ON BAYES TESTS FOR $p \leq 1/2$ VERSUS $p > 1/2$:

ANALYTIC APPROXIMATIONS*

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1. Preface.

Few if any obtain a measure of success in their chosen profession without the beneficient influence of others. The first author is no exception. His interest in optimal stopping problems was ignited as a student in 1965 when Professor Robbins was visiting the University of Minnesota. Perhaps a fitting way of expressing appreciation would be to use this occasion to introduce the second author, Xizhi Wu, a member of a new generation of students whose interest in optimal stopping has been kindled by those who have "made straight the way" - and in particular by the one we seek here to honor.

The present subject is not new. Much of the topic has already been resolved by Wetherill (1961), Moriguti and Robbins (1962), and Lindley and Barnett (1965). What is new is an attempt to approximate the optimal stopping rule analytically. There are several reasons why one should want such an approximation: 1. The exact rule cannot be obtained explicitly. In the present context, the required backward induction is not difficult, and it can

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easily be programmed on an office computer. But there are several continuous parameters associated with the problem, and one is faced with an infinity of different backward induction problems if all values of these parameters are to be considered. 2. A good analytic approximation can indicate how the various parameters influence the optimal stopping rule. 3. A good analytic approximation can be used to describe asymptotic behavior. 4. Most importantly, a good analytic approximation suggests a stopping rule (in an obvious way) which is probably close to optimal - a good substitute for the harder-to-describe optimal stopping rule.

A standard method of approximation is to replace a discrete time optimal stopping problem by a continuous time free boundary problem. A clear introduction to the method has been given by Chernoff (1972, p.88-100). A frequent advantage is that the number of free parameters is reduced. As Chernoff (1965) has shown, a suitable correction is needed when one returns to the discrete time setting. An interesting nontrivial example of this approach has been described by Petkau (1978). Unfortunately, the free boundary problems one encounters are usually insoluble - and one must be content with asymptotic approximations and numerical solutions.

Two alternative techniques have been described by Bather (1983) which do yield analytic approximations in the free-boundary context. One technique approximates the stopping boundary within the continuation region with an "inner approximation". Another produces an "outer approximation" within the stopping region. A pair of approximations completely bounds the optimal stopping boundary. Some of his <u>outer</u> approximations suggest simple stopping rules that may be nearly optimal; their qualities have not been adequately assessed. His inner approximations seem less promising.

One of the authors (Simons (1986)) has recently adapted Bather's techniques to a discrete-time context for a clinical-trials model. A very precise <u>inner</u> approximation was obtained. The stopping rule suggested by this "approximation" is sometimes optimal. And it is always very nearly optimal. No approximation based on a continuous time free boundary problem could be expected to do so well (except possibly for certain values of the free parameters that correspond to large sample sizes). For such approximations depend on the operation of the central limit theorem. And one frequently cannot expect it to be efficacious. In short, one should hope to find better approximations by attacking discrete time problems directly.

While there now exist methods which can produce a good or excellent analytic approximation for a specific discrete time optimal stopping problem, the present technology certainly does not guarantee that one will be found. The study described below is an attempt to apply the existing methods in a nontrivial but relatively easy hypothesis testing context. Some of this work represents a portion of the second author's dissertation, which is not yet complete. The existing results are definitely encouraging; we believe they are interesting. It is our hope that the methodology under development will eventually become widely applicable.

The main objective here will be to exposit the new methodology - particularly as it applies to the specific problem of testing "$p \leq 1/2$" versus "$p > 1/2$".

2. Introduction.

Consider the following hypothesis testing problem: Let $X_1, X_2, \ldots$ be independent Bernoulli random variables with a common mean p, $0 \leq p \leq 1$, and let

$$H_0\colon\ p \leq 1/2, \qquad H_1\colon\ p > 1/2$$

be the null and alternative hypotheses. A unit cost is assigned for each observed random variable. And an additional cost $2A|p-1/2|$ is assessed if the wrong hypothesis is chosen.

