3. Weak convergence of non-BOREL measures on a metric space.*)

Let $S = (S,d)$ be a metric space with metric d and let $\mathcal{B}_b(S) \equiv \mathcal{B}_b(S,d)$ be the σ-algebra in S generated by the open (d-) balls

$$B_d(x,r) \equiv B(x,r) := \{y \in S: d(x,y)<r\},\ x \in S,\ r>0.$$

Clearly, $\mathcal{B}_b(S)$ is a sub-σ-algebra of the Borel σ-algebra $\mathcal{B}(S)$ in S (generated by all open subsets of S).

In this section we will study a mode of weak convergence for nets of finite measures which are defined at least on $\mathcal{B}_b(S)$. Our formulation is a slight modification of a concept which was introduced by R.M. Dudley (1966) and further studied and extended by M.J. Wichura (1968); cf. also D. Pollard (1979), where it is shown that some of the key results in that theory can be deduced directly from the better known weak convergence theory for Borel measures.

As in Wichura, our presentation here is made roughly along the lines of Chapter I of Billingsley (1968) (see also P. Billingsley (1971), S I A M No. 5) which treats similar aspects of the theory of weak convergence of probabilities defined on all of $\mathcal{B}(S)$. The present theory is especially suited to cope with measurability problems arising in the theory of empirical processes as well as to allow for a proper formulation of functional central limit theorems for empirical $\mathcal{C}$-processes (cf. Section 4).

To start with, let us first establish some notation and terminology to be

*) This section represents and extends parts of a first draft of a "Diplomarbeit" by J. Schattauer, University of Munich, 1981.

used throughout this section.

If not stated otherwise, $S = (S,d)$ is always a (possibly non-separable) metric space. Let $\mathcal{A}$ be a σ-algebra of subsets of S such that $\mathcal{B}_b(S) \subset \mathcal{A} \subset \mathcal{B}(S)$; then the following spaces of real valued functions on S will be considered:

$\mathcal{F}_a(S) := \{f: S \to \mathbb{R},\ f\ \mathcal{A},\mathcal{B}\text{-measurable}\}$

$C^b(S) := \{f: S \to \mathbb{R},\ f \text{ bounded and continuous}\}$

$C_a^b(S) := \mathcal{F}_a(S) \cap C^b(S)$

$U^b(S) := \{f: S \to \mathbb{R},\ f \text{ bounded and uniformly continuous}\}$

$U_a^b(S) := \mathcal{F}_a(S) \cap U^b(S).$

In case of $\mathcal{B}_b(S)$ instead of $\mathcal{A}$ we shall write $\mathcal{F}_b(S)$, $C_b^b(S)$ and $U_b^b(S)$ instead of $\mathcal{F}_a(S)$, $C_a^b(S)$ and $U_a^b(S)$, respectively.

The following figure may help to visualize the different spaces, where the largest box represents the class of all $\mathcal{B}(S)$, $\mathcal{B}$-measurable functions $f: S \to \mathbb{R}$ and where the smallest class $U_b^b(S)$ is represented by the shaded area:

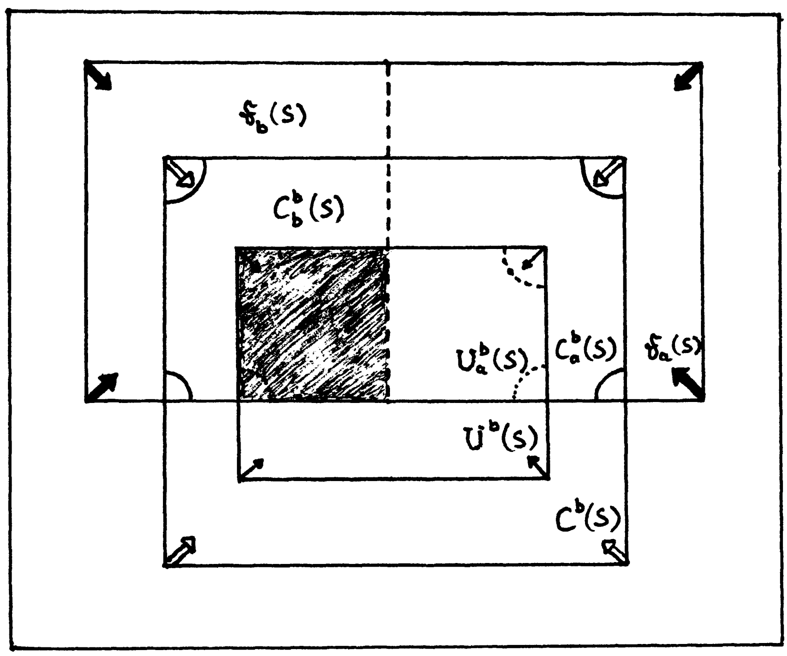

FIGURE 3 ($\mathcal{F}_b(S)$ for example is represented as that part of the $\mathcal{F}_a(S)$-box (marked by the bold arrows) which is left to the dotted line.)

Furthermore, let $M_a(S)$ be the space of all nonnegative finite measures defined on A and write $M_b(S)$ for the space of all nonnegative finite measures on $\mathcal{B}_b(S)$. For $f: S \to \mathbb{R}$, we denote by $D(f)$ the set of discontinuity points of f. Finally, given any $\mu \in M_a(S)$ and any bounded f or $A \subset S$, respectively, let

$\int^* f\,d\mu := \inf\{\int g\,d\mu: g \geq f,\ g \in \mathcal{F}_a(S) \text{ and } g \text{ bounded}\}$,

$\int_* f\,d\mu := \sup\{\int g\,d\mu: g \leq f,\ g \in \mathcal{F}_a(S) \text{ and } g \text{ bounded}\}$,

$\mu_*(A) := \int_* 1_A d\mu$, and $\mu^*(A) := \int^* 1_A d\mu$.

Note that μ_* and μ^* are inner and outer measures, respectively, i.e., one has for every $A \subset S$

$$\text{(23)} \qquad \mu^*(A) = \inf\{\mu(B): B \supset A,\ B \in A\} \text{ and } \mu_*(A) = \sup\{\mu(B): B \subset A,\ B \in A\}.$$

(In fact, as to the first equality in (23), "$\leq$" is obvious, since for any $B \supset A$, $B \in A$, $g := 1_B \geq 1_A$, $g \in \mathcal{F}_a(S)$ and g bounded; as to the other inequality, given any $g \geq 1_A$, $g \in \mathcal{F}_a(S)$ and g bounded, choose for each $\varepsilon > 0$ $B_\varepsilon := \{g \geq 1-\varepsilon\}$ to get $B_\varepsilon \in A$ with $B_\varepsilon \supset A$ and $\mu(B_\varepsilon) \leq \int_{B_\varepsilon} (g+\varepsilon)d\mu \leq \int g d\mu + \varepsilon\mu(S)$; since $\mu(S) < \infty$, we obtain, taking $\varepsilon = \frac{1}{n}$ and letting $n \to \infty$, $B := \bigcap_{n \in \mathbb{N}} B_{1/n} \in A$ with $B \supset A$ and $\mu(B) \leq \int g d\mu$, which proves the other inequality.)

The following lemma comprises some simple but still essential facts to be used later on.

LEMMA 11. (i) $\mathcal{B}_b(S) = \sigma(\{d(\cdot,x): x \in S\})$, where $\sigma(\{d(\cdot,x): x \in S\})$ denotes the smallest σ-algebra in S w.r.t. which all of the functions $d(\cdot,x)$, for each fixed $x \in S$, are measurable.

(ii) Let $S_o \subset S$ be such that $S_o = (S_o,d)$ is a separable metric space, then for $d(\cdot,S_o) := \inf\{d(\cdot,x): x \in S_o\}$ we have $\min(d(\cdot,S_o),n) \in U_b^b(S)$ for each n; in this case also $S_o^\delta := \{x \in S: d(x,S_o) < \delta\} \in \mathcal{B}_b(S)$ for every $\delta > 0$, and $S_o^c \in \mathcal{B}_b(S)$, where S_o^c denotes the closure of S_o in (S,d).

(iii) $K(S) \subset \mathcal{B}_b(S)$, where $K(S)$ denotes the class of all compact subsets of (S,d).

(iv) If (S,d) is separable, then $\mathcal{B}_b(S) = \mathcal{B}(S)$.

Proof. (i) is an immediate consequence of the identity $B(x,r) = \{y\in S: d(y,x)<r\}$, $x\in S$, $r>0$. To verify (ii), since $d(\cdot,A)$ is uniformly continuous for each $A\subset S$, it suffices to show that $d(\cdot,S_o) \in \mathcal{F}_b(S)$; for this, let T_o be a countable dense subset of S_o. Then, since $d(x,S_o) = d(x,T_o) = \inf\{d(x,y): y\in T_o\}$, $d(x,S_o)$ is a countable infimum of $\mathcal{B}_b$, $\mathcal{B}$-measurable functions, hence $d(\cdot,S_o) \in \mathcal{F}_b(S)$. This also shows that $S_o^\delta \in \mathcal{B}_b(S)$ implying that $S_o^c = \bigcap_{n\in\mathbb{N}} S_o^{1/n} \in \mathcal{B}_b(S)$. Since each compact subset of S is closed and separable, (iii) is just a particular case of (ii). Finally, if (S,d) is separable, then (S_o,d) is separable for each $S_o\subset S$, especially for all closed subsets F of S, whence, by (ii), $F\in\mathcal{B}_b(S)$ for all closed $F\subset S$ and therefore $\mathcal{B}(S)\subset\mathcal{B}_b(S)$ which proves (iv). □

Remark. The converse of (iv) is not true, in general: Talagrand (1978) has constructed an example of a non-separable metric space S for which $\mathcal{B}_b(S)$ coincides with $\mathcal{B}(S)$.

Now, our first subsection will be concerned with

SEPARABLE AND TIGHT MEASURES ON $\mathcal{B}_b(S)$:

DEFINITION 2. $\mu\in\mathcal{M}_b(S)$ is called separable iff there exists a separable subset S_o of S (i.e., an $S_o\subset S$ s.t. $S_o = (S_o,d)$ is a separable metric space) with $\mu(S_o^c) = \mu(S)$.
(Note that the closure S_o^c of a separable S_o is also separable.)

(24) REMARK. Let $\mu\in\mathcal{M}_b(S)$ be separable; then there exists a unique extension of μ to an (even τ-smooth) Borel measure $\tilde{\mu}$ on $\mathcal{B}(S)$.

Proof. By assumption there exists a closed and separable $A_o\subset S$ such that $\mu(A_o) = \mu(S)$, where $A_o\in\mathcal{B}_b(S)$ by Lemma 11 (ii). Let $\mathcal{D} := \{B\in\mathcal{B}(S): B\cap A_o\in\mathcal{B}_b(S)\}$; then $\mathcal{D}$ is a σ-algebra in S. But, since each closed subset belongs to $\mathcal{D}$ (cf. Lemma 11 (ii) and notice that $F\cap A_o$ is again closed and separable),

$\mathcal{D}$ equals $\mathcal{B}(S)$ and therefore $\tilde{\mu}(B) := \mu(B\cap A_o)$ is well defined for all $B\in\mathcal{B}(S)$. Furthermore, for every $B\in\mathcal{B}(S)$, $\tilde{\mu}(B) = \mu(B\cap A_o) = \mu(B)-\mu(B\setminus A_o) = \mu(B)$, since $\mu(B\setminus A_o) = 0$ according to $\mu(A_o) = \mu(S)$, showing that $\tilde{\mu}$ is a Borel extension of μ (being even τ-smooth since $\tilde{\mu}$ concentrates on the separable subset A_o of S). As to the uniqueness of $\tilde{\mu}$, suppose that $\tilde{\mu}_i$ are finite measures on $\mathcal{B}(S)$ with $\mathrm{rest}_{\mathcal{B}_b(S)}\tilde{\mu}_i = \mu$, $i=1,2$; then $\tilde{\mu}_1(A_o) = \tilde{\mu}_2(A_o) = \mu(A_o) = \mu(S)$ and therefore $\tilde{\mu}_1(B) = \tilde{\mu}_1(B\cap A_o) = \mu(B\cap A_o) = \tilde{\mu}_2(B\cap A_o) = \tilde{\mu}_2(B)$ for all $B\in\mathcal{B}(S)$ showing that $\tilde{\mu}_1 = \tilde{\mu}_2$. □

(Note: It can be shown by examples that the assumption in (24) of μ being separable cannot be dispensed with, in general.)

DEFINITION 3. $\mu\in M_b(S)$ is called tight iff $\sup\{\mu(K): K\in\mathcal{K}(S)\} = \mu(S)$. (Note that $\mathcal{K}(S)\subset\mathcal{B}_b(S)$ according to Lemma 11 (iii).)

(25) REMARK. Any tight $\mu\in M_b(S)$ is separable.

Proof. Note first that any $K\in\mathcal{K}(S)$ is separable; now, since μ is tight, there exists for every n a $K_n\in\mathcal{K}(S)$ s.t. $\mu(K_n)>\mu(S) - \frac{1}{n}$; then $S_o := \bigcup_{n\in\mathbb{N}} K_n$ is separable and $\mu(S_o^c) \geq \mu(S_o) \geq \mu(K_n) > \mu(S) - \frac{1}{n}$ for all n, whence $\mu(S_o^c)=\mu(S)$. □

As to the converse of (25) one has

(26) REMARK. If $\mu\in M_b(S)$ is separable and if S is topologically complete, then μ is tight.

Proof. Use (24) to get the unique Borel extension $\tilde{\mu}$ of μ and apply Theorem 1, Appendix III, p. 234, in Billingsley (1968). □

(Note: As shown by Billingsley (1968), Remark 2, p. 234, the hypothesis of topological completeness cannot be suppressed in (26).)

WEAK CONVERGENCE/PORTMANTEAU-THEOREM:

As before, let $S = (S,d)$ be a (possibly non-separable) metric space, let $M_a(S)$ be the space of all nonnegative finite measures on a σ-algebra A with $B_b(S) \subset A \subset B(S)$ and let $M_b(S)$ be the space of all nonnegative finite measures on $B_b(S)$.

Then, given a net $(\mu_\alpha)_{\alpha\in A}$ in $M_a(S)$ and a $\mu\in M_b(S)$, we define:

DEFINITION 4. (μ_α) converges weakly to μ (denoted by $\mu_\alpha \xrightarrow[b]{} \mu$) if

(i) μ is separable

(ii) $\lim_\alpha \int f d\mu_\alpha = \int f d\tilde{\mu}$ for all $f\in C_a^b(S)$

(where again $\tilde{\mu}$ is the unique Borel extension of μ, according to (24)).

(27) REMARKS. a) If (S,d) is a separable metric space, then Definition 4 coincides with the usual definition of weak convergence of Borel measures (cf. Lemma 11 (iv)).

b) If (μ_α) converges weakly in the sense of Wichura's (1968) definition, then (μ_α) converges in the sense of our Definition 4 but not vice versa; both definitions are equivalent if (S,d) is topologically complete (cf. (26) and our Portmanteau-Theorem below).

LEMMA 12. Let $f: S \to \mathbb{R}$ be such that $0\leq f<n$ for some $n\in\mathbb{N}$; then, for every $\mu\in M_a(S)$,

$$\int^* f d\mu \leq \mu(S) + \sum_{k=1}^{n} \mu^*(\{f\geq k\}).$$

Proof. Since by (23), for every $A\subset S$, $\mu^*(A) = \inf\{\mu(B): B\supset A, B\in A\}$, it follows that for every $\varepsilon>0$ and every $1\leq k\leq n$ there exists a $B_{\varepsilon,k}\in A$ s.t. $B_{\varepsilon,k}\supset\{f\leq k\}$ and $\mu^*(\{f\geq k\}) \geq \mu(B_{\varepsilon,k}) - \frac{\varepsilon}{n}$. Put $f_\varepsilon := 1_S + \sum_{k=1}^{n} 1_{B_{\varepsilon,k}}$ to get a bounded function belonging to $F_a(S)$ and dominating f ($f \leq \sum_{k=0}^{n} 1_{\{f\geq k\}} \leq 1_S + \sum_{k=1}^{n} 1_{B_{\varepsilon,k}} = f_\varepsilon$), whence

$$\overset{*}{\int} f d\mu = \inf\{\int g d\mu\colon g \geq f,\ g \in \mathcal{F}_a(S) \text{ and } g \text{ bounded}\} \leq \int f_\varepsilon d\mu$$

$$= \mu(S) + \sum_{k=1}^{n} \mu(B_{\varepsilon,k}) \leq \mu(S) + \sum_{k=1}^{n} [\mu^*(\{f \geq k\}) + \frac{\varepsilon}{n}]$$

$$= \mu(S) + \sum_{k=1}^{n} \mu^*(\{f \geq k\}) + \varepsilon,$$

which implies the assertion since $\varepsilon > 0$ was chosen arbitrary. □

In what follows, let $\mathcal{G}(S)$, resp. $\mathcal{F}(S)$, denote the class of all open, resp. closed, subsets of S; also for $A \subset S$ let A^o, A^c and ∂A denote the interior, closure and boundary of A, respectively.

(28) <u>PORTMANTEAU-THEOREM.</u>

Let (μ_α) be a net in $M_a(S)$ and let $\mu \in M_b(S)$ be separable with $\tilde{\mu}$ being its unique Borel extension (cf. (24)).

Then the following assertions (a) - (h') are all equivalent:

(a) $\lim_\alpha \mu_\alpha(S) = \mu(S)$ and $\liminf_\alpha (\mu_\alpha)_*(G) \geq \tilde{\mu}(G)$ for all $G \in \mathcal{G}(S)$

(a') $\lim_\alpha \mu_\alpha(S) = \mu(S)$ and $\liminf_\alpha \mu_\alpha(G) \geq \tilde{\mu}(G)$ for all $G \in \mathcal{G}(S) \cap \mathcal{A}$

(b) $\lim_\alpha \mu_\alpha(S) = \mu(S)$ and $\limsup_\alpha \mu_\alpha^*(F) \leq \tilde{\mu}(F)$ for all $F \in \mathcal{F}(S)$

(b') $\lim_\alpha \mu_\alpha(S) = \mu(S)$ and $\limsup_\alpha \mu_\alpha(F) \leq \tilde{\mu}(F)$ for all $F \in \mathcal{F}(S) \cap \mathcal{A}$

(c) $\liminf_\alpha \underset{*}{\int} f d\mu_\alpha \geq \int f d\tilde{\mu}$ for all bounded lower semicontinuous $f\colon S \to \mathbb{R}$

(c') $\liminf_\alpha \int f d\mu_\alpha \geq \int f d\tilde{\mu}$ for all bounded lower semicontinuous $f \in \mathcal{F}_a(S)$

(d) $\limsup_\alpha \overset{*}{\int} f d\mu_\alpha \leq \int f d\tilde{\mu}$ for all bounded upper semicontinuous $f\colon S \to \mathbb{R}$

(d') $\limsup_\alpha \int f d\mu_\alpha \leq \int f d\tilde{\mu}$ for all bounded upper semicontinuous $f \in \mathcal{F}_a(S)$

(e) $\lim_\alpha \overset{*}{\int} f d\mu_\alpha = \lim_\alpha \underset{*}{\int} f d\mu_\alpha = \int f d\tilde{\mu}$ for all bounded $\mathcal{B}(S)$, $\mathbb{B}$-measurable $f\colon S \to \mathbb{R}$ which are $\tilde{\mu}$-almost everywhere ($\tilde{\mu}$-a.e.) continuous

(e') $\lim_\alpha \int f d\mu_\alpha = \int f d\tilde{\mu}$ for all bounded $f \in \mathcal{F}_a(S)$ which are $\tilde{\mu}$-a.e. continuous

(f) $\lim\limits_{\alpha} (\mu_\alpha)_*(A) = \lim\limits_{\alpha} \mu_\alpha^*(A) = \tilde{\mu}(A)$ for all $A \in \mathcal{B}(S)$ with $\tilde{\mu}(\partial A) = 0$

(f') $\lim\limits_{\alpha} \mu_\alpha(A) = \tilde{\mu}(A)$ for all $A \in \mathcal{A}$ with $\tilde{\mu}(\partial A) = 0$

(g) $\lim\limits_{\alpha} \int^* f d\mu_\alpha = \lim\limits_{\alpha} \int_* f d\mu_\alpha = \int f d\tilde{\mu}$ for all $f \in C^b(S)$

(g') $\lim\limits_{\alpha} \int f d\mu_\alpha = \int f d\tilde{\mu}$ for all $f \in C_a^b(S)$ (cf. Definition 4, (ii))

(h) $\lim\limits_{\alpha} \int^* f d\mu_\alpha = \lim\limits_{\alpha} \int_* f d\mu_\alpha = \int f d\tilde{\mu}$ for all $f \in U^b(S)$

(h') $\lim\limits_{\alpha} \int f d\mu_\alpha = \int f d\mu$ for all $f \in U_b^b(S)$.

<u>Proof.</u> The proof may be divided into 4 steps showing that the following implications hold true, where "$\pmb{\Rightarrow}$" indicates the non-trivial parts.

<u>STEP 1:</u>
$$\left.\begin{array}{ccc} (a) & \Leftrightarrow & (b) \\ \Uparrow & & \pmb{\Downarrow} \\ (c) & \Leftrightarrow & (d) \end{array}\right\} \pmb{\Rightarrow} (e) \Rightarrow (e') \Rightarrow (f') \pmb{\Rightarrow} (g') \Rightarrow (h') \pmb{\Rightarrow} (b)$$

<u>STEP 2:</u> (d) $\Rightarrow$ (d') $\Rightarrow$ (c') $\Rightarrow$ (a') $\Rightarrow$ (b') $\pmb{\Rightarrow}$ (b) ($\Rightarrow$ (d) by STEP 1)

<u>STEP 3:</u> (a) and (b) $\pmb{\Rightarrow}$ (f) $\Rightarrow$ (f') ($\Rightarrow$ (a) by STEP 1)

<u>STEP 4:</u> (e) $\Rightarrow$ (g) $\Rightarrow$ (h) $\Rightarrow$ (h') ($\Rightarrow$ (e) by STEP 1).

We are going to prove the "$\pmb{\Rightarrow}$" parts; the others are either immediate or easy to prove.

<u>(b) $\pmb{\Rightarrow}$ (d):</u> 1. Let $f: S \to \mathbb{R}$ be upper semicontinuous and assume for the moment that $0<f<1$; then, by Lemma 12, we have for every $n \in \mathbb{N}$

$$\limsup_{\alpha} \int^* nf\, d\mu_\alpha \leq \limsup_{\alpha} [\mu_\alpha(S) + \sum_{k=1}^{n} \mu_\alpha^*(\{nf \geq k\})]$$

$$\leq \limsup_{\alpha} \mu_\alpha(S) + \sum_{k=1}^{n} \limsup_{\alpha} \mu_\alpha^*(\{nf \geq k\}) \underset{(b)}{\leq} \mu(S) + \sum_{k=1}^{n} \tilde{\mu}(\{nf \geq k\})$$

$$\leq \mu(S) + \int nf d\tilde{\mu},$$

whence

$$\limsup_{\alpha} \int^* f d\mu_\alpha \leq \frac{\mu(S)}{n} + \int f d\tilde{\mu};$$

thus (for $n\to\infty$) we obtain (d) for all upper semicontinuous f with $0<f<1$.

