 1$ and H all matrices $\text{diag}(1, \Gamma_2)$, $\Gamma_2 \in O(n-1)$. Then the left coset of any $\Gamma \in G$ consists of all orthogonal matrices with the same first column that Γ has. □

If G acts on the left of X, then the subgroups H in which we shall be most interested are those that leave points of X fixed. For arbitrary $x \in X$ define

$$G_x = \{g \in G : gx = x\}. \tag{2.1.4}$$

It is easily checked that G_x is a group. It is called the **isotropy subgroup** (or **stability** subgroup) of G at x. There is a natural correspondence between the points of the orbit Gx of x and the left cosets of G_x: to $gx \in Gx$ corresponds $gG_x \in G/G_x$. It is easily verified that $g_1 x = g_2 x$ if and only if g_1 and g_2 are in the same coset: $g_2^{-1} g_1 \in G_x$. Formally, let $\psi_x : G/G_x \to Gx$ represent this bijection:

$$\psi_x(gG_x) = gx. \tag{2.1.5}$$

2.1.10. EXAMPLE. Let X and G be as in Example 2.1.5 and take $x = (1, 0, \dots, 0)'$. Then G_x consists of all matrices of the form $\text{diag}(1, \Gamma_2)$ with $\Gamma_2 \in O(n-1)$; i.e., G_x here is H of the special case in Example 2.1.9. Take any $y \in X$ with $\|y\| = 1$ so that $y \in Gx$, by Example 2.1.5. The equation $\Gamma x = y$, $\Gamma \in G$, is equivalent with the requirement that the first column of Γ be y. According to Example 2.1.9, all such Γ constitute a coset of G modulo H. Thus, the function ψ_x of (2.1.5) can be described simply as follows: ψ_x assigns to an $n \times n$ orthogonal matrix its first column. □

Still keeping $x \in X$ fixed there is another function of interest. Define $\alpha_x : G \to X$ by

$$\alpha_x(g) = gx. \tag{2.1.6}$$

In (2.1.2) replace H by G_x and write the coset projection as $\pi_x : G \to G/G_x$, defined by

$$\pi_x(g) = gG_x. \tag{2.1.7}$$

Then comparing (2.1.5), (2.1.6), and (2.1.7) we see that

$$\alpha_x = \psi_x \circ \pi_x. \tag{2.1.8}$$

Equation (2.1.8) together with the fact that π_x is onto results in the following two equations:

$$\psi_x^{-1}(A) = \pi_x(\alpha_x^{-1}(A)), \quad A \subset Gx, \tag{2.1.9}$$

$$\psi_x(B) = \alpha_x(\pi_x^{-1}(B)), \quad B \subset G/G_x. \tag{2.1.10}$$

(These two equations are true even without the injective nature of ψ_x.) If G acts freely, then G_x is **trivial**, i.e., $G_x = \{e\}$, and the functions ψ_x and α_x coincide. In that case there is a 1-1 correspondence between the points gx of the orbit Gx and the elements g of G.

Suppose that besides G and X there is another space Y and a function $f : X \to Y$. For the time being suppose G does not act on Y. For $g \in G$ the **g-translate** of f, written gf, is defined by $(gf)(x) = f(g^{-1}x)$ for all $x \in X$ (this definition arises from $(gf)(gx) = f(x)$). If $gf = f$ for every $g \in G$, then f is called **invariant** (under the left action of G on X). An invariant function can also be characterized as a function that is constant on each orbit. If an invariant function assumes different values on distinct orbits, it is called **maximal invariant**.

2.1.11. EXAMPLES. There are many examples of maximal invariants in Lehmann (1986, Chapter 6). Here follow two simple ones. In the situation of Example 2.1.5, we can take $Y = R$ and $f(x) = \|x\|$ since f is constant on each orbit, and different orbits have different values of f. Any other 1-1 function of $\|x\|$ will do just as well for f; for instance, $f(x) = \|x\|^2$. In Example 2.1.6, each orbit is characterized by a direction, which may be taken as a point on the $(n-1)$-dimensional unit sphere $\sum_1^n x_i^2 = 1$. Therefore, the mathematically most natural choice for Y is this unit sphere. In practice, however, Euclidean spaces are more convenient and one usually resorts to the removal from X of a set of n-dimensional Lebesgue measure 0 in order to make it possible to choose Y Euclidean. For instance, one can redefine X to be all $x \in R^n$ with $x_1 \neq 0$. Then one can take Y to be two copies of R^{n-1}, and $f(x) = (\operatorname{sgn} x_1, x_2/x_1, \ldots, x_n/x_1)$. □

Now suppose G acts both on x and on Y to the left. The result of the action of $g \in G$ on $y \in Y$ will also be denoted gy as long as it is clear from the context that the action takes place in Y rather than in X. A function $f : X \to Y$ is called **equivariant** if $g(f(x)) = f(gx)$ for every $g \in G$, $x \in X$. It is seen that if the action of G is trivial, then equivariance reduces to invariance.

