CHAPTER 4

More properties of zonal polynomials

This chapter is a collection of results which are for the most part generalizations and refinements of the basic results given in Chapter 3. A particular emphasis is placed on the coefficients of zonal polynomials. In this respect this chapter contains new results and presumably covers almost all known results. On the other hand we do not survey various known identities involving zonal polynomials. For this purpose the reader is referred to an excellent survey paper by Subrahmaniam (1976). Actually in the discussion of the orthogonally invariant distributions we saw that zonal polynomials satisfy an infinite number of identities. It is a rather frustrating fact that although many identities for zonal polynomials are already known, explicit forms of zonal polynomials are not known.

§ 4.1 MAJORIZATION ORDERING

The proof of Theorem 3.2.1 which played an essential role for the subsequent development in Chapter 3 is not complete as it is. In (3.2.11) we argued that

$$(1) \qquad 0 = \sum_{q,q'} (\lambda_{\nu_0 p} - \lambda_{\nu_0 q}) c_{qq'} \mathcal{Y}_q(\boldsymbol{\Sigma}) \mathcal{Y}_{q'}(\boldsymbol{B}),$$

for all symmetric $\boldsymbol{B}$ and all positive semidefinite $\boldsymbol{\Sigma}$ implies $(\lambda_{\nu_0 p} - \lambda_{\nu_0 q}) c_{qq'} = 0$. One objection may be that $\boldsymbol{\Sigma}$ is restricted to be positive semidefinite.

But this causes no trouble since (1) is a polynomial and a polynomial which is identically zero for nonnegative arguments has to be zero everywhere. A more serious question is that the dimensionality of $\boldsymbol{\Sigma}$ and $\boldsymbol{B}$ is fixed to be $k \times k$ which is the dimensionality of the uniform orthogonal matrix $\boldsymbol{H}$. The same objection applies to the proof of Lemma 3.4.1.

What we have to consider is the space of *n-th* degree homogeneous symmetric polynomials in k variables, where k is fixed. We denote this space by $V_{n,k}$. Let $m < k$. If $f(x_1, \ldots, x_k) \in V_{n,k}$ then $f(x_1, \ldots, x_m, 0, \ldots, 0) \in V_{n,m}$. In this sense $V_{n,m}$ can be considered as a subspace of $V_{n,k}$. Now from the argument in Section 2.2 it follows that if $k \geq n$ then $\{ \mathcal{M}_p, p \in \mathcal{P}_n \}$ forms a basis of $V_{n,k}$. Therefore if $k \geq n$, $V_{n,k}$ are all isomorphic to V_n. However if $k < n$ then for p such that $\ell(p) > k$ $\mathcal{M}_p$ is identically 0. Therefore $\dim V_{n,k} < \dim V_n$. In order to proceed further we have to identify bases of $V_{n,k}$ for $k < n$.

For this purpose we now study homogeneous symmetric polynomials again from the viewpoint of majorization ordering. The following is a refinement of Lemma 2.2.2.

Lemma 1.

$$\mathcal{U}_p = \mathcal{M}_p + \sum_{q \prec p, q \neq p} a_{pq} \mathcal{M}_q, \tag{2}$$

$$\mathcal{M}_p = \mathcal{U}_p + \sum_{q \prec p, q \neq p} a^{pq} \mathcal{U}_q. \tag{3}$$

Proof. For $1 \leq r < k$ let $\alpha = x_1 = \cdots = x_r$. Then the degree of α in $\mathcal{M}_q$ is $q_1 + \cdots + q_r$. The degree of α in $\mathcal{U}_p$ is $p_1 + \cdots + p_r$ because

$$\begin{aligned} &\mathcal{U}_p(\alpha, \ldots, \alpha, x_{r+1}, \ldots) \\ &\quad = (\alpha + \cdots)^{p_1 - p_2} (\alpha^2 + \cdots)^{p_2 - p_3} \cdots \\ &\quad\quad \cdot (\alpha^r + \cdots)^{p_r - p_{r+1}} (\alpha^r x_{r+1} \cdots)^{p_{r+1} - p_{r+2}} \cdots (\alpha^r x_{r+1} \cdots x_\ell + \cdots)^{p_\ell} \\ &\quad = c\alpha^{(p_1 - p_2) + 2(p_2 - p_3) + \cdots + r(p_r - p_{r+1}) + \cdots + r p_\ell} + \cdots \\ &\quad = c\alpha^{p_1 + \cdots + p_r} + \cdots, \end{aligned} \tag{4}$$

where c is a term not containing α. Let $Q_r = \{ q \mid q < p \quad and \quad q_1 + \cdots + q_r > p_1 + \cdots + p_r \}$. Now since the degree of α in (2) is $p_1 + \cdots + p_r$ we have

$$\sum_{q \in Q_r} a_{pq} \mathcal{M}_q(\alpha, \ldots, \alpha, x_{r+1}, \ldots) = 0. \tag{5}$$

Now $\mathcal{M}_q(\alpha, \ldots, \alpha, x_{r+1}, \ldots, x_k)$ are linearly independent if k is sufficiently large. Therefore $a_{pq} = 0$ for $q \in Q_r$. Repeating this argument for $r = 1, 2, \ldots$ we have

$$a_{pq} = 0 \quad if \quad q \in Q_1 \cup Q_2 \cup \cdots. \tag{6}$$

But if q is not majorized by p there exists an r such that $q \in Q_r$. Therefore $a_{pq} = 0$ for every q which is not majorized by p. This proves (2). (3) can be proved similarly. ∎

Lemma 2. *$\{ \mathcal{M}_p, p \in \mathcal{P}_n, \ell(p) \leq k \}$, $\{ \mathcal{U}_p, p \in \mathcal{P}_n, \ell(p) \leq k \}$ are bases of $V_{n,k}$.*

Proof. Note that $\mathcal{M}_p(x_1, \ldots, x_k) = 0$, $\mathcal{U}_p(x_1, \ldots, x_k) = 0$ for p such that $\ell(p) > k$. Let $f \in V_{n,k}$. Then from (2.2.2)

$$\begin{aligned} f(x_1, \ldots, x_k) &= \sum_{p \in \mathcal{P}_n} a_p \mathcal{M}_p(x_1, \ldots, x_k) \\ &= \sum_{p \in \mathcal{P}_n, \ell(p) \leq k} a_p \mathcal{M}_p(x_1, \ldots, x_k). \end{aligned} \tag{7}$$

Therefore any $f \in V_{n,k}$ can be written as a linear combination of $\mathcal{M}_p$'s for which $p \in \mathcal{P}_n, \ell(p) \leq k$. Now suppose

$$\sum_{q \in \mathcal{P}_n, \ell(q) \leq k} a_q \mathcal{M}_q(x_1, \ldots, x_k) = 0. \tag{8}$$

Then differentiating (8) p_i times with respect to x_i, $i = 1, \ldots, \ell(p)$, (note $\ell(p) \leq k$) we obtain $a_p = 0$. Therefore $\{ \mathcal{M}_p, p \in \mathcal{P}_n, \ell(p) \leq k \}$ is linearly independent in $V_{n,k}$. This shows that $\{ \mathcal{M}_p, p \in \mathcal{P}_n, \ell(p) \leq k \}$ is a basis of $V_{n,k}$. To show that $\{ \mathcal{U}_p, p \in \mathcal{P}_n, \ell(p) \leq k \}$ is a basis it suffices to observe

$$\begin{aligned} \mathcal{M}_p(x_1,\dots,x_k) &= \mathcal{U}_p(x_1,\dots,x_k) + \sum_{q<p} a^{pq}\mathcal{U}_q(x_1,\dots,x_k) \\ &= \mathcal{U}_p(x_1,\dots,x_k) + \sum_{q<p,\ell(q)\leq k} a^{pq}\mathcal{U}_q(x_1,\dots,x_k). \end{aligned} \tag{9}$$

This and (7) with f replaced by $\mathcal{U}_p$ shows that $\{ \mathcal{U}_p, p \in \mathcal{P}_n, \ell(p) \leq k \}$ is another basis of $V_{n,k}$. ∎

Remark 1. It is known that a_{pq} in (2) is nonzero and positive if and only if $q \prec p$. This is called the Gale-Ryser theorem. (See Macdonald (1979), Marshall and Olkin (1979).)

Now we prove the following.

Theorem 1.

$$\mathcal{Y}_p = \sum_{q\prec p} a_{pq}\mathcal{U}_q = \sum_{q\prec p} a'_{pq}\mathcal{M}_q \tag{10}$$

for some real numbers a_{pq}, a'_{pq} *and* $\{ \mathcal{Y}_p, p \in \mathcal{P}_n, \ell(p) \leq k \}$ *forms a basis of* $V_{n,k}$.

Proof. We first note that majorization is transitive, i.e. if $p^1 \succ p^2, p^2 \succ p^3$ then $p^1 \succ p^3$. Therefore in view of Lemma 1 the equalities involving $\mathcal{U}$'s and $\mathcal{M}$'s are equivalent. Hence we prove one involving $\mathcal{U}$'s. Now as in the proof of (2), the right hand side of (3.1.10)

$$\mathcal{E}_W\Big(\sum_{i_1} \alpha_{i_1} W(i_1)\Big)^{p_1-p_2}\Big(\sum_{i_1<i_2} \alpha_{i_1}\alpha_{i_2} W(i_1,i_2)\Big)^{p_2-p_3}\cdots \tag{11}$$

has only those monomial terms $\mathcal{M}_q(\boldsymbol{A})$ for which $q \prec p$. Therefore we can write

$$\tau_\nu(\mathcal{U}_p) = \sum_{q\prec p} b_{pq}\mathcal{M}_q. \tag{12}$$

Substituting (3) into (12) and using the transitivity of majorization we obtain

$$\tau_\nu(\mathcal{U}_p) = \sum_{q \prec p} b'_{pq}\mathcal{U}_q. \tag{13}$$

Now let

$$\mathcal{Y}_p = \sum_{q \leq p} a_{pq}\mathcal{U}_q. \tag{14}$$

We want to show that $Q_p = \{ q \mid a_{pq} \neq 0, q \textit{ not majorized by } p \}$ is empty. We argue by contradiction. Suppose that Q_p is nonempty. Let q^* be the highest partition in Q_p. Then $a_{pq} \neq 0$ and $q > q^*$ imply $q \prec p$. For any such q

$$a_{pq}\tau_\nu(\mathcal{U}_q) = a_{pq}\{ \sum_{q' \prec q} b'_{qq'}\mathcal{U}_{q'} \}. \tag{15}$$

Now $q' \prec q$, $q \prec p$ imply $q' \prec p$. Hence the right hand side does not have $\mathcal{U}_{q^*}$ term. It follows that $\mathcal{U}_{q^*}$ does not appear in

$$\sum_{q^* < q \leq p} a_{pq}\tau_\nu(\mathcal{U}_q). \tag{16}$$

Obviously

$$\sum_{q < q^*} a_{pq}\tau_\nu(\mathcal{U}_q) \tag{17}$$

does not involve $\mathcal{U}_{q^*}$ term either. Therefore the coefficient of $\mathcal{U}_{q^*}$ in

$$\begin{aligned} \tau_\nu(\mathcal{Y}_p) &= \sum_{q \leq p} a_{pq}\tau_\nu(\mathcal{U}_q) \\ &= a_{pq^*}\tau_\nu(\mathcal{U}_{q^*}) + (16) + (17) \end{aligned} \tag{18}$$

is $a_{pq^*}\lambda_{\nu q^*}$. On the other hand

$$\tau_\nu(\mathcal{Y}_p) = \lambda_{\nu p}\mathcal{Y}_p. \tag{19}$$

Therefore the coefficient of $\mathcal{U}_{q^*}$ on the right hand side of (19) is $\lambda_{\nu p} a_{pq^*}$. Taking $\nu = \nu_0$ we have a contradiction (see the proof of Lemma 3.1.4 for ν_0). Therefore Q_p is empty. This proves (10).

To prove the second assertion we note that $q \prec p$ implies $\ell(q) \geq \ell(p)$. Otherwise $p_1 + \cdots + p_{\ell(q)} < p_1 + \cdots + p_{\ell(p)} = n = q_1 + \cdots + q_{\ell(q)}$ and this contradicts $q \prec p$. Therefore in (10) we have only those $\mathcal{U}_q$'s for which $\ell(q) \geq \ell(p)$. Now suppose that $\boldsymbol{A}$ is $k \times k$ and $k < \ell(p)$. Then every $\mathcal{U}_q(\boldsymbol{A})$ in (10) vanishes. Hence

$$\mathcal{Y}_p(\boldsymbol{A}) = 0 \quad \textit{if } \boldsymbol{A} \textit{ is } k \times k \textit{ and } k < \ell(p). \tag{20}$$

Now write

$$\mathcal{U}_p = \sum_{q \leq p} a^{pq} \mathcal{Y}_q. \tag{21}$$

Then

$$\begin{aligned} \mathcal{U}_p(x_1, \ldots, x_k) &= \sum_{q \leq p} a^{pq} \mathcal{Y}_q(x_1, \ldots, x_k) \\ &= \sum_{q \leq p, \ell(q) \leq k} a^{pq} \mathcal{Y}_q(x_1, \ldots, x_k). \end{aligned} \tag{22}$$

Similarly

$$\mathcal{Y}_p(x_1, \ldots, x_k) = \sum_{q \leq p, \ell(q) \leq k} a_{pq} \mathcal{U}_q(x_1, \ldots, x_k). \tag{23}$$

In view of Lemma 2, (22) and (23) imply that $\{ \mathcal{Y}_p, p \in \mathcal{P}_n, \ell(p) \leq k \}$ forms a basis of $V_{n,k}$. ∎

In the proofs of Theorem 3.2.1 and Lemma 3.4.1 we replace all the summations by

$$\sum_{p \in \mathcal{P}_n, \ell(p) \leq k} , \qquad \sum_{\substack{q, q' \in \mathcal{P}_n \\ \ell(q) \leq k, \ell(q') \leq k}} \qquad \textit{etc.} \tag{24}$$

Then those proofs are complete. We do not repeat the steps of those proofs. But in later proofs we will be careful.

