Inequalities in Statistics and Probability
IMS Lecture Notes–Monograph Series Vol. 5 (1984), 193–198

MULTIVARIATE LIFE CLASSES AND INEQUALITIES

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In this paper we review some univariate life classes which are useful in reliability theory. Recently some new characterizations of these classes have been given in terms of integral inequalities with respect to certain classes of function. These characterizations and their natural multivariate extensions are discussed. Some moment inequalities are then deduced.

1. Introduction. Various univariate classes of life distributions have been introduced in the context of mathematical reliability theory. Most of these classes have intuitive appeal, possess nice closure properties and lead to useful bound in estimating system reliability. The book by Barlow and Proschan (1975) gives an excellent discussion of these classes and their properties.

Recently there has been much interest in obtaining multivariate versions of these classes. Although there have been many different approaches, this review paper will focus on only three: the multivariate IFR class of Savits (1983); the multivariate IFRA class of Block and Savits (1980); the multivariate NBU class of Marshall and Shaked (1982). All three are based on recent characterizations which are expressable in terms of integral inequalities for certain classes of functions. Also, more importantly, all three classes possess many desirable closure properties.

All functions and sets in this paper are assumed to be Borel measurable. A subset A is said to be an upper set if $\mathbf{x} \in A$ and $\mathbf{y} \geq \mathbf{x}$ implies that $\mathbf{y} \in A$. A nonnegative function h is said to be log concave (on $\mathcal{R}_+^n$) if $h[\lambda\mathbf{x}+(1-\lambda)\mathbf{y}] \geq h^\lambda(\mathbf{x})h^{1-\lambda}(\mathbf{y})$ for all $\mathbf{x},\mathbf{y} \geq \mathbf{0}$ and all $0 < \lambda < 1$. A function ψ is said to be subhomogeneous (on $\mathcal{R}_+^n$) if $\psi(\alpha\, \mathbf{x}) \geq \alpha\psi(\mathbf{x})$ for all $\mathbf{x} \geq \mathbf{0}$ and all $0 < \alpha < 1$.

2. Review of Univariate Life Classes. Let T be a nonnegative random variable with survival function $\bar{F}(t) = P\{T>t\}$. Set $b = \inf\{t \geq 0; \bar{F}(t) = 0\}$ ($\inf \phi = +\infty$). For simplicity we assume $\bar{F}(0) = 1$.

Definition 1. (i) T is said to have an increasing failure rate (IFR) distribution if $\bar{F}(s+t)/\bar{F}(t)$ is nonincreasing in $t \in [0,b)$ for all $s \geq 0$. (ii) T is said to have an increasing failure rate average (IFRA) distribution if $\bar{F}(\alpha t) \geq \bar{F}^\alpha(t)$ for all $t \geq 0$, $0 < \alpha < 1$. (iii) T is said to have a new better than used (NBU) distribution if $\bar{F}(s+t) \leq \bar{F}(s)\bar{F}(t)$ for all $s,t \geq 0$. (iv) T is said to have a new better than used in expectation (NBUE) distribution if $\mu = E[T] < \infty$ and $\int_t^\infty \bar{F}(x)dx \leq \mu\, \bar{F}(t)$ for all $t \geq 0$.

These classes of distribution have been very useful in reliability theory (cf. Barlow and Proschan (1975) for a detailed discussion of their properties). It is known that IFR $\to$ IFRA $\to$ NBU $\to$ NBUE.

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[1] Supported by ONR Contract N00014–76–C–0839.

AMS 1980 subject classifications. Primary 60K10; Secondary 62N05.

Key words and phrases: multivariate, inequalities, IFR, IFRA, NBU, NBUE, upper sets, log concave functions, subhomogeneous functions, log subhomogeneous functions.

The next two theorems are well known and list some useful equivalent conditions (see Barlow and Proschan (1975)).