2.1.12. EXAMPLE. A simple example of an equivariant function is any linear function $f : R^n \to R^m$ with G and action of G on both spaces as in Example 2.1.6. Then $f(cx) = cf(x)$, $c > 0$. □

2.1.13. EXAMPLE. A more sophisticated example is provided by the example in Chapter 1 related to the Wishart distribution. Let X be the space of all $n \times p$ matrices x with linearly independent columns (x was denoted X in Chapter 1) and Y the space of all $n \times p$ matrices U with orthonormal columns. The action of $G = O(n)$ on X and Y is the obvious one: $x \to \Gamma x$, $U \to \Gamma U$ (matrix multiplication), $\Gamma \in G$. The Gram-Schmidt decomposition $x = UT$, $T \in UT(p)$ is unique and we put $U = f(x)$. Then $f(\Gamma x) = \Gamma U$ so that f is equivariant. □

2.1.14. EXAMPLE. Let G be any group and H a subgroup. Put $X = G$, $Y = G/H$; let G act on itself to the left and on G/H

as defined by (2.1.3). Then the function π of (2.1.2) is seen to be equivariant, for (2.1.3) can be put in the form $\pi(g_1 g) = g_1 \pi(g)$ for every $g_1, g \in G$. □

2.2. Topological spaces. Continuous and proper functions. For general information on this topic, see, e.g., Kelley (1955) and Bourbaki (1966b). For local compactness see also Halmos (1950), Chapter X. Since the reference to Bourbaki (1966b) occurs rather often in this section and the next, we shall often abbreviate it by the letter "B."

A **topological space** $(X, \mathcal{T})$ consists of a space X together with a family $\mathcal{T}$ of subsets of X with the property that $X \in \mathcal{T}$ and $\mathcal{T}$ is closed under finite intersections and arbitrary unions. The members of $\mathcal{T}$ are called **open** and $\mathcal{T}$ is called the **topology** of X. The **coarsest** topology of X, also called the **trivial** topology, consists of only X and the empty set. The **finest** topology, also called the **discrete** topology, consists of all subsets of X; in particular, every singleton is open. A familiar example of a topology is the so-called **usual** topology of Euclidean n-space R^n, which consists of the sets in R^n that are common called open.

A subset A of X is called **closed** if its complement A^c is open. The **interior** of a set $A \subset X$, denoted A°, is the union of all open sets contained in A. The **closure** of $A \subset X$, written $\bar{A}$, is the smallest closed set containing A. The **boundary** of A is the set $\partial A = \bar{A} \cap \overline{A^c}$. A set A is **dense** in X if $\bar{A} = X$.

If $A \subset X$ and $\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}$, then $(A, \mathcal{T}_A)$ is a topological space and A is said to receive the **relative topology** of X. Then A is said to be a **subspace** of X. A topological space is called **connected** if it cannot be written as a disjoint union of two nonempty open sets. (If it can be so written, then those open sets are closed as well.) For instance, an interval of R, in the relative topology of R, is connected, but the union of two disjoint open intervals is not. The **component** of a point $x \in X$ is the largest connected subset of X containing x. It is necessarily closed (B, I §11.5, Propos. 9).

If $(X_i, \mathcal{T}_i)$, $i = 1, 2$, are two topological spaces, then on the product space $X_1 \times X_2$ we shall always put the **product topology** $\mathcal{T}_1 \times \mathcal{T}_2$, which is defined as the coarsest topology containing the product sets $A_1 \times A_2$, $A_i \in \mathcal{T}_i$. For instance, the usual topology of $R^2 = R \times R$ is the product topology generated by the usual topology of R.

A **neighborhood** of $x \in X$ is any $V \subset X$ with the property $x \in A \subset V$ for some $A \in \mathcal{T}$. A **base** $\mathcal{B}$ of a topology $\mathcal{T}$ is any subfamily of $\mathcal{T}$ with the property that for every $x \in X$ and neighborhood V of x there exists $B \in \mathcal{B}$ such that $x \in B \subset V$. The topological space $(X, \mathcal{T})$ is said to satisfy the second axiom of countability, or simply to be **second countable**, if $\mathcal{T}$ has a countable base $\mathcal{B}$. For instance, the usual topology of R has a base consisting of all open intervals with rational endpoints.

A topological space is called **Hausdorff** if every two distinct points have disjoint neighborhoods. This is a desirable feature that we usually want our spaces to have. For instance, it prevents the possibility of a sequence of points converging to two distinct points. In a Hausdorff space the complement of a single point is open; therefore, every singleton is closed.