Remark 2. Using the Gale-Ryser theorem (Remark 1), (3.3.2), and (3.3.8) it can be shown that a'_{pq} in (10) is positive if and only if $q \prec p$. This is stronger than Theorem 1.

For future references we record (20) as a corollary.

Corollary 1. *If $\boldsymbol{A}$ is $k \times k$ and $\ell(p) > k$ then $\mathcal{Y}_p(\boldsymbol{A}) = 0$.*

With Theorem 1 we can strengthen the converse part of Theorem 3.1.1.

Theorem 2. *Let integers n, k and a partition $p \in \mathcal{P}_n$ be given with $\ell(p) \leq k$. Suppose that f satisfies the following conditions:*

(i) $f \in V_{n,k}$.

(ii) *The leading term in f is the partition p, i.e.*

$$f = \sum_{q \leq p, \ell(q) \leq k} a_{pq\,1}\mathcal{Y}_q,$$

for some real numbers a_{pq} with $a_{pp} \neq 0$.

(iii) *For some constants c_ν,*

$$\mathcal{E}_W f(\boldsymbol{AW}) = c_\nu f(\boldsymbol{A}),$$

for all $k \times k$ symmetric $\boldsymbol{A}$ and for all sufficiently large degrees of freedom ν.

Then $f = a_{pp\,1}\mathcal{Y}_p$ and $c_\nu = \lambda_{\nu p}$.

Proof. From (ii)

$$\mathcal{E}_W f(\boldsymbol{AW}) = \sum_{q \leq p, \ell(q) \leq k} a_{pq} \lambda_{\nu q\,1}\mathcal{Y}_q(\boldsymbol{A}). \tag{25}$$

On the other hand by (iii)

$$\mathcal{E}_W f(\boldsymbol{AW}) = c_\nu \sum_{q \leq p, \ell(q) \leq k} a_{pq\,1}\mathcal{Y}_q(\boldsymbol{A}). \tag{26}$$

Hence

$$0 = \sum_{q \leq p, \ell(q) \leq k} a_{pq}(c_\nu - \lambda_{\nu q})\,{}_1\mathcal{Y}_q(\boldsymbol{A}) \tag{27}$$

for all $k \times k$ symmetric matrix $\boldsymbol{A}$. Therefore by Theorem 1 we have $a_{pq}(c_\nu - \lambda_{\nu q}) = 0$ for all q. Considering $q = p$ we obtain $c_\nu = \lambda_{\nu p}$. Then for all $q < p$ we have $a_{pq}(\lambda_{\nu p} - \lambda_{\nu q}) = 0$. Taking $\nu = \nu_0$ we have $a_{pq} = 0$ for all $q < p$. Therefore $f = a_{pp}\, {}_1\mathcal{Y}_p$. This completes the proof. ∎

§ 4.2 EVALUATION OF ${}_1\mathcal{Y}_p(\boldsymbol{I}_k)$

In the sequel we often work with a normalization denoted by ${}_1\mathcal{Y}_p$ which has the leading coefficient 1, namely

$$(1) \qquad {}_1\mathcal{Y}_p = \mathcal{U}_p + \sum_{q<p} {}_1a_{pq}\mathcal{U}_q.$$

Advantages of this normalization will become clear soon.

Remark 1. In several places we already have used the expressions "leading term" or "leading coefficient". Here and in the sequel "leading" refers to the highest partition when a homogeneous symmetric polynomial is expressed in terms of bases $\{\mathcal{U}_p\}$, $\{\mathcal{M}_p\}$, or $\{\mathcal{Y}_p\}$.

We shall evaluate ${}_1\mathcal{Y}_p(\boldsymbol{I}_k)$. From Theorem 3.2.4 we know that ${}_1b_p \equiv \lambda_{kp}/{}_1\mathcal{Y}_p(\boldsymbol{I}_k)$ is a constant independent of k. Therefore our goal is to obtain ${}_1b_p$. Now

$$(2) \qquad {}_1b_p\, {}_1\mathcal{Y}_p(\boldsymbol{I}_k) = \lambda_{kp} = \mathcal{Z}_p(\boldsymbol{I}_k).$$

Therefore ${}_1b_p$ is the leading coefficient of $\mathcal{Z}_p$. This was needed for the unique decomposition of the left hand side of (3.4.11). We use the following recursive relation.

Theorem 1. *If $\boldsymbol{A}$ is a $k \times k$ symmetric matrix, then*

$$(3) \qquad |\boldsymbol{A}|\, {}_1\mathcal{Y}_p(\boldsymbol{A}) = {}_1\mathcal{Y}_{p+(1^k)}(\boldsymbol{A}),$$

where $p + (1^k) = (p_1 + 1, p_2 + 1, \ldots, p_k + 1, p_{k+1}, \ldots) \in \mathcal{P}_{n+k}$, $n = |p|$.

Proof. If $\ell(p) > k$ then ${}_1\mathcal{Y}_p(A) = 0$ by Corollary 4.1.1. In this case $\ell(p + (1^k)) = \ell(p) > k$. Hence ${}_1\mathcal{Y}_{p+(1^k)}(A) = 0$. (3) holds trivially in this case. Now let $\ell(p) \leq k$. We can use Theorem 4.1.2. Let $f(A)$ denote the left hand side of (3). Clearly $f \in V_{n+k,k}$ and (i) of Theorem 4.1.2 is verified. With respect to the basis $\{\mathcal{U}_p\}$ the leading term in ${}_1\mathcal{Y}_p$ is $\mathcal{U}_p$. Since $|A|\mathcal{U}_p(A) = \mathcal{U}_{p+(1^k)}(A)$ the leading term in f is $\mathcal{U}_{p+(1^k)}$ and this implies (ii). Now consider

$$\mathcal{E}_W f(AW) = \mathcal{E}_W\{|AW|\,{}_1\mathcal{Y}_p(AW)\} = |A|\mathcal{E}_W\{|W|\,{}_1\mathcal{Y}_p(AW)\}. \tag{4}$$

Note that we can absorb $|W|$ into the Wishart density which is proportional to $|W|^{(\nu-p-1)/2}\exp(-\frac{1}{2}\operatorname{tr} W)$. This changes the degrees of freedom of the Wishart density, but in any case we have $\mathcal{E}_W\{|W|\,{}_1\mathcal{Y}_p(AW)\}= c_\nu\,{}_1\mathcal{Y}_p(A)$ for some c_ν. (Explicit evaluation of c_ν is straightforward, but we do not need it.) Hence

$$\mathcal{E}_W f(AW) = |A|\,c_\nu\,{}_1\mathcal{Y}_p(A) = c_\nu f(A) \tag{5}$$

and (iii) is verified. Therefore by Theorem 4.1.2 $|A|\,{}_1\mathcal{Y}_p(A) = c\,{}_1\mathcal{Y}_{p+(1^k)}(A)$ for some c. Comparing the leading term with respect to the basis $\{\mathcal{U}_q\}$ we obtain c=1. This completes the proof. ∎

Corollary 1. *(Formula(129) in James (1964)) Let $p = (p_1, \ldots, p_\ell)$ and $p - (p_\ell^\ell) = (p_1 - p_\ell, p_2 - p_\ell, \ldots, p_{\ell-1} - p_\ell)$. Then for an $\ell \times \ell$ symmetric A*

$${}_1\mathcal{Y}_p(A) = |A|^{p_\ell}\,{}_1\mathcal{Y}_{p-(p_\ell^\ell)}(A). \tag{6}$$

Proof. $|A|^{p_\ell}\,{}_1\mathcal{Y}_{p-(p_\ell^\ell)}(A) = |A|^{p_\ell - 1}\,{}_1\mathcal{Y}_{p-(p_\ell^\ell)+(1^\ell)}(A) = \cdots = {}_1\mathcal{Y}_p(A)$. ∎

Applying Corollary 1 to the identity matrices of appropriate dimensionalities we can evaluate ${}_1b_p$ in (2).

Theorem 2.

$${}_1b_p = 2^{|p|}\prod_{i=1}^{\ell(p)}\prod_{j=1}^{i}\left(\frac{1}{2}i - \frac{1}{2}(j-1) + p_j - p_i\right)_{p_i - p_{i+1}}, \tag{7}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$.

Proof. We prove this by induction on the length of p. Let $\ell(p) = 1$, namely $p = (p_1)$. Then

$$ {}_1\mathcal{Y}_p(\boldsymbol{I}_1) = \mathcal{U}_p(\boldsymbol{I}_1) = 1^{p_1} = 1. \tag{8}$$

Therefore

$$\begin{aligned} {}_1b_p &= \lambda_{1p} / {}_1\mathcal{Y}_p(\boldsymbol{I}_1) \\ &= 1\cdot 3\cdots(2p_1-1) \\ &= 2^{p_1}\left(\frac{1}{2}\right)_{p_1}, \end{aligned} \tag{9}$$

which is of the form (7). Now suppose that (7) is true for $\ell(p) = k-1$. We want to show that then (7) holds for $\ell(p) = k$. Let $p=(p_1,\dots,p_k)$ and $p-(p_k^k)=(p_1-p_k, p_2-p_k, \dots, p_{k-1}-p_k)$. Note that $\ell(p-(p_k^k)) = k-1$. Putting $\boldsymbol{I}_k$ in (6) we obtain

$$ {}_1\mathcal{Y}_p(\boldsymbol{I}_k) = {}_1\mathcal{Y}_{p-(p_k^k)}(\boldsymbol{I}_k) \tag{10}$$

or

$$ {}_1b_p = {}_1b_{p-(p_k^k)}\frac{\lambda_{kp}}{\lambda_{k,p-(p_k^k)}}. \tag{11}$$

Using the induction hypothesis

(12)

$$
\begin{aligned}
{}_1b_p &= \prod_{i=1}^{k-1}\prod_{j=1}^{i}\left(\frac{1}{2}i-\frac{1}{2}(j-1)+(p_j-p_k)-(p_i-p_k)\right)_{(p_i-p_k)-(p_{i+1}-p_k)} \\
&\quad\cdot 2^{|p-(p_k^k)|}\cdot\frac{\lambda_{kp}}{\lambda_{k,p-(p_k^k)}} \\
&= \prod_{i=1}^{k-1}\prod_{j=1}^{i}\left(\frac{1}{2}i-\frac{1}{2}(j-1)+p_j-p_i\right)_{p_i-p_{i+1}} \\
&\quad\cdot 2^{|p-(p_k^k)|}\cdot\frac{2^{|p|}\prod_{j=1}^{k}\Gamma[p_j+\frac{1}{2}(k+1-j)]/\Gamma[\frac{1}{2}(k+1-j)]}{2^{|p-(p_k^k)|}\prod_{j=1}^{k-1}\Gamma[p_j-p_k+\frac{1}{2}(k+1-j)]/\Gamma[\frac{1}{2}(k+1-j)]} \\
&= 2^{|p|}\prod_{i=1}^{k-1}\prod_{j=1}^{i}\left(\frac{1}{2}i-\frac{1}{2}(j-1)+p_j-p_i\right)_{p_i-p_{i+1}} \\
&\quad\cdot\prod_{j=1}^{k}\left(\left(\frac{1}{2}k-\frac{1}{2}(j-1)+p_j-p_k\right)\right)_{p_k} \\
&= 2^{|p|}\prod_{i=1}^{k}\prod_{j=1}^{i}\left(\frac{1}{2}i-\frac{1}{2}(j-1)+p_j-p_i\right)_{p_i-p_{i+1}}.
\end{aligned}
$$

Therefore (7) holds for $k=\ell(p)$ and the theorem is proved. ∎

There is a curious fact about ${}_1b_p$. Let $k!!$ denote $1\cdot 3\cdots k$ or $2\cdot 4\cdots k$ depending on whether k is odd or even. Then as above it can be shown by induction that

$$
{}_1b_p=\prod_{i<j}\frac{(2p_i-2p_j-i+j-2)!!}{(2p_i-2p_j-i+j-1)!!}\prod_{i=1}^{\ell(p)}(2p_i-i+\ell(p)-1)!!. \tag{13}
$$

Now $(\operatorname{tr}\boldsymbol{A})^n=\sum d_p Z_p(\boldsymbol{A})=\sum d_p\,{}_1b_p\,{}_1\mathcal{Y}_p(\boldsymbol{A})$. From (3.4.12)

$$
\begin{aligned}
d_p\,{}_1b_p &= 2^n n!\prod_{i<j}(2p_i-2p_j-i+j)\Big/\prod_{i=1}^{\ell(p)}(2p_i-i+\ell(p))! \\
&\quad\cdot\prod_{i<j}\frac{(2p_i-2p_j-i+j-2)!!}{(2p_i-2p_j-i+j-1)!!}\prod_{i=1}^{\ell(p)}(2p_i-i+\ell(p)-1)!! \qquad (14)\\
&= 2^n n!\prod_{i<j}\frac{(2p_i-2p_j-i+j)!!}{(2p_i-2p_j-i+j-1)!!}\Big/\prod_{i=1}^{\ell(p)}(2p_i-i+\ell(p))!!.
\end{aligned}
$$

This is very similar to ${}_1b_p^{-1}$ if we ignore the constant $2^n n!$. In Section 5.3 we will see that in the complex case the corresponding quantities $\tilde{d}_p$, ${}_1\tilde{b}_p$ satisfy an exact relation $\tilde{d}_p({}_1\tilde{b}_p)^2 = n!$.

§ 4.3 MORE ON INTEGRAL IDENTITIES

In this section we evaluate the constant c_p in Theorem 3.2.2 for several distributions. The first one is the inverted Wishart distribution. See Khatri (1966), Constantine (1963).

