Inequalities in Statistics and Probability
IMS Lecture Notes–Monograph Series Vol. 5 (1984), 150–155

PROBABILISTIC ORDERING OF SCHEFFÉ POLYHEDRA

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Some inequalities for spherically symmetric distributions are discussed using simple ideas from convex geometry. There are two dual orderings of the size of convex polytopes with respect to "width" in a random direction. One is equivalent to the ordering of content with respect to all spherically symmetric distributions. The other is the stochastic version of the mean width of convex geometry. Dual versions of known results are given and in particular the complete classification of the Platonic solids is listed. Some remarks are made about future developments.

1. Introduction. There is a duality between a problem in statistics of ordering certain regions with respect to probability content and an ordering based on the support function of a convex set. Recent results are discussed in the light of this connection.

The first ordering arises naturally in the statistical theory of multiple comparisons and by now has a considerable literature. Let $\mathcal{C}$ be a class of sets in p-dimensional Euclidean space $\mathcal{E}^p$. Let F be a family of probability measures on $\mathcal{E}^p$ with respect to which every member of $\mathcal{C}$ is measurable. We say that for two members C_1 and C_2 of $\mathcal{C}$, $C_1 > C_2$ if $\mu(C_1) > \mu(C_2)$ for all μ in F. It is usual to specialize $\mathcal{C}$ and F in various ways. Typically C may comprise all convex radially symmetric sets ($x \in C$ implies $-x \in C$) and F may be all unimodal, spherically symmetric distributions or their multivariate normal versions. Much of this material is summarised in Tong (1980).

In this paper we first restrict $\mathcal{C}$ to all closed star-shaped regions: $x \in C$ implies $\lambda x \in C$ for all $0 \leqslant \lambda \leqslant 1$. Thus C contains all points on the ray to the boundary point in any direction. Let F consist of all spherically symmetric distributions: all measures preserved under any rotation about the origin. We refer to the induced ordering as $>_h$. The following simple geometric characterisation comes as Theorem 1 in Bohrer and Wynn (1982). Let s be a random direction in $\mathcal{E}^p$ which may be interpeted as a point distributed with the uniform distribution on the surface of the unit sphere S_{p-1} in $\mathcal{E}^p$. Let $h(C,s)$ be the distance to the boundary of C from the origin in the direction s. Then the result is that, for C_1 and C_2 in $\mathcal{C}$, $C_1 >_h C_2$ if an only if $h(C_1,s) > h(C_2,s)$, where $>$ is stochastic ordering:

$$P[h(C_1,s) \geqslant r] > P[h(C_2,s) \geqslant r] \qquad \text{for all } 0 \leqslant r \leqslant \infty.$$

The proof follows directly from the fact that it is sufficient to prove that the p–1 dimensional area of the intersection with the spherical shell rS_{p-1} of radius r is at least as great for C_1 as for C_2, for all $0 \leqslant r \leqslant \infty$. Then since C_1 and C_2 are star-shaped these intersections are (proportional to) the s-probability of the boundary in the direction s lying outside or on rS_{p-1}.

Measures of the size of convex bodies abound in the field of convex geometry which has had a resurgence in recent years but has been little used in the field of multiple comparisons. The subject arises as a foundation for Minkowski's geometry of numbers and in particular for his theorem on the volume of n-dimensional lattices (see Stewart and Tall (1979) for an elementary treatment). One arm of the subject is loosely called integral geometry

AMS 1980 subject classifications. 62J15.
Key words and phrases: convexity, convex polytopes, multiple comparisons, stochastic geometry.

(see Santalo (1976)). More recently many advances have been made in stochastic geometry and mathematical morphology (Serra (1982), Coleman (1979)) and in describing the combinatorial and geometric properties of the face lattices of convex polyhedra (see Grunbaum (1967), McMullen and Shephard (1971)).