A **cover** of a set $A \subset X$ is a family $\mathcal{F}$ of subsets of X whose union contains A; it is an **open cover** if the sets in $\mathcal{F}$ are open. If $\mathcal{F}_1 \subset \mathcal{F}$ and $\mathcal{F}_1$ is also a cover, then $\mathcal{F}_1$ is called a **subcover**. If the number of sets in $\mathcal{F}_1$ is finite, $\mathcal{F}_1$ is called finite. If X is Hausdorff, a subset A is called **compact** if every open cover of A has a finite subcover. (We follow Bourbaki here in restricting the definition of compactness to Hausdorff spaces; other standard books such as Kelley (1955) or Halmos (1950) do not impose that restriction.) A closed subset of a compact set is compact and the continuous image into a Hausdorff space of a compact set is compact (continuous functions will be defined later in this section). A space is called **σ-compact** if it is a countable union of compact subsets. (For instance, R is the union of the intervals $[n, n+1]$, $n = 0, \pm 1, \ldots$.) A topological space is called **locally compact** if it is Hausdorff and if every point has a compact neighborhood (again we follow Bourbaki by including Hausdorff in the definition). Since the property of local compactness occurs so frequently in this monograph, we shall consistently abbreviate it by **l.c.** If X and Y are both l.c., then so is $X \times Y$. In a l.c. space the compact sets form a base for the topology. A closed subset of a l.c. space is l.c. as a subspace. The real line R is l.c. since each point has a compact neighborhood $[x - a, x + a]$, with any $a > 0$. Similarly, R^n is l.c. By-and-large the spaces to be considered in this monograph are l.c.

Let X and Y be topological spaces and f a mapping $X \to Y$, then f is said to be **continuous** if $f^{-1}(B)$ is open for every open $B \subset Y$. It follows immediately that for f continuous $f^{-1}(B)$ is closed for every closed $B \subset Y$. It also follows from the definition that the composition of two continuous functions is continuous. If $f(A)$ is open for every open $A \subset X$, then f is called an **open mapping**, or simply **open**. Similarly, f is **closed** if $f(A)$ is closed for every closed $A \subset X$. If $X_1 \times X_2$ is a product topological space, then the projection $\mathrm{pr}_i : X_1 \times X_2 \to X_i$, $i = 1, 2$, is both continuous and open, but not closed. For instance, in R^2 the curve $C = \{(x_1, x_2) : x_1 x_2 = 1\}$ is closed but $\mathrm{pr}_1(C) = R - \{0\}$ is not closed.

The following concept is relevant to real valued functions on a topological space and will play an important role in this monograph: the **support** of $f : X \to R$, written $\mathrm{supp}\, f$, is the smallest closed subset of X outside of which f vanishes. A function f such that $\mathrm{supp}\, f$ is compact is called a function with **compact support**. The family of all real valued continuous functions on X with compact support will be denoted $\mathcal{K}(X)$.

A metric space X with a distance function d is made into a topological space by taking as a base for the neighborhoods system of $x \in X$ the open balls $\{y \in X : d(x, y) < r\}$ for all $r \in R_+$. Equivalently, r may be restricted to be positive rational, which shows that in a metric space the neighborhoods system of a point has a countable base. I.e., X is **first countable** (but not necessarily second countable). A **Cauchy sequence** in the metric space X is a sequence $x_n \in X$ $(n = 1, 2, \dots)$ such that $d(x_m, x_n) \to 0$ as $m, n \to \infty$. If for each Cauchy sequence $\{x_n\}$ there is $x \in X$ such that $x_n \to x$, then X is called **complete**. This can be applied in particular to a normed linear space. A **linear topological space** is a linear space in which addition and scalar multiplication are continuous in the pair. A **Banach space** is a complete normed linear topological space. A familiar example is R^n with he Euclidean norm. Other examples will appear in Chapter 6 as spaces of functions. If X and Y are normed linear topological spaces (not necessarily complete) and $f : X \to Y$ linear, then f is called **bounded** if there exists $0 < M < \infty$ such that $\|f(x)\| \leq M\|x\|$ for every $x \in X$. It is easy to prove that a linear function f is continuous if and only if it is bounded (Dunford and Schwartz, 1958, II.3.4). This applies in particular to linear functionals on X.

A real valued function f on a topological space X is called **lower semicontinuous** (l.s.c.) if for every $x \in X$, $\liminf_{y \to x} f(y) \geq f(x)$, and **upper semicontinuous** (u.s.c.) if $\limsup_{y \to x} f(y) \leq f(x)$. Thus, f is u.s.c. if and only if $-f$ is l.s.c. An equivalent definition of f being l.s.c. is that for every $c \in R$, $\{x : f(x) > c\}$ is open. The supremum of an arbitrary family of l.s.c. functions is l.s.c. For proofs of these and other properties, see Taylor (1965, 1985), Section 6-9.

Two topological spaces X and Y are said to be **homeomorphic** if there exists $f : X \to Y$ such that f is bijective and both f and f^{-1} are continuous. It amounts to the same to say that f is bijective, continuous, and open.

If $f : X \to$ Hausdorff Y is continuous and $A \subset X$ is compact, then $f(A)$ is compact (B, I §9.4). The converse is false in general: if $B \subset Y$ is compact, then $f^{-1}(B)$ may fail to be compact. Yet, it is a very desirable property for f to possess, for instance if one wants to induce a measure on Y from a measure on X. This has led Bourbaki to the notion of **proper mapping** (B, I §10).