Lemma 1. *Let W be distributed according to $\mathcal{W}(I_k, \nu)$, $\nu > 2h(p) + k - 1$. Then for symmetric A*

$$\mathcal{E}_W \mathcal{Y}_p(AW^{-1}) = c_p \mathcal{Y}_p(A), \tag{1}$$

where

$$c_p = \prod_{i=1}^{\ell(p)} \frac{\Gamma[\frac{1}{2}(\nu - k + i) - p_i]}{\Gamma[\frac{1}{2}(\nu - k + i)]2^{p_i}}. \tag{2}$$

Proof. Let $A = \text{diag}(\alpha_1, \ldots, \alpha_k)$ without loss of generality. We look at the monomial term $\alpha^{p_1} \cdots \alpha^{p_\ell}$ $(\ell = \ell(p))$. Then as in (3.1.12) its coefficient in (1) is

$$\mathcal{E}_W\{W^{-1}(1)^{p_1-p_2} W^{-1}(1,2)^{p_2-p_3} \cdots W^{-1}(1,\ldots,\ell)^{p_\ell}\}, \tag{3}$$

which has to be equal to c_p. Let $W = T'T$ where T is lower triangular with positive diagonal elements. Then analogous to Lemma 3.1.3 t_{ii}, $i = 1, \ldots, k$, are independently distributed according to $\chi(\nu - k + i)$. Then $W^{-1} = T^{-1}T'^{-1}$ and T^{-1} is lower triangular with diagonal elements reciprocal to the diagonal elements of T. Therefore $W^{-1}(1, \ldots, r) = (t_{11} \cdots t_{rr})^{-2}$. Hence

$$\begin{aligned} c_p &= \mathcal{E}\{t_{11}^{-2p_1} \cdots t_{\ell\ell}^{-2p_\ell}\} \\ &= \prod_{i=1}^{\ell} \frac{\Gamma[\frac{1}{2}(\nu - k + i) - p_i]}{\Gamma[\frac{1}{2}(\nu - k + i)]2^{p_i}} \\ &= \{\prod_{i=1}^{\ell} (\nu - k + i - 2p_i)(\nu - k + i - 2p_i + 2) \cdots (\nu - k + i - 2)\}^{-1}. \end{aligned}$$

∎

Related to Lemma 1 we have the following interesting identity which is briefly mentioned in Constantine (1966). Let $p^*_{s,t}$ be defined by (2.1.5).

Lemma 2. *Let A be a $t \times t$ positive definite matrix. Then*

$$|A|^s \frac{\mathcal{Y}_p(A^{-1})}{\mathcal{Y}_p(I_t)} = \frac{\mathcal{Y}_{p^*_{s,t}}(A)}{\mathcal{Y}_{p^*_{s,t}}(I_t)}, \tag{4}$$

where $s \geq h(p)$, $t \geq \ell(p)$.

Proof. Without loss of generality let $A=\text{diag}(\alpha_1, \ldots, \alpha_t)$. let $f(A)=|A|^s \mathcal{Y}_p(A^{-1})$. We use Theorem 4.1.2. In terms of the basis $\{ \mathcal{M}_q \}$ we can write

$$\mathcal{Y}_p(A^{-1}) = \sum_{q \prec p, \ell(q) \leq t} a_{pq} \mathcal{M}_q(1/\alpha_1, \ldots, 1/\alpha_t). \tag{5}$$

Note that $q \prec p$ implies $h(q) \leq h(p)$. Now the degree of $1/\alpha_i$ in $\mathcal{M}_q(1/\alpha_1, \ldots, 1/\alpha_t)$ is $h(q)$. Hence the degree of $1/\alpha_i$ in $\mathcal{Y}_p(A^{-1})$ is $h(p)$. Now $|A|^s = (\alpha_1 \cdots \alpha_t)^s$ and $s \geq h(p)$. We see that $1/\alpha_i$ is canceled by $|A|^s$ and $f(A) = |A|^s \mathcal{Y}_p(A^{-1})$ is a polynomial in $(\alpha_1, \ldots, \alpha_t)$. Clearly it is symmetric and homogeneous of degree $st - |p|$. Therefore $f \in V_{st-|p|,t}$. This verifies (i) of Theorem 4.1.2. Now

$$\begin{aligned} |A|^s \mathcal{M}_q(A^{-1}) &= (\alpha_1 \cdots \alpha_t)^s \sum_{(i_1,\ldots,i_\ell) \subset (1,\ldots,t)} \frac{1}{\alpha_{i_1}^{q_1} \cdots \alpha_{i_\ell}^{q_\ell}} \\ &= \sum_{(i_1,\ldots,i_\ell) \subset (1,\ldots,t)} \alpha_{i_1}^{s-q_1} \alpha_{i_2}^{s-q_2} \cdots \alpha_{i_\ell}^{s-q_\ell} \cdot \alpha_{j_1}^s \cdots \alpha_{j_{t-\ell}}^s \\ &= \mathcal{M}_{q^*_{s,t}}(A), \end{aligned}$$

where $1 \leq j_1, \ldots, j_{t-\ell} \leq t$ are indices not included in $(i_1, \ldots, i_\ell)$ and $q^*_{s,t} = (s, \ldots, s, s - q_\ell, \ldots, s - q_2, s - q_1)$. Hence by Lemma 2.1.4 the leading term in f is $a_{pp} \mathcal{M}_{p^*_{s,t}}$. This verifies (ii). Now consider

$$\mathcal{E}_W f(AW) = \mathcal{E}_W\{|AW|^s \mathcal{Y}_p(A^{-1}W^{-1})\} = |A|^s \mathcal{E}_W\{|W|^s \mathcal{Y}_p(A^{-1}W^{-1})\}.$$

As in the proof of Theorem 4.2.1 $|W|^s$ can be absorbed into the Wishart density and we have $\mathcal{E}_W\{|W|^s \mathcal{Y}_p(A^{-1}W^{-1})\}=c_\nu \mathcal{Y}_p(A^{-1})$ for some c_ν. Therefore

$$\mathcal{E}_W f(AW) = |A|^s c_\nu \mathcal{Y}_p(A^{-1}) = c_\nu f(A).$$

This verifies (iii) and by Theorem 4.1.2 we have $f(A)=|A|^s \mathcal{Y}_p(A^{-1})=c\mathcal{Y}_{p^*_{s,t}}(A)$ for some c. Putting $A = I_t$ we obtain $c = \mathcal{Y}_p(I_t)/\mathcal{Y}_{p^*_{s,t}}(I_t)$. ∎

The second distribution is a "multivariate F" distribution. There are many ways to generalize the univariate F distribution to the multivariate case. Here we work with the following version. For other generalizations see Johnson and Kotz (1972).

Lemma 3. *Let the columns of* $X_1 : k \times \nu_1$, $X_2 : k \times \nu_2$ $(\nu_2 > 2h(p) + k - 1)$ *be independently distributed according to* $\mathcal{N}(0, \Sigma)$. *Let* $W = X_1'(X_2X_2')^{-1}X_1$. *Then*

$$\mathcal{E}_W\{\mathcal{Y}_p(AW)\} = \lambda_{kp} \prod_{i=1}^{\ell(p)} \frac{\Gamma[\frac{1}{2}(\nu_2 - k + i) - p_i]}{\Gamma[\frac{1}{2}(\nu_2 - k + i)]2^{p_i}} \mathcal{Y}_p(A). \tag{6}$$

Proof. Premultiplying X_1, X_2 by $\Sigma^{-\frac{1}{2}}$ we can take $\Sigma = I_k$ without loss of generality. Then

$$\begin{aligned}\mathcal{E}_W \mathcal{Y}_p(AW) &= \mathcal{E}_{X_1} \mathcal{E}_{X_2} \mathcal{Y}_p(X_1 A X_1'(X_2X_2')^{-1}) \\ &= \prod_{i=1}^{\ell(p)} \frac{\Gamma[\frac{1}{2}(\nu_2 - k + i) - p_i]}{\Gamma[\frac{1}{2}(\nu_2 - k + i)]2^{p_i}} \mathcal{E}_{X_1} \mathcal{Y}_p(AX_1'X_1) \\ &= \prod_{i=1}^{\ell(p)} \frac{\Gamma[\frac{1}{2}(\nu_2 - k + i) - p_i]}{\Gamma[\frac{1}{2}(\nu_2 - k + i)]2^{p_i}} \lambda_{kp} \mathcal{Y}_p(A).\end{aligned} \tag{7}$$

∎

Remark 1. It is more or less obvious to prove Lemma 3 for other definitions of multivariate F distribution.

Our last distribution is multivariate beta distribution (Constantine (1963)). The following derivation is essentially the same as in Constantine (1963), but more probabilistic. Let W_1, W_2 be independently distributed according to $\mathcal{W}(\Sigma, \nu_1), \mathcal{W}(\Sigma, \nu_2)$ $(\Sigma : k \times k)$ respectively. Note that $W = W_1 + W_2 \sim \mathcal{W}(\Sigma, \nu_1 + \nu_2)$. Now the conditional density of W_1 given W is

$$\begin{aligned} & f(W_1 \mid W) \\ &= c\frac{|W_1|^{\frac{\nu_1 - k - 1}{2}} \exp(-\frac{1}{2}\operatorname{tr}\Sigma^{-1}W_1)|W_2|^{\frac{\nu_2 - k - 1}{2}} \exp(-\frac{1}{2}\operatorname{tr}\Sigma^{-1}W_2)}{|W|^{\frac{\nu_1 + \nu_2 - k - 1}{2}} \exp(-\frac{1}{2}\operatorname{tr}\Sigma^{-1}W)} \\ &= c\frac{|W_1|^{\frac{\nu_1 - k - 1}{2}}|W_2|^{\frac{\nu_2 - k - 1}{2}}}{|W|^{\frac{\nu_1 + \nu_2 - k - 1}{2}}},\end{aligned} \tag{8}$$

where

$$c = \frac{\prod_{i=1}^{k} \Gamma[\frac{1}{2}(\nu_1 + \nu_2 - i + 1)]}{\pi^{k(k-1)/4} \prod_{i=1}^{k} \Gamma[\frac{1}{2}(\nu_1 - i + 1)]\Gamma[\frac{1}{2}(\nu_2 - i + 1)]}. \tag{9}$$

Note that terms involving $\boldsymbol{\Sigma}$ cancel out in (8) . Therefore the conditional distribution does not depend on $\boldsymbol{\Sigma}$. When $\boldsymbol{W}=\boldsymbol{I}$, $f(\boldsymbol{W}_1 \mid \boldsymbol{I})$ is called multivariate beta density:

$$f(\boldsymbol{W}_1 \mid \boldsymbol{I}) = c|\boldsymbol{W}_1|^{\frac{\nu_1-k-1}{2}}|\boldsymbol{I} - \boldsymbol{W}_1|^{\frac{\nu_2-k-1}{2}}. \tag{10}$$

Since this density is orthogonally invariant the conditional distribution of $\boldsymbol{W}_1$ given $\boldsymbol{W} = \boldsymbol{I}$ is orthogonally invariant. Now we want to evaluate c_p in

$$\mathcal{E}\{\mathcal{Y}_p(\boldsymbol{A}\boldsymbol{W}_1) \mid \boldsymbol{W} = \boldsymbol{I}\} = c_p \mathcal{Y}_p(\boldsymbol{A}). \tag{11}$$

For a positive definite $\boldsymbol{A}$ let $\boldsymbol{A}^{\frac{1}{2}} = \boldsymbol{\Gamma}\boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\Gamma}'$ where $\boldsymbol{\Gamma}$ is orthogonal and $\boldsymbol{D}$ is diagonal in $\boldsymbol{A}=\boldsymbol{\Gamma}\boldsymbol{D}\boldsymbol{\Gamma}'$. Now the conditional distribution of $\boldsymbol{A}^{\frac{1}{2}}\boldsymbol{W}_1\boldsymbol{A}^{\frac{1}{2}}$ given $\boldsymbol{W} = \boldsymbol{I}$ is the same as the conditional distribution of $\boldsymbol{W}_1$ given $\boldsymbol{W} = \boldsymbol{A}$. This follows from the above mentioned fact that the conditional distribution does not depend on $\boldsymbol{\Sigma}$. Therefore

$$\mathcal{E}\{\mathcal{Y}_p(\boldsymbol{A}\boldsymbol{W}_1) \mid \boldsymbol{W} = \boldsymbol{I}\} = \mathcal{E}\{\mathcal{Y}_p(\boldsymbol{W}_1) \mid \boldsymbol{W} = \boldsymbol{A}\}. \tag{12}$$

Letting $\boldsymbol{A} = \boldsymbol{W}_1 + \boldsymbol{W}_2$ we obtain from (11) and (12)

$$\mathcal{E}\{\mathcal{Y}_p(\boldsymbol{W}_1) \mid \boldsymbol{W}_1 + \boldsymbol{W}_2\} = c_p \mathcal{Y}_p(\boldsymbol{W}_1 + \boldsymbol{W}_2). \tag{13}$$

Now taking unconditional expectation we obtain

$$\lambda_{\nu_1 p} \mathcal{Y}_p(\boldsymbol{\Sigma}) = c_p \lambda_{\nu_1+\nu_2,p} \mathcal{Y}_p(\boldsymbol{\Sigma}). \tag{14}$$

Hence $c_p = \lambda_{\nu_1 p}/\lambda_{\nu_1+\nu_2,p}$. Now we have proved

Lemma 4. *Let W_1 have the density (10). Then*

$$\mathcal{E}_{W_1} \mathcal{Y}_p(AW_1) = \frac{\lambda_{\nu_1 p}}{\lambda_{\nu_1+\nu_2,p}} \mathcal{Y}_p(A). \tag{15}$$

Variations of the above three lemmas can be found in Khatri (1966), Subrahmaniam (1976).