Our second ordering is based, then, on a classical idea from convex geometry. Let $\mathcal{C}$ now be the class of closed convex sets in $\mathcal{E}^p$ containing the origin. They are obviously star-shaped. The support function $H(C,s)$ of C is $\mathcal{C}$ in the direction s is defined to be

$$H(C,s) = \sup\{<y,s>|\, y \in C\}.$$

Note that since the origin is in C, $H(C,s) \geqslant 0$ for C in $\mathcal{C}$. With s random $H(C,s)$ becomes a random variable. Then we define $C_1 >_H C_2$ if and only if $H(C_1,s) > H(C_2,s)$ where again we mean stochastic ordering as defined above.

Both $>_h$ and $>_H$ are orderings of the size of the sets in $\mathcal{C}$. We now show that there is a close connection between them. Let C be a closed convex set containing the origin. The dual set of C which has the same properties is defined as follows

$$C^* = \{y|\, <x,y> \leqslant 1 \text{ for all } x \in C\}.$$

Note that $C^{**} = C$.

THEOREM 1. *Let C_1 and C_2 be closed convex sets containing the origin. Then $C_1 >_h C_2$ if and only if $C_2^* >_H C_1^*$.*

Proof. We can denote a general point $y \in \mathcal{E}^p$ by $y = rs$ where $r \geqslant 0$ and s is a point on the unit sphere. Then for a general closed convex set C containing the origin $h(C,s) = \sup\{r|y = rs \text{ in } C\}$. Then since $C^{**} = C$ this can be written

$$h(C,s) = \sup\{r|\, <x,rs> \leqslant 1 \text{ for all } x \text{ in } C^*\} = r^*, \text{ say,}$$

while

$$H(C^*,s) = \sup\{<x,s> |\text{ for all } x \in C^*\}.$$

Clearly if this supremum is achieved at x^* in C then $r^* = <x^*,s>^{-1}$. Thus $H(C^*,s) = h(C,s)^{-1}$, with the value taken as ∞ when $h(C,s) = 0$. This immediately gives the inverse relationship between the orderings expressed in the theorem. □

There are a number of results using the support function as a measure of the size of a convex set C. Most important of these is that based on the so-called mean width $W(C)$:

$$W(C) = E\{H(C,s) + H(C,-s)\},$$

where E denotes expectation with respect to random s. The quantity $W(C)$ is invariant under change or origin and according to a result due originally to Crofton (see Hadwiger (1957) for the general case) $W(C)$ is proportion to the surface area of C. The constant of proportionality depends only on how one defines the measure of area of the unit sphere. Now in our case C contains the origin so that $E\{H(C,s)\} = ½W(C)$. It is clear that $>_H$ implies the ordering of $E(H)$. Thus Theorem 1 combined with the Crofton result gives the following corollary: $C_1 >_h C_2$ implies that the surface area of C_2^* is at least that of C_1^*. We also have that the volume of C_1 is at least that of C_2. These necessary results can be used to obtain counterexamples to or conjectures about the ordering $>_h$. We shall return to this idea later. For a recent related paper on integral geometry see Enns and Ehlers (1980).

2. Scheffé Polyhedra. In the multiple comparison literature many interesting regions are obtained as one or two-sided confidence regions, or their translation to the origin. Thus in some testing or confidence procedures we may construct intervals based on estimates

$\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_p)$ for parameters $\theta = (\theta_1, \dots, \theta_p)$ which take the form $<c_i, \hat{\theta} - \theta> \leq a_i$, $(i=1, \dots, q)$ where the c_i are vectors of known coefficients. Writing $\mathbf{X} = \hat{\theta} - \theta$ we obtain $<c_i, \mathbf{X}> \leq a_i$, $(i = 1, \dots, q)$. In many applications the vector $\mathbf{X}$ can be assumed to have a spherically symmetric distribution about the origin. Moreover the standardisation may make all the a_i equal. If this is the case then we term the $\mathbf{X}$-region so defined a Scheffé region (or polyhedron): the polyhedron consisting of the intersection of half spaces each of whose boundary hyperplane is tangent to the same sphere centred at the origin. Without loss of generality in what follows we shall take the sphere to be the unit sphere. Thus define for any collection of distinct points $S = \{s_1, \dots, s_q\}$ in the unit sphere a Scheffé region to be

$$C(S) = \{<x, s_i> \leq 1, i = 1, \dots, q\}.$$

For two-sided regions $s_i \in S$ implies $-s_i \in S$ $(i = 1, \dots, q)$. In this case $C(S)$ is centrally symmetric.

