Change-point Problems
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BOUNDARY ESTIMATION FOR STAR-SHAPED OBJECTS

BY MATS RUDEMO AND HENRIK STRYHN
The Royal Veterinary and Agricultural University, Copenhagen

The present paper gives a motivation for the study of star-shaped contour models, surveys briefly some related models in statistical image analysis, and presents the asymptotic distribution of a histogram-like boundary estimator.

We consider a two-region image with a star-shaped contour and continuous nonstationary observations. The image model contains two-dimensional Gaussian white noise and a representation of the contour by a regressogram in radii, i.e., a piecewise circular boundary. The maximum likelihood estimator of arc radii is studied when the noise variance and the regressogram sector width tend to zero, and the asymptotic distribution is specified in terms of the location of the maximum of a two-sided Brownian motion with negative drift.

1. Introduction. The term star-shapedness is being used in several contexts; here we use it to denote the geometrical property of a bounded planar set, that from some reference point within the set all halflines intersect the boundary exactly once. Sometimes this is called wide-sense convexity; a convex set is star-shaped with respect to any reference point. Star-shapedness enables, via polar coordinates, a univariate representation of the boundary and a partial ordering of the set. Considerable simplifications result for the (statistical) image modeling and analysis, compared to wider classes of connected sets.

Our interest in star-shaped objects was initiated by an application with identification of weed seeds from microscope images, see Petersen (1991, 1992). In this work high automatic classification rates have been obtained by incorporating discriminatory features from the boundary as well as from the texture of the seed. The contours of the weed seeds are often to a very good approximation star-shaped, as illustrated in Figure 1, which shows contours of individual seeds for some of the 40 species studied in Petersen (1992).

Segmentation of an image into two regions (by estimating the separating contour) may be seen as a generalization of the one-dimensional change-point

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problem. Optimal rates of convergence – in a minimax sense – for discrete image models, when the number of observations (pixels) tends to infinity, have been given in a series of papers of Korostelev and Tsybakov. A unified treatment is presented in Korostelev and Tsybakov (1992, 1993); of earlier work we mention Tsybakov (1989) and Korostelev (1991). Most detailed results are available for boundary fragments, where the boundary is the graph of a real function of one variable. To a large extent these results carry over to star-shaped models, cf. Korostelev and Tsybakov (1993).

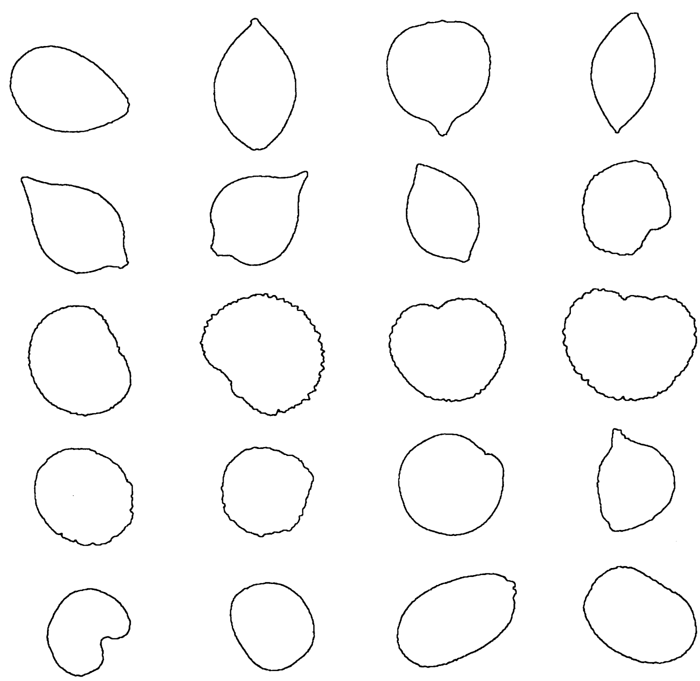

Figure 1: Smoothed contours of individual weed seeds from 20 different species, cf. Petersen (1992), obtained from microscope images by thresholding and use of 3×3-morphological filters (erosion and dilation).