§ 4.4 COEFFICIENTS OF $\mathcal{U}_q$ IN $\mathcal{Y}_p$

In this section we study coefficients of $\mathcal{U}_q$'s when zonal polynomials are expressed as linear combinations of $\mathcal{U}_q$'s. For definiteness we work with ${}_1a_{pq}$ in ${}_1\mathcal{Y}_p = \mathcal{U}_p + \sum {}_1a_{pq}\mathcal{U}_q$. If rank A=1,2 all the relevant coefficients are known and we can compute $\mathcal{Y}_p(A)$ explicitly. We review this first. After that we study several recurrence relations between the coefficients. When rank $A > 2$ these recurrence relations are not enough to compute the values of zonal polynomials $\mathcal{Y}_p(A)$ for all p. Nonetheless they seem to be very useful. Coefficients of $\mathcal{M}_q$'s will be discussed in the next section and $\mathcal{T}_q$'s in Section 4.6. We discuss relative advantages of various bases on the way.

4.4.1 Rank 1 and rank 2 cases

If A is symmetric and rank A=1 then A has only one nonzero root. Let $A = \text{diag}(\alpha_1, 0, \ldots, 0)$ without loss of generality. By Corollary 4.1.1 $\mathcal{Y}_p(A) = 0$ if $\ell(p) \geq 2$. Therefore only onepart partitions $p = (p_1)$ count. Obviously

$${}_1\mathcal{Y}_{(p_1)}(A) = \mathcal{U}_{(p_1)}(A) = \alpha_1^{p_1}. \tag{1}$$

Therefore in this case zonal polynomials reduce to powers of α_1.

Now suppose rank A=2. Let $A = \text{diag}(\alpha_1, \alpha_2, 0, \ldots, 0)$ and $\tilde{A} = \text{diag}(\alpha_1, \alpha_2)$: 2×2. We have to consider only partitions with two parts $p = (p_1, p_2)$. Now we use Corollary 4.2.1:

$${}_1\mathcal{Y}_{(p_1,p_2)}(A) = {}_1\mathcal{Y}_{(p_1,p_2)}(\tilde{A}) = |\tilde{A}|^{p_2}\, {}_1\mathcal{Y}_{(p_1-p_2)}(\tilde{A}), \tag{2}$$

where $(p_1 - p_2)$ is a onepart partition. Therefore it suffices to know the value of a zonal polynomial of onepart partition evaluated at 2×2 matrix $\tilde{A}$. Actually zonal polynomials of onepart partitions are known explicitly and can be derived as follows. If we let $\beta_1 = 1, \beta_2 = \cdots = \beta_k = 0$ in (3.4.6) we obtain

$$(3) \qquad \prod_{i=1}^{k} (1 - 2\theta\alpha_i)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} (\theta^n / n!) d_{(n)} Z_{(n)}(A)\, {}_1b_{(n)},$$

where ${}_1b_{(n)} = Z_{(n)}(I_1)$ is the leading coefficient of $Z_{(n)}$ (see (4.2.2)). Note that $(\operatorname{tr} C)^n = \mathcal{U}_{(n)}(C)$ and with respect to the basis $\{\mathcal{U}_p\}$ only $Z_{(n)}(C)$ contains $\mathcal{U}_{(n)}$. Therefore in (3.4.1) $(\operatorname{tr} C)^n = {}_1\mathcal{Y}_{(n)}(C) + \cdots = (1/{}_1b_{(n)})\ Z_{(n)}(C) + \cdots$. Hence $d_{(n)} = {}_1b_{(n)}^{-1}$. We have

$$(4) \qquad \prod_{i=1}^{k} (1 - 2\theta\alpha_i)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} (\theta^n / n!) Z_{(n)}(A).$$

The left hand side can be expanded as follows.

$$(5) \qquad \begin{aligned} &\prod_{i=1}^{k} (1 - 2\theta\alpha_i)^{-\frac{1}{2}} \\ &= \{1 - (2\theta u_1 - 4\theta^2 u_2 + \cdots)\}^{-\frac{1}{2}} \\ &= \sum_{n=0}^{\infty} (2\theta)^n \sum_{p \in \mathcal{P}_n} \frac{1}{p_1!} \frac{1}{2} \frac{3}{2} \cdots \frac{2p_1 - 1}{2} \binom{p_1}{p_1 - p_2, p_2 - p_3, \ldots, p_{\ell(p)}} \\ &\quad \cdot (-1)^{(p_2 - p_3) + (p_4 - p_5) + \cdots} \mathcal{U}_p(A). \end{aligned}$$

This follows from the fact that $\mathcal{U}_p$ being a product of p_1 terms comes only from p_1-*th* power term in the expansion of $(1 - 2\theta u_1 + \cdots)^{-1/2}$. Comparing (4) and (5) we obtain

$$(6) \qquad Z_{(n)} = 2^n n! \sum_{p \in \mathcal{P}_n} (-1)^{(p_2 - p_3) + (p_4 - p_5) + \cdots} \frac{(\frac{1}{2})_{p_1}}{(p_1 - p_2)! \cdots p_{\ell(p)}!} \mathcal{U}_p.$$

Note that ${}_1b_{(n)} = 1 \cdot 3 \cdots (2n-1)$, $2^n n! = 2 \cdot 4 \cdots (2n)$, $|\tilde{A}|^{p_2} \mathcal{U}_{(q_1, q_2)}(\tilde{A}) = \mathcal{U}_{(q_1 + p_2, q_2 + p_2)}(\tilde{A}) = \mathcal{U}_{(q_1 + p_2, q_2 + p_2)}(A)$. Therefore combining these equalities we obtain

$$\begin{aligned} {}_1\mathcal{Y}_{(p_1,p_2)}(\boldsymbol{A}) &= \frac{2\cdot 4\cdots(2p_1-2p_2)}{1\cdot 3\cdots(2p_1-2p_2-1)} \\ &\cdot \sum_{(q_1,q_2)\in \mathcal{P}_{p_1-p_2}} (-1)^{q_2}\frac{(\frac{1}{2})_{q_1}}{(q_1-q_2)!q_2!}\mathcal{U}_{(q_1+p_2,q_2+p_2)}(\boldsymbol{A}). \end{aligned} \tag{7}$$

if rank $\boldsymbol{A}=2$. See formula (130) in James (1964).

If rank $\boldsymbol{A}$=3 what we have to know are the values of zonal polynomials of twopart partitions evaluated at a rank 3 matrix $\boldsymbol{A}$. Obviously things become more and more complicated as rank $\boldsymbol{A}$ increases. However several useful recurrence relations on the coefficients can be obtained.

4.4.2 Recurrence relations on the coefficients

We present here three recurrence relations. The first one has been already used in deriving (7).

Lemma 1. *If $k \geq \ell(p), k \geq \ell(q)$, then*

$$ {}_1a_{pq} = {}_1a_{p+(1^k),q+(1^k)}. \tag{8}$$

Proof. Let $\boldsymbol{A}$ be $k\times k$. Then

$$\begin{aligned} |\boldsymbol{A}|\,{}_1\mathcal{Y}_p(\boldsymbol{A}) &= |\boldsymbol{A}|\{\mathcal{U}_p(\boldsymbol{A}) + \sum_{q<p,\ell(q)\leq k} {}_1a_{pq}\mathcal{U}_q(\boldsymbol{A})\} \\ &= \mathcal{U}_{p+(1^k)}(\boldsymbol{A}) + \sum_{q<p,\ell(q)\leq k} {}_1a_{pq}\mathcal{U}_{q+(1^k)}(\boldsymbol{A}). \end{aligned} \tag{9}$$

By Theorem 4.2.1

$$\begin{aligned} |\boldsymbol{A}|\,{}_1\mathcal{Y}_p(\boldsymbol{A}) &= {}_1\mathcal{Y}_{p+(1^k)}(\boldsymbol{A}) \\ &= \mathcal{U}_{p+(1^k)}(\boldsymbol{A}) + \sum_{q'<p+(1^k),\ell(q')\leq k} {}_1a_{p+(1^k),q'}\mathcal{U}_{q'}(\boldsymbol{A}). \end{aligned} \tag{10}$$

Comparing (9) and (10) we obtain by Lemma 4.1.2 ${}_1a_{pq} = {}_1a_{p+(1^k),q+(1^k)}$.
∎

Remark 1. Theorem 4.2.1 has been known and Lemma 1 is almost an immediate consequence. However it does not seem to have been explicitly stated.

The next one is in a sense conjugate to Lemma 1. Let $p = (p_1, \dots, p_\ell) \in \mathcal{P}_n$ and $m \geq p_1 = h(p)$. We denote by (m, p) the partition $(m, p_1, p_2, \dots, p_\ell) \in \mathcal{P}_{n+m}$.

Theorem 1. *Let $m \geq h(p)$. Then*

$$\frac{\partial^m}{\partial \alpha_{k+1}^m} \, {}_1\mathcal{Y}_{(m,p)}(\alpha_1, \dots, \alpha_{k+1}) = m! \, {}_1\mathcal{Y}_p(\alpha_1, \dots, \alpha_k). \tag{11}$$

Proof. With respect to the basis $\{ \mathcal{U}_p \}$ let

$$\begin{aligned} &{}_1\mathcal{Y}_{(m,p)}(\alpha_1, \dots, \alpha_{k+1}) \\ &\quad = \sum_{q \leq (m,p), q \in \mathcal{P}_{n+m}} {}_1a_{(m,p),q} \mathcal{U}_q(\alpha_1, \dots, \alpha_{k+1}) \\ &\quad = \sum_{(m,q') \leq (m,p), q' \in \mathcal{P}_n} {}_1a_{(m,p),(m,q')} \mathcal{U}_{(m,q')}(\alpha_1, \dots, \alpha_{k+1}) \\ &\quad\quad + \sum_{q \in \mathcal{P}_{n+m}, h(q) < m} {}_1a_{(m,p),q} \mathcal{U}_q(\alpha_1, \dots, \alpha_{k+1}). \end{aligned} \tag{12}$$

We differentiate (12) m times with respect to α_{k+1}. Now the degree of α_{k+1} in

$$\mathcal{U}_q(\alpha_1, \dots, \alpha_{k+1}) = (\sum \alpha_i)^{q_1 - q_2} (\sum_{i<j} \alpha_i \alpha_j)^{q_2 - q_3} \dots \tag{13}$$

is $q_1 = (q_1 - q_2) + (q_2 - q_3) + \cdots + q_{\ell(q)}$. Therefore the terms in the second summation on the right hand side of (12) drop out. Now $\mathcal{U}_{(m,q')}(\alpha_1, \dots, \alpha_{k+1})$ is a product of $m = (m - q'_1) + \cdots + q'_{\ell(q')}$ elementary symmetric functions $u_r(\alpha_1, \dots, \alpha_{k+1})$ which are linear in α_{k+1}. Therefore differentiating $\mathcal{U}_{(m,q')}$ m times we are left with the term where each u_r is differentiated exactly once. Furthermore

$$\frac{\partial}{\partial \alpha_{k+1}} u_r(\alpha_1, \dots, \alpha_{k+1}) = u_{r-1}(\alpha_1, \dots, \alpha_k). \tag{14}$$

Therefore by the chain rule of differentiation

$$\begin{aligned}(15)\quad &\frac{\partial^m}{\partial\alpha_{k+1}^m}\mathcal{U}_{(m,q')}(\alpha_1,\dots,\alpha_{k+1})\\ &= m!\{\frac{\partial}{\partial\alpha_{k+1}}u_1(\alpha_1,\dots,\alpha_{k+1})\}^{m-q'}\{\frac{\partial}{\partial\alpha_{k+1}}u_2(\alpha_1,\dots,\alpha_{k+1})\}^{q'_1-q'_2}\dots\\ &= m!u_1(\alpha_1,\dots,\alpha_k)^{q'_1-q'_2}u_2(\alpha_1,\dots,\alpha_k)^{q'_2-q'_3}\dots\\ &= m!\mathcal{U}_{q'}(\alpha_1,\dots,\alpha_k).\end{aligned}$$

Let $f(\alpha_1,\dots,\alpha_k)=(\partial^m/\partial\alpha_{k+1}^m)\,{}_1\mathcal{Y}_{(m,p)}(\alpha_1,\dots,\alpha_{k+1})$. Then we have

$$\begin{aligned}(16)\quad f(\alpha_1,\dots,\alpha_k) &= \sum_{q\le p}{}_1a_{(m,p),(m,q)}\frac{\partial^m}{\partial\alpha_{k+1}^m}\mathcal{U}_{(m,q)}(\alpha_1,\dots,\alpha_{k+1})\\ &= m!\sum_{q\le p}{}_1a_{(m,p),(m,q)}\mathcal{U}_q(\alpha_1,\dots,\alpha_k).\end{aligned}$$

We replaced q' by q and $(m,q)\le(m,p)$ by $q\le p$ since $(m,q)\le(m,p)$ if and only if $q\le p$. We have shown that conditions (i) and (ii) of Theorem 4.1.2 are satisfied. We are going to show that condition (iii) of Theorem 4.1.2 is satisfied as well.