Of interest here is the Bayes stopping rule when the parameter p has a beta prior distribution: $\text{Beta}(x_0, y_0)$. After m random variables have been observed, the posterior distribution becomes $\text{Beta}(x_m, y_m)$, where $x_m = x_0 + X_1 + \ldots + X_m$ and $y_m = y_0 + (1-X_1) + \ldots + (1-X_m)$. And the relevant posterior Bayes loss takes the form

$$m+2A.\min(E_m(p-1/2)^+,\ E_m(p-1/2)^-) = m + AE_m|p-1/2| - A|E_m(p-1/2)|,$$

where "E_m" refers to expectation under $B(x_m,y_m)$, and the superscripts "+" and "−" refer to positive and negative parts. The central term $AE_m|p-1/2|$ on the right is a (uniformly integrable) martingale in m, and has no influence on the form of the Bayes stopping rule (only on the values of Bayes risks). Thus one may restrict one's attention to the "Bayes reward" $A|E_m(p-1/2)| - m$ (with a change of sign), which can be written as

$$A\frac{x_m \vee y_m}{x_m + y_m} - (m + \frac{1}{2}A),$$

where $x_m \vee x_m = \max(x_m,y_m)$. Since $m + \frac{1}{2}A = (x_m + y_m) - (x_0 + y_0 - \frac{1}{2}A)$, and the expression within the latter parentheses can be ignored, the problem at hand can be viewed as a Markovian optimal stopping problem with states (x,y), x > 0, y > 0, and with a reward for state (x,y) given by

$$R(x,y) = A\,\frac{x \vee y}{x + y} - x - y. \tag{1}$$

A new observation sends the state (x,y) into state (x+1,y) with (posterior) probability x/(x+y) and into state (x,y+1) with probability y/(x+y). So the dynamic equation assumes the form

$$S(x,y) = \max(R(x,y),\ \frac{x}{x+y}\,S(x+1,y) + \frac{y}{x+y}\,S(x,y+1)). \tag{2}$$

where S(x,y) denotes the optimal stopping reward that can be obtained starting in state (x,y).

Throughout the paper, we shall use the notation n=x+y and k=x−y. Observe that $x = \frac{1}{2}(n+k)$ and $y\ \frac{1}{2}(n-k)$.

A state (x,y) will be called an <u>optimal stopping point</u> if

$S(x,y)=R(x,y)$, and an <u>optimal continuation point</u> if

$$(3) \qquad S(x,y) = \frac{x}{n} S(x+1,y) + \frac{y}{n} S(x,y+1).$$

Some states may have both appellations.

A useful correlate of S is the function $Q=S-R$, which satisfies the recursive equation

$$(4) \qquad Q(x,y) = \left(A \frac{x \wedge y}{n(n+1)} (1 - |k|)^+ - 1 + \frac{x}{n} Q(x+1,y) + \frac{y}{n} Q(x,y+1)\right)^+.$$

where $x \wedge y = \min(x,y)$. Clearly: Q is nonnegative; (x,y) is an optimal stopping point iff $Q(x,y)=0$; and (x,y) is an optimal continuation point iff the expression within the outer parentheses of (4) is nonnegative. Using the fact that

$$(5) \qquad A \frac{x \wedge y}{n(n+1)} (1 - |k|)^+ - 1 \le \frac{A}{2(n+1)} - 1,$$

One can see from (4) that (x,y) is an optimal stopping point whenever $2(n+1) \geq A$. Moreover, since (5) becomes an equality when $x=y$, it follows from (4) that the diagonal point (x,x) is an optimal continuation point whenever $2(n+1) \leq A$. (The latter two results were demonstrated by Moriguti and Robbins (1962) in other ways.) Finally, since, for n fixed, the left side of (5) is continuous and symmetric in k and nonincreasing in $|k|$, it follows again from (4) that the function Q is continuous and symmetric in k and nonincreasing in $|k|$ for each fixed n. (Use backward induction based on n, and include in the induction hypothesis the fact that Q is convex in $|k|$. Note that Q is zero when $2(n+1) \geq A$.) Thus there exists a nonnegative <u>boundary function</u> b defined on $(0,\frac{1}{2}A-1]$ with the property: (x,y) is an optimal continuation point iff $2(n+1) \leq A$ and $|k| \leq b(n)$. The optimal stopping rule for every state is completely specified through the function b.

