CONVERGENCE RATES FOR ITERATIVE SOLUTIONS TO OPTIMAL STOPPING PROBLEMS

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We transform a class of optimal stopping problems to problems of finding fixed points of certain transformations, and show that the classical iterative solutions to the latter yield approximations to the value of the corresponding optimal stopping problem. We give upper and lower bounds on the value as functions of the number of iterations, and numerically treat two examples: the well known S_n/n problem, and the coupon collector problem, which is solved exactly and numerically compared to the iterative approximations.

1. Iterative solutions to fixed point problems.

If Q is an operator which sends a space B into itself, a point x is called a fixed point if $Qx = x$. Many problems, both theoretical and applied, are those of finding the fixed points of certain transformations. For example, the extinction probability and the characteristic functions of some limiting distributions in branching theory, the limiting renewal function in renewal theory, certain optimizing strategies, and the equilibrium states of certain evolving phenomena are fixed points.

The iterative method of finding fixed points consists of taking an initial value x_0, arrived at in some manner, and determining sequentially x_n by setting $x_{n+1} = Qx_n$, $n = 0,1,\ldots$. If, in some fashion, x_n "converges" to a limit x, it is generally possible to show that x is a fixed point for Q. The main questions here are: 1) Does a fixed point exist, and if so is it unique?

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2) With a suitable metric, does the iterative method converge, and if the fixed point is not unique, does it converge to the required one? 3) What is the "optimal" choice of the initial value x_0? 4) How many iterations are needed to approximate x to a given degree of accuracy? Of course there are other questions, such as the feasibility of computations, which must also be answered.

Historically, iterative methods for fixed point problems were used in the fundamental existence proofs for differential equations (Picard-Lindelof) and integral equations (Liouville-Neumann), but when high speed computers were introduced forty years ago it was quickly realized that these purely existential proofs provided a way to obtain numerical results, and with the advent of the microprocessor a few years ago such procedures are now available for the individual do-it-yourself home programmer.

In this paper we shall transform a class of optimal stopping problems to fixed point problems, answering the questions raised above, and in particular study the rate of convergence of the iterates. We shall give some numerical examples and illustrate the accuracy of our bounds.

2. Optimal Stopping as a Fixed Point Problem.

We shall quote extensively the results obtained in Darling (1985), but modulo the proofs; this paper should be largely self contained. The setting is a Markov process $X(n)$, $n = 0,1,\ldots$ with stationary increments, taking values in an arbitrary measurable space $(A,\mathbf{A})$. A payoff function $f(n,x)$ gives the reward for stopping the observations at "time" n when $X(n) = x$. The main goal is to estimate the value $v = v(n,x)$ defined as

$$v(n,x) = \sup_{T>0} E[f(n+T,X(n+T))|X(n) = x)] \tag{2.1}$$

where the sup is taken over all finite valued stopping times $T = T(n,x)$ such that the corresponding expectation in (2.1) does not have the indeterminate form $(\infty-\infty)$. We shall have little to say concerning the existence or properties of stopping times per se.

It is convenient, mainly for notational purposes, to work with the "space-time" process $(n,X(n))$, which has the state space $B = \{(n,x)|n=0,1,2,\ldots, x \in A\}$. We define the "shift" operator P acting on a function $g = g(n,x)$, $(n,x) \in B$, by

$$(2.2) \qquad Pg(n,x) = E[g(n+1,X(n+1))|X(n)=x]$$

Set $P^0 = I$, the identity, and for $n = 1,2,\ldots$ define $P^{n+1} = PP^n$, so that $P^k g(n,x) = E[g(n+k,X(n+k))|X(n)=x]$, $k=0,1,\ldots$. The "potential" operator R is defined as

$$(2.3) \qquad R = I + P + P^2 + P^3 + \ldots$$

We shall define an operator Q acting on functions defined over B and show that the problem of determining the value (2.1) is essentially one of finding a fixed "point" - i.e. a fixed function - for this operator.

