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COMPARISON OF EXPERIMENTS OF SOME MULTIVARIATE DISTRIBUTIONS WITH A COMMON MARGINAL

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In this paper we review some current work on comparison of experiments of some multivariate distributions. First we describe some results regarding comparison of experiments of univariate distributions that belong to two-parameters exponential families and that satisfy the semi-group property. Then we discuss comparison of experiments of vectors that arise from an additive model based on univariate two-parameter exponential families of random variables. These models give rise to vectors of random variables which are positively dependent and these are compared to vectors of independent random variables with the same marginals. It is shown that positively dependent random variables contain less information than independent random variables. Finally we describe some results regarding the comparison of experiments of exchangeable and nonexchangeable normal random vectors. In particular, we show how the majorization ordering can be used to identify various information orderings of multivariate normal random vectors which have a common marginal density.

1. Introduction

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ be two random vectors such that $X_1 =_d X_2 =_d \cdots =_d X_n =_d Y_1 =_d Y_2 =_d \cdots =_d Y_n$, where '$=_d$' denotes equality in law. That is, $\mathbf{X}$ and $\mathbf{Y}$ have the same univariate marginal distributions and all these marginals are equal to each other. Let $\theta \in \Theta$ be an unknown parameter and assume that the distributions of $\mathbf{X}$ and $\mathbf{Y}$ depend on θ. Denote the distributions of $\mathbf{X}$ and $\mathbf{Y}$ by F_θ and G_θ, respectively. In this paper we will be concerned with the amount of information

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about θ that is contained in F_θ and in G_θ. We will review some current work regarding comparison of experiments which are based on vectors which may have dependent random variables. The basic intuitive conjecture, that we can prove in some instances, is that if $X_1, X_2, \ldots, X_n$ are more 'positively dependent' than $Y_1, Y_2, \ldots, Y_n$ then they should contain less information about θ than the Y_j's. This is clearly the case in the extreme case where the Y_j's are independent and identically distributed and the X_j's are all equal to each other with probability one. But we will show that in some cases, even if the X_j's are not necessarily totally dependent, but only positively dependent, then they contain less information about θ than the Y_j's. We will show that in other cases the information content is a monotone function of the 'strength of dependence' of the underlying random variables. In still other cases the information content is shown to be monotone in the amount of 'homogeneity' of the underlying random variables.

2. Background and Preliminaries

The notion of comparison of experiments, as introduced by Blackwell (1951, 1953) and others, concerns a partial ordering of the information contained in the experiments (or in the distributions of the underlying random variables). For a review of the basic ideas and related results see Goel and DeGroot (1979), Lehmann (1988) and Torgersen (1991).

DEFINITION 1 The experiment associated with $\mathbf{Y}$ is said to be *at least as informative* for θ as that associated with $\mathbf{X}$, in symbols $\mathbf{X} \leq_{(i)} \mathbf{Y}$ or $F_\theta \leq_{(i)} G_\theta$, if for every decision problem involving θ and every prior distribution on Θ the expected Bayes risk from F_θ is not less than that from G_θ.

Recently many useful results have been obtained by researchers on the comparison of various types of experiments. For example, Hansen and Torgersen (1974) considered the comparison of normal experiments, Torgersen (1984) and Stepniak, Wang and Wu (1984) studied the comparison of linear experiments, Hollander, Proschan and Sconing (1985, 1987) and Goel (1988) gave results comparing experiments with censored data, and Lehmann (1988) discussed the comparison of location parameter experiments. Recently Eaton (1991) discussed a group action on covariances with applications to the comparison of linear normal experiments.

Lehmann (1959, p. 75) noted the following sufficient condition:

PROPOSITION 2 *The information inequality* $\mathbf{X} \leq_{(i)} \mathbf{Y}$ *holds if there exists a function* $\psi : \mathbb{R}^{n+r} \to \mathbb{R}^n$ *and an* r*-dimensional random vector* $\mathbf{Z}$ $(r \geq 1)$, *which is independent of* $\mathbf{Y}$ *and having a distribution which does not depend on* θ, *such that* $\mathbf{X} =_d \psi(\mathbf{Y}, \mathbf{Z})$.

Proposition 2 is the basic technical tool that we will use throughout this paper. LeCam (1964) noticed that the condition of Proposition 2 is necessary as well when the family $\{F_\theta, \theta \in \Theta\}$ is dominated.

In certain cases an ordering, via comparison of experiments, can be obtained for a given family of distributions. For example:

(a) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed Poisson random variables with mean $t_1\theta$ and let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed Poisson random variables with mean $t_2\theta$, where $t_1 \le t_2$ are known real numbers and θ is an unknown parameter. Then $\mathbf{X} \le_{(i)} \mathbf{Y}$. This easily follows from Lehmann (1959, p. 77).

(b) Let $X_1, X_2, \ldots, X_n$ $[Y_1, Y_2, \ldots, Y_n]$ be independent and identically distributed normal random variables with mean θ and standard deviation σ_2 $[\sigma_1]$, where θ is unknown and σ_1 and σ_2 are fixed such that $0 < \sigma_1 \le \sigma_2$. Then $\mathbf{X} \le_{(i)} \mathbf{Y}$ (Goel and DeGroot (1979)).

(c) Let $X_1, X_2, \ldots, X_n$ $[Y_1, Y_2, \ldots, Y_n]$ be independent and identically distributed gamma random variables with shape parameter a and scale parameter $b\theta^{k_1}$, $[b\theta^{k_2}]$ where $a > 0, b > 0, k_1 > 0$, and $k_2 > 0$ are fixed and $\theta > 0$ is an unknown parameter. If $k_1 \le k_2$ then $\mathbf{X} \le_{(i)} \mathbf{Y}$ (Goel and DeGroot (1979)).