It will be assumed hereafter that $A > 2$; when $A \leq 2$, every point is an optimal stopping point, and the "function" b has a void domain.

Figure 1 of Moriguti and Robbins (1962), as drawn, is misleading in a couple of respects: 1. The function b does not meet the axis tangentially as n approaches the right endpoint $\frac{1}{2}A-1$. It has a negative slope $-2(A-2)/A^2$. 2. The function b is concave - at least for $n > 0$ in the latter interval $[\frac{1}{2}A-2, \frac{1}{2}A-1]$, where it is determined by the hyperbola $\{(n,b): Ab^2-A(n+1)b+n(A-2(n+1))=0\}$ with b in the range $0 \leq b < 1$. (For n in this interval, it is optimal to take no more than one observation.) If b is concave everywhere, then, of course, the set of optimal continuation points is a convex region.

We suspect that the function b is differentiable, even at points that are exactly an integer below the right endpoint $\frac{1}{2}A-1$. The behavior of b as n approaches zero is known: Figure 1 of Moriguti and Robbins (1962) correctly shows that b(n) goes to zero as n goes to zero. And Corollary 2 of Section 5 permits one to conclude that

$$b(n) \leq A^{-1}(A-2)n - 4A^{-2}(A-1)n^2 + o(n^2) \text{ as } n \to 0.$$

This inequality is, in fact, an equality: it can be shown for sufficiently small n that it is optimal to take no more than one observation. (This follows from the simple inequality $b(n) \leq n(A-2(n+1))/A$, based on Corollary 1 of Section 5.) Consequently, the function b is again determined by the hyperbola previously described.

3. Approximation theories.

Two related theories will be described - one for inner approximations, and one for outer approximations. Both depend on the fact that the function S satisfies equation (3) at optimal continuation points (x,y). And both depend explicitly on an arbitrary but specific solution Z of equation (3). Equation (3) is a discrete analog of the heat equation encountered by Chernoff and others when working with continuous time. More will be said in Sections 4 and 5 about how Z should be chosen. For now, it should be viewed as something given.

A. Inner approximations. Let Z be a specific solution of (3). There are many solutions. A point (x,y) will be called "good" if $Z(x,y) \geq R(x,y)$, and called "warm" if $Z(x,y) \leq S(x,y)$. If a point (x,y) is good and if its immediate successors (x+1,y) and (x,y+1) are warm, then it is an optimal continuation point. For then,

$$(6) \qquad \frac{x}{n} S(x+1,y) + \frac{y}{n} S(x,y+1) \geq \frac{x}{n} Z(x+1,y) + \frac{y}{n} Z(x,y+1) = Z(x,y) \geq R(x,y).$$

So (3) holds and (x,y) is an optimal continuation point.

Since there is an explicit formula for R (given by (1)), it is easy to check whether a point (x,y) is good. It can be more difficult to check that a point is warm. Clearly (x,y) is warm if $Z(x,y) \leq R(x,y)$ $(\leq S(x,y))$. But there usually are other warm points: If the immediate successors of (x,y) are warm, then (x,y) is warm. For then

$$(7) \quad S(x,y) \geq \frac{x}{n} S(x+1,y) + \frac{y}{n} S(x,y+1) \geq \frac{x}{n} Z(x+1,y) + \frac{y}{n} Z(x,y+1) = Z(x,y)$$

So (x,y) is warm.

A point (x,y) can be identified as warm, when $Z(x,y) > R(x,y)$, if one can identify an associated finite set of points C which contains the immediate successors of (x,y), and has the additional property that it contains the immediate successors of each point (x',y') in C for which $Z(x',y') > R(x',y')$. Then (x,y) is "trapped" by other warm points, and it can be shown to be warm by backward induction.

B. Outer approximations. There is a parallel theory for finding optimal stopping points. One begins by reversing the directions of the inequalities used to define "good" and "warm": A point (x,y) is "good" if $Z(x,y) \leq R(x,y)$, and is "warm" if $Z(x,y) \geq S(x,y)$.