It is somewhat easier to work not with the functions f and v, defined above, but with the two functions

$$(2.4) \qquad d = Pf - f$$

$$(2.5) \qquad h = v - f$$

and it is clear that the determination of v is equivalent to that of h in (2.5). The functions d and h have obvious interpretations in terms of the stopping problem. We now define the operator Q acting on a function g defined over B as follows

$$(2.6) \qquad Qg = (Pg + d)^+$$

where we use the standard notation $Y^+ = \max(0,y)$, $y^- = \max(0,-y)$. The function h in (2.5) will be a fixed point of the operator Q.

Let the function u be defined by

$$(2.7) \qquad u(n,x) = E[\sup_{k\geq 0} f(n+k, X(n+k))^{-} \mid X(n)=x]$$

We quote the following Theorem from Darling (1985)

THEOREM 1. a) The equation $g = Qg$ has the solution $g = h$, where h is given by (2.5). b) The equation has a minimal solution - i.e., a solution g_0 nowhere exceeding any other solution. c) If $u < \infty$, where u is given by (2.7), then $h = g_0$. d) If $Rd^+ < \infty$, then $g_0 < \infty$, and g_0 is the unique solution to $g = Qg$ satisfying $\inf_{m\geq 0} P^m g = 0$. Combining these statements, if $u < \infty$ and $Rd^+ < \infty$, then $h = g_0$, the minimal solution, and moreover h is the unique solution satisfying $\inf_{m\geq 0} P^m h = 0$.

3. <u>The Iterative Solution</u>.

We assume henceforth that $Rd^+ < \infty$, $u < \infty$, so that the conclusions of d) in Theorem 1 hold. We shall determine functions $h_0 \leq h$, and $h_1 \geq h$ so that $Q^k h_0$ converges to h from below, and $Q^k h_1$ converges to h from above, and we shall find bounds on the deviations from h of these iterates in terms of the transition operator P and the payoff f. We quote from Darling (1985) the following Theorem

THEOREM 2. Let

$$(3.1) \qquad h_0 = \sup_{m\geq 0} (P^m f - f).$$

Then $h_0 < h$ and $Q^m h_0$ converges to h from below, $m \to \infty$. Let

$$(3.2) \qquad h_1 = Rd^+ = d^+ + Pd^+ + P^2 d^+ + \dots$$

Then $h_1 > h$ and $Q^m h_1$ converges to h from above, $m \to \infty$. Moreover

$$(3.3) \qquad 0 \leq h - Q^m h_0 \leq \sum_{k=m}^{\infty} P^k d^+ - \{\sum_{k=m}^{\infty} P^k d\}^+.$$

The series on the right side of (3.3) converge and so we have a measure of the rate of convergence of the iterates $Q^m h_0$ as well as a bound on their deviation from h. It is possible to improve the estimates for the convergence rate in (3.3) and also the initial values h_0 and h_1 in (3.1) and (3.2), at the cost of more complex and less easily computed expressions. In fact if we choose any functions $h_0 \leq h$ and $h_1 \geq h$ such that $\lim_{m\to\infty} P^m h_1 = 0$ their Q iterates will converge respectively from below and above to h. However, it may not be easy to estimate their deviations from h.

In closing this section we should say that in Theorem 1 the assertion that h is a fixed point of Q is a somewhat disguised restatement of the well known fact that the value v is the least superharmonic majorant of the payoff f, and in Theorem 2 the statement that iterates $Q^m h_0$ converge to h is an amplified version of the method of "backward induction" to which it reduces in effect if we take initially $h_0 = 0$; the choice of h_0 as in (3.1) is in general considerably better than choosing $h_0 = 0$.

4. Examples.

a) The S_n/n problem.

