Lecture II. CONTINUATION OF THE BASIC IDEA

I shall first study the specialization of the lower row of the diagram in (I.28) to the case of approximation by a standard normal distribution as treated in Lemmas I.3 and I.4 and the comments below these lemmas. Then I shall return to the proof of Lemma I.2 in the general abstract formulation.

As I have already indicated briefly in the comments below Lemmas I.3 and I.4, the lower row

(1)
$$\mathfrak{F}_0 \underset{U_0}{\overset{T_0}{\rightleftarrows}} \mathcal{X}_0 \underset{\iota_0}{\overset{E_0}{\rightleftarrows}} R$$

of Diagram (I.28) is specialized in the following way for the treatment of the standard normal approximation problem. (In order to emphasize this specialization I shall write N, T_N, and U_N instead of E_0, T_0, and U_0.) Let

(2)
$$Nh = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x)e^{-\frac{1}{2}x^2}\, dx,$$

(3)
$$(T_N f)(w) = f'(w) - wf(w)$$

and

(4)
$$(U_N h)(w) = e^{\frac{1}{2}w^2} \int_{-\infty}^{w} [h(x)-Nh]e^{-\frac{1}{2}x^2}\, dx = -e^{\frac{1}{2}w^2} \int_{w}^{\infty} [h(x)-Nh]e^{-\frac{1}{2}x^2}\, dx.$$

The equality of the two alternative forms given in (4) follows from

(5)
$$\int_{-\infty}^{\infty} [h(x)-Nh]e^{-\frac{1}{2}x^2}\, dx = 0,$$

which is a consequence of (2). I shall also write Φ for the standard normal c.d.f.:

$$\Phi(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{w} e^{-\frac{1}{2}x^2}\, dx. \tag{6}$$

It remains to specify the linear spaces $\mathcal{X}_0$ and $\mathcal{F}_0$ in diagram (1). Choose $\mathcal{X}_0$ to be the linear space of all piecewise continuous h: $R \to R$ such that, for all $k > 0$

$$\int_{-\infty}^{\infty} |x|^k |h(x)| e^{-\frac{1}{2}x^2}\, dx < \infty, \tag{7}$$

and $\mathcal{F}_0$ to be the linear space of all continuous and piecewise continuously differentiable f: $R \to R$ with $f' \in \mathcal{X}_0$. We must verify that

(i) for all $f \in \mathcal{F}_0$, $T_N f \in \mathcal{X}_0$,

and

(ii) for all $h \in \mathcal{X}_0$, $U_N h \in \mathcal{F}_0$.

In order to verify (i) we observe that, for all $f \in \mathcal{F}_0$ and all $k > 0$

$$\int_0^\infty w^{k+1} |f(w)-f(0)| e^{-\frac{1}{2}w^2}\, dw = \int_0^\infty w^{k+1} \left| \int_0^w f'(x)dx \right| e^{-\frac{1}{2}w^2}\, dw \tag{8}$$

$$\leq \int_0^\infty |f'(x)| \left(\int_x^\infty w^{k+1} e^{-\frac{1}{2}w^2}\, dw \right) dx \leq \int_0^\infty |f'(x)| C(1+|x|^k) e^{-\frac{1}{2}x^2}\, dx < \infty,$$

for some positive constant C, and similarly

$$\int_{-\infty}^{0} |w|^{k+1} |f(w)-f(0)| e^{-\frac{1}{2}w^2}\, dw < \infty. \tag{9}$$

It follows that the function $w \mapsto wf(w) \in \mathcal{X}_0$ and thus, by (3), $T_N f \in \mathcal{X}_0$. In order to verify (ii) we observe that, for all $k > 0$,

$$(10)\qquad \int_0^\infty w^k|w(U_Nh)(w)|e^{-\frac{1}{2}x^2}\,dw \le \int_0^\infty w^{k+1}\left(\int_w^\infty |h(x)-Nh|e^{-\frac{1}{2}x^2}\,dx\right)dw$$

$$= \int_0^\infty |h(x)-Nh|\,\frac{x^{k+2}}{k+2}\,e^{-\frac{1}{2}x^2}\,dx < \infty,$$

and similarly

$$(11)\qquad \int_{-\infty}^0 |w|^k|w(U_Nh)(w)|e^{-\frac{1}{2}w^2}\,dw < \infty.$$