2. Let $f\colon S \to \mathbb{R}$ be upper semicontinuous and bounded, say $a<f<b$ for some $-\infty<a<b<\infty$; then $0 < \frac{f-a}{b-a} < 1$, and therefore it follows from part 1 that

$$\limsup_{\alpha} \int^{*} \frac{f-a}{b-a}\, d\mu_{\alpha} \leq \int \frac{f-a}{b-a}\, d\tilde{\mu},$$

which implies (d), since $\lim_{\alpha} \mu_{\alpha}(S) = \mu(S) = \tilde{\mu}(S)$.

<u>[(a)-(d)] $\Longrightarrow$ (e):</u> Let $f\colon S \to \mathbb{R}$ be bounded, $\mathcal{B}(S)$, $\mathbb{B}$-measurable and $\tilde{\mu}$-a.e. continuous. It follows (cf. Gaenssler-Stute (1977), Satz 8.4.3) that

$$\tilde{\mu}(\{f_{*}<f^{*}\}) = 0,$$

where $\quad f_{*} := \sup\{g\colon g\leq f,\ g \text{ lower semicontinuous}\}$ and

$$f^{*} := \inf\{g\colon g\geq f,\ g \text{ upper semicontinuous}\};$$

therefore, since $f_{*} \leq f \leq f^{*}$, we obtain

$$(+) \quad \int f_{*}\, d\tilde{\mu} = \int f\, d\tilde{\mu} = \int f^{*}\, d\tilde{\mu}.$$

Furthermore, since f_{*} and f^{*} are also bounded with f_{*} being lower semicontinuous and f^{*} being upper semicontinuous, respectively, we obtain

$$\int f_{*}\, d\tilde{\mu} \underset{(c)}{\leq} \liminf_{\alpha} \int_{*} f_{*}\, d\mu_{\alpha} \leq \liminf_{\alpha} \int_{*} f\, d\mu_{\alpha}$$

$$\leq \liminf_{\alpha} \int^{*} f\, d\mu_{\alpha} \leq \limsup_{\alpha} \int^{*} f\, d\mu_{\alpha} \leq \limsup_{\alpha} \int^{*} f^{*}\, d\mu_{\alpha} \underset{(d)}{\leq} \int f^{*}\, d\tilde{\mu},$$

whence, by (+), $$\lim_{\alpha} \int^{*} f\, d\mu_{\alpha} = \int f\, d\mu.$$

On the other hand, one obtains in the same way that

$$\int f_{*}\, d\tilde{\mu} \leq \liminf_{\alpha} \int_{*} f_{*}\, d\mu_{\alpha} \leq \limsup_{\alpha} \int_{*} f\, d\mu_{\alpha}$$

$$\leq \limsup_{\alpha} \int^{*} f\, d\mu_{\alpha} \leq \limsup_{\alpha} \int^{*} f^{*}\, d\mu_{\alpha} \leq \int f^{*}\, d\tilde{\mu},$$ whence, again by (+),

$\lim_{\alpha} \int_{*} f\, d\mu_{\alpha} = \int f\, d\tilde{\mu}$, which proves (e).

<u>(f') $\Longrightarrow$ (g'):</u> Given $f\in C_{a}^{b}(S)$, let $f\tilde{\mu}$ ($\equiv \tilde{\mu}\circ f^{-1}$) be the image measure that f induces on $\mathbb{B}$ in $\mathbb{R}$ from $\tilde{\mu}$ (i.e., $f\tilde{\mu}(B) = \tilde{\mu}(\{f\in B\}), B\in\mathbb{B}$). Since f is bounded, we have $f\tilde{\mu}([a,b]) = \tilde{\mu}(S)$ for some $-\infty<a<b<\infty$; furthermore, since $\tilde{\mu}(S)<\infty$, we

have $f\tilde{\mu}(\{t\})>0$ for at most countable many $t\in[a,b]$. Therefore, it follows that for every $\varepsilon>0$ there exist $t_o, t_1,\ldots,t_m$ such that

(1) $a = t_o < t_1 < \ldots < t_m = b$

(2) $a < f(x) < b$ for all $x\in S$

(3) $t_j - t_{j-1} < \varepsilon$ for all $j=1,\ldots,m$

and (4) $\tilde{\mu}(\{x\in S: f(x) = t_j\}) = 0$ for all $j=0,1,\ldots,m$.

Now, let $A_j := \{x\in S: t_{j-1}\leq f(x) < t_j\}$; then $A_j\in\mathcal{A}$, the A_j's being pairwise disjoint with union S, and $\partial A_j\subset\{x\in S: f(x)\in\{t_{j-1},t_j\}\}$, whence (by (4)) $\tilde{\mu}(\partial A_j) = 0$, $j=1,\ldots,m$.

Therefore it follows by (f') that

$$(+)\quad \lim_\alpha \mu_\alpha(A_j) = \tilde{\mu}(A_j) \quad \text{for } j=1,\ldots,m.$$

Now, put $g := \sum_{j=1}^{m} t_{j-1}\, 1_{A_j}$ to get a bounded function $g\in\mathcal{F}_a(S)$ for which (by (3))

$$(++)\quad \sup_{x\in S} |f(x) - g(x)| < \varepsilon.$$

Then, it follows that

$$\begin{aligned}
\left|\int f d\mu_\alpha - \int f d\tilde{\mu}\right| &= \left|\int (f-g)d\mu_\alpha + \int g d\mu_\alpha - \int (f-g)d\tilde{\mu} - \int g d\tilde{\mu}\right| \\
&\leq \int |f-g|d\mu_\alpha + \int |f-g|d\tilde{\mu} + \left|\int g d\mu_\alpha - \int g d\tilde{\mu}\right| \\
&\underset{(++)}{\leq} \varepsilon\mu_\alpha(S) + \varepsilon\tilde{\mu}(S) + \sum_{j=1}^{m} |t_{j-1}|\,|\mu_\alpha(A_j) - \tilde{\mu}(A_j)|,
\end{aligned}$$

whence, by (+),

$$\limsup_\alpha \left|\int f d\mu_\alpha - \int f d\tilde{\mu}\right| \leq 2\varepsilon\mu(S)$$

(note that $S\in\mathcal{A}$ with $\tilde{\mu}(\partial S) = \tilde{\mu}(\emptyset) = 0$, and $\tilde{\mu}(S) = \mu(S)$).

Thus (for $\varepsilon\to 0$) we have shown (g').

<u>(h')$\Rightarrow$(b):</u> Since $f\equiv 1\in U_b^b(S)$, we obtain from (h') at once $\lim_\alpha \mu_\alpha(S) = \mu(S)$. Next, given an arbitrary $F\in\mathcal{F}(S)$ and $\varepsilon>0$, let

$$F^\varepsilon := \{x\in S: d(x,F)<\varepsilon\};$$

then $F^{\varepsilon}\downarrow F$ as $\varepsilon\downarrow 0$, and therefore, for every $n\in\mathbb{N}$ there exists an $F^{\varepsilon_n}\in\mathcal{G}(S)$ such that $\tilde{\mu}(F^{\varepsilon_n}) \leq \tilde{\mu}(F) + \frac{1}{n}$.

Now, since by assumption μ is separable, there exists a separable $S_o\subset S$ with $\mu(S_o^c) = \mu(S)$; put, for each $x\in S$,

$$f_n(x) := \begin{cases} d(x,S_o^c\cap \complement F^{\varepsilon_n})/\varepsilon_n, & \text{if } S_o^c\cap\complement F^{\varepsilon_n} \neq \emptyset \\ 1, & \text{if } S_o^c\cap\complement F^{\varepsilon_n} = \emptyset; \end{cases}$$

then the function $g_n := \min(f_n,1)$ has the following properties for every $n\in\mathbb{N}$:

(1) $g_n\in U_b^b(S)$ (cf. Lemma 11 (ii) and note that $S_o^c\cap\complement F^{\varepsilon_n}$ is separable),

(2) $\operatorname{rest}_{S_o^c\cap\complement F^{\varepsilon_n}} g_n \equiv 0$,

and (3) $\operatorname{rest}_F g_n \equiv 1$.

Therefore, for every $n\in\mathbb{N}$ we obtain

$$\limsup_\alpha \mu_\alpha^*(F) = \limsup_\alpha \int^* 1_F d\mu_\alpha \underset{(3),(1)}{\leq} \limsup_\alpha \int g_n d\mu_\alpha$$

$$\underset{(h'),(1)}{=} \int g_n d\mu = \int_{S_o^c} g_n d\mu = \int_{S_o^c\cap\complement F^{\varepsilon_n}} g_n d\mu + \int_{S_c^o\cap F^{\varepsilon_n}} g_n d\mu \underset{(2)}{=} \int_{S_c^o\cap F^{\varepsilon_n}} g_n d\mu$$

$$\underset{(g_n\leq 1)}{\leq} \tilde{\mu}(S_o^c\cap F^{\varepsilon_n}) = \tilde{\mu}(F^{\varepsilon_n}) \leq \tilde{\mu}(F) + \frac{1}{n},$$

whence (for $n\to\infty$) we obtain $\limsup_\alpha \mu_\alpha^*(F) \leq \tilde{\mu}(F)$, which proves (b).

<u>(b') $\Rightarrow$ (b):</u> Given an arbitrary $F\in\mathcal{F}(S)$, we have as before that for every $n\in\mathbb{N}$ there exists an $F^{\varepsilon_n}\in\mathcal{G}(S)$ s.t. $\tilde{\mu}(F^{\varepsilon_n}) \leq \tilde{\mu}(F) + \frac{1}{n}$.

Let g_n be defined as before and put $F_n := \{x\in S\colon g_n(x)\in[\frac{1}{2},1]\}$; then

$$F_n\in\mathcal{F}(S)\cap\mathcal{B}_b(S),\ F_n\supset F \text{ for all } n\in\mathbb{N}\text{, and}$$

$$(+)\quad F_n\cap S_o^c \subset F^{\varepsilon_n}\cap S_o^c \text{ for all } n\in\mathbb{N}.$$

(As to (+), let $x\in F_n\cap S_c^o$; then, if $S_o^c\cap\complement F^{\varepsilon_n} \neq \emptyset$, we have by construction of g_n, $d(x,S_o^c\cap\complement F^{\varepsilon_n}) \geq \frac{\varepsilon_n}{2} > 0$, whence $x\notin S_o^c\cap\complement F^{\varepsilon_n}$, and therefore $x\in F^{\varepsilon_n}\cap S_o^c$; if $S_o^c\cap\complement F^{\varepsilon_n} = \emptyset$

(and therefore $g_n \equiv 1$), it follows that $F^{\varepsilon_n} \cap S_c^o = S_o^c$ and therefore $F_n \cap S_o^c \subset S_o^c = F^{\varepsilon_n} \cap S_o^c$.)

We thus obtain

$$\limsup_\alpha \mu_\alpha^*(F) \leq \limsup_\alpha \mu_\alpha(F_n) \underset{(b')}{\leq} \mu(F_n)$$

$$= \mu(F_n \cap S_o^c) \underset{(+)}{\leq} \tilde{\mu}(F^{\varepsilon_n} \cap S_o^c) = \tilde{\mu}(F^{\varepsilon_n}) \leq \tilde{\mu}(F) + \frac{1}{n},$$

whence (for $n \to \infty$) $\limsup_\alpha \mu_\alpha^*(F) \leq \tilde{\mu}(F)$, which proves (b).

<u>(a) and (b) $\Rightarrow$ (f):</u> Given an $A \in \mathcal{B}(S)$ with $\tilde{\mu}(\partial A) = 0$, we have

$$\tilde{\mu}(A^o) \underset{(a)}{\leq} \liminf_\alpha (\mu_\alpha)_*(A^o) \leq \liminf_\alpha (\mu_\alpha)_*(A) \leq \liminf_\alpha \mu_\alpha^*(A)$$

$$\leq \limsup_\alpha \mu_\alpha^*(A) \leq \limsup_\alpha \mu_\alpha^*(A^c) \underset{(b)}{\leq} \tilde{\mu}(A^c) = \tilde{\mu}(A^o), \text{ whence}$$

$$\lim_\alpha \mu_\alpha^*(A) = \tilde{\mu}(A).$$

On the other hand, one obtains in the same way that

$$\tilde{\mu}(A^o) \leq \liminf_\alpha (\mu_\alpha)_*(A) \leq \limsup_\alpha (\mu_\alpha)_*(A) \leq \limsup_\alpha \mu_\alpha^*(A^c)$$

$\leq \tilde{\mu}(A^c) = \tilde{\mu}(A^o)$, whence also $\lim_\alpha (\mu_\alpha)_*(A) = \tilde{\mu}(A)$, which proves (f).

This concludes the proof of the Portmanteau theorem. □

<u>IDENTIFICATION OF LIMITS:</u>

Let C be the set of all closed balls in $S = (S,d)$ and let $C^{\cap f}$ denote the class of all subsets of S which are finite intersections of sets in C. Then, since $C^{\cap f}$ is a $\cap$-closed generator of $\mathcal{B}_b(S)$, we have for any two $\mu_i \in M_b(S)$, $i=1,2$, that $\mu_1 = \mu_2$ if $\mu_1(A) = \mu_2(A)$ for all $A \in C^{\cap f}$ (cf. Gaenssler-Stute (1977), Satz 1.4.10).

We will show below that for any net (μ_α) in $M_a(S)$ and any $\mu_i \in M_b(S)$, $\mu_\alpha \xrightarrow[b]{} \mu_i$, $i=1,2$, implies $\mu_1 = \mu_2$.

For this we need the following auxiliary result:

(29) For any $A \subset S$, any $\varepsilon > 0$, and any separable $S_o \subset S$, there exists an $f_\varepsilon \in U_b^b(S)$ such that $0 \leq f \leq 1$,
$\mathrm{rest}_{\complement(A\cap S_o^c)^\varepsilon} f_\varepsilon \equiv 0$ and $\mathrm{rest}_{A\cap S_o^c} f_\varepsilon \equiv 1$.

Proof. It follows from Lemma 11 (ii) that

$$f_\varepsilon(x) := \max\ [(1 - \frac{d(x, A\cap S_o^c)}{\varepsilon}), 0],\ x \in S,$$

has the stated properties. □

LEMMA 13. Let $\mu_i \in M_b(S)$ be separable, i=1,2, and suppose that

$$(+) \quad \int f d\mu_1 = \int f d\mu_2 \text{ for all } f \in U_b^b(S);$$

then $\mu_1 = \mu_2$.

Proof. Let S_i be the separable subsets of S for which $\mu_i(S_i^c) = \mu_i(S)$, i=1,2; put $S_o := S_1^c \cup S_2^c$ to get a separable subset of S for which $\mu_i(S_o^c) = \mu_i(S)$, i=1,2. Now, given an arbitrary $A \in C^{\cap f}$ and $n \in \mathbb{N}$, choose $f_n \equiv f_{1/n}$ according to (29) to get a sequence $(f_n) \subset U_b^b(S)$ for which

$\lim_{n\to\infty} f_n = 1_{A\cap S_c^o}$; from this, by Lebesgue's theorem and (+)

it follows that

$$\mu_1(A) = \mu_1(A\cap S_o^c) = \mu_2(A\cap S_o^c) = \mu_2(A). \quad \square$$

Lemma 13, together with the equivalence of (g') and (h') in (28) implies the result announced above (cf. Definition 4 (i)):

Lemma 14. For any net (μ_α) in $M_a(S)$ with $\mu_\alpha \xrightarrow[b]{} \mu_i$, i=1,2, we have $\mu_1 = \mu_2$.

WEAK CONVERGENCE AND MAPPINGS (Continuous Mapping Theorems):

Let S = (S,d) and S' = (S',d') be two metric spaces and suppose again that A is a σ-algebra of subsets of S such that $\mathcal{B}_b(S) \subset A \subset \mathcal{B}(S)$; let $g: S \to S'$ be $A, \mathcal{B}_b(S')$-measurable and let $\mu_\alpha \in M_a(S)$ and $\mu \in M_b(S)$, respectively, μ separable. Then μ_α and μ induce measures ν_α and ν on $\mathcal{B}_b(S')$, defined by

$\nu_\alpha(B') := \mu_\alpha(g^{-1}(B'))$ and $\nu(B') := \tilde{\mu}(g^{-1}(B'))$ for $B' \in \mathcal{B}_b(S')$, where $g^{-1}(B') = \{x \in S: g(x) \in B'\}$ and where $\tilde{\mu}$ is again the unique Borel extension of μ (cf. (24)).

We are interested in conditions on g under which $\mu_\alpha \xrightarrow[b]{} \mu$ implies $\nu_\alpha \equiv \mu_\alpha \circ g^{-1} \xrightarrow[b]{} \nu \equiv \tilde{\mu} \circ g^{-1}$. It can be shown by examples that measurability of g alone is not sufficient for preserving weak convergence. As we will see, some continuity assumptions on g will be needed. The corresponding theorems are then usually called <u>CONTINUOUS MAPPING THEOREMS</u>.

<u>THEOREM 3.</u>

Let $S = (S,d)$ and $S' = (S',d')$ be metric spaces, let A be a σ-algebra of subsets of S such that $\mathcal{B}_b(S) \subset A \subset \mathcal{B}(S)$, and let $g: S \to S'$ be $A,\mathcal{B}_b(S')$-measurable and continuous. Let (μ_α) be a net in $M_a(S)$ and let $\mu \in M_b(S)$ be separable such that $\mu_\alpha \xrightarrow[b]{} \mu$. Then $\nu_\alpha \equiv \mu_\alpha \circ g^{-1} \xrightarrow[b]{} \nu \equiv \tilde{\mu} \circ g^{-1}$.

Theorem 3 is a special case of the following result where the continuity assumption on g is weakened:

<u>THEOREM 4.</u> Let $S = (S,d)$ and $S' = (S',d')$ be metric spaces, let A be a σ-algebra of subsets of S such that $\mathcal{B}_b(S) \subset A \subset \mathcal{B}(S)$, and let (μ_α) be a net in $M_a(S)$ and $\mu \in M_b(S)$ be separable such that $\mu_\alpha \xrightarrow[b]{} \mu$; let $g: S \to S'$ be $A,\mathcal{B}_b(S')$-measurable such that $\tilde{\mu}(D(g)) = 0$. Then $\nu_\alpha \equiv \mu_\alpha \circ g^{-1} \xrightarrow[b]{} \nu \equiv \tilde{\mu} \circ g^{-1}$.
(Note that $D(g) \in \mathcal{B}(S)$; cf. P. Billingsley (1968), p. 225-226.)

<u>Proof.</u> Note that $\nu_\alpha \in M_b(S')$ and $\nu \in M_b(S')$, whence $\nu_\alpha \xrightarrow[b]{} \nu$ iff

(i) ν is separable and (ii) $\lim_\alpha \int_{S'} f d\nu_\alpha = \int_{S'} f d\nu$ for all $f \in C_b^b(S')$ where (ii) is equivalent to any of the conditions (a)-(h') in (28) (with S replaced by S' and A replaced by $A' = \mathcal{B}_b(S')$).

1.) ν is separable:

since μ is separable, there exists a separable $S_o \subset S$ such that $\mu(S_o^c) = \mu(S)$.

Let $T_o \subset S_o$ be countable and dense in S_o (as well as in S_o^c) and let $T_o' := g(T_o)$; we will show that $S_o' := g(S_o^c \setminus D(g)) \cup T_o'$ is a separable subset of S' with $\nu((S_o')^c) = \nu(S')$.

For this we will show that T_o' (being countable) is dense in S_o': in fact, let (w.l.o.g.) $y \in g(S_o^c \setminus D(g))$, i.e., $y = g(x)$ for some $x \in S_o^c \setminus D(g)$. Since T_o is dense in S_o^c there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset T_o$ such that $x_n \to x$ and therefore, since $x \notin D(g)$, we have $g(x_n) \to g(x) = y$, where $g(x_n) \in T_o'$.

Next, since $g^{-1}((S_o')^c) \supset g^{-1}(g(S_o^c \setminus D(g))) \supset S_o^c \setminus D(g)$ and since $\tilde{\mu}(D(g)) = 0$, we have

$$\nu((S_o')^c) = \tilde{\mu}(g^{-1}((S_o')^c)) \geq \tilde{\mu}(S_o^c \setminus D(g)) = \tilde{\mu}(S_o^c) - \tilde{\mu}(S_o^c \cap D(g))$$

$$= \tilde{\mu}(S_o^c) = \mu(S_o^c) = \mu(S) = \tilde{\mu}(S) = \tilde{\mu}(g^{-1}(S')) = \nu(S').$$

2.) It remains to show (ii) $\lim_\alpha \int_{S'} f d\nu_\alpha = \int_{S'} f d\nu$ for all $f \in C_b^b(S')$.

For this, given any $f \in C_b^b(S')$, we have that $f \circ g: S \to \mathbb{R}$ is a bounded function belonging to $\mathcal{F}_a(S)$ which is $\tilde{\mu}$-a.e. continuous, and therefore it follows from (28) (cf. (e')) that

$$\lim_\alpha \int_{S'} f d\nu_\alpha = \lim_\alpha \int_S (f \circ g) d\mu_\alpha = \int_S (f \circ g) d\tilde{\mu} = \int_{S'} f d\nu, \text{ which proves (ii). } \square$$

The following lemma is in some sense an inverse result:

<u>LEMMA 15.</u> Let $S = (S,d)$ be a metric space, (μ_α) be a net in $M_a(S)$ and let $\mu \in M_b(S)$ be separable such that $\mu_\alpha \circ f^{-1} \xrightarrow[b]{} \tilde{\mu} \circ f^{-1}$ for all $f \in C_a^b(S)$. Then $\mu_\alpha \xrightarrow[b]{} \mu$.

<u>Proof.</u> Note that in the present case $S' = \mathbb{R}$ (a separable metric space), whence $\nu_\alpha \equiv \mu_\alpha \circ f^{-1}$ and $\nu \equiv \tilde{\mu} \circ f^{-1}$ are separable Borel measures on $\mathcal{B} = \mathcal{B}(\mathbb{R})$. Now, for any $f \in C_a^b(S)$ and any $g \in C_b^b(\mathbb{R}) = C^b(\mathbb{R})$ we have

$$\lim_\alpha \int_S (g \circ f) d\mu_\alpha = \lim_\alpha \int_{\mathbb{R}} g d\nu_\alpha = \int_{\mathbb{R}} g d\nu = \int_S (g \circ f) d\tilde{\mu}.$$

Furthermore, for any $f \in C_a^b(S)$ there exists a $c>0$ such that $|f| \leq c$, whence for

$$g(t) := \begin{cases} -c, & \text{if } t<-c \\ t, & \text{if } |t| \leq c, t \in \mathbb{R}, \\ c, & \text{if } t>c \end{cases}$$

we have $g\in C^b(\mathbb{R})$ and $g\circ f = f$. Therefore it follows that $\lim_\alpha \int f d\mu_\alpha = \int f d\tilde{\mu}$ for all $f\in C_a^b(S)$ implying the assertion since μ is, by assumption, separable. □

For the next mapping theorem we need the following auxiliary lemma, the proof of which is left to the reader.

LEMMA 16. Let $S = (S,d)$ and $S' = (S',d')$ be metric spaces; given g_n, $g\colon S \to S'$, $n\in\mathbb{N}$, let

$E \equiv E((g_n),g) := \{x\in S\colon \exists (x_n)_{n\in\mathbb{N}} \subset S \text{ s.t. } x_n \to x \text{ but } g_n(x_n) \not\to g(x)\}$.

Then $x\notin E$ iff for every $\varepsilon>0$ there exists a $k\in\mathbb{N}$ and a $\delta>0$ such that $n\geq k$ and $d(x,y) < \delta$ together imply $d'(g(x),g_n(y)) < \varepsilon$.