Let $\boldsymbol{A}=\mathrm{diag}(\alpha_1,\dots,\alpha_{k+1})$ and $\boldsymbol{A}_1=\mathrm{diag}(\alpha_1,\dots,\alpha_k)$. Then exactly as above we obtain

$$\begin{aligned}(17)\quad &\frac{\partial^m}{\partial\alpha_{k+1}^m}\,{}_1\mathcal{Y}_{(m,p)}(\boldsymbol{AW})\\ &= m!\sum_{q\le p}{}_1a_{(m,p),(m,q)}\boldsymbol{W}(k+1)^{m-q_1}\Big(\sum_{i_1}^{k}\alpha_{i_1}\boldsymbol{W}(i_1,k+1)\Big)^{q_1-q_2}\dots\\ &\quad\cdot\Big(\sum_{i_1<\dots<i_{\ell(q)}}^{k}\alpha_{i_1}\cdots\alpha_{i_{\ell(q)}}\boldsymbol{W}(i_1,\dots,i_{\ell(q)},k+1)\Big)^{q_{\ell(q)}}.\end{aligned}$$

Let $\boldsymbol{W}$ be partitioned as

$$(18)\quad \boldsymbol{W}=\begin{pmatrix}\boldsymbol{W}_{11} & \boldsymbol{w}_{k+1}\\ \boldsymbol{w}'_{k+1} & w_{k+1,k+1}\end{pmatrix},$$

where $w_{k+1,k+1} = W(k+1)$ is a scalar. Let

$$W_{11\cdot k+1} = W_{11} - w_{k+1}w'_{k+1}/w_{k+1,k+1}. \tag{19}$$

Then by the well known identity on the determinant of partitioned matrices we have

$$W(i_1, \ldots, i_r, k+1) = w_{k+1,k+1} W_{11\cdot k+1}(i_1, \ldots, i_r). \tag{20}$$

Therefore in (17) $w_{k+1,k+1}^m = w_{k+1,k+1}^{(m-q_1)+(q_1-q_2)+\cdots}$ comes out as a common factor and we obtain

$$\begin{aligned} \frac{\partial^m}{\partial \alpha_{k+1}^m} {}_1\mathcal{Y}_{(m,p)}(AW) &= m! \sum_{q \le p} {}_1a_{(m,p),(m,q)} w_{k+1,k+1}^m \mathcal{U}_q(A_1 W_{11\cdot k+1}) \\ &= w_{k+1,k+1}^m f(A_1 W_{11\cdot k+1}). \end{aligned} \tag{21}$$

Now if W is distributed according to $\mathcal{W}(I_{k+1}, \nu)$ then $w_{k+1,k+1}$ and $W_{11\cdot k+1}$ are independently distributed according to $\chi^2(\nu)$, $\mathcal{W}(I_k, \nu-1)$ respectively. (See Srivastava and Khatri (1979), Theorem 3.3.5 or Mardia, Kent, and Bibby (1979), Theorem 3.4.6.) Therefore taking expectation with respect to W

$$\begin{aligned} \lambda_{\nu,(m,p)} f(A_1) &= \lambda_{\nu,(m,p)} \frac{\partial^m}{\partial \alpha_{k+1}^m} {}_1\mathcal{Y}_{(m,p)}(\alpha_1, \ldots, \alpha_{k+1}) \\ &= \mathcal{E}_W\{\frac{\partial^m}{\partial \alpha_{k+1}^m} {}_1\mathcal{Y}_{(m,p)}(AW)\} \\ &= \mathcal{E}_W\{w_{k+1,k+1}^m f(A_1 W_{11\cdot k+1})\} \\ &= \mathcal{E}(w_{k+1,k+1}^2) \cdot \mathcal{E}_W\{f(A_1 W_{11\cdot k+1})\}. \end{aligned} \tag{22}$$

Letting $c_{\nu-1} = \lambda_{\nu,(m,p)}/\mathcal{E}(w_{k+1,k+1}^2)$ we have

$$\mathcal{E} f(A_1 W_{11\cdot k+1}) = c_{\nu-1} f(A_1) \tag{23}$$

for all $k \times k$ symmetric A_1 and for all sufficiently large ν. This verifies condition (iii) of Theorem 4.1.2 and we conclude

$$f = c\, {}_1\mathcal{Y}_p \tag{24}$$

for some c. Comparing the leading coefficient with respect to the bases $\{\mathcal{U}_p\}$ we obtain $c = m!$. ∎

Corollary 1. *If $m \geq h(p)$ then*

$$ {}_1a_{pq} = {}_1a_{(m,p),(m,q)}. \tag{25} $$

Proof. From (11) and (15)

$$ \begin{aligned} & m!(\mathcal{U}_p + \sum_{q<p} {}_1a_{pq}\mathcal{U}_q) \\ & = m!\,{}_1\mathcal{Y}_p = \frac{\partial^m}{\partial \alpha_{k+1}^m}\,{}_1\mathcal{Y}_{(m,p)} \\ & = m!(\mathcal{U}_p + \sum_{q<p} {}_1a_{(m,p),(m,q)}\mathcal{U}_q). \end{aligned} \tag{26} $$

Therefore (25) holds. ∎

In terms of the diagram of p Lemma 1 corresponds to adding a column to the left of the diagram and Corollary 1 corresponds to adding a row to the top. In this sense they are "conjugate". It might be interesting to interpret this result from group representation theory.

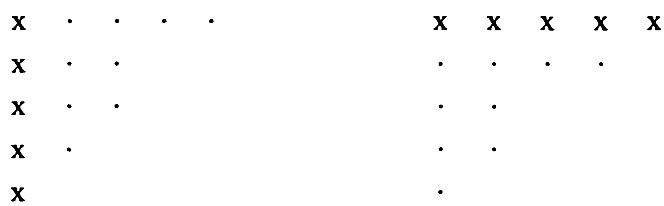

Figure 4.1.

Our third recurrence relation follows from Lemma 4.3.2.

Lemma 2. *If $s \geq h(p)$, $s \geq h(q)$, $t \geq \ell(p)$, $t \geq \ell(q)$, then*

$$ {}_1a_{pq} = {}_1a_{p^*_{s,t},q^*_{s,t}}. \tag{27} $$

Proof. Let $\boldsymbol{A}$ be a $t \times t$ positive definite matrix. From Lemma 4.3.2

$$ |\boldsymbol{A}|^s\,{}_1\mathcal{Y}_p(\boldsymbol{A}^{-1}) = c\,{}_1\mathcal{Y}_{p^*_{s,t}}(\boldsymbol{A}), \tag{28} $$

or

$$(29)\quad \begin{aligned} &|A|^s \mathcal{U}_p(A^{-1}) + \sum_{q<p,\ell(q)\le t} {}_1a_{pq}|A|^s\mathcal{U}_q(A^{-1}) \\ &= c(\mathcal{U}_{p^*}(A) + \sum_{q'<p^*,\ell(q')\le t} {}_1a_{p^*q'}\mathcal{U}_{q'}(A)), \end{aligned}$$

where $p^* = p^*_{s,t}$ and $c = {}_1\mathcal{Y}_p(I_t)/\,{}_1\mathcal{Y}_{p^*}(I_t)$. Now let A=diag$(\alpha_1, \ldots, \alpha_t)$. Then

$$(30)\quad \begin{aligned} |A|u_r(A^{-1}) &= (\alpha_1\cdots\alpha_t)\sum_{i_1<\cdots<i_r}^{t} \frac{1}{\alpha_{i_1}\cdots\alpha_{i_r}} \\ &= \sum_{j_1<\cdots<j_{t-r}}^{t} \alpha_{j_1}\cdots\alpha_{j_{t-r}} \\ &= u_{t-r}(A). \end{aligned}$$

Note that (30) is true for $r = 0, t$ if we define $u_0 = 1$. Therefore

$$(31)\quad \begin{aligned} |A|^s\mathcal{U}_q(A^{-1}) &= |A|^{s-q_1}\{|A|^{q_1}u_1(A^{-1})^{q_1-q_2}\cdots u_{\ell(q)}(A^{-1})^{q_{\ell(q)}}\} \\ &= u_t(A)^{s-q_1}u_{t-1}(A)^{q_1-q_2}\cdots u_{t-\ell(q)}(A)^{q_{\ell(q)}} \\ &= \mathcal{U}_{q^*_{s,t}}(A), \end{aligned}$$

because $q^*_{s,t} = (s, \ldots, s, s-q_{\ell(q)}, \ldots, s-q_2, s-q_1)$ and $\ell(q^*_{s,t}) = t$. Substituting (31) into (29) we obtain

$$(32)\quad \begin{aligned} &\mathcal{U}_{p^*}(A) + \sum_{q<p,\ell(q)\le t} {}_1a_{pq}\mathcal{U}_{q^*_{s,t}}(A) \\ &= c\{\mathcal{U}_{p^*}(A) + \sum_{q'<p^*,\ell(q')\le t} {}_1a_{p^*q'}\mathcal{U}_{q'}(A)\}. \end{aligned}$$

Therefore by Lemma 4.1.2 ${}_1a_{pq} = {}_1a_{p^*_{s,t},q^*_{s,t}}$, c=1. ∎

Remark 2. Again this lemma is much easier to grasp in terms of the diagram. See Figure 2.2.

Looking at Table 2 in Parkhurst and James (1974) we find that the above three recurrence relations give a large number of coefficients without any calculation (except that the table is for Z_p rather than for ${}_1\mathcal{Y}_p$). However it is not clear whether this kind of approach can be further carried out to give all coefficients of zonal polynomials.

§ 4.5 COEFFICIENTS OF $\mathcal{M}_q$

So far we have been mainly working with $\mathcal{U}_p$'s. But in view of Lemma 2.2.2, Lemma 4.1.1 etc. we could have worked with $\mathcal{M}_p$'s as well. We defined zonal polynomials in connection with the Wishart distribution and it was more straightforward to define zonal polynomials in terms of $\mathcal{U}_p$'s in that setting. But when it comes to obtaining coefficients it seems easier to work with $\mathcal{M}_p$'s. In this section we translate every result in Section 4.4 into the coefficients of $\mathcal{M}_p$'s. Another big advantage of working with monomial symmetric functions is a partial differential equation by James (1968), from which he derived a recurrence relation on the coefficients of monomial symmetric functions in a zonal polynomial. (Note that the recurrence relations of Section 4.4.2 were on the coefficients of $\mathcal{U}_q$'s in different zonal polynomials. Here the recurrence relation is on the coefficients in one zonal polynomial.) Actually it is possible to develop a whole theory of zonal polynomials from the partial differential equation. This is done in a recent book by Muirhead (1982) explicitly and illustratively following James (1968). We discuss the partial differential equation and the recurrence relation in Section 4.5.4.

Furthermore Jacob Towber (personal communication) has recently developed a combinatorial method for determining the coefficients. His method involves several steps of counting related to the diagram of a partition. At the moment the combinatorics involved seems to be too complicated to obtain an explicit formula for the coefficients, but it might be carried out.

From the above discussion we see that we have much more information on the coefficients of $\mathcal{M}_p$'s than on the coefficients of $\mathcal{U}_p$'s. Therefore in a sense it is pointless to work with $\mathcal{U}_p$'s any more. However from a computational point of view it is easier to compute $\mathcal{U}_q$'s once we obtain the characteristic roots and the characteristic equation of a matrix $\boldsymbol{A}$. We simply multiply the elementary symmetric functions. In the case of $\mathcal{M}_p$'s we have to multiply the roots in all possible ways and sum them up. The relative advantages of $\mathcal{M}_p$'s and $\mathcal{U}_p$'s should be judged from this viewpoint too.

4.5.1 Rank 1 and rank 2 cases

Let $p = (n)$ be a onepart partition. To express $\mathcal{Z}_{(n)}$ in monomial symmetric functions we can use the integral representation by Kates. This was done by Kates (1980). Letting $r = 1$ in (3.3.8) we obtain

$$(1) \qquad \boldsymbol{UAU}'(1) = \sum_{i=1}^{k} \alpha_i u_{1i}^2,$$

where $\boldsymbol{A} = \mathrm{diag}(\alpha_1, \dots, \alpha_k)$ and u_{1i}, $i = 1, \dots, k$, are independent standard normal variables. Therefore by (3.3.2)

$$(2) \qquad \mathcal{Z}_{(n)} = \mathcal{E}(\sum_{i=1}^{k} \alpha_i u_{1i}^2)^n.$$

Now the coefficient of $\alpha_1^{p_1} \cdots \alpha_\ell^{p_\ell}$ on the right hand side is

$$(3) \qquad \begin{aligned} \binom{n}{p_1, p_2, \dots, p_\ell} \mathcal{E}\{u_{11}^{2p_1} \cdots u_{1\ell}^{2p_\ell}\} &= \frac{n!}{p_1! \cdots p_\ell!} \frac{(2p_1)!}{2^{p_1} p_1!} \cdots \frac{(2p_\ell)!}{2^{p_\ell} p_\ell!} \\ &= n! 2^{-n} \binom{2p_1}{p_1} \cdots \binom{2p_\ell}{p_\ell}. \end{aligned}$$

Therefore

$$(4) \qquad \mathcal{Z}_{(n)} = n! 2^{-n} \sum_{p \in \mathcal{P}_n} \mathcal{M}_p(\boldsymbol{A}) \prod_{i=1}^{\ell(p)} \binom{2p_i}{p_i}.$$

This looks nicer than (4.4.6). ${}_1\mathcal{Y}_p$ has the leading coefficient 1, so

$$(5) \qquad {}_1\mathcal{Y}_{(n)} = \binom{2n}{n}^{-1} \sum_{p \in \mathcal{P}_n} \mathcal{M}_p \prod_{i=1}^{\ell(p)} \binom{2p_i}{p_i}.$$

Now let $\boldsymbol{A} = \mathrm{diag}(\alpha_1, \alpha_2)$. Then for $q=(q_1, q_2)$ ($q_1 \neq q_2$)

$$(6) \qquad \begin{aligned} |\boldsymbol{A}|^k \mathcal{M}_q(\boldsymbol{A}) &= (\alpha_1\alpha_2)^k(\alpha_1^{q_1}\alpha_2^{q_2} + \alpha_2^{q_1}\alpha_1^{q_2}) \\ &= \alpha_1^{q_1+k}\alpha_2^{q_2+k} + \alpha_2^{q_1+k}\alpha_1^{q_2+k} \\ &= \mathcal{M}_{(q_1+k, q_2+k)}(\boldsymbol{A}). \end{aligned}$$

(The equality of the extreme left and the extreme right hand sides holds for $q_1=q_2$ too). Therefore from (4.4.2) we obtain

$$(7) \qquad \begin{aligned} &{}_1\mathcal{Y}_{(p_1,p_2)}(\alpha_1, \alpha_2) \\ &= \binom{2p_1 - 2p_2}{p_1 - p_2}^{-1} \sum_{(q_1,q_2)\in \mathcal{P}_{p_1-p_2}} \binom{2q_1}{q_1}\binom{2q_2}{q_2} \mathcal{M}_{(q_1+p_2, q_2+p_2)}(\alpha_1, \alpha_2). \end{aligned}$$

This takes care of rank 1 and rank 2 cases.