Thus the function $w \mapsto w(U_Nh)(w) \in \mathcal{X}_0$. But, differentiating (4) we easily verify that

$$(12)\qquad (U_Nh)'(w) - w(U_Nh)(w) = h(w) - Nh.$$

It follows that $(U_Nh)' \in \mathcal{X}_0$ and thus $U_Nh \in \mathfrak{F}_0$. Observe that (12) could be rewritten as

$$(13)\qquad T_N \circ U_N = I_{\mathcal{X}_0} - \iota_0 \circ N,$$

that is, condition (I.30) holds in this case.

Lemma 1: In order that the real random variable W have a standard normal distribution it is necessary and sufficient that, for all continuous and piecewise continuously differentiable functions $f\colon R \to R$ with $N|f'| < \infty$, we have

$$(14)\qquad Ef'(W) = EWf(W).$$

Proof of necessity: If W has a standard normal distribution and $N|f'| < \infty$,

$$(15)\qquad Ef'(W) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(w)e^{-\frac{1}{2}w^2}\,dw$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^0 f'(w)\left(\int_{-\infty}^w (-z)e^{-\frac{1}{2}z^2}\,dz\right)dw + \int_0^\infty f'(w)\left(\int_w^\infty ze^{-\frac{1}{2}z^2}\,dz\right)dw\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \{\int_{-\infty}^{0} (\int_{z}^{0} f'(w)dw)(-z)e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} (\int_{0}^{z} f'(w)dw)ze^{-\frac{1}{2}z^2} dz\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(z)-f(0)]ze^{-\frac{1}{2}z^2} dz = EWf(W).$$

Proof of sufficiency: Suppose (14) holds for all continuous and piecewise continuously differentiable functions f with $N|f'| < \infty$. Then it holds, in particular for the functions

(16) $$f_{w_0} = U_N h_{w_0}$$

with h_{w_0} defined by

(17) $$h_{w_0} = \begin{cases} 1 & \text{if } w \leq w_0 \\ 0 & \text{if } w > w_0. \end{cases}$$

Thus

(18) $$0 = E[f'_{w_0}(W)-Wf_{w_0}(W)] = E[h_{w_0}(W)-Nh_{w_0}] = P\{W \leq w_0\} - \Phi(w_0).$$

Thus W has a standard normal distribution.

This lemma is also the starting point for an approach to the estimation of the mean of a multivariate normal distribution. See, for example, Stein (1981).

Before studying the boundedness properties of the linear mapping U_N: $\mathcal{X}_0 \to \mathcal{F}_0$, I shall study the special functions $f_{w_0} = U_N h_{w_0}$ which are given explicitly by

(19) $$f_{w_0}(w) = \begin{cases} \sqrt{2\pi}\, e^{\frac{1}{2}w^2} \Phi(w)[1-\Phi(w_0)] & \text{if } w \leq w_0, \\ \sqrt{2\pi}\, e^{\frac{1}{2}w^2} \Phi(w_0)[1-\Phi(w)] & \text{if } w \geq w_0. \end{cases}$$

It would be desirable to have graphs of these functions for a reasonable range of values of w_0, and also graphs of the f'_{w_0}.

<u>Lemma 2</u>: For the functions f_{w_0} defined by (19) we have

(20) $$0 < f_{w_0}(w) \leq \frac{\sqrt{2\pi}}{4},$$

(21) $$|wf_{w_0}(w)| < 1$$

and

(22) $$|f'_{w_0}(w)| < 1$$

for all real w_0 and w.