THEOREM 5. Let $S = (S,d)$ and $S' = (S',d')$ be metric spaces, A be a σ-algebra of subsets of S such that $B_b(S) \subset A \subset B(S)$, and let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $M_a(S)$ and $\mu\in M_b(S)$ be separable such that $\mu_n \xrightarrow[b]{} \mu$; let g_n, $g\colon S \to S'$ be $A,B_b(S')$-measurable, $n\in\mathbb{N}$, such that

$\tilde{\mu}^*(E) = \inf\{\tilde{\mu}(B)\colon B \supset E,\ B \in A\} = 0$. Then $\nu_n \equiv \mu_n\circ g_n^{-1} \xrightarrow[b]{} \nu \equiv \tilde{\mu}\circ g^{-1}$.

Proof. (cf. P. Billingsley (1968), Proof of Th. 5.5).

1.) ν is separable: this is shown as in the proof of Theorem 4, replacing $g(T_o)$ there by $T_o' := \bigcup_{n\in\mathbb{N}} g_n(T_o)$.

2.) We are going to show

$$(+)\quad \lim_{n\to\infty} \nu_n(S') = \nu(S')$$

and $$(++)\quad \liminf_{n\to\infty} \nu_n(G) \geq \nu(G) \text{ for all } G \in G(S') \cap B_b(S').$$

(Note that (+) together with (++) imply the assertion according to (28) with S replaced by S' and A replaced by $A' = B_b(S')$.)

ad (+): $\mu_n \xrightarrow[b]{} \mu$ implies (cf. (28)) $\mu_n(S) \to \mu(S)$ and therefore

$\nu_n(S') = \mu_n(g_n^{-1}(S')) = \mu_n(S) \to \mu(S) = \tilde{\mu}(S) = \tilde{\mu}(g^{-1}(S')) = \nu(S')$.

ad (++): Given an arbitrary $G \in G(S') \cap B_b(S')$, we have

(a) $\qquad g^{-1}(G) \subset E \cup \bigcup_{k\in\mathbb{N}} T_k^o$, where $T_k := \bigcap_{n\geq k} g_n^{-1}(G) \in \mathcal{A}$,

and

(b) $\qquad \nu(G) = \tilde{\mu}(g^{-1}(G)) \leq \mu(\bigcup_{k\in\mathbb{N}} T_k^o)$.

ad (a): It suffices to show that $x \in \complement E$ and $g(x) \in G$ together imply $x \in T_k^o$ for some k. Now, since $G \in \mathcal{G}(S')$ we have that for some $\varepsilon>0$ $B_{d'}(g(x),\varepsilon) \subset G$; on the other hand, by Lemma 16 $x \in \complement E$ implies that there exists a $k \in \mathbb{N}$ and a $\delta>0$ such that $d'(g(x),g_n(y)) < \varepsilon$ whenever $n\geq k$ and $d(x,y) < \delta$; therefore $g_n(y) \in G$ for all $n\geq k$ and all $y \in S$ with $d(x,y) < \delta$ implying $B_d(x,\delta) \subset g_n^{-1}(G)$ for all $n\geq k$, whence $B_d(x,\delta) \subset T_k$, and therefore $x \in T_k^o$.

ad (b): $\tilde{\mu}(g^{-1}(G)) \underset{(a)}{\leq} \tilde{\mu}^*(E \cup \bigcup_{k\in\mathbb{N}} T_k^o) \leq \tilde{\mu}^*(E) + \tilde{\mu}^*(\bigcup_{k\in\mathbb{N}} T_k^o)$

$= \tilde{\mu}^*(\bigcup_{k\in\mathbb{N}} T_k^o)$; note that for $A \in \mathcal{B}(S)$, $\tilde{\mu}^*(A) \geq \tilde{\mu}(A)$ by (23); we will show that even $\tilde{\mu}^* = \tilde{\mu}$ on $\mathcal{B}(S)$ which proves (b).

For this, let $A \in \mathcal{B}(S)$; it suffices to show that $\tilde{\mu}^*(A) \leq \tilde{\mu}(A)$.

Now, $\tilde{\mu}(A) = \tilde{\mu}(S_o^c \cap A) = \tilde{\mu}(S_o^c \cap A')$ for some $A' \in \mathcal{B}_b(S)$ (noticing that for separable S_o^c one has $S_o^c \cap A \in \mathcal{B}(S_o^c) = \mathcal{B}_b(S_o^c) = S_o^c \cap \mathcal{B}_b(S)$), and therefore $\tilde{\mu}(S_o^c \cap A') = \mu(A' \cup \complement S_o^c) \geq \tilde{\mu}^*(A)$, since $A \subset A' \cup \complement S_o^c \in \mathcal{B}_b(S) \subset \mathcal{A}$.

Now, since $T_k^o \subset T_{k+1}^o$ and therefore $\tilde{\mu}(T_k^o) \uparrow \tilde{\mu}(\bigcup_{k\in\mathbb{N}} T_k^o)$, for every $\varepsilon>0$, there exists a $k_o \in \mathbb{N}$ such that

$$\tilde{\mu}(\bigcup_{k\in\mathbb{N}} T_k^o) \leq \tilde{\mu}(T_k^o) + \varepsilon \text{ for all } k \geq k_o,$$

and therefore, by (b), we obtain

$$\nu(G) \leq \tilde{\mu}(T_k^o) + \varepsilon \text{ for all } k \geq k_o.$$

But $\mu_n \xrightarrow[b]{} \mu$ implies (cf. (28)) that for every $k\in\mathbb{N}$

$$\tilde{\mu}(T_k^o) \leq \liminf_{n\to\infty} (\mu_n)_*(T_k^o),$$

and therefore, noticing that $T_k^o \subset g_n^{-1}(G)$ for sufficiently large n, we obtain

$$\tilde{\mu}(T_k^o) \leq \liminf_{n\to\infty} \mu_n(g_n^{-1}(G)) = \liminf_{n\to\infty} \nu_n(G),$$

whence $\nu(G) \leq \liminf_{n\to\infty} \nu_n(G) + \varepsilon$ for every $\varepsilon>0$, which implies (++). □

WEAK CONVERGENCE CRITERIA AND COMPACTNESS:

As before, let $S = (S,d)$ be a (possibly non-separable) metric space, let $M_a^1(S)$ be the space of all p-measures on a σ-algebra $\mathcal{A}$ with $\mathcal{B}_b(S) \subset \mathcal{A} \subset \mathcal{B}(S)$ and let $M_b^1(S)$ be the space of all p-measures on $\mathcal{B}_b(S)$.

DEFINITION 5. Let $(\mu_\alpha)_{\alpha\in A}$ be a net in $M_a^1(S)$; then $(\mu_\alpha)_{\alpha\in A}$ is called δ-tight iff

(30) $$\sup_{K\in\mathcal{K}(S)} \inf_{\delta>0} \liminf_{\alpha\in A} \mu_\alpha(K^\delta) = 1.$$

(Note that $K^\delta \in \mathcal{B}_b(S) \subset \mathcal{A}$ according to Lemma 11 (ii).)

The following two results were proved by M.J. Wichura (1968), Th. 1.3 and Th. 1.4; in view of (27) b) they can be restated as follows (where in Theorem 7 the assumption of (S,d) being topologically complete cannot be dispensed with, in general).

THEOREM 6 (Wichura). Let $(\mu_\alpha)_{\alpha\in A} \subset M_a^1(S)$ be δ-tight.
Then there exists a subnet $(\mu_{\alpha'})_{\alpha'\in A'}$ of $(\mu_\alpha)_{\alpha\in A}$ and a separable $\mu\in M_b^1(S)$ such that $\mu_{\alpha'} \xrightarrow{b} \mu$.

THEOREM 7 (Wichura). Let $S = (S,d)$ be a topologically complete metric space and (μ_α) be a net in $M_a^1(S)$; then there exists a separable $\mu\in M_b^1(S)$ with $\mu_\alpha \xrightarrow{b} \mu$ iff

(a) $\liminf_\alpha \int f d\mu_\alpha = \limsup_\alpha \int f d\mu_\alpha$ for all $f \in U_b^b(S)$,

and (b) (μ_α) is δ-tight.

We are going to prove here instead the following versions of Theorem 6 and 7 (cf. Remark (31) below):

<u>THEOREM 6*.</u> Let $(\mu_\alpha)_{\alpha\in A}$ be a net in $M_a^1(S)$ fulfilling the following two conditions:

(b_1) For every $(f_n)_{n\in\mathbb{N}} \subset U_b^b(S)$ with $f_n \downarrow 0$ one has

$$\lim_\alpha \sup \int f_n d\mu_\alpha \to 0 \text{ as } n\to\infty.$$

(b_2) There exists a separable $S_o \subset S$ such that

$$\lim_\alpha \inf \int f d\mu_\alpha \geq 1 \text{ for all } f \in U_b^b(S) \text{ with } f \geq 1_{S_o^c}.$$

Then there exists a subnet $(\mu_{\alpha'})_{\alpha'\in A'}$ of $(\mu_\alpha)_{\alpha\in A}$ and a separable $\mu \in M_b^1(S)$ with $\mu(S_o^c) = 1$ such that $\mu_\alpha \xrightarrow[b]{} \mu$.

<u>THEOREM 7*.</u> Let $S = (S,d)$ be an arbitrary metric space and (μ_α) be a net in $M_a^1(S)$; then there exists a separable $\mu \in M_b^1(S)$ with $\mu_\alpha \xrightarrow[b]{} \mu$ iff the following conditions are fulfilled:

(a) as in Theorem 7 and (b_i), i=1,2, as in Theorem 6*, where in this connection the separable S_o with $\mu(S_o^c) = 1$ and the separable S_o occurring in (b_2) coincide.

<u>Proof of Theorem 6*.</u> Let $\mu_\alpha(f) := \int f d\mu_\alpha$ for $f \in U_b^b(S)$ and consider the net

$$\alpha \longmapsto (\mu_\alpha(f))_{f\in U_b^b(S)} \in \prod_{f\in U_b^b(S)} [-\|f\|,\|f\|]$$

where $\|f\| := \sup_{x\in S} |f(x)|$. Since the product space $\prod_{f\in U_b^b(S)} [-\|f\|,\|f\|]$ is compact in the product topology (Tychonov's theorem), there exists a convergent subnet, say $\alpha' \longmapsto (\mu_{\alpha'}(f))_{f\in U_b^b(S)}$, $\alpha'\in A'$. Therefore $\lim_{\alpha'\in A'} \mu_{\alpha'}(f)$ exists for each $f \in U_b^b(S)$.

$$\text{Let } \mu(f) := \lim_{\alpha'\in A'} \mu_{\alpha'}(f) \text{ for } f \in U_b^b(S);$$

then μ: $U_b^b(S) \longrightarrow \mathbb{R}$ is positive, linear, and normed.

We are going to show that μ is also σ-smooth on $U_b^b(S)$:

for this, let $(f_n)_{n\in\mathbb{N}} \subset U_b^b(S)$ with $f_n \downarrow 0$; then it follows by (b_1) that

$$\mu(f_n) = \lim_{\alpha'} \mu_{\alpha'}(f_n) = \limsup_{\alpha'} \int f_n d\mu_{\alpha'}$$

$$\leq \limsup_{\alpha} \int f_n d\mu_\alpha \to 0 \text{ as } n\to\infty.$$

Therefore, according to the Daniell-Stone representation theorem, there exists one and only one $\mu \in M_b^1(S)$ such that

$$\mu(f) = \int f d\mu \text{ for all } f \in U_b^b(S).$$

Hence, in view of (28) (cf. the equivalence of (g') and (h')) it follows that $\mu_{\alpha'} \xrightarrow[b]{} \mu$, if we finally show that $\mu(S_o^c) = 1$ (i.e. μ separable).
For this we use (b_2) according to which

$$(+) \quad \liminf_{\alpha} \int f d\mu_\alpha \geq 1 \text{ for all } f \in U_b^b(S) \text{ with } f \geq 1_{S_o^c};$$

taking

$$f_n(x) := \max\,[1 - nd(x,S_o^c),0],\ x\in S,$$

we obtain a sequence $(f_n)_{n\in\mathbb{N}} \subset U_b^b(S)$ with $0 \leq f_n \leq 1$ and $1_{S_o^c} \leq f_n \leq 1_{(S_o^c)^{1/n}}$, whence by the σ-smoothness of μ (note that $S_o^c, (S_o^c)^{1/n} \in \mathcal{B}_b(S)$ by Lemma 11 (ii)),

$$\mu(S_o^c) = \inf_{n\in\mathbb{N}} \mu((S_o^c)^{1/n}) \geq \inf_n \int f_n d\mu = \inf_n \lim_{\alpha'} \int f_n d\mu_{\alpha'}$$

$$\geq \inf_n \liminf_{\alpha} \int f_n d\mu_\alpha \underset{(+)}{\geq} 1, \text{ whence } \mu(S_o^c) = 1. \quad \square$$

<u>Proof of Theorem 7*.</u> <u>Only if-part:</u> Suppose $\mu_\alpha \xrightarrow[b]{} \mu$; then (a) is a consequence of (28) (cf. the equivalent statements (g') and (h')).

<u>ad (b_1):</u> Let $(f_n)_{n\in\mathbb{N}} \subset U_b^b(S)$ with $f_n \downarrow 0$; then (cf. again the equivalence of (g') and (h') in (28)) $\limsup_{\alpha} \int f_n d\mu_\alpha = \lim_{\alpha} \int f_n d\mu_\alpha = \int f_n d\mu \to 0$ as $n\to\infty$ according to the σ-smoothness of μ on $U_b^b(S)$.

<u>ad (b_2):</u> Since, by assumption, μ is separable, there exists a separable $S_o \subset S$

such that $\mu(S_o^c) = \mu(S)$; therefore, for any $f \in U_b^b(S)$ with $f \geq 1_{S_o^c}$ one has

$$\liminf_\alpha \int f d\mu_\alpha = \lim_\alpha \int f d\mu_\alpha = \int f d\mu \geq \mu(S_o^c) = \mu(S) = 1.$$

If-part: It suffices to show that there exists a $\mu \in M_b^1(S)$ with $\mu(S_o^c) = 1$ such that for any subnet $(\mu_{\alpha'})$ of (μ_α) there exists a further subnet $(\mu_{\alpha''})$ such that $\mu_{\alpha''} \xrightarrow[b]{} \mu$.

For this, let $(\mu_{\alpha'})_{\alpha' \in A'}$ be an arbitrary subnet of $(\mu_\alpha)_{\alpha \in A}$; then it is easy to show that $(\mu_{\alpha'})_{\alpha' \in A'}$ fulfills (b_1) and (b_2) and therefore, by Theorem 6^*, there exists a subnet $(\mu_{\alpha''})_{\alpha'' \in A''}$ of $(\mu_{\alpha'})_{\alpha' \in A'}$ and a $\mu_{A',A''} \in M_b^1(S)$ with $\mu_{A',A''}(S_o^c) = 1$, such that $\mu_{\alpha''} \xrightarrow[b]{} \mu_{A',A''}$.

We are going to show that $\mu_{A',A''}$ in fact does not depend on A' or A'', whence for μ being the common value of all the $\mu_{A',A''}$ we get $\mu_\alpha \xrightarrow[b]{} \mu$, which will conclude the proof.

For this, given any $f \in U_b^b(S)$, we have by (a)

$$\liminf_{\alpha \in A} \int f d\mu_\alpha \leq \liminf_{\alpha'' \in A''} \int f d\mu_{\alpha''} = \int f d\mu_{A',A''}$$

$$= \limsup_{\alpha'' \in A''} \int f d\mu_{\alpha''} \leq \limsup_{\alpha \in A} \int f d\mu_\alpha = \liminf_{\alpha \in A} \int f d\mu_\alpha,$$

whence $\int f d\mu_{A',A''} = \int f d\mu_{\tilde{A}',\tilde{A}''}$, for all $f \in U_b^b(S)$

and any other subnet $(\mu_{\tilde{\alpha}''})_{\tilde{\alpha}'' \in \tilde{A}''}$ of $(\mu_{\tilde{\alpha}'})_{\tilde{\alpha}' \in \tilde{A}'}$

which is a subnet of $(\mu_\alpha)_{\alpha \in A}$, with $\mu_{\tilde{\alpha}''} \xrightarrow[b]{} \mu_{\tilde{A}',\tilde{A}''}$;

therefore Lemma 13 implies the assertion. □

(31) REMARK. Any δ-tight net $(\mu_\alpha) \subset M_a^1(S)$ fulfills (b_i), i=1,2, but not vice versa (look at $\mu_\alpha \equiv \mu$ with a separable $\mu \in M_b^1(S)$ which is not tight).

Proof. ad (b_1): Let $(f_n)_{n \in \mathbb{N}} \subset U_b^b(S)$ with $f_n \downarrow 0$ and assume w.l.o.g. $\sup_n f_n \leq 1$; then for every $n \in \mathbb{N}$, every $\delta > 0$, and every $K \in K(S)$ we have

$$\limsup_\alpha \int f_n d\mu_\alpha = \limsup_\alpha \left(\int_{K^\delta} f_n d\mu_\alpha + \int_{CK^\delta} f_n d\mu_\alpha \right)$$

$$\leq \limsup_\alpha \int_{K^\delta} f_n d\mu_\alpha + \limsup_\alpha \int_{\complement K^\delta} f_n d\mu_\alpha$$

$$\leq (\text{since } f_n \leq 1) \sup_{x \in K^\delta} f_n(x) + \limsup_\alpha \mu_\alpha(\complement K^\delta) \leq \sup_{x \in K^\delta} f_n(x) + \sup_{\delta>0} \limsup_\alpha \mu_\alpha(\complement K^\delta).$$

Now, given any $\varepsilon>0$, there exists by assumption (cf. (30) and look at complements) a $K_\varepsilon \in K(S)$ such that $\sup_{\delta>0} \limsup_\alpha \mu_\alpha(\complement K_\varepsilon^\delta) \leq \varepsilon/3$.

Therefore, for any $\varepsilon>0$ there exists a $K_\varepsilon \in K(S)$ such that for all $n \in \mathbb{N}$ and $\delta>0$

$$\limsup_\alpha \int f_n d\mu_\alpha \leq \sup \{f_n(x): x \in K_\varepsilon^\delta\} + \varepsilon/3.$$

Furthermore, it is easy to show that for any $\varepsilon>0$ and $n \in \mathbb{N}$ there exists a $\delta(\varepsilon,n)$ such that

$$\sup \{f_n(x): x \in K_\varepsilon^{\delta(\varepsilon,n)}\} \leq \sup \{f_n(x): x \in K_\varepsilon\} + \varepsilon/3.$$

We thus obtain that for any $\varepsilon>0$ there exists a $K_\varepsilon \in K(S)$ such that for every $n \in \mathbb{N}$

$$\limsup_\alpha \int f_n d\mu_\alpha \leq \sup \{f_n(x): x \in K_\varepsilon\} + \varepsilon/3 + \varepsilon/3.$$

But, since K_ε is compact, $\sup \{f_n(x): x \in K_\varepsilon\} \to 0$ as $n\to\infty$, whence $\limsup_\alpha \int f_n d\mu_\alpha \leq \varepsilon$ for sufficiently large n, which implies (b_1).

<u>ad (b_2):</u> δ-tightness of (μ_α) implies that for every $n \in \mathbb{N}$ there exists a $K_n \in K(S)$ such that $\inf_{\delta>0} \liminf_\alpha \mu_\alpha(K_n^\delta) \geq 1 - \frac{1}{n}$.

Put $S_o := \bigcup_{n\in\mathbb{N}} K_n$ to obtain a separable $S_o \subset S$; then, given any $f \in U_b^b(S)$ with $f \geq 1_{S_o^c}$, we must show that

$$(+) \quad \liminf_\alpha \int f d\mu_\alpha \geq 1.$$

Since $f \geq 1_{K_n}$ for each n, it follows (by continuity of f) that for every $n \in \mathbb{N}$ and every $\varepsilon>0$ there exists a $\delta_o = \delta_o(\varepsilon,n) > 0$ such that $\inf \{f(x): x \in K_n^{\delta_o}\} \geq 1 - \varepsilon$, whence $\int f d\mu_\alpha \geq (1-\varepsilon)\, \mu_\alpha(K_n^{\delta_o})$. Therefore, for every $\varepsilon>0$ and every $n \in \mathbb{N}$ we have $\liminf_\alpha \int f d\mu_\alpha \geq (1-\varepsilon) \liminf_\alpha \mu_\alpha(K_n^{\delta_o})$

$\geq (1-\varepsilon) \inf_\delta \liminf_\alpha \mu_\alpha(K_n^\delta) \geq (1-\varepsilon)(1-\frac{1}{n})$, which implies (b_2). □

(32) Remark. The proof of Theorem 7^* shows that any net $(\mu_\alpha)_{\alpha\in A} \subset M_a^1(S)$ which fulfills (b_i), $i=1,2$, is a compact net in $M_a^1(S)$ (i.e. for any subnet $(\mu_{\alpha'})_{\alpha'\in A'}$ of $(\mu_\alpha)_{\alpha\in A}$ there exists a further subnet $(\mu_{\alpha''})_{\alpha''\in A''}$ of $(\mu_{\alpha'})_{\alpha'\in A'}$ and a separable $\mu \in M_b^1(S)$ such that $\mu_{\alpha''} \xrightarrow[b]{} \mu$).

The following lemma prepares for the next theorem (cf. M.J. Wichura (1968), Theorem 1.2 (a)).

LEMMA 17. Let (μ_α) be a net in $M_a(S)$ and $\mu \in M_b(S)$ be separable, i.e., $\mu(S_o^c) = \mu(S)$ for some separable $S_o \subset S$; let $\mathcal{C} \subset \mathcal{A}$ be such that

(33) for each $x \in S_o^c$ $\{C \in \mathcal{C}: x \in C^o\}$ is a neighborhood base at x,

and let $\mathcal{C}^{\cap f}$ denote the class of all finite intersections of members of $\mathcal{C}$. Suppose that

$$(+) \qquad \lim_\alpha \mu_\alpha(C) = \tilde{\mu}(C) \text{ for all } C \in \mathcal{C}^{\cap f}.$$

Then

$$\liminf_\alpha \mu_\alpha(G) \geq \tilde{\mu}(G) \text{ for all } G \in \mathcal{G}(S) \cap \mathcal{A}.$$

(Here again $\tilde{\mu}$ denotes the unique Borel extension of μ and $\mathcal{A}$ is a σ-algebra of subsets of S with $\mathcal{B}_b(S) \subset \mathcal{A} \subset \mathcal{B}(S)$.)

Proof. Given any $G \in \mathcal{G}(S) \cap \mathcal{A}$, it follows by (33) that for every $x \in G \cap S_o^c$ there exists a $C_x \in \mathcal{C}$ such that $x \in C_x^o \subset G_x \subset G$, whence $G \cap S_o^c \subset \bigcup_{x\in G\cap S_o^c} C_x^o$, which means that $\{C_x^o \cap S_o^c: x \in G \cap S_o^c\}$ is an open covering of $G \cap S_o^c$ in the separable subspace (S_o^c,d) of (S,d). Therefore (cf. Billingsley (1968), p. 216) there exists a countable subcovering of $G \cap S_o^c$, i.e.,

$$G \cap S_o^c \subset \bigcup_{n\in\mathbb{N}} (C_{x_n}^o \cap S_o^c) \text{ with } x_n \in G \cap S_o^c,\ n \in \mathbb{N}.$$

Put $C_n := C_{x_n}$, $n \in \mathbb{N}$; then $\bigcup_{n\in\mathbb{N}} C_n \subset \bigcup_{x\in G\cap S_o^c} C_x \subset G$, whence

$$\tilde{\mu}(G) \geqq \tilde{\mu}(\bigcup_n C_n) = \tilde{\mu}(\bigcup_n (C_n \cap S_o^c)) \geqq \tilde{\mu}(G \cap S_o^c) = \tilde{\mu}(G),$$

i.e., $\tilde{\mu}(G) = \tilde{\mu}(\bigcup_n C_n)$.

