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PROBABILITY MEASURES ON THE CIRCLE AND THE ISOPERIMETRIC INEQUALITY

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The theory of planar convex sets invokes measures of a certain class. Accordingly, the isoperimetric inequality can be translated into quadratic inequalities for probability measures on the unit circle.

Various tools from convex geometry have been put to highly effective use in probability theory. One example is in the application by Anderson (1955) of the Brunn-Minkowski inequality to the probability content of symmetric, convex sets (see Tong (1980, chapter 4) for this and related results). A second instance is the resolution by Egorychev and Falikman of the van der Waerden conjecture by means of mixed discriminants, which arose originally in the study of mixed volumes (see Lagarias (1982)). The latter have recently been applied to combinatorial questions (Stanley (1981)).

That convex geometry and probability should be linked is not too surprising since each has a strong concern with the notion of positivity. What geometers have appreciated for some time, and what perhaps awaits systematic exploitation by probabilists, is that this link can be made concrete. This goes by the historical name of Minkowski's problem (Busemann (1958, pp. 60–67)). Roughly speaking, each compact, convex set in $\mathscr{R}^n$ can be identified with a bounded, positive measure on the unit sphere in that space. From the probabilist's point of view, a wide class of probability measures on the unit sphere *can be realized as compact, convex sets*. Existence of atoms, modes of convergence, and even statistical procedures have natural geometric analogs.

The author will treat some of these questions elsewhere. Here the flavor of the connection will be given by deriving two inequalities for probability measures μ on the unit circle $C = [0,2\pi)$.

INEQUALITY I.

$$\int_C\int_C g(\theta-\lambda)\mu(d\theta)\mu(d\lambda) \leq (2\pi)^{-1} \tag{I}$$

where $g(\lambda) = [2(\pi-\lambda)\sin\lambda - \cos\lambda]/4\pi$ on $[0,2\pi)$ and is extended 2π periodically. Equality holds iff μ admits the representation

$$\mu(d\lambda) = [(2\pi)^{-1} + c_1\cos\lambda + c_2\sin\lambda]d\lambda$$

for constants c_1, c_2.

INEQUALITY II.

$$\int_C\int_C |\sin(\theta-\lambda)|\mu(d\theta)\mu(d\lambda) \leq 2/\pi \tag{II}$$

with equality iff $1/2[\mu(d\lambda) + \mu(d(\lambda+\pi))] = (2\pi)^{-1}\,d\lambda$.

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Geometrically I and II are versions of the isoperimetric inequality. Once a certain amount of geometric machinery is in place, they fall out immediately. We sketch the arguments.

The point of departure is the notion of the *support function* of a compact, convex subset K of the plane,

$$s_K(\theta) = \max_{x \in K} \langle e_\theta, x \rangle, \; e_\theta = (\cos\theta, \sin\theta).$$

It is possible to show that the class of support functions coincides with functions of the form

$$s(\theta) = a\cos\theta + b\sin\theta + \int_C g(\theta - \lambda) R(d\lambda)$$

where a and b are constants, g is as described above, and R is a positive, bounded measure satisfying

$$(*) \qquad \int_C \sin\lambda\, R(d\lambda) = 0 = \int_C \cos\lambda\, R(d\lambda)$$

(Blaschke (1949, p. 116), Grenander (1976, p. 198), Vitale (1974, 1979)). The trigonometric term amounts to a location parameter so that the final term exhibits the correspondence between sets K and measures of the specified type. If K is a singleton, then R is the zero measure. More generally, the total mass assigned by R is the perimeter of K, $\mathrm{per}(K) = \int_C R(d\lambda)$, and the area of K is quadratic in R

$$\mathrm{area}(K) = 1/2 \int_C \int_C g(\theta - \lambda) R(d\theta) R(d\lambda).$$

Accordingly, the isoperimetric inequality reads

$$\text{(ISO)} \qquad 4\pi \cdot 1/2 \int_C \int_C g(\theta - \lambda) R(d\theta) R(d\lambda) \leq \left[\int_C R(d\lambda)\right]^2.$$

with equality iff R is a constant multiple of Lebesgue measure.

Note that (ISO) can be asserted only for measures which annihilate sin and cos. Inequalities I and II represent two ways of approaching this constraint.

For (I), begin by orthogonalizing a given probability measure μ to sin and cos. This probably yields negative values so add back a constant multiple of Lebesgue measure. Thus

$$\hat{\mu}(d\lambda) = \mu(d\lambda) + \left[\int \pi^{-1}\{1 - \cos(\lambda - \theta)\}\, \mu(d\theta)\right] d\lambda.$$

(ISO) holds for $\hat{\mu}$. It is direct to show that g annihilates sin and cos, so that the quadratic term can be written with the measure $\mu(d\lambda) + (\pi)^{-1}\, d\lambda$. Noting that $\int \hat{\mu}\, (d\lambda) = 3$ and simplifying expressions yields (I).

For (II), we observe that $|\sin\theta|$ is the support function of the line segment $-1 \leq y \leq 1$. Using a probability measure μ for convex combination leads to a support function $\sigma(\theta) = \int |\sin(\theta - \lambda)| \mu(d\lambda)$. Now $|\sin\theta| = \int g(\theta - \lambda) R(d\lambda)$ where R assigns mass two to $\lambda = 0, \pi$. Then

$$\sigma(\theta) = \int_C \int_C g(\theta - \lambda - \gamma) \mu(d\lambda) R(d\gamma).$$

This is the required representation (the bivariate measure integrating the kernel could in principle be reduced to a univariate one). Writing out (ISO) and simplifying yields (II).

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