THEOREM 1. *The following are equivalent: (a) T is IFR. (b) $\bar{F}$ is a Pólya frequency function of order two (PF_2); i.e., $\bar{F} \geqslant 0$ and*

$$\begin{vmatrix} \bar{F}(t_1-s_1) & \bar{F}(t_1-s_2) \\ \bar{F}(t_2-s_1) & \bar{F}(t_2-s_2) \end{vmatrix} \geqslant 0$$

for all $-\infty < t_1 < t_2 < \infty, -\infty < s_1 < s_2 < \infty$. (c) $\log \bar{F}(t)$ *is concave in* $t \geqslant 0$.

THEOREM 2. *The following are equivalent: (a) T is IFRA. (b) $\bar{F}^{1/t}(t)$ is nonincreasing in $t > 0$. (c) $-1/t \log \bar{F}(t)$ is nondecreasing in $t > 0$.*

Remark. If F has a density f, we define the hazard rate by $r(t) = f(t)/\bar{F}(t)$. Then T is IFR if and only if $r(t)$ is nondecreasing in $t \in [0,b)$; T is IFRA if and only if $(1/t)\int_0^t r(u)du$ is nondecreasing in $t \in [0,b)$.

3. Some Recent Characterizations. Within the past several years, various other characterizations of these classes have been discovered. The ones we list below are all expressed via integral inequalities with respect to certain function classes. Many of the known properties of the univariate classes follow easily from these characterizations.

THEOREM 3. *(i) T is IFR if and only if $E[h(\mathbf{x},T)]$ is log concave in $\mathbf{x}$ for all functions $h(\mathbf{x},t)$ which are log concave in $(\mathbf{x},t)$ and nondecreasing in t for each fixed $\mathbf{x}$. (ii) T is IFRA if and only if $E[h(T)] \leqslant E^{1/\alpha}[h^{\alpha}(T/\alpha)]$ for all nonnegative nondecreasing functions h and all $0 < \alpha < 1$. (iii) T is NBU if and only if $E[h(T/(\alpha+\beta))] \leqslant E[h^{\gamma}(T/\alpha)]\, E[h^{1/\gamma}(T/\beta)]$ for all nonnegative nondecreasing functions h, all $\alpha, \beta > 0$ and all $0 < \gamma < 1$.*

The first result is due to Savits (1983); the second to Block and Savits (1976); the third to Marshall and Shaked (1982). The IFRA characterizations given in Theorem 3.1(ii) was particular useful in solving the IFRA convolution problem (Block and Savits (1976)).

Although the original intuitive appeal of the univariate life classes is lost in the abstract characterizations given above, the multivariate extensions that naturally follow enjoy many desirable closure properites, as we shall see in the next section.

4. Multivariate Extensions. There are many different ways of obtaining multivariate extensions of the univariate life classes (e.g., see the review paper of Block and Savits (1981)); however, it is desirable that any such extension satisfy certain properties. We list below some such properties.

Let $\mathcal{C}$ denote a multivariate extension of a univariate class of distributions $\mathcal{C}_0$. By abuse of notation, we say that a random vector $\mathbf{T} \in \mathcal{C}$ if its distribution belongs to $\mathcal{C}$.

Properties.

(P0) A random variable $T \in \mathcal{C}$ if and only if $T \in \mathcal{C}_0$.

(P1) If $\mathbf{T} \in \mathcal{C}$, then all marginals belong to $\mathcal{C}$.

(P2) If $\mathbf{S},\mathbf{T} \in \mathcal{C}$ and are independent, then $(\mathbf{S},\mathbf{T}) \in \mathcal{C}$.

(P3) If $\mathbf{S},\mathbf{T} \in \mathcal{C}$ and are independent, then $\mathbf{S} + \mathbf{T} \in \mathcal{C}$ whenever it makes sense.

(P4) If $\mathbf{T}_n \in \mathcal{C}$ for each n and $\mathbf{T}_n \to \mathbf{T}$ in distribution, then $\mathbf{T} \in \mathcal{C}$.

(P5) If $(T_1, \ldots, T_n) \in \mathcal{C}$ and $a_i \geqslant 0\ (1 \leqslant i \leqslant n)$, then $(a_1T_1, \ldots, a_nT_n) \in \mathcal{C}$.

(P6) If $(T_1, \ldots, T_n) \in \mathcal{C}$ and π is any permutation on $\{1, \ldots, n\}$, then $(T_{\pi(1)}, \ldots, T_{\pi(n)}) \in \mathcal{C}$.