Put $C_1' := C_1$ and $C_n' := C_n \setminus \bigcup_{i=1}^{n-1} C_i$, $n \geqq 2$, to get pairwise disjoint sets $C_n' \in A$ with $\bigcup_{n\in\mathbb{N}} C_n' = \bigcup_{n\in\mathbb{N}} C_n$, for which one can easily show (using the assumption (+)) that

$$\lim_\alpha \mu_\alpha(C_n') = \tilde{\mu}(C_n') \text{ for all } n \in \mathbb{N}.$$

Therefore, for every $n \in \mathbb{N}$ we have

$$(++) \qquad \lim_\alpha \mu_\alpha(\bigcup_{i=1}^n C_i') = \lim_\alpha \sum_{i=1}^n \mu_\alpha(C_i') = \sum_{i=1}^n \tilde{\mu}(C_i') = \tilde{\mu}(\bigcup_{i=1}^n C_i').$$

Since $\tilde{\mu}(G) = \tilde{\mu}(\bigcup_{n\in\mathbb{N}} C_n) = \tilde{\mu}(\bigcup_{n\in\mathbb{N}} C_n')$, there exists for each $\varepsilon>0$, an $n = n(\varepsilon) \in \mathbb{N}$ such that $\tilde{\mu}(\bigcup_{i=1}^n C_i') \geqq \tilde{\mu}(G) - \varepsilon$, and therefore (note also that $G \supset \bigcup_{i\in\mathbb{N}} C_i'$)

$$\liminf_\alpha \mu_\alpha(G) \geqq \liminf_\alpha \mu_\alpha(\bigcup_{i=1}^n C_i') \underset{(++)}{=} \tilde{\mu}(\bigcup_{i=1}^n C_i') \geqq \tilde{\mu}(G) - \varepsilon,$$

which proves the assertion. □

<u>THEOREM 8.</u> Let (μ_α) be a net in $M_a^1(S)$ and $\mu \in M_b^1(S)$ be separable (i.e., $\mu(S_o^c) = \mu(S) = 1$ for some separable $S_o \subset S$).

Suppose that $C \subset \{B \in \mathcal{B}_b(S): \tilde{\mu}(\partial B) = 0\}$ fulfills (33).

Then the following two assertions are equivalent:

(i) $\lim_\alpha \mu_\alpha(C) = \mu(C)$ for all $C \in C^{\cap f}$

(ii) $\mu_\alpha \underset{b}{\rightarrow} \mu$.

<u>Proof.</u> <u>(i) ⇒ (ii):</u> Follows immediately from Lemma 17 and (28) (cf. the equivalence of (a') and (g') there); note that $\lim_\alpha \mu_\alpha(S) = \mu(S)$ is trivially

fulfilled for p-measures μ_α and μ.

(ii) ⇒ (i): Again (28) (cf. the equivalence of (g') and (f')) yields

$\lim_\alpha \mu_\alpha(B) = \mu(B)$ for all $B \in \{B \in \mathcal{B}_b(S): \tilde{\mu}(\partial B) = 0\} =: \mathcal{R}_{\tilde{\mu}} = \mathcal{R}_{\tilde{\mu}}^{\cap f} \supset \mathcal{C}^{\cap f}$. □

We will consider next a Cramér-type result which is useful in applications.

For this, let again $S = (S,d)$ be a (possibly non-separable) metric space, $\mathcal{A}$ be a σ-algebra of subsets of S such that $\mathcal{B}_b(S) \subset \mathcal{A} \subset \mathcal{B}(S)$, and let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of random elements in $(S,\mathcal{A})$ and ξ be a random element in $(S,\mathcal{B}_b(S))$, being all defined on a common p-space $(\Omega,\mathcal{F},\mathbb{P})$. Then

(34) ξ_n is said to converge in law to ξ (denoted by $\xi_n \xrightarrow{\mathcal{L}_b} \xi$) iff $\mathcal{L}\{\xi_n\} \xrightarrow[b]{} \mathcal{L}\{\xi\}$ (in the sense of our Definition 4).

Now, let $(\eta_n)_{n\in\mathbb{N}}$ be another sequence of random elements in $(S,\mathcal{A})$ defined on the same p-space $(\Omega,\mathcal{F},\mathbb{P})$, and let

$$d(\xi_n,\eta_n)(\omega) := d(\xi_n(\omega),\eta_n(\omega)),\ \omega \in \Omega.$$

Note that for non-separable S, $d(\xi_n,\eta_n)$ need not be a random variable.

THEOREM 9a. Suppose that in the setting just decribed

$$\lim_{n\to\infty} \mathbb{P}^*(d(\xi_n,\eta_n) > \delta) = 0 \text{ for every } \delta > 0,$$

where $\mathbb{P}^*$ denotes the outer p-measure pertaining to $\mathbb{P}$.

Then $\xi_n \xrightarrow{\mathcal{L}_b} \xi$ iff $\eta_n \xrightarrow{\mathcal{L}_b} \xi$.

Proof. By symmetry, it suffices to show that

$$\xi_n \xrightarrow{\mathcal{L}_b} \xi \quad \text{implies} \quad \eta_n \xrightarrow{\mathcal{L}_b} \xi.$$

So, assume $\xi_n \xrightarrow{\mathcal{L}_b} \xi$ and let $f \in U_b^b(S)$ be arbitrary but fixed;

then according to (28) (cf. (h')) it suffices to show that

$$(+) \quad \lim_{n\to\infty} | \mathbb{E}(f(\xi_n)) - \mathbb{E}(f(\eta_n))| = 0.$$

(Note that $f(\xi_n)$ and $f(\eta_n)$, as well as $f(\xi)$, are random variables.)

ad (+): Given an arbitrary $\varepsilon>0$ there exists (by uniform continuity of f) a $\delta = \delta(\varepsilon) > 0$ such that $|f(x) - f(y)| \leqq \varepsilon$ whenever $d(x,y) \leqq \delta$; also $\|f\| = \sup_{x\in S} |f(x)| < \infty$.

Therefore,

$$| \mathbb{E}(f(\xi_n)) - \mathbb{E}(f(\eta_n))| \leqq \int |f(\xi_n) - f(\eta_n)| \, d\mathbb{P}$$

$$= \int^* |f(\xi_n) - f(\eta_n)| \, d\mathbb{P} \leqq \int^* 1_{\{d(\xi_n,\eta_n)>\delta\}} |f(\xi_n) - f(\eta_n)| \, d\mathbb{P}$$

$$+ \int^* 1_{\{d(\xi_n,\eta_n)\leqq\delta\}} |f(\xi_n) - f(\eta_n)| \, d\mathbb{P}$$

$$\leqq 2 \|f\| \, \mathbb{P}^*(d(\xi_n,\eta_n) > \delta) + \varepsilon \to \varepsilon \text{ as } n \to \infty,$$

whence $\limsup_{n\to\infty} | \mathbb{E}(f(\xi_n)) - \mathbb{E}(f(\eta_n))| \leqq \varepsilon$ for every $\varepsilon > 0$, which implies (+). □

The following version of Theorem 9a is useful as well.

THEOREM 9a*. Let $(\xi_n)_{n\in\mathbb{N}}$ and $(\eta_n)_{n\in\mathbb{N}}$ be sequences of random elements in $(S,\mathcal{A})$, defined on a common p-space $(\Omega,\mathcal{F},\mathbb{P})$ such that

(a) $\lim_{n\to\infty} \mathbb{P}^*(d(\xi_n,\eta_n) > \delta) = 0$ for every $\delta > 0$.

Let $S' = (S',d')$ be another metric space and

$H: S \to S'$ be $\mathcal{A},\mathcal{B}_b(S')$-measurable,

and such that

(b) $d'(H(x),H(y)) \leqq L\cdot d(x,y)$ for all $x,y \in S$ and some constant $0 < L < \infty$.

Then, for any random element ζ in $(S',\mathcal{B}_b(S'))$,

$$H(\xi_n) \xrightarrow{\mathcal{L}_b} \zeta \text{ iff } H(\eta_n) \xrightarrow{\mathcal{L}_b} \zeta.$$

Proof. $H(\xi_n)$ and $H(\eta_n)$ are random elements in $(S',\mathcal{B}_b(S'))$ for which by (a) and (b)

$$\mathbb{P}^*(d'(H(\xi_n), H(\eta_n)) > \delta) \leq \mathbb{P}^*(d(\xi_n,\eta_n) > \delta/L) \to 0$$

for every $\delta>0$, whence the assertion follows from Theorem 9a. □

REMARK. Instead of (b) it suffices to assume only that H is uniformly continuous.

THEOREM 9b. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of random elements in $(S,\mathcal{A})$ and let ξ be a random element in $(S,\mathcal{B}_b(S))$ being all defined on some common p-space $(\Omega,\mathcal{F},\mathbb{P})$. Suppose that ξ is $\mathbb{P}$-a.s. constant; then $\xi_n \xrightarrow{\mathcal{L}_b} \xi$ implies

$$\lim_{n\to\infty} \mathbb{P}^*(d(\xi_n,\xi) > \delta) = 0 \text{ for every } \delta > 0.$$

Proof. We show first

$$(+) \qquad \lim_{n\to\infty} \mathbb{E}(|f\circ\xi_n - f\circ\xi|) = 0 \text{ for all } f \in U_b^b(S).$$

In fact, for each $f \in U_b^b(S)$ we have (cf. Theorem 3) that $f\circ\xi_n \xrightarrow{\mathcal{L}_b} f\circ\xi$, $f\circ\xi_n$ and $f\circ\xi$ being real random variables such that $f\circ\xi$ is $\mathbb{P}$-a.s. constant, whence (by classical probability theory) $f\circ\xi_n \overset{\mathbb{P}}{\to} f\circ\xi$ (where $\overset{\mathbb{P}}{\to}$ denotes convergence in probability). Since f is bounded, $\{f\circ\xi_n : n\in\mathbb{N}\}$ is uniformly integrable and therefore $f\circ\xi_n \xrightarrow{L^1} f\circ\xi$ which proves (+).

We are going to show that (+) implies

$$\lim_{n\to\infty} \mathbb{P}^*(d(\xi_n,\xi) > \delta) = 0 \text{ for every } \delta > 0.$$

For this, let $\delta>0$ be arbitrary; since $\mathcal{L}\{\xi\}(S_o^c) = 1$ for some separable $S_o \subset S$ there exists a countable and dense subset $\{x_i : i\in\mathbb{N}\}$ of S_o^c and we have

$$(*) \qquad \mathbb{P}(\xi \in S_o^c) = 1.$$

Then, for each $i\in\mathbb{N}$, there exists an $f_i \in U_b^b(S)$ such that $0\leq f_i\leq 1$ and

$$f_i(x) = \begin{cases} 0 \text{ if } x\in B^o(x_i,\delta/4) \\ 1 \text{ if } x\in \complement B^o(x_i,\delta/2), \end{cases}$$

where $B^o(x_i,r)$ denotes the open ball with center x_i and radius r.

In fact, take

$$f_i(x) := 1 - \max\left[\left(1 - \frac{d(x, B^o(x_i,\delta/4)\cap S_o^c)}{\delta/4}\right), 0\right]$$

to get such a function.

Now, let $A_1 := \{\xi\in B^o(x_1,\delta/4)\}$ and for $i\geq 2$ let

$$A_i := \{\xi\in B^o(x_i,\delta/4),\ \xi\in\complement B^o(x_1,\delta/4),\ldots,\xi\in\complement B^o(x_{i-1},\delta/4)\};$$

then $A_i \in F$, the A_i's being pairwise disjoint and such that $\mathbb{P}(\bigcup_{i\in\mathbb{N}} A_i) = 1$ according to (*). Therefore

$$\mathbb{P}^*(d(\xi_n,\xi) > \delta) \leq \sum_{i\in\mathbb{N}} \mathbb{P}^*(\{d(\xi_n,\xi) > \delta\}\cap A_i)$$

$$\leq \sum_{i\in\mathbb{N}} \mathbb{P}^*(\{d(\xi_n,x_i) > \tfrac{3}{4}\delta\}\cap A_i) \leq \sum_{i\in\mathbb{N}} \int^* 1_{A_i} |f_i\circ\xi_n - f_i\circ\xi|\, d\mathbb{P}$$

$$= \sum_{i\in\mathbb{N}} \int_{A_i} |f_i\circ\xi_n - f_i\circ\xi|\, d\mathbb{P},$$

where the last inequality follows from the fact that for all $\omega\in\{d(\xi_n,x_i) > \frac{3}{4}\delta\}\cap A_i$ one has $f_i(\xi_n(\omega)) = 1$ and $f_i(\xi(\omega)) = 0$ by construction of the f_i's .

If we put $\quad g_n(i) := \int_{A_i} |f_i\circ\xi_n - f_i\circ\xi|\, d\mathbb{P}$ and $g(i) := \mathbb{P}(A_i)$

for each $i\in\mathbb{N}$, we obtain functions g_n and g on $\mathbb{N}$ for which

$0\leq g_n\leq g$ and $\sum_{i\in\mathbb{N}} g(i) = \sum_{i\in\mathbb{N}} \mathbb{P}(A_i) = \mathbb{P}(\bigcup_{i\in\mathbb{N}} A_i) = 1,$

i.e., the g_n's are integrable functions on $\mathbb{N}$ (integrable w.r.t. the counting measure on $\mathbb{N}$) being dominated by an integrable function g; since, by assumption (+),

$$\lim_{n\to\infty} g_n(i) = 0 \text{ for all } i\in\mathbb{N},$$

it follows from Lebesgue's dominated convergence theorem that

$$\limsup_{n\to\infty} \mathbb{P}^*(d(\xi_n,\xi) > \delta) \leq \lim_{n\to\infty} \sum_{i\in\mathbb{N}} g_n(i) = 0. \quad \square$$

Finally, concerning the speed of convergence we have the following result:

THEOREM 10. Let ξ_n, $n\in\mathbb{N}$, and η be random elements in $(S,\mathcal{A})$ defined on a common p-space $(\Omega,\mathcal{F},\mathbb{P})$ such that for some sequence $a_n \downarrow 0$

(a') $\mathbb{P}^*(d(\xi_n,\eta) > a_n) = \mathcal{O}(a_n)$.

Let $H: S \to \mathbb{R}$ be $\mathcal{A},\mathcal{B}$-measurable and such that

(b') $|H(x) - H(y)| \leq L \cdot d(x,y)$ for all $x,y \in S$

and some constant $0 < L < \infty$.

Assume further that $\mathcal{L}\{H(\eta)\}$ is absolutely continuous w.r.t. Lebesgue measure λ such that

(c') $\|h\| = \sup_{t\in\mathbb{R}} |h(t)| =: M < \infty$ for $h \in \frac{d\mathcal{L}\{H(\eta)\}}{d\lambda}$.

Then

$$\sup_{t\in\mathbb{R}} |\mathbb{P}(H(\xi_n) \leq t) - \mathbb{P}(H(\eta) \leq t)| = \mathcal{O}(a_n).$$

Proof. Let $t\in\mathbb{R}$ be arbitrary but fixed; then

$$\begin{aligned}
&\mathbb{P}(H(\xi_n) \leq t) - \mathbb{P}(H(\eta) \leq t) \\
&\underset{(a')}{\leq} \mathbb{P}^*(H(\xi_n) \leq t,\ d(\xi_n,\eta) \leq a_n) + \mathcal{O}(a_n) - \mathbb{P}(H(\eta) \leq t) \\
&\underset{(b')}{\leq} \mathbb{P}(H(\xi_n) \leq t,\ |H(\xi_n) - H(\eta)| \leq L\cdot a_n) + \mathcal{O}(a_n) - \mathbb{P}(H(\eta) \leq t) \\
&\leq \mathbb{P}(H(\eta) \leq t + L\cdot a_n) + \mathcal{O}(a_n) - \mathbb{P}(H(\eta) \leq t) \\
&\underset{(c')}{\leq} M \cdot L \cdot a_n + \mathcal{O}(a_n) = \mathcal{O}(a_n).
\end{aligned}$$

In the same way one obtains that

$$\mathbb{P}(H(\xi_n) > t) - \mathbb{P}(H(\eta) > t) = \mathcal{O}(a_n),$$

whence also

$$\mathbb{P}(H(\eta) \leq t) - \mathbb{P}(H(\xi_n) \leq t) = \mathcal{O}(a_n),$$

so that in summary

$$\sup_{t\in\mathbb{R}} |\mathbb{P}(H(\xi_n) \leq t) - \mathbb{P}(H(\eta) \leq t)| = \mathcal{O}(a_n). \quad \square$$

SOME REMARKS ON PRODUCT SPACES:

Let $S' = (S',d')$ and $S'' = (S'',d'')$ be two (possibly non-separable) metric spaces.

Let $S := S' \times S''$ be the Cartesian product of S' and S'' and let $d := \max(d',d'')$, i.e.,

$$d((x',x''),(y',y'')) := \max(d'(x',y'),d''(x'',y''))$$

for $(x',x'') \in S$ and $(y',y'') \in S$.

Then $S = (S,d)$ is again a (possibly non-separable) metric space.

REMARK. (1) $\mathcal{B}_b(S) \subset \mathcal{B}_b(S') \otimes \mathcal{B}_b(S'')$

and (2) $\mathcal{B}(S') \otimes \mathcal{B}(S'') \subset \mathcal{B}(S)$,

the inclusions being strict in general as can be shown by examples.

Let $\mathcal{A}'$ and $\mathcal{A}''$ be σ-algebras of subsets of S' and S'', respectively, such that

$$\mathcal{B}_b(S') \subset \mathcal{A}' \subset \mathcal{B}(S') \quad \text{and} \quad \mathcal{B}_b(S'') \subset \mathcal{A}'' \subset \mathcal{B}(S'').$$

Then

$$\mathcal{B}_b(S) \underset{(1)}{\subset} \mathcal{B}_b(S') \otimes \mathcal{B}_b(S'') \subset \mathcal{A}' \otimes \mathcal{A}'' \subset \mathcal{B}(S') \otimes \mathcal{B}(S'') \underset{(2)}{\subset} \mathcal{B}(S),$$

i.e., putting e.g., $\mathcal{A} := \mathcal{B}_b(S') \otimes \mathcal{B}_b(S'')$ ... (a)

or $\mathcal{A} := \mathcal{A}' \otimes \mathcal{A}''$ ... (a'),

we have again

$\mathcal{B}_b(S) \subset \mathcal{A} \subset \mathcal{B}(S)$ for the product space $S = S' \times S''$.

Now, let ξ_n, $n \in \mathbb{N}$, be random elements in $(S',\mathcal{A}')$,

ξ be a random element in $(S',\mathcal{B}_b(S'))$,

η_n, $n \in \mathbb{N}$, be random elements in $(S'',\mathcal{A}'')$,

and let η be a random element in $(S'',\mathcal{B}_b(S''))$;

suppose that all these random elements are defined on a common p-space $(\Omega,\mathcal{F},\mathbb{P})$.

Then (ξ_n,η_n), $n \in \mathbb{N}$, are random elements in $(S,\mathcal{A})$ (for both choices of $\mathcal{A}$ as in (a) or (a')) and

$$(\xi,\eta) \text{ is a random element in } (S,\mathcal{B}_b(S') \otimes \mathcal{B}_b(S''))$$

as well as in $(S,\mathcal{B}_b(S))$ (cf. (1) in the above remark).

Thus, considering (ξ,η) as a random element in $(S,\mathcal{B}_b(S))$,

$$(\xi_n,\eta_n) \xrightarrow{L_b} (\xi,\eta)$$

is again defined in the sense of (34), i.e., as

$$L\{(\xi_n,\eta_n)\}\ (\in M_a(S)) \xrightarrow[b]{} L\{(\xi,\eta)\}\ (\in M_b(S)).$$

Supplementing the results contained in Theorems 9a and 9b we can prove within the setting just described the following Theorems 9c and 9d:

<u>THEOREM 9c.</u> Suppose that η equals $\mathbb{P}$-a.s. some constant c;

then $\xi_n \xrightarrow{L_b} \xi$ and $\eta_n \xrightarrow{L_b} \eta$ together imply $(\xi_n,\eta_n) \xrightarrow{L_b} (\xi,\eta)$.

<u>Proof.</u> According to Theorem 9b,

$\eta_n \xrightarrow{L_b} \eta$ and $\eta = c$ $\mathbb{P}$-a.s. imply $\lim\limits_{n\to\infty} \mathbb{P}^*(d''(\eta_n,\eta) > \delta) = 0$ for every $\delta>0$.

Since $d((\xi_n,\eta_n),(\xi_n,\eta)) = \max (d'(\xi_n,\xi_n), d''(\eta_n,\eta)) = d''(\eta_n,\eta)$,

we thus have

$$\lim_{n\to\infty} \mathbb{P}^*(d((\xi_n,\eta_n),(\xi,\eta)) > \delta) = 0 \text{ for every } \delta>0.$$

Therefore, by Theorem 9a, the assertion of the present theorem will follow if we show

$$(+) \qquad (\xi_n,\eta) \xrightarrow{L_b} (\xi,\eta).$$

<u>ad (+):</u> 1.) $L\{(\xi,\eta)\}$ is separable:

since $L\{\xi\}$ is separable, there exists a separable $S'_o \subset S'$ such that $L\{\xi\}(S'^c_o) = 1$. Take $S_o := S'^c_o \times \{c\}$ to get a separable and closed subset $S_o = S^c_o$ of S for which

$$L\{(\xi,\eta)\}(S_o) = \mathbb{P}((\xi,\eta)\in S_o) = \mathbb{P}((\xi,\eta)\in S'^c_o \times \{c\})$$
$$= \mathbb{P}((\xi,c)\in S'^c_o \times \{c\}) = \mathbb{P}(\xi \in S'^c_o) = L\{\xi\}(S'^c_o) = 1.$$

2.) According to the Portmanteau theorem (cf. (h') there) it remains to show that

$$\int_S f\, d\mu_n \longrightarrow \int_S f\, d\mu \quad \text{for all } f \in U^b_b(S),$$

where $\mu_n := L\{(\xi_n,\eta)\}$ and $\mu := L\{(\xi,\eta)\}$.

Now, given any $f: S = S' \times S'' \to \mathbb{R}$ being bounded, d-uniformly continuous and $\mathcal{B}_b(S)$-measurable, it follows from (1) in the remark made at the beginning that f is also $\mathcal{B}_b(S') \otimes \mathcal{B}_b(S'')$-measurable,
whence

$$f': S' \to \mathbb{R}, \text{ defined by } f'(x') := f(x',c),\ x' \in S',$$

is $\mathcal{B}_b(S')$-measurable, and thus $f' \in U_b^b(S')$.