2.2.1. DEFINITION. *Let X and Y be topological spaces and f a function $X \to Y$. Then f is said to be proper if f is continuous and for every topological space Z the function $f \times i_Z : X \times Z \to Y \times Z$ is closed.*

An obvious example of a proper function f is a homeomorphism, since then $f \times i_Z$ is a homeomorphism. Furthermore, by taking Z to be a space consisting of a single point one has immediately

2.2.2. COROLLARY. *Every proper mapping is closed.*

Definition 2.2.1 is not very intuitive and it is only through its consequences that it becomes meaningful. For instance, if X and Y are l.c., then the previously mentioned property of compactness of inverse images of compact sets emerges. In fact, this becomes an equivalent definition. This is stated in the next theorem; for the proof the reader is referred to B, I §10.3, Propos. 7.

2.2.3. THEOREM. *Let X and Y be l.c. spaces and $f : X \to Y$ continuous. Then f is proper if and only if $f^{-1}(K)$ is compact for every compact $K \subset Y$.*

One of the important consequences of a function $f : X \to Y$ to be proper is that the image of a measure μ on X under f is a measure on Y (the **induced** measure), denoted $f(\mu)$ or μf^{-1} (see Section 6.3). This is not true in general for an arbitrary continuous f (unless μ is finite). For instance, take $X = R^2$ with $\mu =$ 2-dimensional Lebesgue measure, $Y = R$, and $f = \mathrm{pr}_1$ (i.e., $f(x_1, x_2) = x_1$), and put $\nu = \mu f^{-1}$. Take $B = [a, b]$ with $a < b$, then $\nu(B) = \mu(B \times R) = \infty$ since $B \times R$ is a vertical strip with infinite area. Then ν cannot be a measure in the Bourbaki sense since it is supposed to be finite on compact sets (Section 6.3). Indeed, in this example f is not proper: B is compact, but $f^{-1}(B)$ is not. By changing the example and letting $f : R^2 \to R$ be defined by $f(x_1, x_2) = x_1^2 + x_2^2$ one obtains a proper f, and $f(\mu)$ equals π times Lebesgue measure on R_+.

2.3. Continuous and proper actions of topological groups. For later applications the most important result in this section is Theorem 2.3.13. In order to arrive there we shall develop along the way in small steps a host of other interesting and useful results.

A **topological group** is a group G that is at the same time a topological space in which group multiplication and inversion are continuous. More precisely, the function $G \times G \to G$ defined by $(g_1, g_2) \to g_1 g_2^{-1}$ is required to be continuous, from which it is easy to prove that $(g_1, g_2) \to g_1 g_2$ and $g \to g^{-1}$ are both continuous. A subgroup of a topological group will always tacitly be understood to be a subspace in the topological sense, i.e., receive the relative topology. The component (Section 2.2) of G containing the identity element e of G is called the **identity component** of G. It is a closed (and normal) subgroup of G (Cohn, 1957, Theorem 2.4.1). A neighborhood of e is also called a **nucleus**. The elements of a connected nucleus generate the identity component of G, i.e., the latter is the smallest connected subgroup containing the given nucleus (Cohn, 1957, Theorem 2.4.3). Equivalently, every element of the identity component is a finite product of elements of the nucleus.

A very simple example of a topological group is $G = R$ under addition, with the usual topology of R. Then $e = 0$ and an open interval about 0 can serve as nucleus. In this example, the identity component is the whole of G. Another example is $G = R - \{0\}$ under multiplication. Now $e = 1$ and the identity component is R_+; the other component of G is R_-.

Let a topological group G act on a topological space X to the left. The action is said to be continuous if the function $G \times X \to X$ defined by the action $(g, x) \to gx$ is continuous. In particular, for each fixed $g \in G$, the transformation $x \to gx$ is a homeomorphism $X \to X$. Continuity of the action will be understood throughout. The orbit space X/G is made into a topological space by providing it with the **quotient topology**, which specifies the open sets to be those subsets B of X/G for which $\pi^{-1}(B)$ is open in X, where π is the orbit projection defined in (2.1.1). With this topology on X/G, π is continuous and open. To show that π is open take open $A \subset X$, then $\pi(A) = \pi(GA)$ and GA is open (as a union of open sets gA) and invariant, so that by definition of the quotient topology $\pi(GA)$ is open. (The quotient topology coincides with the finest topology that makes π continuous.) If for $x \in X$ the singleton $\{x\}$ is closed (e.g., if X is Hausdorff), then the isotropy subgroup G_x is closed since it is the inverse image of $\{x\}$ under the continuous function $g \to gx$.

As a particular case, that will come up frequently, of a group acting on a space consider a topological group G and a subgroup H acting on G to the right, as in Section 2.1. The action of G on G/H was defined in (2.1.3).