<u>Proof</u>: Since

(23) $$f_{w_0}(w) = f_{-w_0}(-w),$$

we need only consider the case

(24) $$w_0 \geq 0.$$

To prove (21) we start from the familiar fact that, for $w > 0$,

(25) $$1-\Phi(w) = \frac{1}{\sqrt{2\pi}}\int_w^\infty e^{-\frac{1}{2}x^2}\,dx < \frac{1}{\sqrt{2\pi}}\int_w^\infty \frac{x}{w}\,e^{-\frac{1}{2}x^2}\,dx = \frac{e^{-\frac{1}{2}w^2}}{w\sqrt{2\pi}},$$

and analogously, for $w < 0$

(26) $$\Phi(w) < \frac{e^{-\frac{1}{2}w^2}}{|w|\sqrt{2\pi}}.$$

From (25) and the second case of (19) it follows that, for $w \geq w_0$,

(27) $$0 \leq wf_{w_0}(w) \leq \Phi(w_0) < 1.$$

It also follows from (25) and the first case of (19) that, for $0 \leq w \leq w_0$,

(28) $$0 \leq wf_{w_0}(w) \leq \sqrt{2\pi}\,we^{\frac{1}{2}w^2}\Phi(w)[1-\Phi(w)] < \Phi(w) < 1.$$

Finally, for $w < 0$, we use (26) and the first half of (19) to obtain

(29) $$|wf_0(w)| < 1-\Phi(w_0) < 1.$$

Now let us go on to (22). More precisely we shall see that, for $w < w_0$,

(30) $$0 < f'_{w_0}(w) < 1,$$

while, for $w > w_0$,

(31) $$0 > f'_{w_0}(w) > -1.$$

First I shall verify (30) for $w \leq 0$. By the first half of (19),

(32) $$f'_{w_0}(w) = [1-\Phi(w_0)][1+\sqrt{2\pi}\; we^{\frac{1}{2}w^2}\Phi(w)]$$

and then (26) implies

(33) $$0 < f'_{w_0}(w) < 1-\Phi(w_0) < 1$$

for $w \leq 0$. Again using (32) in the range $0 \leq w \leq w_0$, we have

(34) $$0 < f'_{w_0}(w) \leq [1-\Phi(w_0)]+\Phi(w_0)\sqrt{2\pi}\; we^{\frac{1}{2}w^2}[1-\Phi(w)] < [1-\Phi(w_0)]+\Phi(w_0) = 1,$$

by (25). Finally, for $w > w_0$ the second half of (19) yields

(35) $$f'_{w_0}(w) = \Phi(w_0)\{-1+\sqrt{2\pi}\; we^{\frac{1}{2}w^2}[1-\Phi(w)]\},$$

and then from (25) it follows that, for $w > w_0$,

(36) $$0 > f'_{w_0}(w) > -\Phi(w_0) > -1.$$

In order to prove (20) we first observe that, by (30) and (31) f_{w_0} attains its maximum at w_0. Thus

(37) $$0 < f_{w_0}(w) \leq f_{w_0}(w_0) = F(w_0),$$

where

(38) $$F(w) = \sqrt{2\pi}\, e^{\frac{1}{2}w^2}\Phi(w)[1-\Phi(w)].$$

We want to show that the even function F attains its maximum at 0 so that

(39) $$\sup_{w_0,w} f_{w_0}(w) = \sup F = F(0) = \frac{\sqrt{2\pi}}{4}.$$

From the identity

(40) $$F^{(k)}(w) = wF^{(k-1)}(w) + (k-1)F^{(k-2)}(w) + (\frac{d}{dw})^{k-1}[1-2\Phi(w)]$$

for $k \geq 1$, which is readily proved by induction, we compute the coefficients of the Taylor series for F about 0, obtaining

(41) $$F(w) = \frac{1}{2\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{1}{k!} [\pi+4 \sum_{j=0}^{k-1} \frac{(-1)^{j+1}}{2j+1}](\frac{w^2}{2})^k.$$

Since the coefficients c_k of this power series in $\frac{w^2}{2}$ alternate in sign and

(42) $$|c_{k+1}| \leq \frac{1}{2}|c_k|$$

it follows that, for $|w| \leq 2$,

(43) $$|F(w)| \leq F(0).$$

But, for $|w| \geq 2$, by (38) and (25)

(44) $$|F(w)| \leq \frac{1}{|w|} \leq \frac{1}{2} < \frac{\sqrt{2\pi}}{4} = F(0).$$

This completes the proof of Lemma 2.