This time, if (x,y) is good and its immediate successors are warm, then (x,y) is an optimal stopping point. This is proved by reversing the directions of the inequalities in (6).

Again it is easy to check whether a point (x,y) is good. It is not so easy to check that a point is warm: When one knows that a point (x,y) is an optimal stopping point then one knows that $S(x,y) = R(x,y)$, and one can simply check that $Z(x,y) \geq R(x,y)$. Such an opportunity arises for the current optimal stopping problem when $x+y \geq \frac{1}{2}A - 1$. (See Section 2.)

Here, "warmness" is not automatically inherited from immediate successors. But *if the immediate successors of* (x,y) *are warm, and if* $Z(x,y) \geq R(x,y)$, *then* (x,y) *is warm*. For then

$$\frac{x}{n}\, S(x+1,y) + \frac{y}{n}\, S(x,y+1) \leq \frac{x}{n}\, Z(x+1,y) + \frac{y}{n}\, Z(x,y+1) = Z(x,y),$$

so that

$$Z(x,y) \geq \max(R(x,y), \frac{x}{n}\, S(x+1,y) + \frac{y}{n}\, S(x,y+1)) = S(x,y).$$

Thus (x,y) is warm.

In practice, one shows that a point (x,y) is warm by showing that $Z \geq R$ for a suitable set of successors of (x,y). (A *successor* is a point (x',y') for which x'-x and y'-y are nonnegative integers and $x'+y' > x+y$.) It is always enough to show this for all of its successors (since the current optimal stopping problem has a finite optimal stopping rule starting from each point (x,y)). But one does not need to consider all successors. For instance, if $Z \geq R$ at every successor (x',y') for which $x'+y' < \frac{1}{2}A$, then (x,y) must be warm. For then, the points (x',y') with $x'+y' \geq \frac{1}{2}A - 1$ (being optimal stopping points) must be warm. And, by backward induction, every successor with smaller sum x'+y' must be warm. So (x,y) must be warm.

More generally, any set C of successors of (x,y) is suitable providing the same kind of reasoning is applicable: The set C must contain the immediate successors of (x,y). And it must contain the immediate successors of each point (x',y') in C which is not (known to be) an optimal stopping point. *If* $Z \geq R$ *on* such a set, then (x,y) must be warm.

C. A summary.

I. A point (x,y) is an optimal continuation point if

(i) $Z(x,y) > R(x,y)$, and

(ii) $Z < S$ at $(x+1,y)$ and $(x,y+1)$.

In order to check that $Z < S$ at some point (x',y'), it is enough to show that $Z < R$ at (x',y'), or to show that $Z < S$ at each of the immediate successors of (x',y'). In practice, the latter is accomplished by (backward) induction.

II. A point (x,y) is an optimal stopping point if

(iii) $Z(x,y) < R(x,y)$, and

(iv) $Z > S$ at $(x+1,y)$ and $(x,y+1)$.

Typically, (iv) is verified by showing that $Z > R$ at every successor (x',y') of (x,y). It is enough to show this for every successor for which $x'+y' < \frac{1}{2}A$, or for every successor in a suitable set C (as described in the previous subsection).

4. Inner approximations.

It is perhaps helpful to begin with some intuition: The desired role of the function Z is to approximate S and R simultaneously in the vicinity of a point (x,y) that is just within the optimal continuation region, where one expects S and R to be nearly equal. The fact that Z is required to satisfy equation (3) aids in the approximating of S. Likewise, one naturally wants Z, in some sense, to be compatible with R. The solution Z used in Simons (1986) is compatible with R (i) by exhibiting a type of symmetry possessed by R, and (ii) by having the same growth rate in the state parameter that represents "time to go" (the horizon).

Unfortunately, there doesn't seem to be any way to give a precise meaning to the phrase "compatible with R." The present problem also exhibits a symmetry: $R(y,x) = R(x,y)$. And all of the solutions Z considered in this section possess the same type of symmetry. But in the next section, "simplicity of solution" overrules, and nonsymmetric solutions are used (successfully) to generate outer approximations. It may be that nonsymmetric solutions should be considered here.