Let $X_1, X_2, \ldots$ be i.i.d. random variables having a mean μ, and set $S_0 = 0$, $S_n = X_1 + X_2 + \ldots + X_n$, $n \geq 1$. Let $X(n) = S_n$ and set $f(0,x) = 0$, $f(n,x) = x/n$, $n \geq 1$. This optimal stopping problem has been fairly extensively studied - Chow and Robbins (1962), (1965), (1967); Dvoretsky (1967); Davis (1971), (1973); McCabe and Shepp (1970); Klass (1973); Taylor (1968) among many others - but numerical values do not seem available for any examples prior to Darling (1985).

It is not difficult to calculate the functions d, h_0, h_1 introduced above, and again we cite the results from Darling (1985). Since f(n,x) vanishes for $n = 0$, we have $h(0,0) = v(0,0)$ and we denote this common value by V

$$V = v(0,0) = h(0,0) \tag{4.1}$$

For $n \geq 1$ we have

$$d(n,x) = \frac{1}{n+1}\left[\mu - \frac{n}{x}\right], \quad h_0(n,x) = \left[\mu - \frac{x}{n}\right]^+ \tag{4.2}$$

$$h_1(n,x) = \sum_{m=0}^{\infty} \frac{1}{(m+n)\,(m+n+1)} E\left[(m\mu - S_m + n\mu - x)^+\right]. \tag{4.3}$$

With some loss in precision we can find a simple upper bound for h_1 as follows. If the X_i have a mean of order $b > 1$, there will exist an inequality of the form

$$E[(m\mu - S_m)^+] = \frac{1}{2} E(|S_m - m\mu|) \leq Cm^{1/a} \tag{4.4}$$

for $1 < a < b$ and a suitable constant C (von Bahr and Esseen (1965)). If the X_i are in the domain of normal attraction of a stable law of exponent a, $1 < a \leq 2$, (4.4) will hold for a suitable C. From (4.3) we can deduce

$$h_1(n,x) \leq h_0(n,x) + Ct(n) \tag{4.5}$$

with

$$t(n) = \frac{\pi}{a\,\sin(\pi/a)}\;\frac{1}{n^{1-(1/a)}} \tag{4.6}$$

where a and C are as in (4.4). This result in conjunction with (3.3) leads to the following simple bound for the deviation between the m^{th} iterate $Q^m H_0$ and V

$$0 \leq V - Q^n h_0(0,0) \leq Ct(n) \tag{4.7}$$

where $h_0(n,x)$ is given by (4.2).

As an illustration, suppose the X_i are normal with mean μ and variance σ^2. Then $E[(m\mu-S_m)^+] = \frac{\sigma}{(2\pi)^{1/2}} m^{1/2}$. With n = 50 and a = 2 we have t(50) = .222.. so that $V-Q^{50}h \leq (.0860)\sigma$.

Let us consider the case of coin tossing where the X_i are Bernoulli variables $P(X_i=1) = p$, $P(X_i=0) = q = 1-p$, so that $\mu = p$ and $\sigma^2 = pq$. Even this simple case has apparently previously defied numerical evaluation - cf. Chow, et. al. (1971).

The Q operation is especially simple here

$$Qg(n-1,x) = [pg(n,x+1) + qg(n,x) + \frac{1}{n}(p - \frac{x}{n+1})]^+ \tag{4.8}$$

for $n > 2$, $x = 0,1,\ldots,n-1$, and

$$Qg(0,0) = pg(1,1) + qg(1,0) + p$$

Then from (4.2) and (4.5)

$$h_0(n,x) = (p - \frac{x}{n})^+$$

$$h_1(n,x) < (p - \frac{x}{n})^+ + \frac{\pi}{4}\,[\frac{pq}{n}]^{1/2} \tag{4.9}$$

The following table shows the result of applying the Q operator n = 50 times to the two functions h_0 and h_1 yielding the lower bound $V_L = Q^{50}h_0(0,0)$ and the upper bound $V_U - Q^{50}h_1(0,0)$ on V for various values of p. Table 1 also compares the difference $V_U - V_L$ with the error bound given by (4.7), which in this case is simply the second term on the right hand side of (4.9)