<u>Lemma 3</u>: For bounded absolutely continuous h: $R \to R$;

(45) $$\sup|U_N h| \leq \sqrt{\frac{\pi}{2}} \sup|h-Nh|,$$

(46) $$\sup|(U_N h)'| \leq 2 \sup|h-Nh|,$$

and

(47) $$\sup|(U_N h)''| \leq 2 \sup|h'|.$$

<u>Proof</u>: First let us verify (45). From the definition (4) of $U_N h$, it follows that, for $w \leq 0$,

(48) $$|(U_N h)(w)| \leq [\sup_{x\leq 0}|h(x)-Nh|]e^{\frac{1}{2}w^2}\int_{-\infty}^{w} e^{-\frac{1}{2}x^2}\, dx,$$

and, for $w \geq 0$,

(49) $$|(U_N h)(w)| \leq [\sup_{x \geq 0} |h(x) - Nh|] e^{\frac{1}{2}w^2} \int_w^\infty e^{-\frac{1}{2}x^2}\, dx.$$

Then (45) follows from (48) and (49) since

(50) $$\frac{d}{dw} e^{\frac{1}{2}w^2} \int_{-\infty}^{w} e^{-\frac{1}{2}x^2}\, dx = 1 + we^{\frac{1}{2}w^2} \int_{-\infty}^{w} e^{-\frac{1}{2}x^2}\, dx > 0$$

by (26), so that the right hand side of (48) and (49) attain their maxima at 0.

In order to verify (46) for $w \geq 0$ we use

(51) $$(U_N h)'(w) = h(w) - Nh - we^{\frac{1}{2}w^2} \int_w^\infty [h(x) - Nh] e^{-\frac{1}{2}x^2}\, dx,$$

which follows from the differential equation (12) for $U_N h$. Then

(52) $$\sup_{w \geq 0} |(U_N h)'(w)|$$

$$\leq [\sup|h - Nh|][1 + \sup_{w \geq 0} we^{\frac{1}{2}w^2} \int_w^\infty e^{-\frac{1}{2}x^2}\, dx \leq 2 \sup|h - Nh|$$

by (25). This implies (46) because, with

(53) $$h^*(w) = h(-w),$$

we have

(54) $$(U_N h^*)(w) = (U_N h)(-w).$$

A similar remark applies to (45) and (47).

Now let us prove (47). First we differentiate (12), obtaining

(55) $$(U_N h)''(w) = (U_N h)(w) + w(U_N h)'(w) + h'(w)$$

$$= (1 + w^2)(U_N h)(w) + w[h(w) - Nh] + h'(w).$$

Then we need to express $(U_N h)''$ explicitly in terms of h'. From

(56) $$h(x)-Nh = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [h(x)-h(y)]e^{-\frac{1}{2}y^2}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \{\int_{-\infty}^{x} (\int_{y}^{x} h'(z)dz)e^{-\frac{1}{2}y^2}\, dy - \int_{x}^{\infty}(\int_{x}^{y} h'(z)dz)e^{-\frac{1}{2}y^2}\, dy\}$$

$$= \frac{1}{\sqrt{2\pi}} \{\int_{-\infty}^{x} h'(z)(\int_{-\infty}^{z} e^{-\frac{1}{2}y^2}\, dy) - \int_{x}^{\infty} h'(z)(\int_{z}^{\infty} e^{-\frac{1}{2}y^2}\, dy)dz\}$$

$$= \int_{-\infty}^{x} h'(z)\Phi(z)dz - \int_{x}^{\infty} h'(z)[1-\Phi(z)]dz,$$

it follows that

(57) $$(U_N h)(w) = e^{\frac{1}{2}w^2}\int_{-\infty}^{w} [h(x)-Nh]e^{-\frac{1}{2}x^2}\, dx$$