2.3.1. PROPOSITION (B, III §2.5, Propos. 12). *Let H be a subgroup of a topological group G, then the action of G on G/H is continuous.*

PROOF. Recall the definitions of π and $[g]$ from (2.1.1) and (2.1.2). We have to show that the function $f : G\times G/H \to G/H$ given by $f(g_1, [g_2]) = g_1[g_2]$ is continuous. Observe that $g_1[g_2] = [g_1 g_2]$ by (2.1.3). Define $h : G \times G \to G$ by $h(g_1, g_2) = g_1 g_2$. The function $G \times G \to G/H$ given by $(g_1, g_2) \to [g_1 g_2]$ can be obtained as the composition of two functions in two different ways: (a) $(g_1, g_2) \to g_1 g_2 \to [g_1 g_2]$, and (b) $(g_1, g_2) \to (g_1, [g_2]) \to [g_1 g_2]$. This yields the equation $\pi \circ h = f \circ (i_G \times \pi)$. (The proof is followed easier if one draws a commuting diagram.) Take an arbitrary open subset U of G/H, then we have to show that $f^{-1}(U)$ is open in $G \times G/H$. Put $V = (\pi \circ h)^{-1}(U)$, then V is open in $G\times G$ since π and h are continuous. Now $i_G \times \pi$ is onto, which enables one to write $f^{-1}(U) = (i_G \times \pi)(V)$ (with the same reasoning that led to (2.1.9)). Here both i_G and π are open, so that $(i_G \times \pi)(V)$ is open. □

2.3.2. PROPOSITION (B, III §2.5, Propos. 13). *If H is a subgroup of a topological group G, then G/H is Hausdorff if and only if H is closed in G.*

PROOF. Let π be defined by (2.1.2), put $Z = G/H$, and $z_0 = [e] = \pi(H) \in Z$. Then $H = \pi^{-1}(z_0)$. If Z is Hausdorff, then $\{z_0\}$ is closed so that $\pi^{-1}(z_0)$ is closed since π is continuous. Conversely, assume that H is closed. If $H = G$, then G/H consists of the single point z_0 and there is nothing to prove. Therefore, assume that H is properly contained in G so that Z has more than one point. Consider the mapping $\pi \times \pi : G \times G \to Z \times Z$ and define $C = \{(g_1, g_2) \in G \times G : g_1 \text{ and } g_2 \text{ lie on the same orbit}\} = \{(g_1, g_2) \in G \times G : g_1^{-1} g_2 \in H\}$. This is the inverse image under the continuous function $(g_1, g_2) \to g_1^{-1} g_2$ of the closed set H, so that C is closed. Let z_1, z_2 be

any distinct points of Z and take any $g_1, g_2 \in G$ such that $\pi(g_i) = z_i$, $i = 1, 2$. Since $z_1 \neq z_2$, $(g_1, g_2) \notin C$. Since C is closed, there is a product neighborhood $A_1 \times A_2$ of (g_1, g_2), with A_i a neighborhood of g_i, such that $A_1 \times A_2$ is disjoint from C. This implies that the two saturations GA_i are disjoint, and therefore $\pi(A_1)$ and $\pi(A_2)$ are disjoint subsets of Z. Furthermore, $\pi(A_i)$ is a neighborhood of z_i since $z_i \in \pi(A_i^\circ) \subset \pi(A_i)$, and $\pi(A_i^\circ)$ is open in Z because π is open.
$\square$

For example, if $G = R$ under addition and H is the set of integers (so that H is closed in G), then G/H is the unit circle with its usual topology, which is Hausdorff. In contrast, if H consists of all rationals, then H is dense in $G$and not closed. Let B be an arbitrary nonempty open subset of G/H, then $\pi^{-1}(B)$ is both open in G and saturated under the right action of H on G. Since H is dense, $\pi^{-1}(B)$ must be all of G so that $B = G/H$. In other words, G/H receives the trivial topology and is therefore not Hausdorff.

2.3.3. PROPOSITION (B, I §10.4, Propos. 10). *Let the topological group G act continuously on the l.c. space X in such a way that X/G is Hausdorff. Then* (i) *X/G is l.c., and* (ii) *if K' is any compact subset of X/G, then there exists a compact set $K \subset X$ such that $\pi(K) = K'$.*

PROOF. (i) For $z \in X/G$ choose any $x \in X$ such that $\pi(x) = z$. Take a compact neighborhood V of x, then $\pi(V)$ is a compact neighborhood of z as in the proof of Proposition 2.3.2. (ii) For each $x \in \pi^{-1}(K')$ choose a compact neighborhood $V(x)$. Then $\{\pi(V(x)^\circ) : x \in \pi^{-1}(K')\}$ is an open cover of K'. Since K' is compact, there is a finite subcover $\{\pi(V(x)^\circ) : x = x_1, \dots, x_n\}$ with the $x_i \in \pi^{-1}(K')$. Put $K_1 = V(x_1) \cup \cdots \cup V(x_n)$, then K_1 is compact and $\pi(K_1) \supset K'$. Since π is continuous, $\pi^{-1}(K')$ is closed so that $K_1 \cap \pi^{-1}(K') = K$ is compact. Then observe that $\pi(K) = K'$. $\square$

2.3.4. COROLLARY. *Let H be a closed subgroup of a l.c. group G. Then G/H is l.c. and for any compact $K' \subset G/H$ there exists a compact $K \subset G$ such that $\pi(K) = K'$.*

PROOF. In Proposition 2.3.3 take $X = G$, $G = H$ and observe that G/H is Hausdorff by Proposition 2.3.2. $\square$