4.5.2 Again on the generating function of zonal polynomials

To express $\mathcal{T}_p$ in terms of $\mathcal{M}_q$'s we can simply expand $\mathcal{T}_p$ and count various monomial terms. Therefore it seems easier to express the right hand side of (3.4.9) in $\mathcal{M}_q$'s than in $\mathcal{U}_q$'s. Then we decompose the resulting positive definite coefficient matrix as $\boldsymbol{LL}'$ where $\boldsymbol{L}$ is lower triangular with positive diagonal elements. The elements of $\boldsymbol{L}$ give the desired coefficients of zonal polynomials. The development on Section 3.4 goes through in exactly the same way except that we order $\{ \mathcal{T}_p, p \in \mathcal{P}_n \}$ according to the lexicographic ordering of the conjugate partition p' (see Remark 2.2.2). We do not repeat it here.

Rather we notice here the similarity between two generating functions (3.4.6) and (4.4.4). Let $\gamma_1, \ldots, \gamma_{k^2}$ denote the k^2 numbers $\alpha_i\beta_j$, $i = 1, \ldots, k$, $j = 1, \ldots, k$. Let $\boldsymbol{C}=\mathrm{diag}(\gamma_1, \ldots, \gamma_{k^2})$. Then from (3.4.6) and (4.4.4) we have

$$(8) \qquad \sum_{n=0}^{\infty}(\theta^n/n!) \sum_{p\in\mathcal{P}_n} d_p Z_p(\boldsymbol{A})Z_p(\boldsymbol{B}) = \sum_{n=0}^{\infty}(\theta^n/n!)Z_{(n)}(\boldsymbol{C}).$$

Hence

$$(9) \qquad \sum_{p\in\mathcal{P}_n} d_p Z_p(\boldsymbol{A})Z_p(\boldsymbol{B}) = Z_{(n)}(\boldsymbol{C}).$$

Now we can substitute (4) into the right hand side. Then it reduces to expressing $\mathcal{M}_p(\boldsymbol{C})$ as a sum of products $\mathcal{M}_q(\boldsymbol{A})\mathcal{M}_{q'}(\boldsymbol{B})$. This seems nicer than directly expanding the right hand side of (3.4.9).

Finally we prove that the coefficient of $\mathcal{M}_{(1^k)}$ in Z_p , $p \in \mathcal{P}_k$ is $k!$. This is stated in James (1968).

Lemma 1. *Let $p \in \mathcal{P}_k$ and $\boldsymbol{A} = \mathrm{diag}(\alpha_1, \ldots, \alpha_k)$. Then*

$$\frac{\partial^k}{\partial\alpha_1\partial\alpha_2\cdots\partial\alpha_k} Z_p(\boldsymbol{A}) = k!. \tag{10}$$

Hence the coefficient of $\mathcal{M}_{(1^k)}$ in Z_p is $k!$.

Proof.

$$\prod_{i,j}^{k}(1-2\theta\alpha_i\beta_j)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}(\theta^n/n!)\sum_{p\in\mathcal{P}_n} d_p Z_p(\boldsymbol{A})Z_p(\boldsymbol{B}). \tag{11}$$

Differentiating this by $\alpha_1, \alpha_2, \ldots, \alpha_k$ we obtain

$$\begin{aligned} &\prod_{i=1}^{k}\left(\sum_{j=1}^{k}\frac{\theta\beta_j}{1-2\theta\alpha_i\beta_j}\right)\prod_{i,j}^{k}(1-2\theta\alpha_i\beta_j)^{-\frac{1}{2}} \\ &\quad = \sum_{n=0}^{\infty}(\theta^n/n!)\sum_{p\in\mathcal{P}_n} d_p Z_p(\boldsymbol{B})\frac{\partial^k}{\partial\alpha_1\cdots\partial\alpha_k}Z_p(\boldsymbol{A}). \end{aligned} \tag{12}$$

Now $\theta\beta_j/(1-2\theta\alpha_i\beta_j) = \theta\beta_j$ + *higher order in* θ. Hence

$$\prod_{i=1}^{k}\left(\sum_{j=1}^{k}\frac{\theta\beta_j}{1-2\theta\alpha_i\beta_j}\right) = \theta^k\left(\sum_{j=1}^{k}\beta_j\right)^k + \textit{higher order in } \theta. \tag{13}$$

Comparing the coefficients of θ^k we obtain

$$\left(\sum_{j=1}^{k}\beta_j\right)^k = \sum_{p\in\mathcal{P}_k}\frac{d_p}{k!}Z_p(\boldsymbol{B})\frac{\partial^k}{\partial\alpha_1\cdots\partial\alpha_k}Z_p(\boldsymbol{A}). \tag{14}$$

But by (3.4.1)

$$(15)\qquad (\sum_{j=1}^{k} \beta_j)^k = (\operatorname{tr} \boldsymbol{B})^k = \sum_{p \in \mathcal{P}_k} d_p Z_p(\boldsymbol{B}).$$

Comparing (14) and (15) we obtain

$$\frac{\partial^k}{\partial \alpha_1 \cdots \partial \alpha_k} Z_p(\boldsymbol{A}) = k!$$

∎

4.5.3 Recurrence relations of Section 4.4.2

Here we work again with the normalization ${}_1\mathcal{Y}_p$. Let

$$(16)\qquad {}_1\mathcal{Y}_p = \mathcal{M}_p + \sum_{q<p} {}_1b_{pq} \mathcal{M}_q.$$

Lemma 2. *If $k \geq \ell(p), k \geq \ell(q)$, then*

$$(17)\qquad {}_1b_{pq} = {}_1b_{p+(1^k),q+(1^k)}.$$

Proof. Let $\boldsymbol{A} = \operatorname{diag}(\alpha_1, \ldots, \alpha_k)$ where $k \geq \ell(p)$. Then

$$(18)\qquad \begin{aligned} |\boldsymbol{A}| \mathcal{M}_q(\boldsymbol{A}) &= (\alpha_1 \cdots \alpha_k) \sum_{(i_1,\ldots,i_\ell) \subset (1,\ldots,k)} \alpha_{i_1}^{q_1} \alpha_{i_2}^{q_2} \cdots \alpha_{i_\ell}^{q_\ell} \\ &= \sum_{(i_1,\ldots,i_\ell) \subset (1,\ldots,k)} \alpha_{i_1}^{q_1+1} \alpha_{i_2}^{q_2+1} \cdots \\ &= \mathcal{M}_{q+(1^k)}(\boldsymbol{A}). \end{aligned}$$

Note that this equality does not hold for augmented monomial symmetric functions. The equality above holds because the summation is over distinguishable terms and $\alpha_{i_1}^{q_1} \cdots \alpha_{i_\ell}^{q_\ell}$ is distinguishable from $\alpha_{j_1}^{q_1} \cdots \alpha_{j_\ell}^{q_\ell}$ if and only if $(\alpha_1 \cdots \alpha_k)\alpha_{i_1}^{q_1} \cdots \alpha_{i_\ell}^{q_\ell}$ is distinguishable from $(\alpha_1 \cdots \alpha_k)\alpha_{j_1}^{q_1} \cdots \alpha_{j_\ell}^{q_\ell}$. For augmented monomial functions refer to (2.2.6). Now the lemma can be proved just as Lemma 4.4.1 if we replace $\mathcal{U}_p, \mathcal{U}_q$, ${}_1a_{pq}$ in (4.4.9) by $\mathcal{M}_p, \mathcal{M}_q$, ${}_1b_{pq}$ respectively.

∎

Lemma 3. *Let $h(p) \leq m$. Then*

$$ {}_1b_{pq} = {}_1b_{(m,p),(m,q)}. \tag{19} $$

Proof. The degree of α_{k+1} in $\mathcal{M}_q(\alpha_1,\ldots,\alpha_{k+1})$ is $h(q)$. Hence if $h(q) < m$, then

$$ \frac{\partial^m}{\partial\alpha_{k+1}^m}\mathcal{M}_q(\alpha_1,\ldots,\alpha_{k+1}) = 0. \tag{20} $$

If $h(q)=m$ let $q=(m,q')$. Then clearly

$$ \frac{\partial^m}{\partial\alpha_{k+1}^m}\mathcal{M}_q(\alpha_1,\ldots,\alpha_{k+1}) = m!\mathcal{M}_{q'}(\alpha_1,\ldots,\alpha_k). \tag{21} $$

(Again this equality does not hold for $\mathcal{A}\mathcal{M}_q$.) Now (4.4.26) holds with $\mathcal{M}_p,\mathcal{M}_q$, ${}_1b_{pq}$ replacing $\mathcal{U}_p,\mathcal{U}_q$, ${}_1a_{pq}$ respectively. This proves the lemma. ∎

Lemma 4. *If $h(p) \leq s, h(q) \leq s, \ell(p) \leq t, \ell(q) \leq t$, then*

$$ {}_1b_{pq} = {}_1b_{p^*_{s,t},q^*_{s,t}}. \tag{22} $$

Proof. Let $\boldsymbol{A} = \operatorname{diag}(\alpha_1,\ldots,\alpha_t)$, $q=(q_1,\ldots,q_\ell)$, $q_1 \leq s$, $\ell \leq t$. Then

$$ |\boldsymbol{A}|^s\mathcal{M}_q(\boldsymbol{A}^{-1}) = \mathcal{M}_{q^*_{s,t}}(\boldsymbol{A}). \tag{23} $$

See the proof of Lemma 4.3.2. (Again (23) dos not hold for $\mathcal{A}\mathcal{M}_q$). Now (4.4.29), (4.4.31), (4.4.32) hold with $\mathcal{M}_p,\mathcal{M}_q, {}_1b_{pq}$ replacing $\mathcal{U}_p,\mathcal{U}_q, {}_1a_{pq}$ respectively. This proves the lemma. ∎

We have shown that the recurrence relations of Section 4.4.2 hold in exactly the same way for the coefficients of $\mathcal{U}_q$'s as for the coefficients of $\mathcal{M}_q$'s.

In the next section we discuss James' partial differential equation and a recurrence relation derived from it. The mathematical development will be somewhat sketchy.

4.5.4 James' partial differential equation and recurrence relation

James (1968) derived a partial differential equation satisfied by a zonal polynomial from the fact that a zonal polynomial is an "eigenfunction of the Laplace-Beltrami operator." Let $A=\text{diag}(\alpha_1, \ldots, \alpha_k)$, $p=(p_1, \ldots, p_\ell) \in \mathcal{P}_n$.Then his partial differential equation is

$$(24)\quad \sum_{i=1}^{k} \alpha_i^2 \frac{\partial^2}{\partial \alpha_i^2} \mathcal{Y}_p(A) + \sum_{i \neq j}^{k} \frac{\alpha_i^2}{\alpha_i - \alpha_j} \frac{\partial}{\partial \alpha_i} \mathcal{Y}_p(A) = \Big(\sum_{i=1}^{\ell} p_i(p_i - i + k - 1)\Big)\mathcal{Y}_p(A).$$

This might seem a little bit strange because it depends on the number of variables (k appears in the summation on the right hand side). Let

$$(25)\qquad a_1(p) = \sum_{i=1}^{\ell} p_i(p_i - i).$$

Then the right hand side of (24) can be written as

$$(26)\qquad a_1(p)\mathcal{Y}_p(A) + n(k-1)\mathcal{Y}_p(A), \qquad n = |p|.$$

To get rid of $n(k-1)\mathcal{Y}_p(A)$ we notice the fact that for any $f \in V_n$, $\sum_{i=1}^{k} \alpha_i (\partial/\partial\alpha_i) f = nf$. Therefore

$$(27)\qquad (k-1)n\mathcal{Y}_p(A) = (k-1)\sum_{i=1}^{k} \alpha_i \frac{\partial}{\partial \alpha_i} \mathcal{Y}_p(A).$$

But we can write

$$(28)\qquad (k-1)\sum_{i=1}^{k} \alpha_i \frac{\partial}{\partial \alpha_i} \mathcal{Y}_p(A) = \sum_{j=1}^{k} \sum_{i \neq j} \alpha_i \frac{\partial}{\partial \alpha_i} \mathcal{Y}_p(A).$$

Now subtracting (28) from both sides of (24) and using the relation $\alpha_i^2/(\alpha_i - \alpha_j) - \alpha_i = \alpha_i\alpha_j/(\alpha_i - \alpha_j)$ we obtain

$$(29)\qquad \sum_{i=1}^{k} \alpha_i^2 \frac{\partial^2}{\partial \alpha_i^2} \mathcal{Y}_p(A) + \sum_{i \neq j}^{k} \frac{\alpha_i\alpha_j}{\alpha_i - \alpha_j} \frac{\partial}{\partial \alpha_i} \mathcal{Y}_p(A) = a_1(p)\mathcal{Y}_p(A),$$

which does not involve k as a coefficient and is valid for any number of variables. (29) was derived by Sugiura (1973) in an elementary way. Because his exposition is clear and readable (except that there are complications like a higher order partial differential equation and differential equations for complex zonal polynomials) we do not derive it here. Let

$$\mathcal{Y}_p(A) = \sum_{q \le p} b_{pq} \mathcal{M}_q(A). \tag{30}$$

Substituting this into (29) we obtain

$$\begin{aligned}\sum_{q \le p} b_{pq} \left\{ \sum_{i=1}^{k} \alpha_i^2 \frac{\partial^2}{\partial \alpha_i^2} + \sum_{i \ne j} \frac{\alpha_i \alpha_j}{\alpha_i - \alpha_j} \frac{\partial}{\partial \alpha_i} \right\} \mathcal{M}_q(A) \\ = a_1(p) \sum_{q \le p} b_{pq} \mathcal{M}_q(A).\end{aligned} \tag{31}$$

Now

$$\sum_{i=1}^{k} \alpha_i^2 \frac{\partial^2}{\partial \alpha_i^2} \mathcal{M}_q(A), \qquad \sum_{i \ne j}^{k} \frac{\alpha_i \alpha_j}{\alpha_i - \alpha_j} \frac{\partial}{\partial \alpha_i} \mathcal{M}_q(A) \tag{32}$$

can be expressed as sums of monomial symmetric functions. Then comparing both sides of (31) we can determine the coefficients b_{pq}. It is hard to visualize what is going on here unless one works out some examples. Muirhead (1982) does that very carefully using (24) rather than (29) following James (1968). Therefore we only sketch the procedure here.