A useful technique is to consider a linear family of solutions Z of (3), and to establish the conditions required for an inner approximation, where possible, by adjusting the linear coefficients. One simply hopes that this can be accomplished for most optimal continuation points. As might be expected, certain linear families work better than others.

While the simplicity of a linear family is quite important when it comes to establishing an analytic description of the optimal continuation points, more complicated families can be studied - for their potential - by using a small computer. The emphasis in this section is on feasibility; the results described are computer generated.

Equation (3) has many solutions. One easily accessible family takes the form

$$Z(x,y) = \frac{B(x+\mu,y+\nu)}{B(x,y)}, \quad \mu > 0,\ \nu > 0, \tag{8}$$

where B is the beta function defined by the integral

$$B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}du, \quad x > 0,\ y > 0.$$

Z is symmetric in x and y when $\nu=\mu$. For $\mu=\nu=0,1,2,\ldots$, one obtains:

$$Z_0(x,y) = 1,\ Z_1 = \frac{x\ y}{n(n+1)},\ Z_2 = \frac{x(x+1)y(y+1)}{n(n+1)(n+2)(n+3)},\ldots,$$

The solution $Z_\infty(x,y)=2^{-n}/B(x,y)$ is obtained as a limit of (8), suitably scaled, as $\mu=\nu \to \infty$. Another solution is obtained by evaluating (twice) the expectation of $|p-1/2|$ when p is distributed Beta(x,y). It takes the form $Z_*(x,y) = W(x,y)/n$, where W is defined recursively for positive integers x and y by

$$\left.\begin{aligned} w(x,0) &= W(0,x) = x \\ W(x,y) &= \frac{1}{2}W(x-1,y) + \frac{1}{2}W(x,y-1) \end{aligned}\right\} \quad ,x,y=1,2,\ldots .$$

The form for nonintegers is unknown. The linear families that have been considered are linear combinations of Z_0 (a constant) and one other of the subscripted Z's.

Computer studies have shown that the linear family of the form $\alpha + \beta Z_1$ (where the coefficients α and β are arbitrary) works well <u>only</u> when A is small; it is incapable of finding optimal continuation points (x,y) for which $|x-y| > 1$. Performance improves for families of the form $\alpha + \beta_i$ as i increases. The best performance is provided by $\alpha + \beta Z_\infty$. It (apparently) is capable of finding optimal continuation points for arbitrarily large values of $|x-y|$ (for arbitrarily large values of A). The family of the form $\alpha+\beta Z_*$ achieves only slightly better results than does the family $\alpha+\beta Z_1$.

Two types of comparisons have been tabulated <u>for integer-valued pairs</u> x and y: for A=200, 500 and 1000, and for the linear families of the forms $\alpha + \beta Z_1$, $\alpha + \beta Z_*$ and $\alpha + \beta Z_\infty$, the <u>increase</u> in Bayes risk is shown in Table 1. The comparison is between the minimal Bayes risk at (x,y) and the Bayes risk at (x,y) that results from continuing according to the optimal continuation points that can be discovered using the linear family in question (stopping at any point <u>not discovered</u> to be an optimal continuation point). Table 2 makes the same types of comparisons, except the <u>ratio</u> of the larger Bayes risk to the smaller is shown instead of the difference. The superiority of the family $\alpha + \beta Z_\infty$ is evident. Unfortunately, it does not appear to be a family with which one can easily work to produce an analytic approximation.