(P7) If $\mathbf{T} \in \mathcal{E}$ and $\psi_1, \ldots, \psi_m$ are nonnegative nondecreasing subhomogeneous functions, then $(\psi_1(\mathbf{T}), \ldots, \psi_m(\mathbf{T})) \in \mathcal{E}$.
(P8) If $\mathbf{T} \in \mathcal{E}$ and $\psi_1, \ldots, \psi_m$ are nonnegative nondecreasing concave functions, then $(\psi_1(\mathbf{T}), \ldots, \psi_m(\mathbf{T})) \in \mathcal{E}$.

Property (P3) is included since all the univariate life classes described in Section 2 are closed under convolution. For the IFRA and NBU classes, property (P7) is natural since these classes are closed under the formation of coherent systems, which are included within the class of subhomogeneous functions. This is not true for the IFR class, however. On the other hand, the IFR class is closed under minimums and these are special examples of concave functions. Properties (P5) and (P6) are not as essential as the others.

Definition 2. (i) $\mathbf{T}$ is MIFR (in the sense of Savits (1983)) if $E[h(\mathbf{x},\mathbf{T})]$ is log concave in $\mathbf{x}$ for all log concave functions $h(\mathbf{x},\mathbf{t})$ which are nondecreasing and continuous in $\mathbf{t}$ for each $\mathbf{x}$. (ii) $\mathbf{T}$ is MIFRA (in the sense of Block and Savits (1980)) if $E[h(\mathbf{T})] \leq E^{1/\alpha}(h^{\alpha}(\mathbf{T}/\alpha)]$ for all continuous nonnegative nondecreasing functions h and all $0 < \alpha < 1$. (iii) $\mathbf{T}$ is MNBU (in the sense of Marshall and Shaked (1982)) if $\mathrm{E}[h(\mathbf{T}/(\alpha+\beta))] \leq E[h^{\gamma}(\mathbf{T}/\alpha)]\, E[h^{1-\gamma}(\mathbf{T}/\beta)]$ for all continuous nonnegative nondecreasing functions h and all $\alpha,\beta > 0$, $0<\gamma<1$.

Remark. It is shown in the above papers that the continuity assumption on h is not necessary.

THEOREM 4. *(i) The MIFR class satisfies properties (P0)–(P6) and (P8). (ii) The MIFRA and the MNBU class satisfy properties (P0)–(P7).*

The proofs of this theorem and related results are contained in the above cited papers. In particular, some useful equivalent formulations are given.

If $\mathbf{T}$ is a random vector, let $\mu(d\mathbf{y}) = P(\mathbf{T} \in d\mathbf{y})$ be its induced measure.

THEOREM 5. *(i)* $\mathbf{T}$ *is MIFR if and only if* $\mu[\lambda A+(1-\lambda)B] \geq \mu^{\lambda}(A)\mu^{1-\lambda}(B)$ for all upper convex sets A,B and all $0 < \lambda < 1$. *(ii)* $\mathbf{T}$ *is MIFRA iff* $\mu(\alpha A) \geq \mu^{\alpha}(A)$ for all upper sets A and all $0 < \alpha < 1$. *(iii)* $\mathbf{T}$ *is MNBU if and only if* $\mu((\alpha+\beta)A) \leq \mu(\alpha A)\mu(\beta A)$ *for all upper sets A and all* $\alpha, \beta > 0$.

It is known that MIFRA → MNBU, but the implication MIFR → MIFRA remains a conjecture.

5. Some Moment Inequalities.

Before we consider some multivariate moment inequalities, let us first discuss the univariate case. If T is a nonnegative random variable, let $\mu_r = E[T^r]$ for $r > 0$.

Case (i). T is IFR. We consider functions of the form $h(r,t) = t^r/\phi(r)$ where ϕ is to be suitably chosen. In order to make use of Theorem 3(i) we need that h be log concave in (r,t). If ϕ is twice continuously differentiable, then a necessary and sufficient condition is that

$$r \cdot d/dr[\phi'(r)/\phi(r)] \geq 1. \tag{5.1}$$

One can easily check that this is true for $\phi(r) = r^r e^{-r}$. Thus we conclude that the "normalized moments" $\rho_r = \mu_r/(r^r e^{-r})$ are log concave in $r > 0$.