But now, with $\mu_n' := L\{\xi_n\}$ and $\mu' := L\{\xi\}$, we obtain from $\xi_n \xrightarrow{L_b} \xi$ (using again the Portmanteau theorem):

$$\int_S f\, d\mu_n = \int_\Omega f \circ (\xi_n,\eta)d\mathbb{P} = \int_\Omega f \circ (\xi_n,c)d\mathbb{P} = \int_\Omega f' \circ \xi_n\, d\mathbb{P}$$

$$= \int_{S'} f' d\mu_n' \longrightarrow \int_{S'} f' d\mu' = \int_\Omega f' \circ \xi\, d\mathbb{P} = \int_\Omega f \circ (\xi,c)\, d\mathbb{P}$$

$$= \int_\Omega f \circ (\xi,\eta)d\mathbb{P} = \int_S f\, d\mu. \quad \square$$

For sequences of <u>independent</u> random elements one gets

<u>THEOREM 9d.</u> Suppose that ξ_n and η_n are independent for each $n \in \mathbb{N}$ and suppose also that ξ and η are independent. Then the following two statements are equivalent:

(i) $\xi_n \xrightarrow{L_b} \xi$ and $\eta_n \xrightarrow{L_b} \eta$

(ii) $(\xi_n,\eta_n) \xrightarrow{L_b} (\xi,\eta)$.

<u>Proof.</u> <u>(i) ⇒ (ii):</u> 1.) $L\{(\xi,\eta)\}$ is separable:
since both, $L\{\xi\}$ and $L\{\eta\}$ are separable, there exist $S_o' \subset S'$ and $S_o'' \subset S''$ such that (S_o',d') and (S_o'',d'') are separable and
$L\{\xi\}(S_o'^{c}) = L\{\eta\}(S_o''^{c}) = 1$.

Put $S_o := S_o'^{c} \times S_o''^{c}$ to get a separable and closed subspace of $S = (S,d)$ $(S = S' \times S'',\ d = \max(d',d''))$ for which
$L\{(\xi,\eta)\}(S_o^c) = (L\{\xi\} \times L\{\eta\})(S_o'^{c} \times S_o''^{c}) = L\{\xi\}(S_o'^{c}) \cdot L\{\eta\}(S_o''^{c}) = 1.$

2.) According to the Portmanteau theorem (cf. (a') there) it remains to show

$$(+)\qquad \liminf_{n\to\infty} \mathcal{L}\{(\xi_n,\eta_n)\}(G) \geq \widetilde{\mathcal{L}\{(\xi,\eta)\}}(G) \quad \text{for all} \quad G \in \mathcal{G}(S) \cap \mathcal{A}$$

(where $\mathcal{A} = \mathcal{A}' \otimes \mathcal{A}''$). For this, let $\mu' := \mathcal{L}\{\xi\}$ and $\mu'' := \mathcal{L}\{\eta\}$ and let

$$\mathcal{C} := \{A' \times A'' \colon A' \in \mathcal{A}',\ \widetilde{\mu'}(\partial A') = 0,\ A'' \in \mathcal{A}'',\ \widetilde{\mu''}(\partial A'') = 0\};$$

then $\mathcal{C}$ is closed under finite intersections, i.e. $\mathcal{C} = \mathcal{C}^{\cap f}$, and (33) holds which means that

for each $x \in S_o^c$ $\{C \in \mathcal{C}\colon x \in C^o\}$ is a neighborhood base at x.

Furthermore, by assumption and the Portmanteau theorem (cf. (f') there), we have for $\mu_n' = \mathcal{L}\{\xi_n\}$ and $\mu_n'' = \mathcal{L}\{\eta_n\}$

$$\lim_{n\to\infty} (\mu_n' \times \mu_n'')(A' \times A'') = \lim_{n\to\infty} \mu_n'(A')\cdot\mu_n''(A'') = \widetilde{\mu'}(A')\cdot\widetilde{\mu''}(A'')$$

$$= (\widetilde{\mu'} \times \widetilde{\mu''})(A' \times A'') = \widetilde{\mu' \times \mu''}(A' \times A'') \quad \text{for all} \quad A' \times A'' \in \mathcal{C} = \mathcal{C}^{\cap f},$$

whence (+) follows from Lemma 17.

<u>(ii) ⇒ (i):</u> 1.) Both $\mathcal{L}\{\xi\}$ and $\mathcal{L}\{\eta\}$ are separable:

since $\mathcal{L}\{(\xi,\eta)\}$ is separable there exists a separable $S_o \subset S = S' \times S''$ such that $\mathcal{L}\{(\xi,\eta)\}(S_o^c) = 1$. Put

$$S_o' := \{x \in S'\colon \exists y \in S'' \text{ such that } (x,y) \in S_o\}$$

to get a separable $S_o' \subset S'$ for which $S_o^c \subset \pi'^{-1}(S_o'^c)$, whence

$$\mathcal{L}\{\xi\}(S_o'^c) = (\mathcal{L}\{\xi\} \times \mathcal{L}\{\eta\})(\pi'^{-1}(S_o'^c)) = \mathcal{L}\{(\xi,\eta)\}(\pi'^{-1}(S_o'^c))$$

$$\geq \mathcal{L}\{(\xi,\eta)\}(S_o^c) = 1, \text{ i.e., } \mathcal{L}\{\xi\}(S_o'^c) = 1;$$

here π' denotes the projection of $S = S' \times S''$ onto S'.

In the same way one shows that $\mathcal{L}\{\eta\}$ is separable.

2.) According to the Portmanteau theorem (cf. (a') there) it remains to show that for $\mu_n' = \mathcal{L}\{\xi_n\}$ and $\mu' = \mathcal{L}\{\xi\}$

$$(+)\qquad \liminf_{n\to\infty} \mu_n'(G') \geq \widetilde{\mu'}(G') \quad \text{for all} \quad G' \in \mathcal{G}(S') \cap \mathcal{A}'$$

and that for $\mu_n'' = \mathcal{L}\{\eta_n\}$ and $\mu'' = \mathcal{L}\{\eta\}$

(++) $\liminf_{n\to\infty} \mu_n''(G'') \geq \widetilde{\mu''}(G'')$ for all $G'' \in \mathcal{G}(S'') \cap \mathcal{A}''$.

We will show (+); the proof of (++) runs analogously.

ad (+): Let $G' \in \mathcal{G}(S') \cap \mathcal{A}'$ be arbitrary but fixed; then

$$\pi'^{-1}(G') = G' \times S'' \in \mathcal{A} \cap \mathcal{G}(S) \qquad (\mathcal{A} = \mathcal{A}' \otimes \mathcal{A}'')$$

and $\mu_n'(G') = (\mu_n' \times \mu_n'')(\pi'^{-1}(G')) = (\mu_n' \times \mu_n'')(G' \times S'')$ for each $n \in \mathbb{N}$.

By assumption and the Portmanteau theorem (cf. (a') there) we therefore obtain

$$\liminf_{n\to\infty} \mu_n'(G') = \liminf_{n\to\infty} (\mu_n' \times \mu_n'')(G' \times S'')$$

$$\geq \widetilde{(\mu' \times \mu'')}(G' \times S'') = \widetilde{\mu'}(G') \cdot \widetilde{\mu''}(S'') = \widetilde{\mu'}(G'). \quad \square$$

Remark. Using the continuous mapping theorem (Theorem 3) one easily gets an alternative proof of "(ii) ⇒ (i)" in Theorem 9d, even without imposing the independence assumptions.

SEQUENTIAL COMPACTNESS:

We have shown before (cf. (32)) that any net $(\mu_\alpha) \subset M_a^1(S)$ which fulfills (b_i), i=1,2, is a compact net in $M_a^1(S)$. At this point we ask the question whether the same is true for sequences instead of nets, i.e., whether for any sequence $(\mu_n)_{n\in\mathbb{N}} \subset M_a^1(S)$ fulfilling (b_i), i=1,2, there exists a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ and a separable $\mu \in M_b^1(S)$ such that $\mu_{n_k} \xrightarrow[b]{} \mu$ (as $k\to\infty$).

(Note that a subnet of a sequence need not be a sequence!)

If (b_i), i=1,2, is replaced by the (stronger) assumption of $(\mu_n)_{n\in\mathbb{N}}$ being δ-tight (cf. (31)), then it follows that the answer is affirmative; in fact, as shown by Dudley (1966), Theorem 1, the following is true:

(35) For any δ-tight sequence $(\mu_n)_{n\in\mathbb{N}} \subset M_a^1(S)$ there exists a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$ and a Borel p-measure $\tilde{\mu}$ (on $\mathcal{B}(S)$) such that

(36) $\lim_{k\to\infty} \int^* f d\mu_{n_k} = \lim_{k\to\infty} \int_* f d\mu_{n_k} = \int f d\tilde{\mu}$ for all $f \in C^b(S)$.

Based on this result we obtain in a first step the following theorem:

THEOREM 11. Let $(\mu_n)_{n\in\mathbb{N}} \subset M_a^1(S)$ be δ-tight; then there exists a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$ and a separable $\mu \in M_b^1(S)$ such that $\mu_{n_k} \xrightarrow[b]{} \mu$.

Proof. Apply (35) to get a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$ and a Borel p-measure $\tilde{\mu}$ for which (36) holds true. Then it can be shown as in the "(h') $\Rightarrow$ (b)" part of the proof of (28) that (36) implies

$$(+) \qquad \limsup_{k\to\infty} \mu^*_{n_k}(F) \leq \tilde{\mu}(F) \quad \text{for all} \quad F \in F(S).$$

(In fact, given an arbitrary $F \in F(S)$, there exists for every $n\in\mathbb{N}$ an $\varepsilon_n>0$ such that $\tilde{\mu}(F^{\varepsilon_n}) \leq \tilde{\mu}(F) + \frac{1}{n}$; taking then

$$f_n(x) =: \begin{cases} d(x, \complement F^{\varepsilon_n})/\varepsilon_n & \text{if } \complement F^{\varepsilon_n} \neq \emptyset, \\ 1 \text{ if } \complement F^{\varepsilon_n} = \emptyset, \end{cases} \qquad x \in S,$$

and $g_n := \min(f_n, 1)$, we obtain a sequence of functions g_n having the following properties for every $n\in\mathbb{N}$:

$$\text{①} \quad g_n \in C^b(S), \qquad \text{②} \quad \operatorname{rest}_{\complement F^{\varepsilon_n}} g_n \equiv 0 \quad \text{and} \quad \text{③} \quad \operatorname{rest}_F g_n \equiv 1.$$

Therefore, for every $n\in\mathbb{N}$ we obtain

$$\limsup_{k\to\infty} \mu^*_{n_k}(F) = \limsup_{k\to\infty} \int^* 1_F \, d\mu_{n_k} \underset{③}{\leq} \limsup_{k\to\infty} \int^* g_n \, d\mu_{n_k}$$

$$\underset{(36),①}{=} \int g_n \, d\tilde{\mu} \underset{②}{=} \int_{F^{\varepsilon_n}} g_n \, d\tilde{\mu} \leq \tilde{\mu}(F^{\varepsilon_n}) \leq \tilde{\mu}(F) + \frac{1}{n}, \text{ which implies (+).)}$$

Now, we are going to show that (due to the δ-tightness of (μ_n))

$$(++) \qquad \tilde{\mu} \text{ is necessarily tight,}$$

whence $\mu := \operatorname{rest}_{\mathcal{B}_b(S)} \tilde{\mu}$ is also tight and therefore separable (cf. (25)) and thus (noticing also (24)) we can apply (28) (cf. the equivalent statements (g) and (g')) to obtain the result, i.e., $\mu_{n_k} \xrightarrow[b]{} \mu$.

ad (++): Since $(\mu_n)_{n\in\mathbb{N}}$ is δ-tight, it follows that for every $n\in\mathbb{N}$ there exists a $K_n \in K(S)$ such that

$$(+++) \quad \liminf_{k\to\infty} \mu_{n_k}(K_n^{1/2m}) \geq 1 - \frac{1}{n} \quad \text{for all } m \in \mathbb{N}.$$

W.l.o.g. we may assume $K_n \uparrow$ and therefore

$$\tilde{\mu}\Big(\bigcup_{n\in\mathbb{N}} K_n\Big) = \lim_{n\to\infty} \tilde{\mu}(K_n) = \lim_{n\to\infty} \big(\lim_{m\to\infty} \tilde{\mu}(K_n^{1/m})\big)$$

$$\geq \lim_{n\to\infty} \big(\limsup_{m\to\infty} \tilde{\mu}((K_n^{1/2m})^c)\big) \underset{(+)}{\geq} \lim_n \limsup_m \limsup_{k\to\infty} \mu_{n_k}^*((K_n^{1/2m})^c)$$

$$\geq \lim_n \limsup_m \liminf_k \mu_{n_k}(K_n^{1/2m}) \underset{(+++)}{\geq} 1. \quad \square$$

The proof of Theorem 11 also shows that the following result holds true:

<u>THEOREM 11*.</u> If $S_o = (S_o,d)$ is a separable subspace of S and if $(\mu_n)_{n\in\mathbb{N}} \subset M_a^1(S)$ is <u>δ-tight w.r.t. S_o</u> (i.e., if $\sup_{K\in K(S_o)} \inf_{\delta>0} \liminf_{n\to\infty} \mu_n(K^\delta) = 1$), then there exists a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$ and a $\mu \in M_b^1(S)$ with $\mu(S_o^c) = 1$ such that $\mu_{n_k} \xrightarrow[b]{} \mu$.

As to our question raised at the beginning, it was shown by J. Schattauer (1982) that the assertion of Theorem 11 even holds if the assumption of δ-tightness of $(\mu_n)_{n\in\mathbb{N}}$ is replaced by the (weaker) conditions (b_i), $i=1,2$:

<u>THEOREM 11a).</u> Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $M_a^1(S)$ fulfilling the following two conditions:

(b_1) For every sequence $(f_m)_{m\in\mathbb{N}} \subset U_b^b(S)$ with $f_m \downarrow 0$ one has

$\limsup_n \int f_m d\mu_n \to 0$ as $m \to \infty$.

(b_2) There exists a separable $S_o \subset S$ such that

$\liminf_n \int f\mu_n \geq 1$ for all $f \in U_b^b(S)$ with $f \geq 1_{S_o^c}$.

Then there exists a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$ and a $\mu \in M_b^1(S)$ with $\mu(S_o^c) = 1$ such that $\mu_{n_k} \xrightarrow[b]{} \mu$ (as $k \to \infty$).

For the proof of this theorem we need an auxiliary result which is based on the following

DEFINITION. Let $S = (S,d)$ be a metric space and $A_i \subset S$, $i=1,2$; A_1 and A_2 are said to be d-strictly separated if either $A_1^{\delta_1} \cap A_2 = \emptyset$ for some $\delta_1 > 0$ or $A_1 \cap A_2^{\delta_2} = \emptyset$ for some $\delta_2 > 0$,

where $A_i^{\delta_i} := \{x \in S: d(x,A_i) < \delta_i\}$, $i=1,2$.

PROPOSITION (cf. E. Hewitt (1947), Theorem 1).

Let $S = (S,d)$ be a metric space, and let G be a subset of $U^b(S)$ such that for every d-strictly separated pair $F_1,F_2 \in F(S)$ there exists a function $g \in G$ such that $\sup_{x\in F_1} g(x) < \inf_{x\in F_2} g(x)$ (or $\sup_{x\in F_2} g(x) < \inf_{x\in F_1} g(x)$). Then G is an "analytic generator" of $U^b(S)$, i.e., for every $f \in U^b(S)$ and every positive real number ε there exist functions $f_1,\ldots,f_k \in G$ and a polynomial $P(z_1,\ldots,z_k) \equiv$

$$\sum_{\ell_1=0}^{L_1} \cdots \sum_{\ell_k=0}^{L_k} \alpha_{\ell_1\ldots\ell_k} z_1^{\ell_1}\cdots z_k^{\ell_k} \quad \text{(with real coefficients } \alpha_{\ell_1\ldots\ell_k})$$

such that

$$\|f - P(f_1,\ldots,f_k)\| := \sup_{x\in S} |f(x) - P(f_1,\ldots,f_k)(x)| < \varepsilon.$$

Proof of the proposition. This follows along the same lines as in Hewitt (1947) noticing that the functions $\psi,\varphi,h,h_1,\ldots,h_n$ and φ, $\frac{3}{2}(\varphi - h_1)$, $(\frac{3}{2})^i(\varphi - h_1 - \frac{2}{3}h_2 - \ldots - (\frac{2}{3})^{i-1}h_i)$, $2 \leq i \leq n$, respectively, occurring there are uniformly continuous which implies that the sets

$F_1^f := \{x \in S: f(x) \leq -\frac{1}{3}\}$ and $F_2^f := \{x \in S: f(x) \geq \frac{1}{3}\}$,

for $f \in \{\varphi, (\frac{3}{2})^i(\varphi - h_1 - \frac{2}{3}h_2 - \ldots - (\frac{2}{3})^{i-1}h_i), 1 \leq i \leq n\}$,

are d-strictly separated:

In fact, $f \in U^b(S)$ implies that for $\varepsilon = \frac{1}{6}$ there exists a $\delta>0$ such that $|f(x) - f(y)| < \frac{1}{6}$ whenever $d(x,y) < \delta$; thus given any $x \in (F_1^f)^\delta$, we have $d(x,x_o) < \delta$ for some $x_o \in F_1^f$ and therefore $|f(x) - f(x_o)| < \frac{1}{6}$ and $f(x_o) \leq -\frac{1}{3}$ (since $x_o \in F_1^f$) which implies $f(x) \leq -\frac{1}{6}$ for all $x \in (F_1^f)^\delta$; since $f(x) \geq \frac{1}{3}$ for all $x \in F_2^f$, we thus have $(F_1^f)^\delta \cap F_2^f = \emptyset$. $\square$

Proof of Theorem 11a). According to (b_2), let $T \subset S_o$ be countable and dense in S_o (as well as in S_o^c), say $T = \{x_1, x_2, \ldots\}$, and let

$G_1 := \{\min(d(\cdot, x_n), 1): n \in \mathbb{N}\}$,

$G_2 := \{f: S \to \mathbb{R}: f = \min_{1 \leq i \leq n} g_i,\ g_i \in G_1,\ i=1,\ldots,n,\ n \in \mathbb{N}\}$, and

$G_3 := \{f: S \to \mathbb{R}: f = g_1^{\ell_1} \ldots g_k^{\ell_k}$ for some $g_i \in G_2,\ \ell_1, \ldots, \ell_k \in \mathbb{N} \cup \{0\},$
$k \in \mathbb{N}\}$;

then $G_1 \subset G_2 \subset G_3$ with G_3 being a countable class of functions in $U_b^b(S)$ (cf. Lemma 11 (ii)).

Therefore, by the diagonal method, there exists a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$ such that

(i) $\lim_{k\to\infty} \int f d\mu_{n_k}$ exists for all $f \in G_3$.

Let $G_4 := \{f: S \to \mathbb{R}: f = \min(d(\cdot, F^\varepsilon \cap S_o^c), 1)$ for some $F \in F(S_o^c)$, $\varepsilon > 0\}$;
then (cf. Lemma 11 (ii)) $G_4 \subset U_b^b(S)$, and

(ii) For any $f \in G_4$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset G_2$ such that $f_n \downarrow f$ as $n \to \infty$.

ad (ii): Let $f \in G_4$, i.e., $f = \min(d(\cdot, F^\varepsilon \cap S_o^c), 1)$, $F \in F(S_o^c)$, $\varepsilon > 0$.

It is easy to show that $T \cap F^\varepsilon$ is countable and dense in $F^\varepsilon \cap S_o^c$; let $T \cap F^\varepsilon = \{z_1, z_2, \ldots\}$; then $d(\cdot, F^\varepsilon \cap S_o^c) = \inf_n d(\cdot, z_n)$, and therefore

$g_n := \inf_{1\leq i\leq n} d(\cdot, z_i) \downarrow \inf_i d(\cdot, z_i)$, i.e., $f_n := \min(g_n, 1) \downarrow f$,

where $f_n = \min_{1\leq i\leq n} (\min(d(\cdot, z_i), 1)) \in G_2$ for each n which proves (ii).

Now, let $G_5 := \{f: S \to \mathbb{R}: f = g_1^{\ell_1} \ldots g_k^{\ell_k}$ for some $g_i \in G_4$,
$\ell_1, \ldots, \ell_k \in \mathbb{N} \cup \{0\},\ k \in \mathbb{N}\}$;

then, since $G_4 \subset U_b^b(S)$, we have also

(iii) $G_5 \subset U_b^b(S)$.

On the other hand, it follows from (ii) that

(iv) For any $f \in G_5$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset G_3$ such that

$f_n \downarrow f$ as $n \to \infty$.

Furthermore,

(v) $\lim_{k\to\infty} \int f d\mu_{n_k}$ exists for all $f \in G_5$.

<u>ad (v):</u> Let $f \in G_5$; then, by (iv), there exists a sequence $(f_m)_{m\in\mathbb{N}} \subset G_3$ such that $f_m \downarrow f$ as $m \to \infty$. Since $f_m - f \downarrow 0$ and $f_m - f \in U_b^b(S)$, we obtain by (b_1) that $\limsup_{n\to\infty} \int (f_m - f)d\mu_n \to 0$ as $m \to \infty$, and therefore also

$\limsup_{k\to\infty} \int (f_m - f)d\mu_{n_k} \to 0$ as $m \to \infty$. Since

$$\limsup_k \int (f_m - f)d\mu_{n_k} \geq \limsup_k \int f_m d\mu_{n_k} - \limsup_k \int f d\mu_{n_k} \geq 0,$$

it follows that $\limsup_k \int f_m d\mu_{n_k} - \limsup_k \int f d\mu_{n_k} \to 0$ as $m \to \infty$,

and therefore we obtain by (i)

a $$\lim_{m\to\infty} \lim_{k\to\infty} \int f_m d\mu_{n_k} = \limsup_{k\to\infty} \int f d\mu_{n_k}.$$

On the other hand, since $\liminf_k \int (f - f_m)d\mu_{n_k} = -\limsup_k \int (f_m - f)d\mu_{n_k}$,

we have $\liminf_k \int (f - f_m)d\mu_{n_k} \to 0$ as $m \to \infty$. Since

$$\liminf_k \int (f - f_m)d\mu_{n_k} \leq \liminf_k \int f d\mu_{n_k} - \liminf_k \int f_m d\mu_{n_k} \leq 0,$$

we thus obtain in the same way as before, using (i), that

$\lim_{m\to\infty} \lim_{k\to\infty} \int f_m d\mu_{n_k} = \liminf_{k\to\infty} \int f d\mu_{n_k}$, whence together with a the assertion in (v) follows.

Finally, let $G_6 := \{f: S \to \mathbb{R}: f = P(g_1,\dots,g_k)$ for some $g_i \in G_4$, $1 \leq i \leq k$, $k \in \mathbb{N}\}$;

then, by (iii), $G_6 \subset U_b^b(S)$, and it can be easily shown that (v) implies

(vi) $\lim_{k\to\infty} \int f d\mu_{n_k}$ exists for all $f \in G_6$.

Now, let $U^b(S_o^c) := \{f: S_o^c \to \mathbb{R}: f$ bounded and uniformly (d-)continuous$\}$ and consider $G_4' := \{\mathrm{rest}_{S_o^c} f: f \in G_4\} \subset U^b(S_o^c)$.

Let $F_1, F_2 \in \mathcal{F}(S_o^c)$ be a d-strictly separated pair of closed subsets in the metric space $S_o^c = (S_o^c, d)$, i.e.,(w.l.o.g.) there exists a $\delta>0$ such that $(F_1^\delta \cap S_o^c) \cap F_2 = \emptyset$.

Put $f := \min(d(\cdot, F_1^{\delta/2} \cap S_o^c), 1)$; then $f \in G_4$ and $g := \text{rest}_{S_o^c} f \in G_4'$; we will show that

(b) $\quad \sup_{x \in F_1} g(x) < \inf_{x \in F_2} g(x)$.

ad (b): $x \in F_1$ implies $d(x, F_1^{\delta/2} \cap S_o^c) = 0$ since $F_1 \subset S_o^c$, and therefore $f(x) = 0$ for all $x \in F_1$ whence $\sup_{x \in F_1} g(x) = 0$.