Let $q = (q_1, \ldots, q_\ell)$. It is fairly straightforward to verify that

$$\sum_{i=1}^{k} \alpha_i^2 \frac{\partial^2}{\partial \alpha_i^2} \mathcal{M}_q = \sum_{i=1}^{\ell} q_i(q_i - 1) \mathcal{M}_q, \tag{33}$$

$$\sum_{i \ne j} \frac{\alpha_i \alpha_j}{\alpha_i - \alpha_j} \frac{\partial}{\partial \alpha_i} \mathcal{M}_q = - \sum_{i=1}^{\ell} q_i(i-1) \mathcal{M}_q + \textit{lower order terms}. \tag{34}$$

Adding (33) and (34) we obtain

$$DM_q = a_1(q)M_q + \textit{lower order terms}, \tag{35}$$

where

$$D = \sum_{i=1}^{k} \alpha_i^2 \frac{\partial^2}{\partial \alpha_i^2} + \sum_{i \neq j}^{k} \frac{\alpha_i \alpha_j}{\alpha_i - \alpha_j} \frac{\partial}{\partial \alpha_i}.$$

It is fortunate to get only lower order terms by the differential operation. It is this triangular nature of the differential operation that enables one to determine b_{pq} recursively starting from the (arbitrary) leading coefficient b_{pp}. If one works out "lower order terms" (which is not hard) one arrives at the following rule by James (1968):

$$b_{pq} = \sum_{q < q' \leq p} \frac{((q_i + r) - (q_j - r))b_{pq'}}{a_1(p) - a_1(q)}, \tag{36}$$

where q' is an unordered partition of the form $q' = (q_1, \ldots, q_i + r, \ldots, q_j - r, \ldots, q_{\ell(q)})$, $(1 \leq r \leq q_j)$. The summation is over all (i, j, r) where $i < j$, $r \geq 1$ such that when the unordered partition q' is ordered we have $q < q' \leq p$.

Actually in view of Theorem 4.1.1 we have to consider only partitions q, q' which are majorized by p.

The advantage of this method is that it is self-contained. It gives all coefficients of a single zonal polynomial without computing others. Therefore it is by far the best method if one is interested in computing few zonal polynomials. On the other hand if one wants to compute many zonal polynomials then relying exclusively on this method seems to involve a great deal of redundant computations in view of recurrence relations of Section 4.5.3.

Remark 1. Logically the recurrence relation (36) is not complete until one shows that the denominator $a_1(p) - a_1(q)$ is never zero. James (1968) states that (36) gives rise to positive b_{pq}'s. Since the numerator $(q_i + r) - (q_j - r) = (q_i - q_j) + 2r$ is positive he seems to claim that $a_1(p) - a_1(q) > 0$ for all relevant pairs p, q. By Theorem 4.1.1 it is enough to prove

$$a_1(p) - a_1(q) > 0 \qquad \textit{for } p \succ q. \tag{37}$$

Then this ensures that (36) works for all cases and nonzero b_{pq}'s are positive. (36) can be easily proved using techniques from the theory of majorization. See Marshall and Olkin (1979). We do not go into this here.

§ 4.6 COEFFICIENTS OF $\mathcal{T}_q$ IN Z_p

In this section we study the coefficients of $\mathcal{T}_p$. The normalization Z_p seems to be most advantageous. An important fact about the coefficients of $\mathcal{T}_p$ is their orthogonality. We derive this first. See formulas (117) and (118) in James (1964), and Problem 13.3.9 in Farrell (1976).

For $p \in \mathcal{P}_n$ let

$$Z_p = \sum_{q \in \mathcal{P}_n} g_{pq} \mathcal{T}_q. \tag{1}$$

Let $\boldsymbol{Z}=(Z_{(n)}, Z_{(n-1,1)}, \ldots, Z_{(1^n)})'$,$\boldsymbol{G}=(g_{pq})$. Then (1) can be expressed in a matrix form as

$$\boldsymbol{Z} = \boldsymbol{G}\boldsymbol{\mathcal{T}}. \tag{2}$$

Now we recall that the transition matrix $\boldsymbol{F}$ in $\boldsymbol{\mathcal{T}}=\boldsymbol{F}\boldsymbol{\mathcal{U}}$ is lower triangular (see (2.2.28)). Substituting this into (2) we obtain

$$\boldsymbol{Z} = \boldsymbol{G}\boldsymbol{F}\boldsymbol{\mathcal{U}}. \tag{3}$$

But $\boldsymbol{Z} = \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{U}}$. Hence

$$\boldsymbol{G} = \boldsymbol{\mathcal{B}}\boldsymbol{F}^{-1} \tag{4}$$

Now (3.4.11) shows

$$\boldsymbol{\mathcal{B}}'\boldsymbol{D}\boldsymbol{\mathcal{B}} = \boldsymbol{F}'\boldsymbol{C}\boldsymbol{F}, \tag{5}$$

where $\boldsymbol{C} = \mathrm{diag}(c_p, p \in \mathcal{P}_n)$ is obtained in (3.4.8) and $\boldsymbol{D}=\mathrm{diag}(d_p,\ p \in \mathcal{P}_n)$ is known as (3.4.12). From (4) and (5) we obtain

$$\boldsymbol{G}'\boldsymbol{D}\boldsymbol{G} = \boldsymbol{C}. \tag{6}$$

Inverting this

$$GC^{-1}G' = D^{-1}. \tag{7}$$

Coordinatewise

$$\sum_p d_p g_{pq} g_{pq'} = \delta_{qq'} c_q, \qquad (\textit{column orthogonality}), \tag{8}$$

$$\sum_q g_{pq} g_{p'q} / c_q = \delta_{pp'} / d_p, \qquad (\textit{row orthogonality}), \tag{9}$$

where $\delta_{pp'}$ is Kronecker's delta.

Actually $c_q, q \in \mathcal{P}_n$ coincide with the elements of the first row of G. To see this let $\beta_1 = 1$, $\beta_2 = \cdots = \beta_k = 0$ in (3.4.7). Then clearly $\mathcal{T}_p(I_1) = 1$ for every p and we have

$$\prod_{i=1}^{k} (1 - 2\theta\alpha_i)^{-1/2} = \sum_{n=0}^{\infty} (\theta^n / n!) \sum_{p \in \mathcal{P}_n} c_p \mathcal{T}_p(A). \tag{10}$$

Comparing this to (4.4.4) we obtain $Z_{(n)}(A) = \sum c_p \; \mathcal{T}_p(A)$ and hence $c_p = g_{(n),p}$. Therefore (9) can be written alternatively as

$$\sum_q g_{pq} g_{p'q} / g_{(n),q} = \delta_{pp'} / d_p. \tag{11}$$

One obvious advantage of working with $\mathcal{T}_p$'s is that the coefficient matrix is readily invertible. From (6)

$$G^{-1} = C^{-1} G' D. \tag{12}$$

Therefore once we express zonal polynomials in terms of $\mathcal{T}_p$'s then it is easy to express $\mathcal{T}_p$'s (and their linear combinations) in zonal polynomials.

(11) was used to compute zonal polynomials in Parkhurst and James (1974) as follows. (i) $\mathcal{U}_p$'s are expressed in $\mathcal{T}_p$'s. (ii) They are Gram-Schmidt orthogonalized relative to the orthogonality relation (11) starting from the lowest

partition (1^n) upwards. Because of the triangularity of $\mathcal{B}$ this clearly results in zonal polynomials.

Some g_{pq}'s can be explicitly obtained using the fact $Z_p(I_k)=\lambda_{kp}$. We regard λ_{kp} as a function in k. Then by (3.1.15) it is a polynomial in k of degree $|p| = n$. Now since $t_r(I_k)=k$ for any r we obtain $\mathcal{T}_p(I_k)=k^{p_1-p_2}k^{p_2-p_3}\cdots=k^{p_1}= k^{h(p)}$. Therefore putting I_k in (1) we obtain

$$\lambda_{kp} = \sum_{q\in P_n} g_{pq}k^{h(q)}. \tag{13}$$

This uniquely determines g_{pq} for $q = (n)$, $q = (n-1,1)$, $q = (1^n)$ because these are the only partitions in P_n with $h(q) = n, n-1, 1$ respectively. Now the coefficient of k^n in λ_{kp} is 1, hence

$$g_{p,(n)} = 1. \tag{14}$$

(14) was originally used by James to determine the normalization Z_p. Now let us look at the coefficient of k^{n-1} in λ_{kp}. It is

$$\begin{aligned} &\sum_{i=1}^{\ell(p)}\{(-i+1)+(-i+3)+\cdots+(-i+1+2p_i-2)\} \\ &= \sum_{i=1}^{\ell(p)} \frac{1}{2}p_i\{(-i+1)+(-i+1+2p_i-2)\} \\ &= \sum_{i=1}^{\ell(p)} p_i(p_i-i) \\ &= a_1(p). \end{aligned} \tag{15}$$

$a_1(p)$ already appeared in (4.5.25). Therefore

$$g_{p,(n-1,1)} = a_1(p). \tag{16}$$

This is mentioned in the introductory part of Parkhurst and James (1964) in a somewhat different form

$$g_{p,(n-1,1)} = \sum_{i=1}^{\ell(p)} p_i(p_i-1) - \frac{1}{2}\sum_{j=1}^{h(p)} p'_j(p'_j-1), \tag{17}$$

where $p' = (p'_1, \ldots, p'_{h(p)})$ is the conjugate partition of p. Using (2.1.4) it is easy to check that (16) and (17) are equivalent.

Now the coefficient of k in λ_{kp} is

(18)
$$\{2\cdot 4\cdots(2p_1-2)\}\{(-1)(1)\cdots(2p_2-3)\}\cdots = 2^{n-1}(p_1-1)!\prod_{i=2}^{\ell(p)}\left(-\frac{i-1}{2}\right)_{p_i}.$$

Hence

(19)
$$g_{p,(1^n)} = 2^{n-1}(p_1-1)!\prod_{i=2}^{\ell(p)}\left(-\frac{i-1}{2}\right)_{p_i}.$$

This does not seem to have been noticed.

Now from (4)

(20)
$$\boldsymbol{\Xi} = \boldsymbol{G}\boldsymbol{F}, \quad \boldsymbol{\Xi} = (\xi_{pq}), \quad \boldsymbol{F} = (f_{pq}).$$

$\boldsymbol{F}$ is lower triangular. Therefore the last column of $\boldsymbol{\Xi}$ is $f_{(1^n)(1^n)}$ times the last column of $\boldsymbol{G}$. By (2.2.22)

(21)
$$\mathcal{T}_{(1^n)} = t_n = (-1)^{n-1}n(\mathcal{U}_{(1^n)} + \cdots).$$

Hence $f_{(1^n)(1^n)} = (-1)^{n-1}n$. Therefore

(22)
$$\xi_{p,(1^n)} = f_{(1^n)(1^n)}g_{p,(1^n)} = (-2)^{n-1}n(p_1-1)!\prod_{i=2}^{\ell(p)}\left(-\frac{i-1}{2}\right)_{p_i}.$$

Making use of (12) gives another set of identities.

(23)
$$\boldsymbol{\mathcal{T}} = \boldsymbol{G}^{-1}\boldsymbol{Z} = \boldsymbol{C}^{-1}\boldsymbol{G}'\boldsymbol{D}\boldsymbol{Z}.$$

Now $\boldsymbol{DZ} = (d_{(n)}Z_{(n)}, \ldots, d_{(1^n)}Z_{(1^n)})'$ and d_pZ_p is denoted by C_p (3.4.13). Therefore

(24)
$$\boldsymbol{\mathcal{T}} = \boldsymbol{C}^{-1}\boldsymbol{G}'(C_{(n)}, \ldots, C_{(1^n)})'.$$

Comparing the second element we obtain

$$
\begin{aligned}
\mathcal{T}_{(n-1,1)} &= t_1^{n-2} t_2 \\
&= \frac{1}{c_{(n-1,1)}} \sum_{p \in \mathcal{P}_n} a_1(p) \mathcal{C}_p \\
&= \frac{1}{n(n-1)} \sum_{p \in \mathcal{P}_n} a_1(p) \mathcal{C}_p,
\end{aligned} \tag{25}
$$

where $c_{(n-1,1)} = n(n-1)$ is given by (3.4.8). (25) was given by Sugiura and Fujikoshi (1959) by a different method. They derive more identities of this kind. See Sugiura (1971) too. Now looking at the last element we obtain

$$
\begin{aligned}
\mathcal{T}_{(1^n)} &= t_n = \mathcal{M}_{(n)} \\
&= \frac{1}{c_{(1^n)}} 2^{n-1} \sum_{p \in \mathcal{P}_n} \mathcal{C}_p \cdot (p_1 - 1)! \prod_{i=2}^{\ell(p)} \left(-\frac{i-1}{2} \right)_{p_i} \\
&= \frac{1}{(n-1)!} \sum_{p \in \mathcal{P}_n} \mathcal{C}_p \cdot (p_1 - 1)! \prod_{i=2}^{\ell(p)} \left(-\frac{i-1}{2} \right)_{p_i} .
\end{aligned} \tag{26}
$$

What are advantages and disadvantages of working with $\mathcal{T}_p$'s? One advantage is that we do not have to compute characteristic roots of $\boldsymbol{A}$ to compute $\mathcal{T}_p(\boldsymbol{A})$. (One only needs traces of powers of $\boldsymbol{A}$.) Another advantage is the orthogonality discussed above. A serious drawback of $\mathcal{T}_p$ is that we have to compute $\mathcal{T}_p(\boldsymbol{A})$ for all $p \in \mathcal{P}_n$ even if the rank of $\boldsymbol{A}$ is small. In usual statistical computations rank $\boldsymbol{A}$ is fixed and not too large. It is a covariance matrix for example. Since the number of partitions grows very fast as n increases if one wants to compute $\mathcal{Z}_p(\boldsymbol{A})$ for $|p|$ large it seems better to use $\mathcal{U}_q$'s or $\mathcal{M}_q$'s. The growth of the number of partitions p with $\ell(p) \leq k$ (k : fixed) is much smaller than the growth of the number of all partitions. See Table 4.1 in David, Kendall, and Barton (1966).