2.3.5. PROPOSITION (B, III §4.1, Cor. 2 to Propos. 1). *Let H be a compact subgroup of a l.c. group G. Then π of (2.1.2) is proper.*

PROOF. By Corollary 2.3.4 G/H is l.c. Take $K' \subset G/H$ arbitrary. We have to show that $\pi^{-1}(K')$ is compact, using Theorem 2.2.3. By Corollary 2.3.4 there is a compact set $K \subset G$ such that $\pi(K) = K'$. Then $\pi^{-1}(K') = \{gH : g \in K\} = KH$. Let $f : G \times G \to G$ be the function $f(g_1, g_2) = g_1 g_2$, then $KH = f(K \times H)$. Since $K \times H$ is compact and f is continuous, KH is compact. $\square$

The example in Chapter 1 of the irrational flow on the torus shows that the action of G on X can be so bad that X/G fails to be Hausdorff even if G and X are l.c. Therefore, some regularity of the action has to be imposed. One way of doing this is to introduce the notion of a **proper action** (B, III §4.1, Definition 1). First introduce the function $\theta : G \times X \to X \times X$ defined by

$$\theta(g, x) = (gx, x). \tag{2.3.1}$$

(Note that in Bourbaki the definition is $\theta(g, x) = (x, gx)$.)

2.3.6. DEFINITION. *The action of the topological group G on the topological space X is said to be* **proper** *if the function θ defined in (2.3.1) is proper.*

If an action is proper, then it is certainly continuous since the function $(g, x) \to gx$ is the composition of θ and pr_1. An easy example of proper action is the left (and right) action of G on itself since the function $(g_1, g_2) \to (g_1 g_2, g_2)$ is continuous and has a continuous inverse so that it is a homeomorphism of $G \times G$ with itself. For l.c. groups and spaces there is a very useful criterion, stated below as Proposition 2.3.8, to decide whether an action is proper. The following notation, taken from Palais (1961) will be introduced: for any two subsets A, B of X define

$$((A, B)) = \{g \in G : gA \cap B \neq \emptyset\} \tag{2.3.2}$$

(this is denoted $P(A,B)$ in B). It follows easily from (2.3.2) that $((B,A)) = ((A,B))^{-1}$. Since $g \to g^{-1}$ is a homeomorphism of G with itself it follows that $((A,B))$ and $((B,A))$ have the same topological properties, such as being closed, compact, etc. The set on the right-hand side of (2.3.2) is the projection pr_1 of the set $\{(g,x) \in G \times X : x \in A, gx \in B\} = \{(g,x) \in G \times X : \theta(g,x) \in B \times A\}$, using θ of (2.3.1). Therefore,

$$((A,B)) = \mathrm{pr}_1(\theta^{-1}(B \times A)). \tag{2.3.3}$$

2.3.7. PROPOSITION (B, III §4.5, Theorem 1(a)). *Let the l.c. group G act continuously on the l.c. space X. If A and B are two subsets of X of which one is closed, the other compact, then $((A,B))$ is closed.*

PROOF. Suppose A is compact and B is closed. We show first that $\mathrm{pr}_1 : G \times A \to G$ is proper. Let $K \subset G$ be compact, then $\mathrm{pr}_1^{-1}(K) = K \times A$ which is compact. Therefore, pr_1 is proper by Theorem 2.2.3. On the right-hand side of (2.3.3) the set $\theta^{-1}(B \times A)$ is closed since θ is continuous. Then use Corollary 2.2.2. $\square$

2.3.8. PROPOSITION (B, III §4.5, Theorem 1(b)(c)). *If a l.c. group G acts continuously on a l.c. space X, then the action is proper if and only if for every pair of compact subsets A, B of X, $((A,B))$ has compact closure.*

PROOF. If the action is proper, then by Definition 2.3.6 and Theorem 2.2.3 the set $\theta^{-1}(B \times A)$ on the right-hand side of (2.3.3) is compact. Hence the left-hand side is compact since pr_1 is continuous. Conversely, suppose $((A,B))$ has compact closure so that there is a compact $K \subset G$ such that $((A,B)) \subset K$. Put $C = \theta^{-1}(B \times A)$, then by (2.3.1), $C \subset G \times A$. On the other hand, by (2.3.3), $\mathrm{pr}_1(C) \subset K$ so that $C \subset K \times X$. These two inclusions of C together show $C \subset K \times A$. Since $K \times A$ is compact and C is closed, it follows that C is compact. Then use Theorem 2.2.3. $\square$

2.3.9. REMARK. If $((A,B))$ has compact closure, then it is in fact compact since $((A,B))$ is closed in any case (whether the action of

G is proper or not) by Proposition 2.3.7. Therefore, Proposition 2.3.8 could have been stated simpler, with "has compact closure" replaced by "is compact." However, in applications where one wants to establish the properness of the action it is sometimes a convenience to have to show only that $((A, B))$ is contained in a compact set. □

2.3.10. COROLLARY. *If G is a compact group acting continuously on a l.c. space X, then the action is proper.*

PROOF. This is trivial by Proposition 2.3.8 since $((A, B)) \subset G$ which is compact. □

The conclusion of Corollary 2.3.10 remains true if X is Hausdorff but not necessarily l.c. See B, III §4.1, Propos. 2.