The asymptotic distribution in a discrete observation model with a regressogram estimator was studied in Rudemo and Stryhn (1991); pixel values are assumed to be independent and identically distributed within regions. Asymptotically, the number of pixels tends to infinity, whereas the sector width and the difference between the region distributions tend to zero. The limiting distribution of the maximum likelihood estimator is expressed in terms of the location of the maximum of a two-sided Brownian motion with parabolic-triangular drift, i.e., parabolic drift near the origin and linear drift further out.

Here we derive a similar result involving a two-sided Brownian motion with the same type of drift; however, the image model is different. We have continuous observations with additive Gaussian white noise and allow for non-stationary regional pixel value distributions, which are not neccessarily drawn together in the limit.

Related asymptotic results, when the noise variance tends to zero, have been obtained for a continuous Gaussian white noise model by Hasminskii and Lebedev (1990). Special emphasis is given to cases where a reduction of the asymptotics to the one-dimensional change-point problem is possible, which leads to asymptotic distributions expressed in terms of a two-sided Brownian motion with triangular drift. Models with a one-dimensional contour parameter and some binary image models with rectangular and circular contours are shown to permit such a reduction. One marked difference of the present analysis is the occurrence of a regressogram sector width tending to zero along with the noise variance.

Although one in practice is typically faced with discrete observations, we expect the continuous observation model to provide good approximations for a wide range of sampling models. Precise results in this direction for one-dimensional diffusion processes are given in Laredo (1990).

2. Image Model. Let $T = [0,1] \times [0,1] \subseteq \mathcal{R}^2$ be divided into disjoint subsets R_1 and R_2 by a star-shaped curve C. Thus, with respect to a reference point t_0 in the interior of the inner region R_1 the contour C has a representation

$$C = \{t \in T : t = t_0 + \gamma(u)\,(\cos(2\pi u), \sin(2\pi u)),\ 0 \le u \le 1\}, \tag{1}$$

where $\gamma : [0,1] \to (0,\infty)$ is continuous and satisfies the periodic boundary condition $\gamma(0) = \gamma(1)$. For square integrable real-valued functions f_1 and f_2 on T, put

$$f(t) = f_1(t)\,\mathbf{1}\{t \in R_1\} + f_2(t)\,\mathbf{1}\{t \in R_2\}, \tag{2}$$

and consider the continuous image model (cf. Hasminskii and Lebedev (1990) or Korostelev and Tsybakov (1993, Section 8.2))

$$dV(t) = f(t)\,dt + \sigma\,dW(t), \qquad t \in T. \tag{3}$$

Here $(W(t), t \in T)$ is the Brownian sheet, so that dW is two-dimensional Gaussian white noise. For the statistical analysis we assume the noise variance σ^2, the reference point t_0, and the regression functions f_1 and f_2 to be known.

To estimate the unknown contour C we employ a (circular) regressogram estimator; in a parametric setup this corresponds to candidate curves, which are star-shaped and determined via (1) from piecewise constant functions with a fixed number κ of breakpoints. Specifically, for an equidistant division, $u_i = i/\kappa$, $i = 0, \ldots, \kappa$, of the unit interval the candidate curve $C(\theta)$, $\theta \in \Theta \subseteq \mathcal{R}^\kappa$, is given by $\gamma = \gamma(\cdot;\theta)$ in (1), where

$$\gamma(u;\theta) = \sum_{i=1}^{\kappa} \theta_i \, \mathbf{1}\{u_{i-1} \leq u < u_i\}, \tag{4}$$

with the requirement $C(\theta) \subseteq T$.

Let P^f denote the probability measure generated by the image model (3), and write $f = f_C$ to emphasize the dependence of f in (2) on $C = C(\theta)$. For the model corresponding to the true curve C_0 we similarly use the notation f_0 and γ_0. The log-likelihood ratio of (3), cf. Park (1970), is

$$\log \frac{dP^{f_C}}{dP^{f_0}}(V) = \frac{1}{\sigma^2} \int_T (f_C(t) - f_0(t)) \, dV(t) - \frac{1}{2\sigma^2} \int_T (f_C^2(t) - f_0^2(t)) \, \lambda(dt),$$

where λ denotes Lebesgue measure on T. With regard to the true model, $dV(t) = f_0(t)\,dt + \sigma\,dW(t)$, one obtains

$$\log \frac{dP^{f_C}}{dP^{f_0}}(V) = \frac{1}{\sigma} \int_T (f_C - f_0) \, dW - \frac{1}{2\sigma^2} \int_T (f_C - f_0)^2 \, d\lambda. \tag{5}$$

A derivation of the corresponding expression for the one-dimensional Gaussian white noise model can be found in Ibragimov and Hasminskii (1981, Appendix II).