Because we choose to use the unit sphere, the dual of a Scheffé region $C(S)$ based on $S = \{s_1, \dots, s_q\}$ is just the convex hull of S, $C(S)^* = \text{conv}(S)$, and every s_i is an extreme point of $C(S)^*$. While $C(S)$ circumscribes the unit sphere $C(S)^*$ inscribes it.

From a statistical point of view the main interest is in giving conditions or examples for which $C(S_1) >_h C(S_2)$ holds. We can then make claims about the size of confidence level of the relevant procedure. Sometimes $C(S_1)$ will arise as a non-standard case whereas $C(S_2)$ may have a well known form and the μ content be tabulated. The claim then would be that the test based on S_1 may be conservative so that $\mu\{C(S_2)\}$ provides a lower bound to the confidence level or size. We now interpret some known results in the light of the geometric considerations of the last section.

Let $p = 2$ and order the s_i around the unit circle so that the angles subtended at the origin between adjacent s_i are given by

$$\cos^{-1}<s_i, s_{i+1}> = \theta_i, (i = 1, \dots, q-1), \cos^{-1}<s_q, s_1> = \theta_q.$$

THEOREM 2. *Let $p = 2$ and $C(S_1)$ and $C(S_2)$ with angles (as above) $\theta = (\theta_1, \dots, \theta_q)$ and $\phi = (\phi_1, \dots, \phi_q)$. Then $C(S_1) >_h C(S_2)$ if and only if $\theta \succ \phi$ in the sense of majorization.*

Note that if S_2 has more points s_i than S_1 then we merely extend θ by adjoining the requisite number of zeros. Theorem 2 appears in Marshall and Olkin (1979, Chapter 8, Proposition E7) based on the earlier work of Wynn (1975); see also Bohrer and Wynn (1982). None of this work mentions the duality of the last section. The consequence of Theorem 1 is that $\theta \succ \phi$ is also equivalent to $C(S_2)^* >_H C(S_1)^*$ but the $C(S)^*$ are now the inscribed polygons. The Croften result then shows that the perimeter of S_2 is greater than that of S_1. The results of Marshall and Olkin in the same section (1979, Chapter 8, Proposition E1 to E6) concerning the area and perimeter of the inscribed figure are very close to this.

One of the best known results is that of Das Gupta, Eaton, Olkin, Perlman, Savage and Sobel (1972) and can be rewritten with a more geometric flavour. A p-pyramid is defined to be the convex hull of the union of a convex set K in $\mathcal{E}^{p-1}$ containing the origin and a line segment $[0, x]$. For a discussion of the general case see Grunbaum (1967). Three dimensional visualisation is useful with K being the base of the pyramid and $[0, x]$ the (not necessarily vertical) axis. Thus we define the pyramid as $P = \text{conv}(K \cup [0, x])$. A bipyramid is $P = \text{conv}(K \cup [-x, x])$. The results of Das Gupta, Eaton, Olkin, Perlman, Savage and Sobel (1972) can be restated as follows: For any spherically symmetric measure μ, the μ content of the dual P^* increases as x rotates towards the plane of K while lying in a fixed two dimensional "vertical" plane, when K is radially symmetric.