On the other hand, $x \in F_2$ together with $(F_1^\delta \cap S_o^c) \cap F_2 = \emptyset$ implies $d(x, F_1) \geq \delta$ and therefore $d(x, F_1^{\delta/2}) \geq \delta/2$; thus $d(x, F_1^{\delta/2} \cap S_o^c) \geq \delta/2$ for all $x \in F_2$, i.e.,

$$\inf_{x \in F_2} g(x) \geq \min(\delta/2, 1) \text{ which proves (b).}$$

Therefore, by our proposition, G_4' is an analytic generator of $U^b(S_o^c)$, i.e., for every $h \in U^b(S_o^c)$ there exists a sequence $(g_n)_{n \in \mathbb{N}}$ such that $g_n = P_n(g_{n1}, g_{n2}, \ldots, g_{nk_n})$ with $g_{ni} \in G_4'$, $1 \leq i \leq k_n$, and $\sup_{x \in S_c^o} |h(x) - g_n(x)| \to 0$ as $n \to \infty$.

Since $g_{ni} \in G_4'$, $g_{ni} = \text{rest}_{S_o^c} f_{ni}$ for some $f_{ni} \in G_4$, whence $f_n := P_n(f_{n1}, f_{n2}, \ldots, f_{nk_n}) \in G_6$ with $\text{rest}_{S_o^c} f_n = g_n$ for each $n \in \mathbb{N}$.

We thus obtain that

(vii) For any $f \in U_b^b(S)$ there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset G_6$ such that

$$\sup_{x \in S_o^c} |f(x) - f_n(x)| \to 0 \text{ as } n \to \infty.$$

Furthermore, we will show that

(viii) $\quad \lim_{k \to \infty} \int f d\mu_{n_k}$ exists for all $f \in U_b^b(S)$.

ad (viii): Let $f \in U_b^b(S)$; then, by (vii), there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset G_6$ such that $\sup_{x \in S_o^c} |f(x) - f_n(x)| \to 0$ as $n \to \infty$; therefore, given an arbitrary but fixed $m \in \mathbb{N}$ there exists an $n_o = n_o(m) \in \mathbb{N}$ such that $\sup_{x \in S_o^c} |f(x) - f_{n_o}(x)| < \frac{1}{m}$. Since f and f_{n_o} are uniformly continuous, there exists a $\delta_m > 0$ such that $|f(x) - f(y)| < \frac{1}{m}$ and $|f_{n_o}(x) - f_{n_o}(y)| < \frac{1}{m}$ whenever $d(x,y) < \delta_m$. Now, let $\tilde{S} := (S_o^c)^{\delta_m}$; then, for any $x \in \tilde{S}$ there exists a $y \in S_o^c$ such that $d(x,y) < \delta_m$, and therefore

$$|f_{n_o}(x) - f(x)| \leq |f_{n_o}(x) - f_{n_o}(y)| + |f_{n_o}(y) - f(y)| + |f(y) - f(x)| < \frac{3}{m}$$

for all $x \in \tilde{S}$, whence $f(x) \leq f_{n_o}(x) + \frac{3}{m}$ and $f(x) \geq f_{n_o}(x) - \frac{3}{m}$ for all $x \in \tilde{S}$.

Since $\tilde{S} \in \mathcal{B}_b(S)$ (cf. Lemma 11 (ii)), it follows that

c $\quad \int f d\mu_{n_k} \leq \int_{\tilde{S}} (f_{n_o} + \frac{3}{m})\, d\mu_{n_k} + \int_{\complement \tilde{S}} f d\mu_{n_k}$ for all $k \in \mathbb{N}$, and

d $\quad \int f d\mu_{n_k} \geq \int_{\tilde{S}} (f_{n_o} - \frac{3}{m})\, d\mu_{n_k} + \int_{\complement \tilde{S}} f d\mu_{n_k}$ for all $k \in \mathbb{N}$.

Furthermore, it follows from (b_2) that

e $\quad \limsup_k \mu_{n_k}(\complement \tilde{S}) = 0.$

<u>ad e:</u> $1_{\complement \tilde{S}} = 1_{\complement (S_o^c)^{\delta_m}} \leq \min\left(\frac{d(\cdot, S_o^c)}{\delta_m}, 1\right) =: f^o \in U_b^b(S)$, whence

$\limsup_k \mu_{n_k}(\complement \tilde{S}) \leq \limsup_k \int f^o d\mu_{n_k} = \limsup_k \left(1 - \int (1 - f^o) d\mu_{n_k}\right)$

$= 1 - \liminf_k \int (1 - f^o) d\mu_{n_k} \leq 0$, since $1 - f^o \geq 1_{S_o^c}$ and thus

$\liminf_k \int (1 - f^o) d\mu_{n_k} \geq 1$ by (b_2). This proves e.

Next, it can be easily shown that e implies

f $\quad \lim_{k \to \infty} \int_{\complement \tilde{S}} f d\mu_{n_k} = 0$ for all $f \in U_b^b(S)$.

But then, it follows from c, d and f that

$$\limsup_k \int f d\mu_{n_k} \leq \limsup_k \int_{\tilde{S}} f_{n_o} d\mu_{n_k} + \frac{3}{m}, \text{ and}$$

$$\liminf_k \int f d\mu_{n_k} \geq \liminf_k \int_{\tilde{S}} f_{n_o} d\mu_{n_k} - \frac{3}{m}.$$

Furthermore, (f) together with (vi) imply easily that

$$\limsup_k \int_{\tilde{S}} f_{n_o} d\mu_{n_k} = \liminf_k \int_{\tilde{S}} f_{n_o} d\mu_{n_k} = \lim_{k\to\infty} \int_{\tilde{S}} f_{n_o} d\mu_{n_k}$$

and therefore $\limsup_k \int f d\mu_{n_k} - \liminf_k \int f d\mu_{n_k} \leq \frac{6}{m}$

which implies the assertion in (viii) since we started with an arbitrary m.
But now, putting $\mu(f) := \lim_{k\to\infty} \int f d\mu_{n_k}$ for $f \in U_b^b(S)$,
the assertion of Theorem 11 a) follows as in the proof of Theorem 6* applying the Daniell-Stone representation theorem (cf. H. Bauer (1978), 3. Auflage, S. 188) noticing that $\mathcal{B}_b(S)$ coincides with the smallest σ-algebra with respect to which all $f \in U_b^b(S)$ are measurable.
This concludes the proof of Theorem 11 a). □

SKOROKHOD-DUDLEY-WICHURA REPRESENTATION THEOREM:

Let again S = (S,d) be a (possibly non-separable) metric space and suppose that $\mathcal{A}$ is a σ-algebra of subsets of S such that $\mathcal{B}_b(S) \subset \mathcal{A} \subset \mathcal{B}(S)$; let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of random elements in $(S,\mathcal{A})$ and ξ be a random element in $(S,\mathcal{B}_b(S))$ such that $\xi_n \xrightarrow{\mathcal{L}_b} \xi$ (cf. (34)).
Then the Skorokhod-Dudley-Wichura Representation Theorem states:

THEOREM 12. $\xi_n \xrightarrow{\mathcal{L}_b} \xi$ implies that there exists a sequence $\hat{\xi}_n$, $n \in \mathbb{N}$, of random elements in $(S,\mathcal{A})$ and a random element $\hat{\xi}$ in $(S,\mathcal{B}_b(S))$ being all defined on an appropriate p-space $(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}})$ such that $\mathcal{L}\{\hat{\xi}_n\} = \mathcal{L}\{\xi_n\}$ (on $\mathcal{A}$) for all $n \in \mathbb{N}$, $\mathcal{L}\{\hat{\xi}\} = \mathcal{L}\{\xi\}$ (on $\mathcal{B}_b(S)$) and $\hat{\xi}_n \to \hat{\xi}$ $\hat{\mathbb{P}}$-almost surely as $n \to \infty$ (i.e., there exists an $\hat{\Omega}_o \subset \hat{\Omega}$ with $\hat{\Omega}_o \in \hat{\mathcal{F}}$ and $\hat{\mathbb{P}}(\hat{\Omega}_o) = 1$ such that for all $\hat{\omega} \in \hat{\Omega}_o$ $\lim_{n\to\infty} d(\hat{\xi}_n(\hat{\omega}), \hat{\xi}(\hat{\omega})) = 0$).

For complete and separable metric spaces this result was proved by A.V. Skorokhod (1956); it was generalized to arbitrary separable metric spaces by R.M. Dudley (1968), and in its present form (for arbitrary metric spaces) it was first proved by M.J. Wichura (1970); cf. also R.M. Dudley (1976), Lectures 19 and 24.

Our proof will be based on the one given by Dudley (1976). For this, we need the following proposition.

Proposition. Let $S = (S,d)$ be a metric space, $\mu \in M_b^1(S)$ be separable, i.e., $\mu(S_o^c) = 1$ for some separable $S_o \subset S$; then, given any $\varepsilon>0$, there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint subsets A_n of S having the following properties:

(i) $S_o^c \subset \bigcup_{n\in\mathbb{N}} A_n$

(ii) $\tilde{\mu}(\partial A_n) = 0$ for all $n \in \mathbb{N}$ (where $\tilde{\mu}$ denotes the unique Borel extension of μ (cf. (24)))

(iii) $\operatorname{diam}(A_n) := \sup_{x,y\in A_n} d(x,y) < \varepsilon$ for all $n \in \mathbb{N}$, and

(iv) $A_n \in \mathcal{B}_b(S)$ for all $n \in \mathbb{N}$.

Proof. Let $\{x_1,x_2,\ldots\}$ be dense in S_o^c. For each $n \in \mathbb{N}$, the open ball $B(x_n,\delta)$ is a $\tilde{\mu}$-continuity set (i.e., $\tilde{\mu}(\partial B(x_n,\delta))=0$) except for at most countably many values of δ; hence, given any $\varepsilon>0$, for each $n \in \mathbb{N}$ there exists an ε_n such that $\varepsilon/4 < \varepsilon_n < \varepsilon/2$ and $\tilde{\mu}(\partial B(x_n,\varepsilon_n)) = 0$. Now, let $A_1 := B(x_1,\varepsilon_1)$, and recursively for $n>1$ $A_n := B(x_n,\varepsilon_n) \setminus \bigcup_{j<n} B(x_j,\varepsilon_j)$. Then (i) - (iv) are fulfilled:

In fact, (iii) and (iv) follow at once by construction, (ii) holds since the class of all $\tilde{\mu}$-continuity sets forms an algebra containing each $B(x_n,\varepsilon_n)$; finally, given any $x \in S_o^c$ there exists an x_k such that $d(x,x_k) < \varepsilon/4$ whence $x \in B(x_k,\varepsilon/4) \subset B(x_k,\varepsilon_k) \subset \bigcup_{n=1}^{k} A_n$, implying (i). □

Proof of Theorem 12. Let us start by giving a description of the basic steps along which the proof will go, postponing some details to its end. For this, let $P := \mathcal{L}\{\xi\}$ on $\mathcal{B}_b(S)$ with $P(S_o^c) = 1$ for some separable $S_o \subset S$, and let

$P_n := \mathcal{L}\{\xi_n\}$ on $\mathcal{A}$, $n \in \mathbb{N}$.

STEP 1. For each $k \in \mathbb{N}$, by the proposition take a sequence $(A_{kj})_{j\in\mathbb{N}}$ of disjoint $\tilde{P}$-continuity sets $A_{kj} \in \mathcal{B}_b(S)$ such that $\operatorname{diam}(A_{kj}) < \frac{1}{k}$ for all $j \in \mathbb{N}$. Since $\bigcup_{j\in\mathbb{N}} A_{kj} \supset S_o^c$ and $P(S_o^c) = 1$, there exists a $J_k < \infty$ such that

(a) $\sum_{1\le j\le J_k} P(A_{kj}) > 1 - 2^{-k}$ (where w.l.o.g. we may assume

$P(A_{kj}) > 0$ for all $1 \leq j \leq J_k$).

Applying (28) (cf. (f') there) we obtain

(b) For each $k \in \mathbb{N}$ there exists an $n_k \in \mathbb{N}$ such that for

$1 \leq j \leq J_k$ $|P_n(A_{kj}) - P(A_{kj})| < 2^{-k} \min_{1\le j\le J_k} P(A_{kj})$ for

all $n \geq n_k$.

We may assume w.l.o.g. $1 < n_1 < n_2 < \ldots$.

STEP 2. For each $n \in \mathbb{N}$ let $S_n := S$, $I_n := I := [0,1]$, $T_n := S_n \times I_n$, $\mathcal{B}_n := \mathcal{A} \otimes \mathcal{B}(I_n)$ (with $\mathcal{B}(I_n) := I_n \cap \mathcal{B}$), and $Q_n := P_n \times \lambda$, λ being Lebesgue measure on $\mathcal{B}(I_n)$; furthermore, let $T_o := S \times I$, $\mathcal{B}_o := \mathcal{B}_b(S) \otimes \mathcal{B}(I)$ and $Q_o := P \times \lambda$.

For each $k \in \mathbb{N}$, $1 \leq j \leq J_k$ and $n \geq n_k$ let

$$f(n,k,j) := \begin{cases} \dfrac{P(A_{kj})}{P_n(A_{kj})}, & \text{if } P_n(A_{kj}) > P(A_{kj}) \ (> 0) \\ 1 \text{ otherwise,} & \end{cases}$$

$$g(n,k,j) := \begin{cases} \dfrac{P_n(A_{kj})}{P(A_{kj})}, & \text{if } P_n(A_{kj}) < P(A_{kj}), \\ 1 \text{ otherwise,} & \end{cases}$$

$B_{nkj} := A_{kj} \times [0,f(n,k,j)]$, considered as a subset of T_n, and
$C_{nkj} := A_{kj} \times [0,g(n,k,j)]$, considered as a subset of T_o, i.e., $B_{nkj} \in \mathcal{B}_n$ and $C_{nkj} \in \mathcal{B}_o$; then, by the definition of f and g, we have

(c) $Q_n(B_{nkj}) = Q_o(C_{nkj}) = \min(P_n(A_{kj}), P(A_{kj}))$; furthermore,

it follows from (b) that

(d) $\min(g(n,k,j), f(n,k,j)) \geq 1 - 2^{-k}$.

Let $B_{nko} := T_n \setminus \bigcup_{1 \leq j \leq J_k} B_{nkj}$ and $C_{nko} := T_o \setminus \bigcup_{1 \leq j \leq J_k} C_{nkj}$,

For $k = 0$ let $J_o := 0$, $B_{noo} := T_n$ and $C_{noo} := T_o$.

Let $n_o := 1$ and for each $n \in \mathbb{N}$, let $k(n) \in \mathbb{N} \cup \{0\}$ be the unique k such that $n_k \leq n < n_{k+1}$; then T_n is the disjoint union of sets $D_{nj} := B_{nk(n)j}$, i.e.,

(e) $T_n = \sum_{0 \leq j \leq J_{k(n)}} D_{nj}$; likewise $T_o = \sum_{0 \leq j \leq J_{k(n)}} E_{nj}$ with

$$E_{nj} := C_{nk(n)j}.$$

It follows from (c) and (d) that

(f) $Q_n(D_{nj}) = Q_o(E_{nj}) > 0$ if $j \geq 1$.

STEP 3. For each $n \in \mathbb{N}$, given any $x \in T_o$, let $j = j(n,x)$ be the j such that $x \in E_{nj}$ (cf. (e)). Let

$$A := \{x \in T_o\colon Q_o(E_{nj(n,x)}) > 0 \quad \forall n \in \mathbb{N}\}.$$

Then $\complement A = \bigcup_{n \in \mathbb{N}} \{x \in T_o\colon Q_o(E_{nj(n,x)}) = 0\} \overset{(f)}{=} \bigcup_{\{n \in \mathbb{N}, Q_o(E_{no})=0\}} E_{no} \in \mathcal{B}_o$

and $Q_o(\complement A) = 0$, whence

(g) $A \in \mathcal{B}_o$ and $Q_o(A) = 1$.

Therefore, $\tilde{Q}_o(A \cap B) := Q_o(B)$, $B \in \mathcal{B}_o$, is a well defined p-measure on $A \cap \mathcal{B}_o$.

For $x \in A$ and any $B \in \mathcal{B}_n$ let

$$P_{nx}(B) := Q_n(B \cap D_{nj(n,x)})/Q_o(E_{nj(n,x)}).$$

It follows from (f) that

(h) the P_{nx}, $x \in A$, are p-measures on $\mathcal{B}_n$, belonging to a finite set $\{P_{nj}\}$, where $P_{nj} := P_{nx}$ if $x \in A \cap E_{nj} \in A \cap \mathcal{B}_o$, $0 \leq j \leq J_{k(n)}$.

For $x \in A$ let

$\mu_x := \times_{n\in\mathbb{N}} P_{nx}$ be the product measure of the P_{nx} on the product σ-algebra

$\mathcal{B} := \bigotimes_{n\in\mathbb{N}} \mathcal{B}_n$ in the product space $T := \times_{n\in\mathbb{N}} T_n$.

Let $\hat{\mu}: A \times \mathcal{B} \to [0,1]$ be defined by $\hat{\mu}(x,B) := \mu_x(B)$ for $x \in A$ and $B \in \mathcal{B}$; then $\hat{\mu}$ is a "transition probability (or Markov kernel) from $(A, A \cap \mathcal{B}_o)$ into $(T,\mathcal{B})$", i.e.,

(i) (1) For each $x \in A$ $\hat{\mu}(x,\cdot)$ is a p-measure on $\mathcal{B}$, and

(2) For each $B \in \mathcal{B}$ $\hat{\mu}(\cdot,B): A \to I = [0,1]$ is $A \cap \mathcal{B}_o$ $\mathcal{B}(I)$-measurable.

Of course, (1) holds true here and (2) will be shown later.

Therefore (cf. Gaenssler-Stute (1977), Satz 1.8.10)

$\hat{\mathbb{P}} := \tilde{Q}_o \times \hat{\mu}$ defines a p-measure on

$\hat{F} := (A \cap \mathcal{B}_o) \otimes \mathcal{B}$ in

$\hat{\Omega} := A \times T$, where (cf. Gaenssler-Stute (1977), 1.8.7 and 1.8.9)

(j) $$\hat{\mathbb{P}}(C) = \int_A \int_T 1_C(x,y)\hat{\mu}(x,dy)\tilde{Q}_o(dx) = \int_A \hat{\mu}(x,C_x)\tilde{Q}_o(dx)$$

for $C \in \hat{F}$; note that $C_x := \{y \in T: (x,y) \in C\} \in \mathcal{B}$.

STEP 4. $(\hat{\Omega},\hat{F},\hat{\mathbb{P}})$ as obtained before being the desired p-space, let, for $n \in \mathbb{N}$, $\hat{\xi}_n: \hat{\Omega} \to S$ be the natural projection of $\hat{\Omega} = A \times [\times_{n\in\mathbb{N}} (S_n \times I_n)]$ onto $S_n = S$; then the $\hat{\xi}_n$'s are random elements in $(S,\mathcal{A})$ and

(k) $$L\{\hat{\xi}_n\} = L\{\xi_n\} \quad (\text{on } \mathcal{A}) \text{ for all } n \in \mathbb{N}.$$

In fact, for any $A' \in \mathcal{A}$, $L\{\hat{\xi}_n\}(A') = \hat{\mathbb{P}}(\hat{\xi}_n^{-1}(A'))$

$$\underset{(j)}{=} \int_A \mu_x(T_1 \times \ldots \times T_{n-1} \times (A' \times I_n) \times T_{n+1} \times \ldots)\tilde{Q}_o(dx)$$

$$= \int_A P_{nx}(A' \times I_n)\tilde{Q}_o(dx) \underset{(h)}{=} \sum_{0\le j\le J_{k(n)}} P_{nj}(A' \times I_n)\tilde{Q}_o(A \cap E_{nj})$$

$$= \sum_j P_{nj}(A' \times I_n)Q_o(E_{nj}) = \sum_j Q_n((A' \times I_n) \cap D_{nj}) \underset{(e)}{=} Q_n(A' \times I_n)$$

$$= P_n(A') = L\{\xi_n\}(A').$$

Next, let $\hat{\xi} := \Pi_S^{T_o} \circ i(A) \circ \Pi_A^{\hat{\Omega}}: \hat{\Omega} \to S$, where $\Pi_A^{\hat{\Omega}}$ is the natural projection of $\hat{\Omega} = A \times T$ onto A, i(A) is the injection of A into T_o, and $\Pi_S^{T_o}$ is the natural projection of $T_o = S \times I$ onto S; then $\hat{\xi}$ is a random element in $(S, \mathcal{B}_b(S))$ and

(ℓ) $$L\{\hat{\xi}\} = L\{\xi\} \quad (\text{on } \mathcal{B}_b(S)).$$

In fact, for any $B \in \mathcal{B}_b(S)$, $L\{\hat{\xi}\}(B) = \hat{\mathbb{P}}(\hat{\xi}^{-1}(B))$

$= \hat{\mathbb{P}}((\Pi_A^{\hat{\Omega}})^{-1} \circ (i(A))^{-1} \circ (\Pi_S^{T_o})^{-1}(B)) = \hat{\mathbb{P}}((\Pi_A^{\hat{\Omega}})^{-1} \circ (i(A))^{-1}(B \times I))$

$= \hat{\mathbb{P}}((\Pi_A^{\hat{\Omega}})^{-1}(A \cap (B \times I))) = \hat{\mathbb{P}}([A \cap (B \times I)] \times T) \underset{(j)}{=} \int_{A\cap(B\times I)} \mu_x(T)\tilde{Q}_o(dx)$

$= \tilde{Q}_o(A \cap (B \times I)) = Q_o(B \times I) = P(B) = L\{\xi\}(B)$.

Now, let $\hat{\Omega}_o := \liminf_{n\to\infty} \hat{\Omega}_{o,n}$, where

$$\hat{\Omega}_{o,n} := \sum_{1\leq j\leq J_{k(n)}} ([A \cap E_{nj}] \times T_1 \times \ldots \times T_{n-1} \times D_{nj} \times T_{n+1} \times \ldots) \in \hat{F};$$

then $\hat{\Omega}_o \in \hat{F}$ and

(m) $$\lim_{n\to\infty} d(\hat{\xi}_n(\hat{\omega}), \hat{\xi}(\hat{\omega})) = 0 \quad \text{for all } \hat{\omega} \in \hat{\Omega}_o.$$

In fact, for any $\hat{\omega} \in \hat{\Omega}_o$ there exists an $n_o \in \mathbb{N}$ such that for all $n \geq n_o$ there exists a $j(n)$, $1 \leq j(n) \leq J_{k(n)}$, such that

$\hat{\xi}_n(\hat{\omega}) \in A_{k(n)j(n)}$ and $\hat{\xi}(\hat{\omega}) \in A_{k(n)j(n)}$ (note that $\Pi_A^{\hat{\Omega}}(\hat{\omega}) \in A \cap E_{nj(n)}$ implies $i(A)(\Pi_A^{\hat{\Omega}}(\hat{\omega})) \in E_{nj(n)}$ whence $\Pi_S^{T_o}(i(A)(\Pi_A^{\hat{\Omega}}(\hat{\omega}))) \in A_{k(n)j(n)}$);

cf. the definition of the sets D_{nj} and E_{nj}, respectively.

Therefore, for all $n \geq n_o$, $d(\hat{\xi}_n(\hat{\omega}), \hat{\xi}(\hat{\omega})) \leq \operatorname{diam}(A_{k(n)j(n)}) \leq \frac{1}{k(n)} \to 0$ as $n \to \infty$ (since $k(n) \to \infty$ as $n \to \infty$).