It is interesting to note that (5.1) is true for $\phi(r) = \Gamma(r)$ but false for $\phi(r) = \Gamma(r+1)$, which is the classical normalization factor. The class of ϕ which satisfy (5.1) with inequality replaced by equality is given by $\phi(r) = a\,r^r e^{-br}$ for $a > 0, -\infty < b < \infty$.

Case (ii). T is IFRA. If we let $h(x) = x^r$ in Theorem 3.1(ii), it can be easily shown that $(\rho_r)^{1/r}$ is nonincreasing in $r > 0$, where ρ_r are the same normalized moments given above.

Case (iii). T is NBU. Again letting $h(x) = x^r$ in Theorem 3.1(ii), we conclude that $\rho_{r+s} \leq \rho_r \rho_s$ for all $r,s > 0$.

It is convenient at this point to introduce some further definitions. A nonnegative function h is said to be log subhomogeneous (on $\mathcal{R}^n_+$) if $h(\alpha\mathbf{x}) \geq h^\alpha(\mathbf{x})$ for all $\mathbf{x} \geq \mathbf{0}$ and $0 < \alpha < 1$; it is said to be log subadditive (on $\mathcal{R}^n_+$) if $h((\alpha+\beta)\mathbf{x}) \leq h(\alpha\mathbf{x})h(\beta\mathbf{x})$ for all $\mathbf{x} \geq \mathbf{0}$, and $\alpha,\beta > 0$.

Using these definitions we summarize the above univariate results on ρ_r below.

(5.2)
(i) T IFR $\rightarrow \rho_r$ is log concave in r.
(ii) T IFRA $\rightarrow \rho_r$ is log subhomogeneous in r.
(iii) T NBU $\rightarrow \rho_r$ is log subadditive in r.

A particularly interesting special case of (5.2) is the following. First note that in (5.2) we may replace ρ_r with $\rho^*_r = \rho_r e^{-r}$ since they have exactly the same properties. Now consider an exponential random variable with mean one. In this case $\rho^*_r = \Gamma(r+1)/r^r$. Since the exponential is in all life classes we deduce that

(5.3)
(i) $\Gamma(r+1)/r^r$ is log concave in r.
(ii) $\Gamma(r+1)/r^r$ is log subhomogeneous in r.
(iii) $\Gamma(r+1)/r^r$ is log subadditive in r.

(Actually (i) $\rightarrow$ (ii) $\rightarrow$ (iii) but it is useful to list them separately). The result (5.3)(ii) was already proven in Marshall, Olkin and Proschan (1967), but our proof is much simpler.

We now contrast the results in (5.2) with the known classical univariate results given in Barlow and Proschan (1975). Let $\lambda_r = \mu_r/\Gamma(r+1)$. Then:

(5.4)
(i) T IFR $\rightarrow \lambda_r$ is log concave in r.
(ii) T IFRA $\rightarrow \lambda_r$ is log subhomogeneous in r.
(iii) T NBU $\rightarrow \lambda_r$ is log subadditive in r.

Since $\rho_r = \lambda_r \cdot [\Gamma(r+1)/r^r] \cdot e^r$, the results (5.2) follow by combining the results (5.3) and (5.4). Hence the univariate results (5.2) are weaker than the univariate results (5.4). However, in some sense, they are asymptotically equivalent because, e.g., $(\rho_r/\lambda_r)^{1/r} \rightarrow 1$ as $r \rightarrow \infty$. This is the reason the irrelevant factor e^{-r} was introduced into the normalized moments ρ_r.

Although in the univariate case the results (5.2) are weaker than those of (5.4), there are no known generalizations of (5.4) in the multivariate setting. However, the results of (5.2) do generalize. The following multivariate moment relations follow from Definition 2 in exactly the same way as those derived from Theorem 3. If $\mathbf{r} = (r_1, \ldots, r_n)$, we let $\mu_\mathbf{r} = E[T_1^{r_1} \ldots T_n^{r_n}]$ and set $\rho_\mathbf{r} = \mu_\mathbf{r}/(\Pi_{i=1}^n r_i^{r_i} e^{-r_i})$.