THEOREM 3. *Let K be a convex radially symmetric set in $\mathcal{E}^{p-1}$. Let a be a fixed unit vector in $\mathcal{E}^{p-1}$ and e_p the unit vector orthogonal to $\mathcal{E}^{p-1}$. Let $c > 0$ be a fixed constant and define a general vector $x = c\{(1-\lambda)^{1/2} e_p + \lambda^{1/2} a\}$. Let $P^*(\lambda)$ be the dual of the bipyramid $P(\lambda) = \mathrm{conv}(K \cup [-x, x])$. Then the ordering $>_h$ is increasing in λ, for $\lambda > 0$, in the sense that $\lambda_1 \geqslant \lambda_2 > 0$ implies $P^*(\lambda_1) >_h P^*(\lambda_2)$.*

The dual result that $P(\lambda)$ is decreasing in λ with respect to $>_H$ seems to be a new result in convex geometry although the implied decrease in surface area follows from an element argument. The case $p = 2$ follows by elementary geometry. A purely geometric proof in the case $p = 3$ has been given by the authors (Bohrer and Wynn (1983)). The special case when K has $2(p-1)$ vertices so that $P^*(\lambda)$ is p-dimensional parallelogram is the generalisation of a result due to Šidák (1968) for normal distributions.

The paper of Wynn and Bohrer (1982) classifies the platonic solids C_4 tetrahedron, C_6 cube, C_8 octahedron, C_{12} pentagonal dodecahedron and C_{20} icosahedron so that $C(S_i) >_h C(S_j)$ when $i < j$ when all the figures are incident to the same (unit) sphere centred at the origin. The dual result is that $C_{12} >_H C_{20} >_H C_6 >_H C_8 >_H C_4$ when all the solids inscribe the same sphere. This is because (with lazy notation) $C_4 \equiv C_4^*$, $C_6 \equiv C_8^*$ and $C_{12} = C_{20}^*$. The implied ordering for volume and surface area must have been known from antiquity but the statement for the orderings $>_h$ and $>_H$ seem to be new. The work also studied the semi-regular rhomboidal dodecahedron C'_{12}. This arises out of Studentised range test with four means. It is a 12-sided solid each of whose faces is a rhombus with semi-axes in the ratio $1: \sqrt{2}$. We showed that $C_8 >_h C'_{12} >_h C_{12}$. The dual result for the inscribed solids is that $C_{20} >_H C'^{\star}_{12} >_H C_6$. The solid $C'^{\star}_{12}$ is called the cuboctahedron and is obtain by suitably cutting off the corners of the unit cube (see Coxeter (1948) for a full description of all the solids).

4. Counter-Examples and Conjectures. As mentioned above $E\{H(C,s)\}$ is proportional to the surface area for radially symmetric convex sets. Clearly $E(h(C,s)^2)$ is proportional to the volume. In any case uniform μ giving volume content is spherically symmetric. A useful property of $H(C,s)$ is that it is additive with respect to direct sums. Thus

$$H(C_1 + C_2, s) = H(C_1, s) + H(C_2, s),$$

so that $E\{H(C_1 + C_2, s)\} = E\{H(C_1, s)\} + E\{H(C_2, s)\}$. It is not clear that $>_H$ is preserved under the direct sum operation but one can certainly use the result for expectations to eliminate any reversal of the ordering. That is to say if $C_1 >_H C'_1$ and $C_2 >_H C'_2$ then it is impossible for $(C_1' + C_2') >_H (C_1 + C_2)$ to hold strictly. Thus by Theorem 1 if $C_1' >_h C_1$ and $C_2' >_h C_2$ it is impossible for $(C_1 + C_2) >_h (C_1' + C'_2)$ to hold strictly. This is a general indication that direct sums tend to preserve the direction of the ordering. The authors are engaged on a programme to search among regions generated under direct sums from the known results in the last section to establish a wide range of new examples. One interesting case is when C_1 and C'_1 are non centrally symmetric regions for which $C_1 >_h C_1$ and we put $C_1 = -C_2$ and $C'_1 = C'_2$. The direct sums then, statistically, are the regionsob tained from all pairwise contrasts among the defining linear functions of $C_1(C'_1)$. That is to say if $C_1 = C(S)$ where $S = s_1, \ldots, s_p$ then $C_1 + C_2 = C(S^-)$ where $S^- = s_i - s_j \mid i,j = 1, \ldots, q$.