Next we will show that $\hat{\mathbb{P}}(\hat{\Omega}_o) = 1$. For this we will prove later that

(n) $$Q_o(\limsup_{n\to\infty} E_{no}) = 0.$$

Now,

$$\complement\hat{\Omega}_{o,n} = \sum_{1\leq j\leq J_{k(n)}} ([A \cap E_{nj}] \times T_1 \times \ldots \times T_{n-1} \times \complement D_{nj} \times T_{n+1} \times \ldots)$$

$$+ ([A \cap E_{no}] \times T_1 \times \ldots \times T_n \times \ldots)$$

$$= \hat{\Omega}_{1,n} + \hat{\Omega}_{2,n}, \text{ say,}$$

where $\hat{\mathbb{P}}(\hat{\Omega}_{1,n}) = \sum_j \int_{A \cap E_{nj}} P_{nx}(\complement D_{nj})\tilde{Q}_o(dx) = \sum_j P_{nj}(\complement D_{nj})\tilde{Q}_o(A \cap E_{nj})$

$$= \sum_j \frac{Q_n((\complement D_{nj}) \cap D_{nj})}{Q_o(E_{nj})} Q_o(E_{nj}) = 0 \quad \text{for all } n \in \mathbb{N}, \text{ and therefore}$$

$$\hat{\mathbb{P}}(\limsup_{n\to\infty} \complement\hat{\Omega}_{o,n}) = \hat{\mathbb{P}}(\limsup_{n\to\infty} \hat{\Omega}_{2,n}) = \hat{\mathbb{P}}([A \cap \limsup_{n\to\infty} E_{no}] \times T)$$

$$= \tilde{Q}_o(A \cap \limsup_{n\to\infty} E_{no}) = Q_o(\limsup_{n\to\infty} E_{no}) = 0 \quad \text{by (n).}$$

It follows that $\hat{\mathbb{P}}(\hat{\Omega}_o) = \hat{\mathbb{P}}(\liminf_{n\to\infty} \hat{\Omega}_{o,n}) = 1 - \hat{\mathbb{P}}(\limsup_{n\to\infty} \complement\hat{\Omega}_{o,n}) = 1.$

It remains to show (i) (2) and (n):

<u>ad (i) (2):</u> We have to show that

(+) $\mathcal{D} := \{B \in \mathcal{B}: \hat{\mu}(\cdot,B): A \to I = [0,1], \text{ is } A \cap \mathcal{B}_o, \mathcal{B}(I)\text{-measurable}\} = \mathcal{B}.$

If $B = T_1 \times \ldots \times T_{n-1} \times B_n \times T_{n+1} \times \ldots$ for some $B_n \in \mathcal{B}_n$, then for each $t \in I$

$$\{x \in A: \hat{\mu}(x,B) \le t\} = \{x \in A: \mu_x(B) \le t\} = \{x \in A: P_{nx}(B_n) \le t\} \underset{(h)}{=} \bigcup_{\{j: P_{nj}(B_n) \le t\}} A \cap E_{nj} \in A \cap \mathcal{B}_o,$$ whence $B \in \mathcal{D}$ for all these sets.

From this and the product form of μ_x it follows that the class $\mathcal{C}$ of finite intersections of sets B just considered is also contained in $\mathcal{D}$. Since $\mathcal{C}$ is a $\cap$-closed generator of $\mathcal{B}$, we get (+) as in Gaenssler-Stute (1977), 1.8.5.

<u>ad (n):</u> It follows from (a) that $\sum_k P(S \setminus \bigcup_{1\le j\le J_k} A_{kj}) \le \sum_k 2^{-k} < \infty$,

whence, by the Borel-Cantelli lemma

$$P(\limsup_{k\to\infty}(S \setminus \bigcup_{1\le j\le J_k} A_{kj})) = 0, \text{ i.e., } P(\liminf_{k\to\infty} \bigcup_{1\le j\le J_k} A_{kj}) = 1$$

and thus

$$P(\liminf_{n\to\infty} \bigcup_{1\le j\le J_{k(n)}} A_{k(n)j}) = 1 \quad \text{as} \quad k(n) \to \infty \text{ for } n \to \infty.$$

Furthermore,

$$\lambda(\liminf_{n\to\infty} [0, \min_{1\leq j\leq J_{k(n)}} g(n,k(n),j)]) = 1, \text{ since for any } t \in I = [0,1]$$

with $t < 1$, $\min_{1\leq j\leq J_{k(n)}} g(n,k(n),j) \underset{(d)}{\geq} 1 - 2^{-k(n)} > t$ for all large enough n.

Since $\liminf_{n\to\infty} \bigcup_{1\leq j\leq J_{k(n)}} E_{nj} \supset (\liminf_{n\to\infty} \bigcup_{1\leq j\leq J_{k(n)}} A_{k(n)j}) \times$

$$\times (\liminf_{n\to\infty} [0, \min_{1\leq j\leq J_{k(n)}} g(n,k(n),j)]),$$

we thus obtain $Q_o(\liminf_{n\to\infty} \bigcup_{1\leq j\leq J_{k(n)}} E_{nj}) = 1$ which implies (n) since

$$\complement(\bigcup_{1\leq j\leq J_{k(n)}} E_{nj}) = E_{no}.$$

This concludes the proof of Theorem 12. □

Following a suggestion of Ron Pyke, let us demonstrate at this place the usefulness of the representation theorem for proving the following version of Theorem 5 (cf. Lemma 16 for the definition of the set E).

THEOREM 5'. Let $S = (S,d)$ and $S' = (S',d')$ be metric spaces, and $\mathcal{A}$ a σ-algebra of subsets of S such that $\mathcal{B}_b(S) \subset \mathcal{A} \subset \mathcal{B}(S)$. For $n \in \mathbb{N}$ let $g_n: S \to S'$ be $\mathcal{A},\mathcal{B}_b(S')$-measurable and let $g: S \to S'$ be $\mathcal{B}_b(S)$, $\mathcal{B}_b(S')$-measurable. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of random elements in $(S,\mathcal{A})$ and ξ be a random element in $(S,\mathcal{B}_b(S))$ such that $\xi_n \xrightarrow{\mathcal{L}_b} \xi$ and $\widetilde{\mathcal{L}\{\xi\}}^*(E) = 0$. Then

$g_n(\xi_n) \equiv g_n \circ \xi_n \xrightarrow{\mathcal{L}_b} g(\xi) \equiv g \circ \xi$.

(Note that $g_n(\xi_n)$ and $g(\xi)$ are random elements in $(S',\mathcal{B}_b(S'))$.)

Proof. As in the proof of Theorem 4 it is shown that $\mathcal{L}\{g(\xi)\} = \mathcal{L}\{\xi\} \circ g^{-1}$ ($= \widetilde{\mathcal{L}\{\xi\}} \circ g^{-1}$) is separable. Now according to Theorem 12, there exists a p-space $(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}})$ and on it random elements $\hat{\xi}_n$ in $(S,\mathcal{A})$ and a random element $\hat{\xi}$ in $(S,\mathcal{B}_b(S))$ such that

$L\{\hat{\xi}_n\} = L\{\xi_n\}$ (on A) for all $n \in \mathbb{N}$, $L\{\hat{\xi}\} = L\{\xi\}$ (on $\mathcal{B}_b(S)$),

and

$\hat{\xi}_n(\hat{\omega}) \to \hat{\xi}(\hat{\omega})$ (as $n \to \infty$) for all $\hat{\omega} \in \hat{\Omega}_o$, where $\hat{\Omega}_o \in \hat{F}$ with $\hat{\mathbb{P}}(\hat{\Omega}_o) = 1$.

Let $\hat{\Omega}_1 := \{\hat{\xi} \in \complement E\}$ and $\hat{\Omega}_2 := \hat{\Omega}_o \cap \hat{\Omega}_1$; then for all $\hat{\omega} \in \hat{\Omega}_2$,

$g_n(\hat{\xi}_n(\hat{\omega})) \to g(\hat{\xi}(\hat{\omega}))$ (as $n \to \infty$). Since $\hat{\mathbb{P}}(\hat{\Omega}_o) = 1$ and $\hat{\mathbb{P}}_*(\hat{\Omega}_1) = 1$ (note that $\widetilde{L\{\hat{\xi}\}}_* (\complement E) = \widetilde{L\{\xi\}}_* (\complement E) = 1$) we have $\hat{\mathbb{P}}_*(\hat{\Omega}_2) = 1$, whence there exists $\hat{\Omega}_3 \in \hat{F}$ such that $\hat{\Omega}_3 \subset \hat{\Omega}_2$ and $\hat{\mathbb{P}}(\hat{\Omega}_3) = 1$. It follows that for $\hat{\omega} \in \hat{\Omega}_3$ and each $f \in U_b^b(S')$

$$f \circ g_n \circ \hat{\xi}_n(\hat{\omega}) \to f \circ g \circ \hat{\xi}(\hat{\omega}) \quad (\text{as } n \to \infty)$$

whence, by Lebesgue's theorem,

$\hat{\mathbb{E}}(f \circ g_n \circ \hat{\xi}_n) \to \hat{\mathbb{E}}(f \circ g \circ \hat{\xi})$, i.e., $\int f dL\{g_n(\hat{\xi}_n)\} \to \int f dL\{g(\hat{\xi})\}$ (as $n \to \infty$).

Since $L\{g_n(\hat{\xi}_n)\} = L\{\hat{\xi}_n\} \circ g_n^{-1} = L\{\xi_n\} \circ g_n^{-1} = L\{g_n(\xi_n)\}$

and $L\{g(\hat{\xi})\} = L\{\hat{\xi}\} \circ g^{-1} = L\{\xi\} \circ g^{-1} = L\{g(\xi)\}$, the assertion follows by (28) (cf. (h') there). □

Next, we want to make some specific remarks concerning the special case $S = D[0,1]$ reviewing at the same time some of the key results from Billingsley's (1968) book (cf. Appendix A in G. Shorack (1979)).

THE SPACE D[0,1]:

Let $D \equiv D[0,1]$ be the space of all right continuous functions on the unit interval $[0,1]$ that have left hand limits at all points $t \in (0,1]$. Cf. P. Billingsley (1968), Lemma 1, p. 110, and its consequences concerning specific properties of functions $x \in D$; among others, $\sup_{t\in[0,1]} |x(t)| < \infty$ for all $x \in D$. If not stated otherwise, the space D will be equipped with the supremum metric ρ, i.e.

$$\rho(x,y) := \sup_{t\in[0,1]} |x(t) - y(t)| \quad \text{for } x,y \in D.$$

(By the way, (D,ρ) is a linear topological space whereas (D,s), with s being the Skorokhod metric, is not (cf. P. Billingsley (1968), p. 123, 3.)). Note that $(S,d) = (D,\rho)$ is a non-separable metric space (in fact, look at $x_s := 1_{[s,1]}$, $s \in (0,1)$, to obtain an uncountable set of functions in D for which $\rho(x_s,x_{s'}) = 1$ for $s \neq s'$).

Also, as pointed out by D.M. Chibisov (1965), (cf. P. Billingsley (1968), Section 18), the empirical df U_n (based on independent random variables (on some p-space $(\Omega,\mathcal{F},\mathbb{P})$) being uniformly distributed on [0,1]) cannot be considered as a random element in $(D,\mathcal{B}(D,\rho))$ (i.e. $U_n: \Omega \to D$ is not $\mathcal{F},\mathcal{B}(D,\rho)$-measurable), where $\mathcal{B}(D,\rho)$ denotes the Borel σ-algebra in (D,ρ). But, considering instead the smaller σ-algebra $\mathcal{B}_b(D) \equiv \mathcal{B}_b(D,\rho)$ generated by the open (ρ-) balls we have

(37) $\mathcal{B}_b(D) = \sigma(\{\pi_t: t \in [0,1]\})$,

where $\sigma(\{\pi_t: t \in [0,1]\})$ denotes the σ-algebra generated by the coordinate projections $\pi_t \equiv \pi_t(D)$ from D onto $\mathbb{R}$, defined by $\pi_t(x) := x(t)$ for $x \in D$.

(Note that (37) implies that U_n is $\mathcal{F},\mathcal{B}_b(D)$-measurable since $\mathcal{F}$, $\sigma(\{\pi_t: t \in [0,1]\})$-measurability of U_n is equivalent with $\mathcal{F},\mathcal{B}$-measurability of $\pi_t(U_n) = U_n(t)$ for each fixed $t \in [0,1]$ where the latter is satisfied since $U_n(t)$ is a random variable.)

Proof of (37). Let $T := \mathbb{Q} \cap [0,1]$ be the set of rational numbers in [0,1]; then, by the right continuity of each $x \in D$ one has

(a) $\rho(x_1,x_2) = \sup_{t\in T} |x_1(t) - x_2(t)|$ for every $x_1,x_2 \in D$.

Therefore, for any $x_o \in D$ and any $r > 0$

$$\{x \in D: \rho(x,x_o) \leq r\} = \bigcap_{t\in T} \{x \in D: |x(t) - x_o(t)| \leq r\}$$

$$= \bigcap_{t\in T} \pi_t^{-1}([x_o(t) - r, x_o(t) + r]) \in \sigma(\{\pi_t: t \in [0,1]\}); \text{ thus}$$

$\mathcal{B}_b(D) \subset \sigma(\{\pi_t: t \in [0,1]\})$; (note that $\{x \in D: \rho(x,x_o) < r\}$

$$= \bigcup_{m\in\mathbb{N}} \{x \in D: \rho(x,x_o) \leq r - \frac{1}{m}\}).$$

To verify the other inclusion it suffices to show that for every fixed $t \in [0,1]$ and $r \in \mathbb{R}$ one has

(b) $$\{x \in D\colon \pi_t(x) < r\} \in \mathcal{B}_b(D).$$

For this we define, given the fixed t and r, for any $n, k \in \mathbb{N}$ and $s \in [0,1]$

$$x_n^k(s) := \begin{cases} 0, \text{ if } s < t \\ r - \frac{1}{k} - n, \text{ if } s \in [t, t + \frac{1}{k}) \cap [0,1] \\ 0, \text{ if } s \in [t + \frac{1}{k}, \infty) \cap [0,1]. \end{cases}$$

Then $x_n^k \in D$ and it follows that

(c) $$\{x \in D\colon \pi_t(x) < r\} = \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \{x \in D\colon \rho(x, x_n^k) \leq n\},$$

which proves (b).

As to (c), let $x_o \in \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \{x \in D\colon \rho(x,x_n^k) \leq n\}$; then

$\rho(x_o, x_n^k) \leq n$ for some n and k, whence

$$n \geq |x_o(t) - x_n^k(t)| = |x_o(t) - r + \frac{1}{k} + n|$$

$$\geq x_o(t) - r + \frac{1}{k} + n, \text{ and therefore}$$

$$x_o(t) \leq r - \frac{1}{k} < r, \text{ i.e. } \pi_t(x_o) < r.$$

On the other hand, if $x_o(t) < r$, $x_o \in D$, choose $n_o \in \mathbb{N}$ such that $\sup_{t\in[0,1]} |x_o(t)| \leq \min(n_o, n_o - r)$; then it can be easily shown that

$$x_o \in \bigcup_{k\in\mathbb{N}} \{x \in D\colon \rho(x, x_{n_o}^k) \leq n_o\},$$

which proves (c). □

<u>(38) REMARK.</u> Comparing (37) with the known result that the Borel σ-algebra $\mathcal{B}(D,s)$ in (D,s), equipped with the Skorokhod metric s (cf. P. Billingsley (1968), Chapter 3), coincides also with $\sigma(\{\pi_t\colon t \in [0,1]\})$, we obtain that

$$\mathcal{B}(D,s) = \mathcal{B}_b(D,\rho).$$

It is also known, that for any sequence $(x_n)_{n\in\mathbb{N}} \subset D$ and $x \in D$, $\lim_{n\to\infty} \rho(x_n, x) = 0$ always implies $\lim_{n\to\infty} s(x_n,x) = 0$;

on the other hand, if $\lim_{n\to\infty} s(x_n,x) = 0$ for some continuous x, then $\lim_{n\to\infty} \rho(x_n,x) = 0$ (hence the Skorokhod topology relativized to the space of all continuous functions on $[0,1]$ coincides with the uniform topology there).

Let $C \equiv C[0,1]$ be the space of all continuous functions on $[0,1]$ and consider again the supremum metric ρ on C.

Then $(S_o,d) = (C,\rho)$ is a separable metric space being here a closed subspace of (D,ρ), i.e., we have $S_o^c = S_o$ in the present situation. (Note that $x \in C$ is even uniformly continuous.)

Therefore, denoting by $\mathcal{B}(C,\rho)$ the Borel σ-algebra in (C,ρ) we have (cf. Lemma 11)

$$C \cap \mathcal{B}(D,\rho) = \mathcal{B}(C,\rho) = \mathcal{B}_b(C,\rho) \subset C \cap \mathcal{B}_b(D,\rho) \subset C \cap \mathcal{B}(D,\rho),$$
$$\text{and} \quad C \in \mathcal{B}_b(D,\rho), \text{ i.e.,}$$

$$\mathcal{B}(C,\rho) = \mathcal{B}_b(C,\rho) = C \cap \mathcal{B}_b(D,\rho) \quad \text{and} \quad C \in \mathcal{B}_b(D,\rho). \tag{39}$$

In what follows let ξ_n, $n \in \mathbb{N}$, and ξ be random elements in $(D,\mathcal{B}_b(D,\rho))$ which are all defined on a common p-space $(\Omega,\mathcal{F},\mathbb{P})$. Following (34) we write $\xi_n \xrightarrow{\mathcal{L}_b} \xi$ iff $\mathcal{L}\{\xi_n\} \underset{b}{\rightarrow} \mathcal{L}\{\xi\}$ in which case (by our definition of $\underset{b}{\rightarrow}$ - convergence) $\mathcal{L}\{\xi\}$ is assumed to be separable.

On the other hand, in view of (38), $\mathcal{L}\{\xi_n\}$ and $\mathcal{L}\{\xi\}$ may also be considered as Borel measures on $\mathcal{B}(D,s)$, whence the usual concept of weak convergence of Borel measures can also be used, which means that

$\xi_n \xrightarrow{\mathcal{L}} \xi$ iff, by definition, $\mathcal{L}\{\xi_n\}$ on $\mathcal{B}(D,s)$ converges weakly to $\mathcal{L}\{\xi\}$ on $\mathcal{B}(D,s)$ in the sense of Billingsley (1968).

LEMMA 18. If $\xi_n \xrightarrow{\mathcal{L}_b} \xi$, then $\xi_n \xrightarrow{\mathcal{L}} \xi$; on the other hand, if $\xi_n \xrightarrow{\mathcal{L}} \xi$ and $\mathcal{L}\{\xi\}(C) = 1$, then $\xi_n \xrightarrow{\mathcal{L}_b} \xi$.

Proof. Note first that (D,s) is a separable metric space whence we can use (28) with $\mathcal{B}_b(D,s) = \mathcal{A} = \mathcal{B}(D,s)$, which gives us

(+) $\xi_n \xrightarrow{L} \xi \Leftrightarrow \lim_{n\to\infty} \mathbb{E}(f(\xi_n)) = \mathbb{E}(f(\xi))$ for all bounded $\mathcal{B}(D,s)$, $\mathcal{B}$-measurable functions $f: D \to \mathbb{R}$ which are $L\{\xi\}$-a.e. continuous.

1.) Consider an $f: D \to \mathbb{R}$; then:
if f is s-continuous, it is also ρ-continuous and (cf. (38)) $\mathcal{B}_b(D,\rho)$, $\mathcal{B}$-measurable. Therefore
$\xi_n \xrightarrow{L_b} \xi$ implies that $\lim_{n\to\infty} \mathbb{E}(f(\xi_n)) = \mathbb{E}(f(\xi))$ for all bounded s-continuous $f: D \to \mathbb{R}$, whence $\xi_n \xrightarrow{L} \xi$.

2.) $\xi_n \xrightarrow{L} \xi$ implies, according to (+), that $\lim_{n\to\infty} \mathbb{E}(f(\xi_n)) = \mathbb{E}(f(\xi))$ for all bounded $\mathcal{B}(D,s)$, $\mathcal{B}$-measurable $f: D \to \mathbb{R}$ which are $L\{\xi\}$-a.e. continuous. Since $L\{\xi\}(C) = 1$ implies (cf. (38)) that any ρ-continuous f is also $L\{\xi\}$-a.e. s-continuous, we obtain, using again that $\mathcal{B}(D,s) = \mathcal{B}_b(D,\rho)$, that $\lim_{n\to\infty} \mathbb{E}(f(\xi_n)) = \mathbb{E}(f(\xi))$ for all bounded, ρ-continuous, and $\mathcal{B}_b(D,\rho)$, $\mathcal{B}$-measurable $f: D \to \mathbb{R}$; furthermore, since $C = (C,\rho)$ is a closed separable subspace of (D,ρ) with $L\{\xi\}(C) = 1$, we finally obtain (cf. (28)(h')) that
$\xi_n \xrightarrow{L_b} \xi$. □

Now we are going on in reviewing here some of the key results of Billingsley's (1968) book. The following lemma is well known (cf. Yu.V. Prohorov (1956)):

LEMMA 19. Let $F: [0,1] \to \mathbb{R}$ be a continuous function and $a>1$, $b>0$ be constants such that for some random element ξ in

$$(\mathbb{R}^{[0,1]}, \mathcal{B}_{[0,1]}) \quad (\mathcal{B}_{[0,1]} := \bigotimes_{t\in[0,1]} \mathcal{B}_t \text{ with } \mathcal{B}_t \equiv \mathcal{B})$$

(40) $\mathbb{E}(|\xi(t) - \xi(s)|^b) \leq |F(t) - F(s)|^a$ for all $0 \leq s \leq t \leq 1$;

then there exists a random element $\hat{\xi}$ in $(D,\mathcal{B}_b(D,\rho))$ such that $L\{\hat{\xi}\}|\mathcal{B}_{[0,1]} = L\{\xi\}$ and $(L\{\hat{\xi}\}|\mathcal{B}_b(D,\rho))(C) = 1$. (Note that $D \cap \mathcal{B}_{[0,1]} = \mathcal{B}_b(D,\rho)$ (cf. (37)) and

$C \in \mathcal{B}_b(D,\rho)$ (cf. (39)).)

In what follows we shall write $\xi_n \xrightarrow[\text{f.d.}]{L} \xi$, if the finite dimensional distributions (fidis) of ξ_n converge weakly to the corresponding fidis of ξ. (Recall that, given a r.e. ξ in $(D,\mathcal{B}_b(D,\rho))$, the <u>fidis</u> of ξ (or $L\{\xi\}$, respectively) are defined as the image measures that $\pi_{t_1,\ldots,t_k}(D)\colon D \to \mathbb{R}^k$ induce on $\mathbb{B}_k$ from $L\{\xi\}$ on $\mathcal{B}_b(D,\rho)$ $(= \sigma(\{\pi_t(D)\colon t \in [0,1]\}))$ for each fixed $t_1,\ldots,t_k \in [0,1]$, $k \geq 1$, where $\pi_{t_1,\ldots,t_k}(D)\,(x) := (x(t_1),\ldots,x(t_k))$ for $x \in D$; note that $\pi_{t_1,\ldots,t_k}$ is $\mathcal{B}_b(D,\rho)$, $\mathbb{B}_k$-measurable.)