3. Asymptotics for the ML Regressogram Estimator with Equidistant Knots. Consider the statistical model of Section 2 and the maximum likelihood estimator $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_\kappa)$ of θ. We shall study the asymptotic distribution of $\hat{\theta}_i$, when $\kappa \to \infty$ and simultaneously $\sigma^2 \to 0$.

By the independence of the white noise process in disjoint regions, the components of $\hat{\theta}$ are mutually independent, and $\hat{\theta}_i$ is determined solely from the sector

$$T_i = \{t \in T : t = t_0 + r\,(\cos(2\pi u), \sin(2\pi u)),\ u_{i-1} \leq u < u_i,\ r > 0\}. \tag{6}$$

Introduce the subsets $A_i(y)$ and A_{0i} of T_i by

$$\begin{aligned} A_i(y) &= \{t \in T : t = t_0 + r(\cos(2\pi u), \sin(2\pi u)),\ u_{i-1} \leq u < u_i,\ 0 < r \leq y\}, \\ A_{0i} &= \{t \in T : t = t_0 + r(\cos(2\pi u), \sin(2\pi u)),\ u_{i-1} \leq u < u_i,\ 0 < r \leq \gamma_0(u)\}. \end{aligned}$$

For identifiability of θ_i, i.e., uniqueness (a.s.) of the ML estimator $\hat{\theta}_i$, we assume f_1 and f_2 to be continuous on T_i as well as the following condition, corresponding to condition C in Hasminskii and Lebedev (1990), to hold for all y and $h > 0$

$$\lambda(\{t \in A_i(y+h) \setminus A_i(y) : f_1(t) \neq f_2(t)\}) > 0. \tag{7}$$

One should note, however, that the crucial assumption for the following asymptotic result is local, namely that $f_1 \neq f_2$ in a neighbourhood of the true curve; if the ML estimator in case of multiple maxima is defined suitably, the assertion in Theorem 1 is valid also without (7).

Let $F(\cdot; D)$ denote the distribution (function) of the a.s. well-defined location of the maximum of the Brownian motion with continuous drift D, i.e., $(B_s + D(s), s \in \mathcal{R})$, where $(B_s, s \in \mathcal{R})$ is a standard two-sided Brownian motion ($\mathrm{E}B_s = 0$, $\mathrm{E}B_s^2 = |s|$), and $D(s)$ tends to minus infinity such that $B_s + D(s) \to -\infty$ a.s. when $|s| \to \infty$. Up to a constant, the drift functions considered below are parabolic-triangular, cf. Rudemo and Stryhn (1991), defined as

$$D_b(t) = \begin{cases} -t^2/(2b) & \text{for } |t| < b \\ -b/2 - (|t| - b) & \text{for } |t| \geq b \end{cases} \tag{8}$$

where $b \geq 0$ is a constant.

THEOREM 1. *Consider a division of the unit square into inner and outer regions by the star-shaped continuous curve $\gamma = \gamma_0$ in (1), and the maximum likelihood estimator $\hat{\theta}$ corresponding to the regressogram (4). Assume that $\sigma^2 \to 0$ and $\kappa \to \infty$, such that*

$$\sigma^2 \kappa \to 0 \quad \text{and} \quad \frac{1}{\sigma^2 \kappa^2} \to \beta_0, \tag{9}$$

where $\beta_0 \in [0, \infty)$ is a constant.

For a given $u \in [0, 1)$, put $t_u = t_0 + \gamma_0(u)\,(\cos(2\pi u), \sin(2\pi u))$ and $i = [\kappa u] + 1$, where $[\cdot]$ denotes integer part. Assume that γ_0' exists and is continuous in an open interval containing u. Assume further that f_1 and f_2 are continuous on T_i in (6) and satisfy the identifiability condition (7), and that $f_1(t_u) \neq f_2(t_u)$. Finally, let $\bar{u}_i = (u_i + u_{i-1})/2$, the interval midpoint. Then

$$\frac{1}{\sigma^2 \kappa}\{\hat{\theta}_i - \gamma_0(\bar{u}_i)\} \xrightarrow{w} F(\cdot; D), \tag{10}$$

where

$$D(s) = D(s; b_0) = (\pi \gamma_0(u)/2)^{1/2}\, |f_1(t_u) - f_2(t_u)|\, D_{b_0}(s),\ b_0 = \frac{|\gamma_0'(u)|}{2}\beta_0, \tag{11}$$

with D_b given by (8).