2.3.11. PROPOSITION. *If H is a compact subgroup of a l.c. group G, then the action of G on G/H is proper.*

PROOF. By Corollary 2.3.4 G/H is l.c., and by Proposition 2.3.1 the action of G on G/H is continuous. Take A, B compact $\subset G/H$ and put $A_1 = \pi^{-1}(A)$ $B_1 = \pi^{-1}(B)$ with π of (2.1.2). Then A_1 and B_1 are compact subsets of G by Proposition 2.3.5 and Theorem 2.2.3. From the definition (2.1.3) of the action of G on G/H and (2.3.2) it follows that $((A, B)) = ((A_1, B_1))$. The latter is compact by Proposition 2.3.8 since the action of G on itself is proper. Thus, $((A, B))$ is compact. Apply Proposition 2.3.8 again, but in the other direction, and the result follows. □

2.3.12. PROPOSITION (B, III §4.2, PROPOS. 3). *If a topological group G acts properly on a topological space X, then X/G is Hausdorff.*

PROOF. Let C be the pairs of points in X that are on the same orbit, i.e., C is the subset of $X \times X$ defined by $C = \{(gx, x) : g \in G,\ x \in X\} = \theta(G \times X)$ with θ of (2.3.1). Since θ is proper by hypothesis, it is closed (Corollary 2.2.2). Hence C is closed. The remainder of the proof is identical to that of Proposition 2.3.2 with G and H there being X and G here. □

2.3.13. THEOREM (B, III §4.2, Propos. 4). *Let G be a l.c. group acting properly on a l.c. space X and let x be an arbitrary point of X. Then*

(a) X/G is l.c.;
(b) the mapping $\alpha_x : G \to X$ defined by (2.1.6) is proper;
(c) G_x is compact;
(d) Gx is closed in X;
(e) the bijection $\psi_x : G/G_x \to Gx$ defined by (2.1.5) is a homeomorphism.

PROOF. Part (a) follows from Propositions 2.3.12 and 2.3.3(i). For part (b) use the function θ defined by (2.3.1), then θ is proper by hypothesis. Using Theorem 2.2.3 we have to prove that $\alpha_x^{-1}(K)$ is compact for every compact $K \subset X$. Now $K \times \{x\}$ is a compact subset of $X \times X$ and therefore $\theta^{-1}(K \times \{x\}) = C$ is compact by Theorem 2.2.3. From $\alpha_x^{-1}(K) = \{g \in G : gx \in K\}$ and $C = \{(g, x) \in G \times X : gx \in K\}$ it follows that $\alpha_x^{-1}(K) = \mathrm{pr}_1(C)$. Since C is compact and pr_1 continuous, $\alpha_x^{-1}(K)$ is compact. For part (c) write $G_x = \alpha_x^{-1}(\{x\})$ and observe that $\{x\}$ is compact. Then use part (b) and Theorem 2.2.3. For part (d) write $Gx = \alpha_x(G)$ and use part (b) and Corollary 2.2.2. For part (e) we use equations (2.1.9) and (2.1.10). In (2.1.9) take A open in Gx, then $\alpha_x^{-1}(A)$ is open since α_x is continuous and then use the openness of π_x to conclude that $\psi_x^{-1}(A)$ is open. Therefore, ψ_x is continuous. In (2.1.10) take B closed in G/G_x, then $\pi_x^{-1}(B)$ is closed in G since π_x is continuous. Then observe that α_x is closed by part (b) and Corollary 2.2.2, from which it follows that $\psi_x(B)$ is closed. Hence ψ_x^{-1} is continuous. □

2.3.14. REMARK. Parts (b)–(e) of Theorem 2.3.13 are valid without assuming G and X to be l.c. See B, III §4.2, Propos. 4. □

2.3.15. COROLLARY. *Let G be a l.c. group acting properly and transitively on a l.c. space X and let x be an arbitrary point of X. Then the bijection $\psi_x : G/G_x \to X$ is a homeomorphism.*

PROOF. Apply Theorem 2.3.13(e), where now $Gx = X$. □

It follows at once from Theorem 2.3.13(c) that the action of $GL(n)$ on R^n is not proper. For, take $x = (1, 0, \ldots, 0)'$, then G_x consists of all $n \times n$ nonsingular matrices whose first column is x, and that group is not compact. The same holds for the action of $LT(n)$ or $UT(n)$ on R^n. On the other hand, the action of $O(n)$ is proper, by Corollary 2.3.10, since $O(n)$ is compact. An example of Corollary 2.3.15 is the action of $GL(n)$ on $PD(n)$, as in Example 2.1.7. Take x in Corollary 2.3.15 to be the identity matrix I_n, then $G_x = O(n)$. Since the action is transitive, the conclusion of Corollary 2.3.15 is that $GL(n)/O(n)$ is homeomorphic to $PD(n)$.