<u>DEFINITION 6.</u> Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of random elements in $(D,\mathcal{B}_b(D,\rho)) = (D,\mathcal{B}(D,s))$;

(i) (ξ_n) is said to be <u>relatively L-sequentially compact</u>, iff for any subsequence $(L\{\xi_{n'}\})$ of $(L\{\xi_n\})$ there exists a further subsequence $(L\{\xi_{n''}\})$ of $(L\{\xi_{n'}\})$ and a p-measure μ on $\mathcal{B}(D,s)$ such that $L\{\xi_{n''}\}$ converges weakly to μ in the sense of Billingsley (1968).

(ii) (ξ_n) is said to be <u>relatively L_b-sequentially compact</u>, iff for any subsequence $(L\{\xi_{n'}\})$ of $(L\{\xi_n\})$ there exists a further subsequence $(L\{\xi_{n''}\})$ of $(L\{\xi_{n'}\})$ and a separable p-measure μ on $\mathcal{B}_b(D,\rho)$ (in (D,ρ)!) such that $L\{\xi_{n''}\} \xrightarrow[b]{} \mu$.

The following theorem is well known (cf. P. Billingsley (1968), Th. 15.1).

<u>THEOREM 13.</u> Let (ξ_n) be relatively L-sequentially compact and suppose that $\xi_n \xrightarrow[\text{f.d.}]{L} \xi$; then $\xi_n \xrightarrow{L} \xi$.

The next theorem gives sufficient conditions for (ξ_n) to be relatively L-sequentially compact.

For this, given any $x \in D$ and $B \in [0,1] \cap \mathbb{B}$, let

$$||x|| := \sup_{t\in[0,1]} |x(t)|,$$

and

$$w_x(B) := \sup_{s,t\in B} |x(t) - x(s)|.$$

THEOREM 14. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of random elements in $(D,\mathcal{B}_b(D,\rho))$, all defined on a common p-space $(\Omega,\mathcal{F},\mathbb{P})$, and satisfying the following set of conditions (A)-(D):

(A): $$\limsup_{n\to\infty} \mathbb{P}(\|\xi_n\| > m) \to 0 \text{ as } m \to \infty.$$

(B): For every $\varepsilon>0$, $\limsup_{n\to\infty} \mathbb{P}(w_{\xi_n}([0,\delta)) \geq \varepsilon) \to 0$ as $\delta \to 0$

and $\limsup_{n\to\infty} \mathbb{P}(w_{\xi_n}([\delta,1)) \geq \varepsilon) \to 0$ as $\delta \to 1$.

(C): There exist constants $a>1$, $b>0$ and, for every $n \in \mathbb{N}$ there exist monotone increasing functions F_n: $[0,1] \to \mathbb{R}$ such that for every $\varepsilon>0$ and any $0 \leq r \leq s \leq t \leq 1$

$$\mathbb{P}(|\xi_n(s) - \xi_n(r)| \geq \varepsilon,\ |\xi_n(t) - \xi_n(s)| \geq \varepsilon) \leq \varepsilon^{-b}(F_n(t) - F_n(r))^a.$$

(D): There exists a monotone increasing and continuous function F: $[0,1] \to \mathbb{R}$ such that for the F_n's occurring in (C) and any $0 \leq s \leq t \leq 1$

$$\limsup_{n\to\infty} (F_n(t) - F_n(s)) \leq F(t) - F(s).$$

Then (ξ_n) is relatively $\mathcal{L}$-sequentially compact.

(41) REMARK. Given any $x \in D$ and $\delta>0$, let

$$w''_x(\delta) := \sup_{\substack{0\leq r\leq s\leq t\leq 1\\ t-r\leq\delta}} \min\{|x(s) - x(r)|,\ |x(t) - x(s)|\}.$$

Then (C) and (D) together imply

(C''): For every $\varepsilon>0$, $\limsup_{n\to\infty} \mathbb{P}(w''_{\xi_n}(\delta) \geq \varepsilon) \to 0$ as $\delta \to 0$.

As to Theorem 14, it is shown in Billingsley (1968), Theorem 15.3 that (A), (B) and (C'') together imply the assertion of Theorem 14.

So we will prove here only the statement made in (41).

For notational convenience we shall write $\xi_n(s,t]$ instead of $\xi_n(t) - \xi_n(s)$

for $0 \leqq s \leqq t \leqq 1$.

a) Given an arbitrary $\varepsilon>0$, $t \in [0,1)$ and $\delta \leqq 1 - t$, it follows from Theorem 12.5 in Billingsley (1968) together with C that for every $n \in \mathbb{N}$ and every $m \in \mathbb{N}$

$$\mathbb{P}\Big(\bigcup_{0\leqq i\leqq j\leqq k\leqq 2^m}\{\min\,[\,|\xi_n(t + \tfrac{i}{2^m}\,\delta, t + \tfrac{j}{2^m}\,\delta]\,|,\ |\xi_n(t + \tfrac{j}{2^m}\,\delta, t + \tfrac{k}{2^m}\,\delta]\,|\,] \geqq \varepsilon\}\Big)$$

$\leqq K(a,b)\cdot\varepsilon^{-b}(F_n(t + \delta) - F_n(t))^a$, where $K(a,b)$ is a constant depending only on a and b.

Therefore, due to the right-continuity of the sample paths of ξ_n, putting

$$w''_x([t,t + \delta]) := \sup_{t\leqq r\leqq s\leqq t'\leqq t+\delta} \min\,\{|x(r,s]|,|x(s,t']|\}$$

for $x \in D$, $\delta>0$ and $t \leqq 1 - \delta$,

it follows that $\mathbb{P}(w''_{\xi_n}([t,t + \delta]) \geqq \varepsilon)$

$$\leqq K(a,b)\cdot\varepsilon^{-b}(F_n(t + \delta) - F_n(t))^a.$$

b) Let, for any $\delta>0$, $m = m(\delta) := [\frac{1}{2\delta}]$ (where $[x]$ stands for the integer part of x); then, for every $n \in \mathbb{N}$,

$$\mathbb{P}(w''_{\xi_n}(\delta) \geqq \varepsilon) \leqq \sum_{i=0}^{m-1} \mathbb{P}(w''_{\xi_n}([\tfrac{i}{m},\tfrac{i+1}{m}]) \geqq \varepsilon) + \sum_{i=0}^{m-2} \mathbb{P}(w''_{\xi_n}([\tfrac{2i+1}{2m}, \tfrac{2i+3}{2m}]) \geqq \varepsilon)$$

$$\underset{a)}{\leqq} K(a,b)\cdot\varepsilon^{-b}\Big[\sum_{i=0}^{m-1}(F_n(\tfrac{i+1}{m}) - F_n(\tfrac{i}{m}))^a + \sum_{i=0}^{m-2}(F_n(\tfrac{2i+3}{2m}) - F_n(\tfrac{2i+1}{2m}))^a\Big],$$

which implies by D that

$$\limsup_{n\to\infty} \mathbb{P}(w''_{\xi_n}(\delta) \geqq \varepsilon) \leqq K(a,b)\cdot\varepsilon^{-b}\cdot 2(w_F(\tfrac{1}{m}))^{a-1}\cdot(F(1) - F(0)),$$

where $w_F(\frac{1}{m}) := \sup\,\{F(t) - F(s)\colon s \leqq t,\ t - s \leqq \frac{1}{m}\}$

$= w_F(\frac{1}{m(\delta)}) \to 0$ as $\delta \to 0$ (since F is uniformly continuous).

This proves C" . □

(42) <u>REMARK.</u> Let us consider in Theorem 14 instead of B and C the following conditions B' and C' , respectively:

B': For every $\varepsilon>0$, $\limsup_{n\to\infty} \mathbb{P}(|\xi_n(\delta) - \xi_n(0)| \geq \varepsilon) \to 0$ as $\delta \to 0$

and $\limsup_{n\to\infty} \mathbb{P}(|\xi_n(1) - \xi_n(\delta)| \geq \varepsilon) \to 0$ as $\delta \to 1$;

C': There exist constants $a_i, b_i > 0$, $i=1,2$, such that $a_1 + a_2 > 1$ and, for every $n \in \mathbb{N}$ there exist monotone increasing functions $F_n:[0,1] \to \mathbb{R}$ such that for any $0 \leq r \leq s \leq t \leq 1$

$$\mathbb{E}(|\xi_n(s) - \xi_n(r)|^{b_1} \cdot |\xi_n(t) - \xi_n(s)|^{b_2}) \leq (F_n(s) - F_n(r))^{a_1} \cdot (F_n(t) - F_n(s))^{a_2};$$

then C" together with B' imply B, and C' implies C.

THEOREM 15. Let ξ_n, $n \in \mathbb{N}$, and ξ be random elements in $(D,\mathcal{B}_b(D,\rho))$, all defined on a common p-space $(\Omega,\mathcal{F},\mathbb{P})$, and suppose that C (or C') and D together with the following conditions E and F are fulfilled:

E: $\quad \xi_n \xrightarrow[f.d.]{\mathcal{L}} \xi$;

F: $\quad \mathcal{L}\{\xi\}(\{x \in D: x(1) \neq x(1-0)\}) = 0$;

then $\xi_n \xrightarrow{\mathcal{L}} \xi$.

Proof. As remarked in (42), C' implies C which together with D implies C" according to (41). But E together with F and C" imply the assertion according to Theorem 15.4 in Billingsley (1968) (cf. also Gaenssler-Stute (1977), Satz 8.5.6.). □

In view of Lemma 18 we thus obtain the following $\mathcal{L}_b$-convergence theorem:

THEOREM 16. Let ξ_n, $n \in \mathbb{N}$, and ξ be random elements in $(D,\mathcal{B}_b(D,\rho))$, all defined on a common p-space $(\Omega,\mathcal{F},\mathbb{P})$, and suppose that $\mathcal{L}\{\xi\}(C) = 1$. Then C (or C') together with D and E imply $\xi_n \xrightarrow{\mathcal{L}_b} \xi$.

The following result is used in G. Shorack's (1979) paper concerning ξ_n's of a special nature.

THEOREM 17. Let, for every $n \in \mathbb{N}$, $T_n := \{t_o^n, t_1^n, \ldots, t_{m_n}^n\}$ be such that $0 = t_o^n \leq t_1^n \leq \ldots \leq t_{m_n}^n = 1$. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of random elements in $(D, \mathcal{B}_b(D,\rho))$ such that for all n and $i \leq m_n$ ξ_n is constant on $[t_{i-1}^n, t_i^n)$, i.e.,

$$\omega_{\xi_n}([t_{i-1}^n, t_i^n)) = 0 \quad \text{a.s.}$$

Furthermore, assume that the following conditions (i) - (iii) are fulfilled:

(i) $\max\limits_{i\leq m_n} (t_i^n - t_{i-1}^n) \to 0$ as $n \to \infty$;

(ii) There exists a sequence $(F_n)_{n\in\mathbb{N}}$ of monotone increasing functions F_n: $[0,1] \to \mathbb{R}$ such that for some $a>1$ and $b>0$

$$\mathbb{P}(|\xi_n(s) - \xi_n(r)| \geq \varepsilon,\ |\xi_n(t) - \xi_n(s)| \geq \varepsilon) \leq \varepsilon^{-b}(F_n(t) - F_n(r))^a$$

for every $\varepsilon>0$ and any set $\{r,s,t\} \subset T_n$ with $r \leq s \leq t$;

(iii) There exists a monotone increasing and continuous function F: $[0,1] \to \mathbb{R}$ such that for the F_n's occurring in (ii)

either (a) $F_n(t) - F_n(s) \leq F(t) - F(s)$ for every n and any $0 \leq s \leq t \leq 1$

or (b) $F_n(t) \to F(t)$ as $n \to \infty$ for every $t \in [0,1]$.

Then $(\xi_n)_{n\in\mathbb{N}}$ satisfies (C) and (D).

Proof. Let, for each $n \in \mathbb{N}$, φ_n: $[0,1] \to T_n$ be defined by

$$\varphi_n(t) := \max\{r \leq t: r \in T_n\},\ t \in [0,1].$$

Then, according to (i), $\lim\limits_{n\to\infty} \varphi_n(t) = t$ for every $t \in [0,1]$.

Now, put $F_n' := F_n \circ \varphi_n$, $n \in \mathbb{N}$, to get a sequence of monotone increasing functions on $[0,1]$; we are going to show that (C) and (D) are satisfied with F_n' (instead of F_n there):

As to (C), by the assumed nature of the ξ_n's, we have for any $0 \leq t_1 \leq t_2 \leq 1$

$$\xi_n(t_2) - \xi_n(t_1) = \xi_n(\varphi_n(t_2)) - \xi_n(\varphi_n(t_1)),$$

which implies by (ii) that for every $\varepsilon>0$ and any $0 \leq r \leq s \leq t \leq 1$

$$\mathbb{P}(|\xi_n(s) - \xi_n(r)| \geq \varepsilon,\ |\xi_n(t) - \xi_n(s)| \geq \varepsilon) \leq \varepsilon^{-b}(F_n'(t) - F_n'(r))^a,$$

which proves C.

As to D, we have to show that for any $0 \leq s \leq t \leq 1$

$$(+) \quad \limsup_{n\to\infty} (F_n'(t) - F_n'(s)) \leq F(t) - F(s).$$

But this follows easily from (iii); in fact, (iii) a implies that for any $0 \leq s \leq t \leq 1$ $\quad F_n'(t) - F_n'(s) = F_n(\varphi_n(t)) - F_n(\varphi_n(s))$

$\leq F(\varphi_n(t)) - F(\varphi_n(s)) \to F(t) - F(s)$ as $n \to \infty$, which implies (+).

On the other hand, (iii) b implies by the Polya-Cantelli theorem that

$$\sup_{t\in[0,1]} |F_n(t) - F(t)| \to 0 \text{ as } n \to \infty \text{ and therefore,}$$

for any $t \in [0,1]$, $\quad |F_n'(t) - F(t)| \leq |F(t) - F(\varphi_n(t))|$

$+ |F(\varphi_n(t)) - F_n(\varphi_n(t))| \to 0$ as $n \to \infty$, which implies (+). □

This concludes our short review of some of the key results in Billingsley's (1968) book to be used in Section 4 when proving functional central limit theorems for weighted empirical processes along the lines of Shorack's (1979) paper; concerning the L_b-statements there (cf. Theorem 18 and 19 in Section 4) it is possible to modify the above mentioned criteria in Billingsley's book in such a way that they allow for proofs working totally within the theory of L_b-convergence (cf. Remark (73)(b) in Section 4) as it will be the case for the following example concerning Donsker's functional central limit theorem for the uniform empirical process $\alpha_n \equiv (\alpha_n(t))_{t\in[0,1]}$, defined by

$$\alpha_n(t) := n^{1/2}(U_n(t) - t), \; t \in [0,1],$$

where U_n is the empirical distribution function based on independent random variables having uniform distribution on $[0,1]$.

According to (37), α_n can be considered as a random element in $(D,\mathcal{B}_b(D,\rho))$ as well as in $(D,\mathcal{B}(D,s))$ (cf. (38)) and it follows from the multidimensional Central Limit Theorem that

(43) $\alpha_n \xrightarrow[\text{f.d.}]{L} B^o$,

where $B^o \equiv (B^o(t))_{t\in[0,1]}$ is the Brownian bridge.

As to B^o, having all its sample paths in the separable and closed subspace $C = (C,\rho)$ of $D = (D,\rho)$, it follows from (39) that $L\{B^o\}$, being originally defined on $\mathcal{B}(C,\rho)$, may be considered as well on $\mathcal{B}_b(D,\rho)$ having the additional property that $L\{B^o\}(C) = 1$. Therefore, B^o may be considered as a random element in $(D,\mathcal{B}_b(D,\rho))$, too, with $L\{B^o\}$ being concentrated on C, whence by Lemma 18 one has

(44) (i) $\alpha_n \xrightarrow{L} B^o$ iff (ii) $\alpha_n \xrightarrow{L_b} B^o$.

It was conjectured by J.L. Doob (1949) and shown by M.D. Donsker (1952) that (44)(i) holds true. There are various ways of proving this result which is known as <u>Donsker's functional central limit theorem for the uniform empirical process:</u>

One may e.g. use Theorem 15 by showing that the hypotheses C and D are fulfilled (cf. Gaenssler-Stute (1977), Lemma 10.2.2) or one may apply Theorem 15.5 in Billingsley's (1968) book; as to the latter one has to show that

(45) For each positive ε and η there exist a δ, $0 < \delta < 1$, and an integer n_o such that for all $n \geq n_o$

$$\mathbb{P}(w_{\alpha_n}(\delta) > \varepsilon) < \eta,$$

where $w_x(\delta) := \sup_{\substack{|t-s|<\delta \\ t,s\in[0,1]}} |x(t) - x(s)|$ for $x \in D$.

(By the way, it follows from Theorem 15.5 in Billingsley (1968) together with Lemma 18 that (45) is a sufficient condition for $(\alpha_n)_{n\in\mathbb{N}}$ to be relatively L_b-sequentially compact.)

As to (45), this can be shown either by using Donker's invariance principle for partial sum processes (in case of independent exponential random variables) (cf. L. Breiman (1968), problem 9, p. 296) or by more direct computations using the structural properties of empirical measures as presented in Section 1 (cf. W. Stute (1982)) yielding at the same time an independent procf of (44)(ii) within the theory of L_b-convergence in (D,ρ): in fact, it can be shown (cf. Proposition B_2 in Section 4) that (45) implies δ-tightness of $(L\{\alpha_n\})_{n\in\mathbb{N}}$ w.r.t. $S_o = C[0,1]$, and therefore Theorem 11* together with an application of Theorem 3 yields (44)(ii) in view of (43). This also indicates the way to prove Functional Central Limit Theorems for more general empirical processes (empirical C-processes indexed by classes C of sets) in the setting of L_b-convergence of random elements in appropriately chosen metric spaces.

Before doing this in the next section we want to supplement the present one by some remarks on random change of time (cf. Billingsley (1968), Chapter 3,17.).

RANDOM CHANGE OF TIME:

Following Billingsley (1968) we will briefly indicate here that so-called random change of time arguments are valid also within the context of L_b-convergence (even with simplified proofs not relying on Skorokhod's topology); in this connection the reader should remind our remarks on product spaces.

For this, let D_o consist of those elements φ of $D \equiv D[0,1]$ that are increasing and satisfy $0 \leqq \varphi(t) \leqq 1$ for all t. Such a φ represents a transformation of the time interval $[0,1]$.
We topologize D_o by relativizing the uniform topology of D.
Then (37) implies that $D_o \in \mathcal{B}_b(D)$ and therefore

$$\mathcal{B}_b(D_o) \subset A_o := D_o \cap \mathcal{B}_b(D) = \{B \subset D_o: B \in \mathcal{B}_b(D)\} \subset \mathcal{B}(D_o).$$

For $x \in D$ and $\varphi \in D_o$, let

$$x \circ \varphi: [0,1] \to \mathbb{R}$$

be defined by $(x \circ \varphi)(t) := x(\varphi(t))$, $t \in [0,1]$. Then $x \circ \varphi$ lies in D and, if

$$\psi: D \times D_o \to D$$

is defined by $\psi(x,\varphi) := x \circ \varphi$, then ψ is $\mathcal{B}_b(D) \otimes \mathcal{A}_o$, $\mathcal{B}_b(D)$-measurable, i.e. one has

$$(+) \quad \psi^{-1}(\mathcal{B}_b(D)) \subset \mathcal{A} := \mathcal{B}_b(D) \otimes \mathcal{A}_o$$

where $\mathcal{A}$ is a σ-algebra in the product space $S = D \times D_o$ (being equipped with the maximum metric d (cf. our remarks on product spaces)) such that

$$\mathcal{B}_b(S) \subset \mathcal{A} \subset \mathcal{B}(S).$$

ad (+): cf. Billingsley (1968) p. 232 for a proof being based on the fact that $\mathcal{B}_b(D) = \sigma(\{\pi_t : t \in [0,1]\})$ by (37). □

Now, let ξ_n, $n \in \mathbb{N}$, and ξ be random elements in $(D,\mathcal{B}_b(D))$ and, in addition, let η_n, $n \in \mathbb{N}$, and η be random elements in $(D_o,\mathcal{A}_o)$ all defined on a common p-space $(\Omega,\mathcal{F},\mathbb{P})$.

Then (ξ_n,η_n), $n \in \mathbb{N}$, and (ξ,η) are random elements in

$$(S,\mathcal{A}) = (D \times D_o,\ \mathcal{B}_b(D) \otimes \mathcal{A}_o)$$

and so, by (+),

$\xi_n \circ \eta_n = \psi(\xi_n,\eta_n)$, $n \in \mathbb{N}$, and $\xi \circ \eta = \psi(\xi,\eta)$ are random elements in $(D,\mathcal{B}_b(D))$ resulting from subjecting ξ_n and ξ to the random change of time represented by η_n and η, respectively.

Concerning a "$(\xi_n,\eta_n) \xrightarrow{\mathcal{L}_b} (\xi,\eta)$"-statement, (ξ,η) may be considered as a random element in $(S,\mathcal{B}_b(S))$, since $\mathcal{B}_b(S) \subset \mathcal{A}$, thus being in accordance with our definition of $\mathcal{L}_b$-convergence.

When asking for conditions under which

$$(++) \qquad (\xi_n,\eta_n) \xrightarrow{\mathcal{L}_b} (\xi,\eta) \ \text{ implies } \ \xi_n \circ \eta_n \xrightarrow{\mathcal{L}_b} \xi \circ \eta$$

we know from the continuous mapping theorem (Theorem 4) that (++) holds if ψ is $\mathcal{A},\mathcal{B}_b(D)$-measurable and $\widetilde{\mathcal{L}\{(\xi,\eta)\}}$-a.e. d-continuous.

Now, the required measurability of ψ is guaranteed by (+) and it follows as in Billingsley (1968), p. 145, that ψ is also $\widetilde{\mathcal{L}\{(\xi,\eta)\}}$-a.e. d-continuous if $\mathcal{L}\{\xi\}(C) = \mathcal{L}\{\eta\}(C) = 1$ for $C \equiv C[0,1]$; in fact, if $\mathcal{L}\{\xi\}$ and $\mathcal{L}\{\eta\}$ concentrate on C, then $L\{(\xi,\eta)\}(C \times (C \cap D_o)) = 1$, and it is easy to show that ψ is d-continuous on $C \times (C \cap D_o)$.

It remains of course the question of when

$$(\xi_n,\eta_n) \xrightarrow{L_b} (\xi,\eta)$$

holds and here Theorem 9c can be used leading to the following result on stability of L_b-convergence in $D \equiv D[0,1]$ under random change of time:

<u>THEOREM.</u> Suppose that ξ_n, $n \in \mathbb{N}$, and ξ are random elements in $(D,\mathcal{B}_b(D))$ such that $\xi_n \xrightarrow{L_b} \xi$ and $L\{\xi\}(C) = 1$. Let η_n, $n \in \mathbb{N}$, and η be random elements in $(D_o,\mathcal{A}_o)$ such that $\eta_n \xrightarrow{L_b} \eta$ and η equals $\mathbb{P}$-a.s. some function belonging to $C \equiv C[0,1]$*).

Then $\xi_n \circ \eta_n$, $n \in \mathbb{N}$, and $\xi \circ \eta$ are random elements in $(D,\mathcal{B}_b(D))$ for which $\xi_n \circ \eta_n \xrightarrow{L_b} \xi \circ \eta$.

*) This last assumption may be omitted by considering instead the set $C \times \{c\}$ as separable support of $L\{(\xi,\eta)\}$ if $\eta = c$ $\mathbb{P}$-a.s.