$$= e^{\frac{1}{2}w^2}\int_{-\infty}^{w} [\int_{-\infty}^{x} h'(z)\Phi(z)dz - \int_{x}^{\infty} h'(z)[1-\Phi(z)]dz]e^{-\frac{1}{2}x^2}\, dx$$

$$= e^{\frac{1}{2}w^2}\{\int_{-\infty}^{w} h'(z)\Phi(z)(\int_{z}^{w} e^{-\frac{1}{2}x^2}\, dx)dz - \int_{-\infty}^{w} h'(z)[1-\Phi(z)](\int_{-\infty}^{z} e^{-\frac{1}{2}x^2}\, dx)dz$$

$$- \int_{w}^{\infty} h'(z)[1-\Phi(z)](\int_{-\infty}^{w} e^{-\frac{1}{2}x^2}\, dx)dz\}$$

$$= - \sqrt{2\pi}\, e^{\frac{1}{2}w^2}\{[1-\Phi(w)]\int_{-\infty}^{w} h'(z)\Phi(z)dz + \Phi(w)\int_{w}^{\infty} h'(z)[1-\Phi(z)]dz\}.$$

From (55) - (57) we obtain

(58) $$(U_N h)''(w) = (1+w^2)(U_N h)(w) + w[h(w)-Nh] + h'(w)$$

$$= h'(w) + [w-\sqrt{2\pi}\,(1+w^2)e^{\frac{1}{2}w^2}(1-\Phi(w))]\int_{-\infty}^{w} h'(z)\Phi(z)dz$$

$$+ [-w-\sqrt{2\pi}\,(1+w^2)e^{\frac{1}{2}w^2}\Phi(w)]\int_{w}^{\infty} h'(z)[1-\Phi(z)]dz.$$

I shall need the fact that, for all w,

$$w + \sqrt{2\pi}\,(1+w^2)e^{\frac{1}{2}w^2}\Phi(w) > 0 \tag{59}$$

and also

$$-w + \sqrt{2\pi}\,(1+w^2)e^{\frac{1}{2}w^2}[1-\Phi(w)] > 0. \tag{60}$$

See Gnedenko (1967), Ch. II, Ex. 13. Finally, using

$$\int_{-\infty}^{w} \Phi(z)dz = w\Phi(w) + \frac{e^{-\frac{1}{2}w^2}}{\sqrt{2\pi}} \tag{61}$$

and

$$\int_{w}^{\infty} [1-\Phi(z)]dz = -w[1-\Phi(w)] + \frac{e^{-\frac{1}{2}w^2}}{\sqrt{2\pi}}, \tag{62}$$

and the fact that these are obviously both positive, we obtain from (58),

$$\begin{aligned}\sup|U_N''h| &\le (1+\sup_w\{[-w+\sqrt{2\pi}(1+w^2)e^{\frac{1}{2}w^2}(1-\Phi(w))][w\Phi(w) + \frac{e^{-\frac{1}{2}w^2}}{\sqrt{2\pi}}]\\ &\quad + [w+\sqrt{2\pi}(1+w^2)e^{\frac{1}{2}w^2}\Phi(w)][-w(1-\Phi(w)) + \frac{e^{-\frac{1}{2}w^2}}{\sqrt{2\pi}}]\})\sup|h'|\\ &= 2\sup|h'|,\end{aligned} \tag{63}$$

which is (47). This completes the proof of Lemma 3, which will be used in the third lecture to obtain bounds for the remainder in Lemma I.4.

