Chapter 1. Introduction

This monograph delves into the uses of group theory, particularly noncommutative Fourier analysis, in probability and statistics. It presents useful tools for applied problems and develops familiarity with one of the most active areas in modern mathematics.

Groups arise naturally in applied problems. For instance, consider 500 people asked to rank 5 brands of chocolate chip cookies. The rankings can be treated as permutations of 5 objects, leading to a function on the group of permutations S_5 (how many people choose ranking π). Group theorists have developed natural bases for the functions on the permutation group. Data can be analyzed in these bases. The "low order" coefficients have simple interpretations such as "how many people ranked item i in position j." Higher order terms also have interpretations and the benefit of being orthogonal to lower order terms. The theory developed includes the usual spectral analysis of time series and the analysis of variance under one umbrella.

The second half of this monograph develops such techniques and applies them to partially ranked data, and data with values in homogeneous spaces such as the circle and sphere. Three classes of techniques are suggested — techniques based on metrics (Chapter 6), techniques based on direct examination of the coefficients in a Fourier expansion (spectral analysis, Chapter 8), and techniques based on building probability models (Chapter 9).

All of these techniques lean heavily on the tools and language of group representations. These tools are developed from first principles in Chapter 2. Fortunately, there is a lovely accessible little book — Serre's *Linear Representations of Finite Groups* — to lean on. The first third of this may be read while learning the material.

Classically, probability precedes statistics, a path followed here. Chapters 3 and 4 are devoted to concrete probability problems. These serve as motivation for the group theory and as a challenging research area. Many of the problems have the following flavor: how many times must a deck of cards be shuffled to bring it close to random? Repeated shuffling is modeled as repeatedly convolving a fixed probability on the symmetric group. As usual, the Fourier transform turns the analysis of convolutions into the analysis of products. This can lead to very explicit results as described in Chapter 3. Chapter 4 develops some "pure probability" tools - the methods of coupling and stopping times - for random walk problems.

Both card shuffling and data analysis of permutations require detailed knowledge of the representation theory of the symmetric group. This is developed in Chapter 7. Again, a friendly, short book is available: G. D. James' *Representation*

Theory of the Symmetric Group. This is also must reading for a full appreciation of the issues encountered.

Most of the chapters begin with examples and a self-contained introduction. In particular, it is possible to read the statistically oriented Chapters 5 and 6 as a lead in to the theory of Chapters 2 and 7.

A BRIEF ANNOTATED BIBLIOGRAPHY

Group representations is one of the most active areas of modern mathematics. There is a vast literature. Basic supplements are:

J. P. Serre (1977). *Linear Representation of Finite Groups.* Springer-Verlag: New York.

G. D. James (1978). *Representation Theory of the Symmetric Groups.* Springer Lecture Notes in Mathematics 682, Springer-Verlag: New York.

There is however the inevitable tendency to browse. I have found the following sources particularly interesting. Journal articles are referenced in the body of the text as needed.

ELEMENTARY GROUP THEORY

Herstein, I. N. (1975). *Topics in Algebra*, 2nd edition. Wiley: New York.
- The classic, best undergraduate text.

Rotman, J. (1973). *The Theory of Groups: An Introduction*, 2nd edition. Allyn and Bacon: Boston.
- Contains much hard to find at this level; the extension problem, generators and relations, and the word problem.

Suzuki, M. (1982). *Group Theory I, II.* Springer-Verlag: New York.
- Very complete, readable treatise on group theory.

BACKGROUND, HISTORY, CONNECTIONS WITH LIFE AND THE REST OF MATHEMATICS.

Weyl, H. (1950). *Symmetry.* Princeton University Press: New Jersey.
- A wonderful introduction to symmetry.

Mackey, G. (1978). *Unitary Group Representations in Physics, Probability, and Number Theory.* Benjamin/Cummings.

Mackey, G. (1980). Harmonic analysis as the exploitation of symmetry. *Bull.*

Amer. Math. Soc. **3**, 543–697.
- A historical survey.

GENERAL REFERENCES

Hewitt, E. and Ross, K. A. (1963, 1970). *Abstract Harmonic Analysis,* Vols. I, II. Springer-Verlag.
- These are encyclopedias on representation theory of abelian and compact groups. The authors are analysts. These books contain hundreds of carefully worked out examples.

Kirillov, A. A. (1976). *Elements of the Theory of Representations.*
- A fancy, very well done introduction to all of the tools of the theory. Basically a set of terrific, hard exercises. See, for example, Problem 4, Part 1, Section 2; Problem 8, Part 1, Section 3; Example 16.1, Part 3. Part 2 is readable on its own and filled with nice examples.

Pontrijagin, K. S. (1966). *Topological Groups.* Gordon and Breach.
- A chatty, detailed, friendly introduction to infinite groups. Particularly nice introduction to Lie theory.

Naimark, M. A. and Stern, D. I. (1982). *Theory of Group Representations.* Springer-Verlag: New York. Similar to Serre (1977) but also does continuous groups.

GROUP THEORY IN PROBABILITY AND STATISTICS.

Grenander, U. (1963). *Probability on Algebraic Structures.* Wiley: New York.
- Fine, readable introduction in "our language." Lots of interesting examples.

Hannen, E. J. (1965). *Group Representations and Applied Probability.* Methuen. Also in *Jour. Appl. Prob.* **2**, 1–68.
- A pioneering work, full of interesting ideas.

Heyer, H. (1977). *Probability Measures on Locally Compact Groups.* Springer-Verlag: Berlin.

Heyer has also edited splendid symposia on probability on groups. These are a fine way to find out what the latest research is. The last 3 are in *Springer Lecture Notes in Math* nos. 928, 1064, 1210.

SPECIFIC GROUPS.

Curtis, C. W. and Reiner, I. (1982). *Representation of Finite Groups and Asso-*

ciative Algebra, 2nd edition. Wiley: New York.
- The best book on the subject. Friendly, complete, long.

Littlewood, D. E. (1958). *The Theory of Group Characters*, 2nd edition. Oxford.
- "Old-fashioned" representation theory of the symmetric group.

James, G. D. and Kerber, A. (1981). *The Representation Theory of the Symmetric Group.* Addison-Wesley, Reading, Massachusetts.
- A much longer version of our basic text. Contains much else of interest.