In some instances we may be interested only in whether ψ_x in the above corollary is a homeomorphism, but not whether G acts properly. Such a homeomorphism is guaranteed by Lemme 2, Appendice I, of Bourbaki (1963), which is reproduced as Lemma 2.3.17 below. Its conditions are that G be second countable (Section 2.2) and that X be a **Baire space**. For the definition of the latter one first defines a subset A of a topological space to be **nowhere dense** if $\bar{A}$ has empty interior. Then X is a Baire space if the complement of any countable union of nowhere dense sets is dense in X. In particular, a Baire space cannot be a countable union of nowhere dense sets. Therefore, if a countable union of closed sets contains a Baire space, then one of the sets must have a nonempty interior. For us the most important example of a Baire space will be a l.c. space.

2.3.16. THEOREM. *If a space is l.c., then it is a Baire space.*

PROOF. Kelley (1955), Chapter 6, no. 34; or B, IX §5.3, Theorem 1. □

2.3.17. LEMMA (Bourbaki). *Let a second countable l.c. group G act continuously and transitively on the left of a Baire space X. Then for any $x \in X$, the 1-1 correspondence $\psi_x : G/G_x \to X$ is a homeomorphism. In particular, the conclusion holds if G is a subspace of R^n and X is l.c.*

PROOF. The function α_x of (2.1.6) is continuous (since G acts continuously) and maps G onto X. We can write $\alpha_x = \psi_x \circ \pi$ with

π of (2.1.1). If U is an open subset of X, so that $\alpha_x^{-1}(U)$ is an open subset of G, then $\psi_x^{-1}(U) = \pi(a_x^{-1}(U))$ is an open subset of G/G_x since π is open. Hence, ψ_x is continuous. It remains to be shown that ψ_x^{-1} is continuous, or, equivalently, that ψ_x is open. This is equivalent to α_x being open, for, if U is an open subset of G/G_x, then $\psi_x(U) = \alpha_x(\pi^{-1}(U))$. In order to show α_x open it suffices to show that for every $g \in G$ and neighborhood V of g, $\alpha_x(V)$ is a neighborhood of $\alpha_x(g)$ $(= gx)$ in X (since for open $U \subset G$ each point $\alpha_x(g)$, $g \in U$, of $\alpha_x(U)$ has $\alpha_x(U)$ as a neighborhood). It is sufficient to do this at $g = e$ since for fixed $g \in G$ the left translations $G \to G$ and $X \to X$ with g are both homeomorphisms. Thus, it remains to be shown that if V is a neighborhood of e in G, then $\alpha_x(V) = V_x$, say, is a neighborhood of x in X. Since G is l.c. there is a compact neighborhood K_1 of e such that $K_1 \subset V$. By continuity of the function $(g_1, g_2) \to g_1^{-1}g_2$ of $G \times G \to G$ and local compactness of $G \times G$ there is in $G \times G$ a compact neighborhood $K_2 \times K_3$ of (e, e) such that $K_2^{-1}K_3 \subset K_1$. Then $K = K_2 \cap K_3$ is a compact neighborhood of e in G such that $K^{-1}K \subset K_1 \subset V$. By assumption the topology of G has a countable base $\mathcal{B}$. Let $\mathcal{B}_1$ be the subfamily of $\mathcal{B}$ consisting of the members of $\mathcal{B}$ that have compact support, and let $F_1, F_2, \ldots$ be their supports. Then $\bigcup_1^\infty F_n = G$ because for each $g \in G$ there is a compact neighborhood K_g and a member of $\mathcal{B}$, say U_g, such that $g \in U_g \subset K_g$, so that $U_g \subset \mathcal{B}_1$. By compactness each F_n can be covered by a finite family of sets of the form gK, $g \in F_n$. Thus, G has a countable cover by compact sets $g_1K, g_2K, \ldots$, $g_i \in G$. By continuity of α_x, $\alpha_x(g_iK) = g_iKx$ is compact in X, and since G is transitive, $g_1Kx, g_2Kx, \ldots$ is a countable cover of X by compact, therefore closed, sets. Since X is Baire, one of the g_iKx must have nonempty interior; say g_nkx is an interior point of g_nKx, $k \in K$. Then x is an interior point of $k^{-1}Kx \subset K^{-1}Kx \subset Vx$ so that Vx is a neighborhood of x. $\square$

For later reference we list the following proposition, which is half of Proposition 14 in B, III §2.5.

2.3.18. PROPOSITION. *Let H be an open subgroup of a topological* *group G. Then G/H is discrete.*

PROOF. An arbitrary point of G/H is of the form $\pi(g_1)$ with $g_1 \in G$ and π of (2.1.2). For fixed g_1, the left translation $g \to g_1 g$ is a homeomorphism $G \to G$, so that $g_1 H$ is open. Since π is open, $\pi(g_1) = \pi(g_1 H)$ is open. □

An open subgroup H is also closed because it is the complement of the union of all (open) cosets gH other than H itself. An open subgroup is therefore a union of components. An example of such an H is the component of the identity. An example of Proposition 2.3.18 is $G = R - \{0\}$ under multiplication and $H = R_+ =$ identity component. Then G/H consists of the two elements R_+ and R_-.