Table 1

Number of points with a Bayes risk difference in [a,b)

A	Family	[.0001,.001)	[.001,.01)	[.01,∞)
200	$\alpha+\beta Z_1$	7	15	0
	$\alpha+\beta Z_*$	7	15	0
	$\alpha+\beta Z_\infty$	2	5	0
500	$\alpha+\beta Z_1$	100	8	0
	$\alpha+\beta Z_*$	98	6	0
	$\alpha+\beta Z_\infty$	47	6	0
1000	$\alpha+\beta Z_1$	137	102	0
	$\alpha+\beta Z_*$	102	102	0
	$\alpha+\beta Z_\infty$	84	14	0

Table 2

Number of points with a Bayes risk ratio in [a,b)

A	Family	[1.01,1.1)	[1.1,1.2)	[1.2,∞)
200	$\alpha+\beta Z_1$	16	4	0
	$\alpha+\beta Z_*$	16	4	0
	$\alpha+\beta Z_\infty$	3	2	0
500	$\alpha+\beta Z_1$	89	4	0
	$\alpha+\beta Z_*$	76	4	0
	$\alpha+\beta Z_\infty$	44	4	0
1000	$\alpha+\beta Z_1$	139	37	38
	$\alpha+\beta Z_*$	111	37	38
	$\alpha+\beta Z_\infty$	72	6	8

The total number of integer-pairs (x,y) with $x+y < \frac{1}{2}A - 1$ is 4851 for $A = 200$, 30,876 for $A = 500$ and 124,251 for $A = 1000$. So the sizes of all of the entries in Tables 1 and 2 are quite modest.

The main difficulty in applying the theory of inner approximations is the demonstration that the immediate successors of "good" points are "warm". It is easy to make a particular point (x,y) into a good point - by simply adjusting the linear coefficients α and β (of the linear family) so that $Z(x,y)$ and $R(x,y)$ are equal. This does not uniquely determine these coefficients. If the remaining freedom in the coefficients can be utilized to make $Z < R$ at $(x+1,y)$ and $(x,y+1)$, then the immediate successors of (x,y) are warm, so that (x,y) is an optimal continuation point. But this "simple state of affairs" is too easy: It only arises when it is optimal to continue because the rule "take exactly one more observation" is at least as good as "stop immediately." There are many optimal continuation points which must be discovered a different way (except when A is small): At least one of the immediate successors $(x+1,y)$ and $(x,y+1)$ has to be shown to be warm by using a backward induction argument, i.e., by identifying a set of successors C as described in Section 3A. Basically, it must be shown that there is no way of evolving from the point (x,y) without eventually reaching a point (x',y') at which $Z < R$. To show that a point (x,y) is "trapped" in this way typically requires a careful study of the set of points on which $Z < R$ (which depends on the values of the linear coefficients). An illustration of a typical "trapping") is shown in Figure 1.

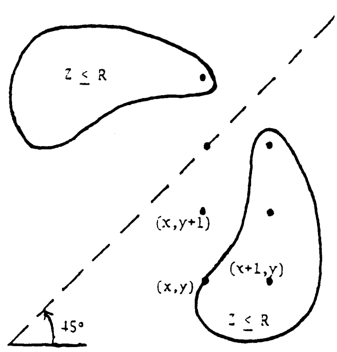

Figure 1

5. Outer approximations.

It is shown in Section 2, that (x,y) is an optimal stopping point whenever $A < 2(n+1)$. The following outer approximation is a stronger result.

THEOREM 1. The point (x,y) is an optimal stopping point whenever

$$(9) \qquad A(1 - |s|) < 2(n+1) + (1 - |s|).\min\{i - 1 + An|s|/i : i=1,2,\ldots\},$$

where $s = k/n$. The minimum occurs when

$$i = [\tfrac{1}{2} + \sqrt{An|s| + 1/4}],$$

where the brackets denote "integer part".

Proof. The second assertion is easily checked. So only the sufficiency of (9) needs to be demonstrated. Express $R(x,y)$ as

$$R(x,y) = \frac{1}{2} A(1 + |s|) - n,$$

and consider solutions of (3) of the general form

$$Z(x,y) = \alpha + \beta s,$$

where α and β are arbitrary constants. For the sake of definiteness, assume that $x > y$ $(s > 0)$, and suppose (9) holds. The task is to find appropriate values for α and β so that $Z < R$ at (x,y) and $Z > R$ at each successor (x',y') of (x,y). Thus one wants:

$$(10) \qquad \alpha + \beta\, s < A(1 + s) - n$$

and

$$(11) \qquad \alpha + \beta\, s' > A(1 + |s'|) - n'$$

for all successor pairs (n',s'), where $n' = x' + y'$ and $s' = (x' - y')/n'$. Let