Now let us return to the proof of Lemma I.2, which asserts that in the abstract formalism introduced at the beginning of the first lecture, provided the underlying sample space Ω_1 is finite and the connectedness condition of that lemma is satisfied, every random variable $g(X)$ determined by X whose expectation is 0 can be expressed as

$$g(X) = E^X F(X,X') \tag{64}$$

with F antisymmetric. I shall derive this from

Lemma 4: Let Ω be a finite set and $\mathcal{J}$ a non-empty set of two-element subsets of Ω such that $(\Omega,\mathcal{J})$ is a connected graph, that is, for every $(x,x^*) \in \Omega^2$ there exists a sequence $x_1,\ldots,x_k$ with $x_1 = x$ and $x_k = x^*$ such that, for every $j \in \{1,\ldots,k-1\}$, $\{x_j,x_{j+1}\} \in \mathcal{J}$. Also let $\bar{\mathcal{J}}$ be the set of all ordered pairs $(x,x') \in \Omega^2$ such that $\{x,x'\} \in \mathcal{J}$, and let $\mathcal{G}$ be an additive abelian group, $\mathcal{X}^*$ the set of all functions $h: \Omega \to \mathcal{G}$ and $\mathfrak{J}^*$ the set of all functions $\phi: \mathcal{J} \to \mathcal{G}$ that are antisymmetric in the sense that, for all $(x,x') \in \bar{\mathcal{J}}$

(65) $$\phi(x,x') = -\phi(x',x).$$

Let T^*: $\mathfrak{J}^* \to \mathcal{X}^*$ and E^*: $\mathcal{X}^* \to \mathcal{G}$ be defined by

(66) $$(T^*\phi)(x) = \sum_{(x,x')\in\bar{\mathcal{J}}} \phi(x,x')$$

and

(67) $$E^*h = \sum_{x\in\Omega} h(x).$$

Then

(68) $$\text{im } T^* = \ker E^*.$$

For completeness I shall give a proof of this lemma although it is the well-known fact that the zeroth reduced homology group of a connected augmented simplical complex is 0. See for example Hilton and Wylie (1965), pp. 62-3. Clearly im $T^* \subset$ ker E^* since

(69) $$E^*(T^*\phi) = \sum_{x\in\Omega}(T^*\phi)(x) = \sum_{x\in\Omega}\sum_{(x,x')\in\bar{\mathcal{J}}} \phi(x,x')$$

$$= \sum_{\{x,x'\}\in\mathcal{J}} [\phi(x,x')+\phi(x',x)] = \sum_{\{x,x'\}\in\mathcal{J}} [\phi(x,x')-\phi(x,x')] = 0.$$

I shall prove that im $T^* \supset$ ker E^* by induction on

(70) $$\nu = |\Omega|$$

We want to prove that for any $h \in \mathcal{X}^*$ such that $E^*h = 0$ there exists $\phi \in \mathcal{J}^*$ such that $T^*\phi = h$. This is obvious for $\nu = 2$ since, with $\Omega = \{x_1,x_2\}$ and $\mathcal{J} = \{x_1,x_2\}$ we need only take

(71) $$\phi(x_1,x_2) = h(x_1)$$

and

(72) $$\phi(x_2,x_1) = -h(x_1) = h(x_2).$$

The first expression for $\phi(x_2,x_1)$ makes ϕ antisymmetric and the second (equivalent because $E^*h = 0$) gives $(T^*\phi)(x_2)$ the value $h(x_2)$. Let us assume the result true for $\nu = \nu_0 \geq 2$ and suppose

(73) $$\Omega = \{x_1,x_2,\ldots,x_{\nu_0+1}\} \text{ with the } x_i \text{ distinct.}$$

Choose $k \in \{1,\ldots,\nu_0\}$ so that $\{x_k,x_{\nu_0+1}\} \in \mathcal{J}$ and, for given $h \in \mathcal{X}$ such that $E^*h = 0$ define $h^*\colon \{x_1,\ldots,x_{\nu_0}\} \to \mathcal{G}$ by

(74) $$h^*(x) = \begin{cases} h(x) & \text{if } x \neq x_k \\ h(x_k) + h(x_{\nu_0+1}) & \text{if } x = x_k. \end{cases}$$

Let $\mathcal{J}^*$ be the set of all $(x,x') \in \mathcal{J}$ with $x_{\nu_0+1} \notin \{x,x'\}$. Then $E^*h^* = 0$ and consequently, by the induction assumption, there exists $\phi^*\colon \mathcal{J}^* \to \mathcal{G}$, antisymmetric, such that $T^*\phi^* = h^*$. Now define $\phi\colon \mathcal{J} \to \mathcal{G}$ by