The intuition for higher dimensions from Theorem 2 and the results on the Platonic solids is that in some general sense for a fixed number of $(p-1)$-dimensional faces the Scheffé region which is most regular is a minimal member of the ordering $>_h$. It appears that this is the case for the Platonic solids although the only one for which this is properly established is the cube with the added restriction of radial symmetry for which it follows from Theorem 3 above (Das Gupta, Olkin, Perlman, Savage and Sobel (1972)). The minimal member idea is closely related to minimal packing problems. Indeed it is well known that the p-simplex, p-cube and the p-dimensional generalisation of C'_{12} can be packed into $\mathcal{E}^p$. The use of orderings rather than volumes or other mean-size measures in packing theory may be new and will be the subject of a further paper. Another development which would be valuable would be a characterisation of these minimal regions in terms of their fundamental groups, that is the groups under which they remain invariant, where of course such a group exists. There must surely be a relationship between the structure of the finite subgroups of the full orthogonal group $0(p)$ and the $>_h$ ordering of their corresponding invariant Scheffé polyhedra. It was hoped to give some simple results in this paper but these too must wait for further developments.

Acknowledgements. We are grateful for conversations with Richard Vitale at the Nebraska Conference on Inequalities which partly led to the use of the support function, see his paper in this volume (Vitale (1983)). This is one reason why this paper differs in some degree from the one presented there. We are also grateful for discussion on stochastic geometry with Rodney Coleman.

REFERENCES

BOHRER, R. and WYNN, H. P. (1982). Spherically symmetric probability orderings useful in multiple comparisons. *Ann. Statist.* *10* 1253–1260.

BOHRER, R. and WYNN, H. P. (1983). A geometric proof of the DEOPSS inequality for spherically symmetric distributions in 3 dimensions. In: *Festschrift for E. L. Lehmann,* Wadsworth, Belmont, CA.

COLEMAN, R. (1979). An introduction to mathematical stereology. Memoirs 3. Department of Theoretical Statistics, University of Aarhus.

COXETER, H. S. M. (1948). *Regular Polytopes*. Methuen, London.

DAS GUPTA, S., EATON, M., OLKIN, I., PERLMAN, M. D., SAVAGE, L. J., and SOBEL, M. (1972). Inequalities on the probability content of convex regions for elliptically contoured distributions. *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* *2* 241–256.

ENNS, E. G. and EHLERS, D. F. (1980). Random paths originating within a convex region and terminating on its surface. *Austal. J. Statist.* *22* 60–68.

GRUNBAUM, B. (1967). *Convex Polytopes*. Wiley, New York.

HADWIGER, H. (1952). Uber zwei quadratische Distanzintegrale fur Eikorper. *Arch. Math.* (Basel) *3* 142–144.

MARSHALL, A. W. and OLKIN, I. (1979). *Inequalities: Theory of Majorization and Its Applications*. Academic Press, New York.

MCMULLEN, P. and SHEPHARD, G. C. (1971). *Convex Polytopes and the Upper Bound Conjecture*. London Math. Soc., Lecture Notes 3.

SANTALO, L. A. (1976). *Integral Geometry and Geometric Probability*. Addison-Wesley, Mass.

SERRA, J. (1982). *Image Analysis and Mathematical Morphology*. Academic Press, New York.

SIDÁK, Z. (1968). On multivariate normal probability of rectangles: their dependence on correlations. *Ann. Math. Statist.* *39* 1425–1434.

STEWART, I. N. and TALL, D. O. (1979). *Algebraic Number Theory*. Chapman and Hall, London.

TONG, Y. L. (1980). *Probability Inequalities in Multivariate Distributions*. Academic Press, New York.

VITALE, R. A. (1984). Probability measures on the circle and the isoperimetric inequality. In *Inequalities in Statistics nad Probability,* Y. L. Tong, ed., Institutes of Mathematical Statistics, Haywood, CA, 109–111.

WYNN, H. P. (1977). An inequality for certain bivariate probability integrals. *Biometrika 64* 411–414.