THEOREM 6. *Let* $\mathbf{T}$ *be a nonnegative random vector.*

(5.5)
(i) *If* $\mathbf{T}$ *is MIFR, then* $\rho_\mathbf{r}$ *is log concave in* $\mathbf{r}$.
(ii) *If* $\mathbf{T}$ *is MIFRA, then* $\rho_\mathbf{r}$ *is log subhomogeneous in* $\mathbf{r}$.
(iii) *If* $\mathbf{T}$ *is MNBU, then* $\rho_\mathbf{r}$ *is log subadditive in* $\mathbf{r}$.

The results (5.5) are the best available at present. In particular they are valid for the MVE of Marshall and Olkin (1967) since the MVE is MIFR, MIFRA and MNBU.

6. Some Other Classes. The recent successful use of log concave functions to characterize the IFR class has suggested other variations on this theme. Although the full ramifications of this approach are being currently investigated, we illustrate with one interesting example.

Recall that in section five we defined a nonnegative function h to be log subhomogeneous if $h(\alpha \mathbf{x}) \geq h^{\alpha}(\mathbf{x})$ for all $0 < \alpha < 1$.

THEOREM 7. *$\mathbf{T}$ is MIFRA if and only if $E[h(\mathbf{x},\mathbf{T})]$ is log subhomogeneous in $\mathbf{x}$ for all functions $h(\mathbf{x},\mathbf{t})$ which are log subhomogeneous in $(\mathbf{x},\mathbf{t})$ and are nondecreasing in $\mathbf{t}$ for each fixed $\mathbf{x}$.*

Proof. Suppose $\mathbf{T}$ is MIFRA and let $h(\mathbf{x},\mathbf{t})$ be a log subhomogeneous function which is nondecreasing in $\mathbf{t}$ for each fixed $\mathbf{x}$. Then

$$E^{\alpha}[h(\mathbf{x},\mathbf{t})] \leq E[h^{\alpha}(\mathbf{x},\mathbf{T}/\alpha)] \leq E[h(\alpha\mathbf{x},\mathbf{T})].$$

The first inequality follows since $\mathbf{T}$ is MIFRA and the second follows since h is log subhomogeneous.

On the other hand suppose $E[h(\mathbf{x},\mathbf{T})]$ is log subhomogeneous in $\mathbf{x}$ for all log subhomogeneous functions $h(\mathbf{x},\mathbf{t})$ which are nondecreasing in $\mathbf{t}$ for each fixed $\mathbf{x}$. Let $h(\mathbf{t})$ be a nondecreasing function in $\mathbf{t}$ and define $H(r,\mathbf{t}) = h^{r}(\mathbf{t}/r)$ for $r > 0$. Then $H(r,\mathbf{t})$ is log subhomogeneous in $(r,\mathbf{t})$ and is nondecreasing in $\mathbf{t}$ for each fixed r. Hence $E[H(r,\mathbf{T})]$ is log subhomogeneous in r, i.e.,

$$E[H(\alpha r,\mathbf{T})] \geq E^{\alpha}[H(r,\mathbf{T})] \quad \text{or}$$

$$E[h^{\alpha r}(\mathbf{T}/\alpha r)] \geq E^{\alpha}[h^{r}(\mathbf{T}/r)].$$

Now set $r = 1$ to conclude that $\mathbf{T}$ is MIFRA.

In Block and Savits (1978), a new characterization of the NBUE class was given.

THEOREM 8. *T is NBUE if and only if $\mu = E[T]$ is finite and*

$$\int_0^{\infty} h(z)\bar{F}(z)dz \leq \mu \int_0^{\infty} h(z)dF(z) \tag{6.1}$$

for all nonnegative nondecreasing functions h, where $\bar{F}(t) = P(T>t)$.

The author has recently proposed a multivariate extension of (6.1) and has shown that the resulting multivariate class satisfies properties (P0)–(P6).

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