(75) $$\phi(x,x') = \begin{cases} \phi^*(x,x') & \text{if } \{x,x'\} \subset \{x_1,\ldots,x_{\nu_0}\} \\ h(x_{\nu_0+1}) & \text{if } x = x_{\nu_0+1},\ x' = x_k \\ -h(x_{\nu_0+1}) & \text{if } x = x_k,\ x' = x_{\nu_0+1} \\ 0 & \text{Otherwise.} \end{cases}$$

Then ϕ is antisymmetric and

(76) $$(T^*\phi)(x) = \begin{cases} (T^*\phi^*(x) = h^*(x) = h(x) & \text{if } x \notin \{x_k,x_{\nu_0+1}\} \\ h(x_{\nu_0+1}) & \text{if } x = x_{\nu_0+1} \\ (T^*\phi^*)x - h(x_{\nu_0+1}) = h^*(x_k) - h(x_{\nu_0+1}) = h(x_k) & \text{if } x = x_k. \end{cases}$$

Thus $T^*\phi = h$ and the proof by induction of Lemma 4 is completed.

Now let us prove Lemma I.2 by applying Lemma 4 with $\mathcal{G} = R$. For $g: \Omega \to R$ with $Eg(X) = 0$ define $h: \Omega \to R$ by

$$h(x) = g(x)P\{X=x\} \tag{77}$$

and, using the fact that

$$0 = Eg(X) = \sum_{x\varepsilon\Omega} h(x), \tag{78}$$

choose an antisymmetric ϕ, in accordance with Lemma 4, such that, for all $x \varepsilon \Omega$

$$h(x) = \sum_{(x,x')\varepsilon\bar{\mathcal{J}}} \phi(x,x') \tag{79}$$

where $\bar{\mathcal{J}}$ is the set of all ordered pairs (x,x') of elements of Ω such that

$$P\{X=x \;\&\; X'=x'\} > 0. \tag{80}$$

We can rewrite (79) as

$$g(x) = \sum_{(x,x')\varepsilon\bar{\mathcal{J}}} \frac{\phi(x,x')}{P\{X=x\}} = \sum_{(x,x')\varepsilon\bar{\mathcal{J}}} \frac{\phi(x,x')}{P\{X=x \;\&\; X'=x'\}} P\{X'=x'|X=x\} \tag{81}$$

$$= E\{F(X,X')|X=x\}$$

where $F: \Omega^2 \to R$ is the antisymmetric function defined by

$$F(x,x') = \begin{cases} \dfrac{\phi(x,x')}{P\{X=x \;\&\; X'=x'\}} & \text{if } (x,x') \varepsilon \bar{\mathcal{J}} \\ 0 & \text{otherwise.} \end{cases} \tag{82}$$

Writing (81) in the form

$$g(X) = E^X F(X,X'), \tag{83}$$

we see that Lemma I.2 has been proved.

In the first part of this lecture I have studied the auxiliary functions and linear mappings that are useful for bounding the error in normal approximation problems by this method. Lemma 1 provides the characterization of a standard normal random variable W by the identity $E[f'(W)-Wf(W)] = 0$. Lemma 2 provides bounds for the special function f_{w_0} and its derivative.

Lemma 3 studies the boundedness properties of the linear mapping U_N in supremum norms for functions or their derivatives. This lemma is used in Lecture III, and elsewhere, in obtaining crude bounds in normal approximation problems.

The last part of this lecture, beginning with Lemma 4, is devoted to the proof of Lemma I.2 which shows that, in the case of a finite sample space, subject to the obviously necessary connectedness condition, any random variable whose expectation is zero is obtained as conditional expectation, given X, of an antisymmetric function of the exchangeable pair (X,X'), that is, ker E = im T. Although this result is never needed when applying this method, one may ask, for the sake of completeness, whether it also holds in the countable case, and whether an appropriate analogue can be formulated for the general case.