PROOF. Essentially, the line of action is to show that a suitably normalized and time-rescaled version of the log-likelihood ratio process converges weakly (on compacta) to the two-sided Brownian motion with drift. Loosely stated, the 'two-phase' drift function D_b occurs in the limit because the mean of the log-likelihood ratio is expressed in terms of sets, whose areas are approximated in different ways for small and large standardized arguments, see (12) below.

If we let $C(y)$ denote the arc with center t_0 and radius y in the sector T_i, the log-likelihood ratio contribution to (5) from T_i is

$$\begin{aligned}\ell_i(y) &= \frac{1}{\sigma}\int_{T_i}(f_{C(y)} - f_0)\,dW - \frac{1}{2\sigma^2}\int_{T_i}(f_{C(y)} - f_0)^2\,d\lambda \\ &= \frac{1}{\sigma}\int_{T_i}(f_2 - f_0)\,dW + \frac{1}{\sigma}\int_{A_i(y)}(f_1 - f_2)\,dW \\ &\quad - \frac{1}{2\sigma^2}\int_{A_i(y)\triangle A_{0i}}(f_2 - f_1)^2\,d\lambda.\end{aligned}$$

Put $y_0 = \gamma_0(\bar{u}_i)$, and define the process (X_s) by

$$X_s = c(\ell_i(y) - \ell_i(y_0)), \text{ when } s = \frac{y - y_0}{\sigma^2\kappa} \text{ and } y > 0,$$

where $c > 0$ is a constant to be chosen such that $\mathrm{Var}(X_s) \to |s|$. By the definition of the stochastic integral with respect to W,

$$\begin{aligned}\mathrm{Var}(\ell_i(y) - \ell_i(y_0)) &= \sigma^{-2}\int_{A_i(y)\triangle A_i(y_0)}(f_1 - f_2)^2\,d\lambda \\ &\to 2\pi(f_1(t_u) - f_2(t_u))^2\gamma_0(u)\,|s|.\end{aligned}$$

Therefore, an appropriate choice of c is $c = \left\{2\pi(f_1(t_u) - f_2(t_u))^2\gamma_0(u)\right\}^{-1/2}$.

Next, we turn to $\mathrm{E}X_s = -c\,(2\sigma^2)^{-1}(\int_{A_i(y)\triangle A_{0i}} - \int_{A_i(y_0)\triangle A_{0i}}(f_1 - f_2)^2\,d\lambda)$. As above, the integrand is approximated by $(f_1(t_u) - f_2(t_u))^2$. The areas of the argument sets of the integration are most easily evaluated in polar coordinates, and a first order expansion of γ_0 at $\bar{u}_i$ yields

$$|A_i(y) \triangle A_{0i}| - |A_i(y_0) \triangle A_{0i}| = -2\pi\,\gamma_0(u)\,\sigma^2\,D_b(s) + o(\kappa^{-2}), \qquad (12)$$

where $b = |\gamma_0'(u)|/(2\sigma^2\kappa^2)$. This calculation is analogous to the proof of Theorem 1 in Rudemo and Stryhn (1991). Passing to the limit, we obtain

$$\mathrm{Var}(X_s) \to |s| \text{ and } \mathrm{E}X_s \to D(s; b_0),$$

and, in fact, this convergence is uniform for s-values in a bounded interval. Hereby the stated convergence of the process (X_s) is ensured, and the theorem follows. ∎

Remark 1. Further asymptotic results for the regressogram estimator. For completeness, some related continuous observation models should be considered also, for instance the contiguous case, where region distributions get close in the limit, as well as corresponding problems for the boundary fragment model. The asymptotic distributions are, however, readily obtainable using the same type of calculations as in the proof of Theorem 1. An overview of the results is given in Stryhn (1993).

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DEPARTMENT OF MATHEMATICS AND PHYSICS
THE ROYAL VETERINARY AND AGRICULTURAL UNIVERSITY
1871 COPENHAGEN, DENMARK