§ 4.7 VARIATIONS OF THE INTEGRAL REPRESENTATION OF ZONAL POLYNOMIALS

In this section we explore various variations of the integral representation (Theorem 3.3.1) discussed in Section 3.3. We first replace $\boldsymbol{U}$ by the $k \times k$ uniform orthogonal matrix $\boldsymbol{H}$.

Theorem 1. *(James, 1973) For $k \times k$ symmetric $\boldsymbol{A}$*

$$\begin{aligned}\frac{\mathcal{Y}_p(\boldsymbol{A})}{\mathcal{Y}_p(\boldsymbol{I}_k)} &= \mathcal{E}_H\{\Delta_1^{p_1-p_2}\cdots\Delta_\ell^{p_\ell}\} \\ &= \mathcal{E}_H\{\prod_{i=1}^{\ell}[\boldsymbol{HAH}'(1,\ldots,i)]^{p_i-p_{i+1}}\},\end{aligned} \tag{1}$$

where $p=(p_1,\ldots,p_\ell)\in \mathcal{P}_n$, $k\geq \ell$, and the $k\times k$ orthogonal $\boldsymbol{H}$ is uniformly distributed.

Proof. As in Lemma 3.1.2 it is easy to check that $\mathcal{E}_H\{\Delta_1^{p_1-p_2}\cdots\Delta_\ell^{p_\ell}\}\in V_{n,k}$. Therefore we can write

$$\mathcal{E}_H\{\Delta_1^{p_1-p_2}\cdots\Delta_\ell^{p_\ell}\} = \sum_{q\in\mathcal{P}_n,\ell(q)\leq k} a_q Z_q(\boldsymbol{A}). \tag{2}$$

Replacing $\boldsymbol{A}$ by $\boldsymbol{UAU}'$ where $k\times k$ $\boldsymbol{U}$ is as in Theorem 3.3.1 and taking expectation with respect to $\boldsymbol{U}$ we obtain

$$Z_p(\boldsymbol{A}) = \sum_{q\in\mathcal{P}_n,\ell(q)\leq k} a_q\lambda_{kq}Z_q(\boldsymbol{A}). \tag{3}$$

This being true for any symmetric $k\times k$ $\boldsymbol{A}$ we conclude from Theorem 4.1.1

$$a_q=0,\ q\neq p,\ a_p=\frac{1}{\lambda_{kp}}=\frac{1}{Z_p(\boldsymbol{I}_k)}.$$

Since (1) is independent of normalization we can have $\mathcal{Y}_p$ instead of Z_p in (1).∎

Corollary 1. *(Kates) Let $\boldsymbol{X}: k\times k$ have an orthogonally biinvariant distribution then*

$$\mathcal{E}_X\{\Delta_1^{p_1-p_2}\cdots\Delta_\ell^{p_\ell}\} = \frac{\mathcal{Y}_p(\boldsymbol{A})\mathcal{E}_X\{\mathcal{Y}_p(\boldsymbol{X}'\boldsymbol{X})\}}{\{\mathcal{Y}_p(\boldsymbol{I}_k)\}^2}. \tag{4}$$

where $\Delta_i=\boldsymbol{XAX}'(1,\ldots,i)$ and $\boldsymbol{A}$ is symmetric.

Proof. We replace $\boldsymbol{X}$ by $\boldsymbol{H}_1\boldsymbol{X}\boldsymbol{H}_2$ where $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are independently uniformly distributed. The distribution of $\boldsymbol{X}$ is unchanged. Now taking expectation with respect to $\boldsymbol{H}_1$ (Theorem 1) and $\boldsymbol{H}_2$ (Theorem 3.2.1) successively we obtain

(5)

$$\begin{aligned}\mathcal{E}_X\{\Delta_1^{p_1-p_2}\cdots\Delta_\ell^{p_\ell}\} &= \mathcal{E}_{H_1,X,H_2}\{\prod_{i=1}^{\ell}[\boldsymbol{H}_1\boldsymbol{X}\boldsymbol{H}_2\boldsymbol{A}\boldsymbol{H}_2'\boldsymbol{X}'\boldsymbol{H}_1'(1,\ldots,i)]^{p_i-p_{i+1}}\} \\ &= \mathcal{E}_{X,H_2}\{\mathcal{Y}_p(\boldsymbol{X}\boldsymbol{H}_2\boldsymbol{A}\boldsymbol{H}_2'\boldsymbol{X}')\}/\mathcal{Y}_p(\boldsymbol{I}_k) \\ &= \mathcal{Y}_p(\boldsymbol{A})\mathcal{E}_X\{\mathcal{Y}_p(\boldsymbol{X}'\boldsymbol{X})\}/\{\mathcal{Y}_p(\boldsymbol{I}_k)\}^2.\end{aligned}$$

∎

Remark 1. As in Remark 3.2.5 $\boldsymbol{X}$ can be rectangular. If $\boldsymbol{X}$ is $m\times k$, then $\{\mathcal{Y}_p(\boldsymbol{I}_k)\}^2$ on the right hand side of (4) is replaced by $\mathcal{Y}_p(\boldsymbol{I}_k)\ \mathcal{Y}_p(\boldsymbol{I}_m)$.

An easy modification of the above formulas produces another set of identities.

Theorem 2. *Let $\boldsymbol{U}_1,\boldsymbol{U}_2$ be $k\times k$ matrices whose entries are independent standard normal variables. Then for $k\times k$ $\boldsymbol{A}$*

$$ {}_1b_p Z_p(\boldsymbol{A}\boldsymbol{A}') = \mathcal{E}_{U_1,U_2}\{\prod_{i=1}^{\ell}[\boldsymbol{U}_1\boldsymbol{A}\boldsymbol{U}_2(1,\ldots,i)]^{2p_i-2p_{i+1}}\}, \tag{6}$$

where ${}_1b_p$ is given by (4.2.7).

Proof. Let the singular value decomposition of $\boldsymbol{A}$ be $\boldsymbol{A}=\boldsymbol{\Gamma}_1\boldsymbol{D}\boldsymbol{\Gamma}_2$ where $\boldsymbol{\Gamma}_1,\boldsymbol{\Gamma}_2$ are orthogonal, $\boldsymbol{D}=\mathrm{diag}(\delta_1,\ldots,\delta_k)$ and $\delta_1^2,\ldots,\delta_k^2$ are the characteristic roots of $\boldsymbol{A}\boldsymbol{A}'$. Since the order of $\delta_1,\ldots,\delta_k$ and the sign of each δ_i can be arbitrary in the singular value decomposition we see that (6) is a homogeneous symmetric polynomial in $\delta_1^2,\ldots,\delta_k^2$. Denote the right hand side of (6) by $f(\boldsymbol{A}\boldsymbol{A}')$. We use the converse part of Theorem 3.1.1. We want to show that if $\boldsymbol{W}\sim\mathcal{W}(\boldsymbol{I}_k,\nu)$ then

$$\mathcal{E}_W f(\boldsymbol{A}\boldsymbol{A}'\boldsymbol{W}) = \lambda_{\nu p} f(\boldsymbol{A}\boldsymbol{A}') \tag{7}$$

for all sufficiently large ν. Now fix ν and let $\boldsymbol{A}$, $\boldsymbol{U}_1$, $\boldsymbol{U}_2$ be augmented to $\nu\times\nu$ as in the proofs of Theorem 3.2.4 or Theorem 3.3.1. (Here we do not place $\sim$

for notational simplicity.) Note that $f(\boldsymbol{AA}')$ does not change by this change of dimensionality. Also in (7) $\boldsymbol{W}$ can be augmented to be $\nu \times \nu$ by considering $\boldsymbol{W}$ as $k \times k$ upper left submatrix of a Wishart matrix $\mathcal{W}(\boldsymbol{I}_\nu, \nu)$. Now let U_3 $(\nu \times \nu)$ be distributed independently of $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$. Then

$$
\begin{aligned}
\mathcal{E}_W f(\boldsymbol{AA}'\boldsymbol{W}) &= \mathcal{E}_{U_3} f(\boldsymbol{AA}'\boldsymbol{U}_3\boldsymbol{U}_3{}') = \mathcal{E}_{U_3} f(\boldsymbol{U}_3\boldsymbol{AA}'\boldsymbol{U}_3{}') \\
&= \mathcal{E}_{U_1,U_2,U_3}\{\prod_{i=1}^{\ell} [\boldsymbol{U}_1\boldsymbol{U}_3\boldsymbol{A}\boldsymbol{U}_2(1,\ldots,i)]^{2p_i-2p_{i+1}}\} \\
&= \mathcal{E}_{U_1,U_2,U_3}\{\prod_{i=1}^{\ell} [\boldsymbol{U}_3\boldsymbol{U}_1\boldsymbol{A}\boldsymbol{U}_2(1,\ldots,i)]^{2p_i-2p_{i+1}}\} \\
&= \mathcal{E}_{U_1,U_2,T,H}\{\prod_{i=1}^{\ell} [\boldsymbol{THU}_1\boldsymbol{A}\boldsymbol{U}_2(1,\ldots,i)]^{2p_i-2p_{i+1}}\} \\
&= \lambda_{\nu p}\mathcal{E}_{U_1,U_2}\{\prod_{i=1}^{\ell} [\boldsymbol{U}_1\boldsymbol{A}\boldsymbol{U}_2(1,\ldots,i)]^{2p_i-2p_{i+1}}\} \\
&= \lambda_{\nu p} f(\boldsymbol{AA}'),
\end{aligned} \tag{8}
$$

where $\boldsymbol{T}, \boldsymbol{H}$ are as in Lemma 3.2.2. Therefore by Theorem 3.1.1 $f = cZ_p$ for some c. To obtain c we put $\boldsymbol{A}=\boldsymbol{I}_\nu$. Then

$$
\begin{aligned}
c\lambda_{\nu p} &= \mathcal{E}_{U_1,U_2}\{\prod_{i=1}^{\ell} [\boldsymbol{U}_1\boldsymbol{U}_2(1,\ldots,i)]^{2p_i-2p_{i+1}}\} \\
&= \mathcal{E}_{T,U_2}\{\prod_{i=1}^{\ell} [\boldsymbol{T}\boldsymbol{U}_2(1,\ldots,i)]^{2p_i-2p_{i+1}}\} \\
&= \lambda_{\nu p}\mathcal{E}_{U_2}\{\prod_{i=1}^{\ell} \boldsymbol{U}_2(1,\ldots,i)^{2p_i-2p_{i+1}}\}.
\end{aligned} \tag{9}
$$

Hence $c = \mathcal{E}_U\{\prod_{i=1}^{\ell} \boldsymbol{U}(1,\ldots,i)^{2p_i-2p_{i+1}}\}$. Now consider (3.3.2) and (3.3.8). Then we see that the coefficient of $\alpha_1^{p_1}\cdots\alpha_\ell^{p_\ell}$ is given just by $\mathcal{E}_U\{\prod_{i=1}^{\ell} \boldsymbol{U}(1, \ldots, i)^{2p_i-2p_{i+1}}\}$. Therefore it is the leading coefficient of Z_p and is equal to ${}_1b_p$ given in (4.2.7). ∎

Corollary 2.

$$
\frac{{}_1b_p Z_p(\boldsymbol{AA}')}{Z_p(\boldsymbol{I}_k)} = \mathcal{E}_{H_1,U_2}\{\prod_{i=1}^{\ell} [\boldsymbol{H}_1\boldsymbol{A}\boldsymbol{U}_2(1,\ldots,i)]^{2p_i-2p_{i+1}}\}. \tag{10}
$$

$$(11)\qquad \frac{{}_1b_p Z_p(\boldsymbol{AA'})}{\{Z_p(\boldsymbol{I}_k)\}^2} = \mathcal{E}_{H_1,H_2}\{\prod_{i=1}^{\ell}[\boldsymbol{H}_1\boldsymbol{A}\boldsymbol{H}_2(1,\ldots,i)]^{2p_i-2p_{i+1}}\}.$$

Proof. (10) and (11) can be proved successively as in the proof of Theorem 2. ∎

Corollary 3. *Let $\boldsymbol{X}_1, \boldsymbol{X}_2$ be independent and have orthogonally biinvariant distributions. Then*

$$(12)\qquad \begin{aligned} Z_p(\boldsymbol{AA'})&\frac{{}_1b_p\mathcal{E}_{X_1}\{Z_p(\boldsymbol{X}_1\boldsymbol{X}_1')\}\mathcal{E}_{X_2}\{Z_p(\boldsymbol{X}_2\boldsymbol{X}_2')\}}{\{Z_p(\boldsymbol{I}_k)\}^4} \\ &= \mathcal{E}_{X_1,X_2}\{\prod_{i=1}^{\ell}[\boldsymbol{X}_1\boldsymbol{A}\boldsymbol{X}_2(1,\ldots,i)]^{2p_i-2p_{i+1}}\}. \end{aligned}$$

Proof. Replace $\boldsymbol{X}_1$ by $\boldsymbol{H}_1\boldsymbol{X}_1\boldsymbol{H}_3$ and $\boldsymbol{X}_2$ by $\boldsymbol{H}_4\boldsymbol{X}_2\boldsymbol{H}_2$. Then taking expectation with respect to $\boldsymbol{H}_1,\boldsymbol{H}_2$, $\boldsymbol{H}_3,\boldsymbol{H}_4$ successively we obtain (12). ∎

Remark 2. Generalization to rectangular matrices is straightforward.