$\gamma = \beta - \frac{1}{2}A$. Then (10) and (11) yield

$$\gamma(s' - s) \geq A(x')^{-} - (n' - n). \tag{12}$$

It is enough that (12) should hold for all pairs (n',s') and some constant γ (depending on (x,y)). When $s' \geq 0$, (12) becomes

$$\gamma(s' - s) \geq -(n' - n).$$

Thus (12) holds when $s' = s$ (since $s > 0$ and $n' > n$). When $s' > s$, (12) requires

$$\gamma \geq -\frac{n'-n}{s'-s}. \tag{13}$$

For a fixed value of n', the latter is most stringent when s' is chosen as large as possible: $s'=(k+n'-n)/n'$. Then $s'-s=(n'-n)(1-s)/n'$, so that (13) requires $\gamma \geq -n'/(1-s)$. Since $n' \geq n+1$, (12) holds for all (n',s') with $s'>s$ providing $\gamma \geq -(n+1)/(1-s)$.

The conditions (12) impose <u>upper</u> bounds for γ when $s'<s$. So the best choice for γ is

$$\gamma = -(n + 1)/(1 - s);$$

it will work if anything will. (The special value $s=1$ is a trivial case and will be ignored.) Inequality (12) easily follows, for this value of γ, when $s > s' \geq 0$. When $s' < 0$, (12) requires

$$(A(1-s)-(n+1))s'+(1-s)(n'-n)+(n+1)s \geq 0, \tag{14}$$

which automatically holds unless $A(1-s) > n+1$. For a fixed n' and $A(1-s) > n+1$, (14) is most stringent when s' is chosen as small as possible:

$s'=(k-n'+n)/n'=(ns-n'+n)/n'$, $s-s'=(n'-n)(1+s)/n'$. Thus one is lead to the following quadratic inequality expressed in terms of $m'=n'-n$:

$$(1-s)m'^2 - \{A(1-s)-2(n+1)+(1-s)\}m' + Ans(1-s) \geq 0. \tag{15}$$

The requirement that this should hold for $m'=1,2,\ldots$ is equivalent to (9) when $s > 0$. So the theorem holds whenever (9) holds.

A simple corollary states a previously unknown result:

COROLLARY 1. The point (x,y) is an optimal stopping point whenever $A(1-|s|) \leq 2(n+1)$. The latter may be written as $A(x\wedge y) \leq n(n+1)$.

A second corollary recaptures some of the strength of Theorem 1 that is lost in Corollary 1:

COROLLARY 2. The point (x,y) is an optimal stopping point whenever $A(1-|s|) \leq 2(n+1)+(1-|s|).\min(2,An|s|)$.

Proof. The second term in the minimum applies when the minimizing index i in (9) is i=1. The first term in the minimum applies when $i=2,3,\ldots$.

Applications for Corollaries 1 and 2 have already been described in Section 2.

Theorem 1 handles intermediate values of n much better than do Corollaries 1 and 2. It can be improved further by restricting attention to a "suitable set of alternatives, as discussed in the last paragraph of Section 3B. For instance, one only needs to consider m' in (15) for which $n'=m'+n$ are less than $\frac{1}{2}A$. Also, since certain pairs (n',s') are now known to be optimal stopping point on the basis of Theorem 1, one does not need to consider some of the successor pairs (n',s') that were considered when (15) was derived. By omitting these, one should be able to derive an inequality which is less tight than (9), i.e., which captures more optimal stopping points. The task may not be an easy one - nor worth the effort. Alternatively, one could choose to work with a different family of functions Z than that used when deriving (9). It was used for the sake of simplicity. But another family might work substantially better.

It remains to assess the quality of Theorems 1 and its corollaries from the standpoint of Bayes risk performance. The question is: If one uses the stopping rule which says to continue until one reaches one of the optimal stopping points described by (a particular) one of these results, how does its Bayes risks compare with the minimal Bayes risks at various points (x,y)? The answer to this question is not known yet.

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