Table 1

Upper and lower bounds for V, n=50

P	V_L	V_U	V_U-V_L	error bound
.2	.44901	.46575	.01674	.0444
.3	.58203	.59745	.01542	.0509
.4	.69632	.71357	.01725	.0544
.5	.78943	.80512	.01569	.0555
.6	.86571	.87887	.01316	.0544
.7	.92449	.93428	.00979	.0509
.8	.96617	.97258	.00641	.0444

Table 2 shows how the approximants V_L and V_U behave as functions of n, when p = .5

Table 2

Bounds as functions of n, p = .5

n	V_L	V_U
10	.78119	.83119
15	.78523	.82661
20	.78636	.81658
25	.78744	.81334
30	.78797	.81053
50	.78943	.80512

b) The coupon collector problem.

We have chosen this problem as one for which the value V can be explicitly calculated and compared with the approximations we have developed. X(n), n = 0,1, is the following pure birth process. Let numbers be drawn independently and uniformly with replacement from the finite set [1,2,...,N] - after n of them have been drawn let X(n) be the number of distinct numbers obtained. Let the payoff be $f(n,x) = \alpha^n x$, where $0 < \alpha < 1$ is a discount factor. It is comparatively simple to solve the corresponding optimal stopping problem, which we leave to the reader, as it is a "monotone case" problem - cf. Chow, et. al. (1971). Let

$$a = \frac{N\alpha}{N(1-\alpha) + \alpha} \tag{4.10}$$

$$k = \text{least integer not less than } a. \tag{4.11}$$

Then $T = \{\text{least } n \text{ such that } X(n) = k\}$ is the optimal stopping time and

$$V = kE(\alpha^T) = \frac{k\alpha^k N!}{(N-k)!N(N-\alpha)(N-2\alpha)\ldots(N-[k-1]\alpha)} \tag{4.12}$$

is the value.

The Q operation is

$$Qg(n,x) = [g(n+1,x)\frac{x}{n} + g(n+1,x+1)(1 - \frac{x}{n}) + \alpha^{n+1}(1 - \frac{x}{a})]^+$$

With

$$h_0 = d^+ = \alpha^{n+1}(1 - \frac{x}{a})^+,$$

we obtain,

$$V - V_L < \sum_{j=m}^{\infty} P^j d^+ = \sum_{j=m}^{\infty} \alpha^{j+1} E[(1 - \frac{X(j)}{a})^+] \tag{4.13}$$

$$= \alpha \sum_{j=0}^{k} (1-\frac{j}{a})\binom{n}{j} \sum_{\xi=0}^{j} (-1)^{j+\xi}\binom{j}{\xi} \frac{[\alpha\xi/N]^m}{1 - [\alpha\xi/N]},$$

with the last expression following from well known results -cf. Feller (1967), Ch. 4.

Table 3 compares the value V with our approximation $V_L = Q^m h_0$ for various values of m, and for several values of α and N. We denote by D the difference $V - V_L$ and by B the upper bound on D calculated in (4.13). The number k is, as above, the optimal stopping number.

Table 3

Error bounds on V.

$N = 50 \quad \alpha = .95 \quad V = 6.1103 \quad k = 14$

m	V_L	D	B
5	4.1958	1.9145	2.3874
10	5.6672	.4431	.6341
15	6.0851	.0252	.0534
20	6.1101	.0002	.0006
25	6.1103	.0000	.0000

$N = 100 \quad \alpha = .9 \quad V = 3.3566 \quad k = 9$

m	V_L	D	B
5	3.1100	.2466	.4626
10	3.3564	.0002	.0009

$N = 20 \quad \alpha = .99 \quad V = 11.9587 \quad k = 17$

m	V_L	D	B
15	9.5342	2.4245	2.7267
20	10.6794	1.2793	1.4647
25	11.3467	.6128	.7151
30	11.6935	.2652	.3158

It appears that in this problem excellent results can be obtained with very few iterations of the Q operator, and (4.13) offers a very good upper bound on the error D.

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