Note that in each one of these examples the distributions of the X_i's and of the Y_i's belong to a particular univariate family of distributions. More explicitly, in each of the examples we have a family of univariate densities $f_{t,\theta}$, with respect to the Lebesgue or the counting measure, depending on a parameter $(t, \theta) \in \Theta_1 \times \Theta_2 \subset \mathbb{R}^2$ of the form

$$f_{t,\theta}(x) = c(t,\theta)\alpha(x,t)e^{\phi(\theta)x} \tag{1}$$

where c, α, and ϕ are some real-valued Borel-measurable functions defined on $\Theta_1 \times \Theta_2$, $\mathbb{R} \times \Theta_1$, and Θ_2 respectively. In addition to the Poisson, normal and gamma densities also the binomial and the negative binomial densities, among others, are of the form (1). See Shaked and Tong (1990) for the particular explicit expressions of the functions c, α, and ϕ for these densities.

The families of density functions mentioned above also have the semi-group property. Formally a family of density functions $f_{t,\theta}$, with respect to the measure μ, is said to have the semi-group property in the parameter $t \in \Theta_1$, where $\Theta_1 = (0, \infty)$ or $\Theta_1 = \{1, 2, \ldots\}$, if

$$f_{t_1,\theta} * f_{t_2,\theta} = f_{t_1+t_2,\theta}, \qquad t_1 \in \Theta_1, \quad t_2 \in \Theta_1$$

Here $*$ denotes the convolution operation:

$$f_{t_1,\theta} * f_{t_2,\theta}(x) = \int f_{t_1,\theta}(y) f_{t_2,\theta}(x-y)\,d\mu(y).$$

Roughly speaking, if $f_{t,\theta}$ has the representation (1) and also satisfies the semi-group property then the parameter t can be thought of as a 'sample size'. The following result then is not surprising (for a proof see Shaked and Tong (1990)):

PROPOSITION 3 *Let $f_{t,\theta}$ be a density of the form* (1) *with $\Theta_1 = (0, \infty)$ or $\Theta_1 = \{1, 2, \ldots\}$, which has the semi-group property in t. Let $F_{t,\theta}$ denote the distribution function associated with the density $f_{t,\theta}$. Then*

$$F_{t_2,\theta} \geq_{(i)} F_{t_1,\theta} \text{ whenever } t_2 \geq t_1.$$

By Blackwell (1953), $(X_1, X_2, \ldots, X_n) \leq_{(i)} (Y_1, Y_2, \ldots, Y_n)$ when $X_j \leq_{(i)} Y_j$, $j = 1, 2, \ldots, n$, and the coordinates of these two vectors are independent. Thus from Proposition 3 we obtain:

COROLLARY 4 *Let X_j and Y_j have the densities $f_{s,\theta}$ and $f_{\tau,\theta}$ of the form* (1) *($j = 1, 2, \ldots, n$) for some functions c, α, and ϕ with $\Theta_1 = (0, \infty)$ or $\Theta_1 = \{1, 2, \ldots\}$. Suppose that the X_j's are independent and that the Y_j's are independent. If $s_j \leq \tau_j$, $j = 1, 2, \ldots, n$, and $f_{t,\theta}$ satisfies the semi-group property in t, then*

$$(X_1, X_2, \ldots, X_n) \leq_{(i)} (Y_1, Y_2, \ldots, Y_n).$$

It follows, using Proposition 2, that $X_1 + X_2 + \cdots + X_n \leq_{(i)} Y_1 + Y_2 + \cdots + Y_n$.

3. Comparison of Vectors of Independent and Dependent Random Variables

In examples (a) - (c), or more generally in Corollary 4, the vectors $\mathbf{X}$ and $\mathbf{Y}$ consist of independent random variables. The main thrust of this section is to obtain results in which the assumption of mutual independence of the X_j's is relaxed. In the next section we will also relax the assumption of independence of the Y_j's.

It is well known that in certain statistical applications the assumption of independence is not realistic. For example, in many reliability problems the lifetimes of components in a coherent system are positively dependent. This happens when the system involves several common units or when the components are subjected to the same set of stresses. In statistical decision theory, if the random variables $X_1, X_2, \ldots, X_n$ are conditionally independent and identically distributed given some random quantity Γ, and if the distribution of Γ is nonsingular, then, after unconditioning, the joint distribution of $X_1, X_2, \ldots, X_n$ is positively dependent by mixture (Shaked (1977)). Thus the random variables are not independent.

In this section we let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ be two random vectors, the distribution of each depending on some parameter θ. We consider random vectors such that marginally $X_j =_d Y_j$ whatever the value of θ is. We suppose that for each θ the Y_j's are independent but the X_j's may be positively dependent in a certain fashion. Then one would expect $\mathbf{Y}$ to be at least as informative as $\mathbf{X}$. This can be easily seen in the extreme case where $P\{X_1 = X_2 = \cdots = X_n\} = 1$. Then the information contained in $\mathbf{X}$ is the same as the information contained in one observation, Y_1, say, whereas the information contained in $\mathbf{Y}$ is larger, since it is based on n observations.

Shaked and Tong (1990) proved that this is indeed the case for a special model of dependence. They considered the following model. Let

$$\mathbf{X} = \begin{pmatrix} U_1' + V_1' \\ U_2' + V_1' \\ \vdots \\ U_n' + V_1' \end{pmatrix} \text{ and } \mathbf{Y} = \begin{pmatrix} U_1 + V_1 \\ U_2 + V_2 \\ \vdots \\ U_n + V_n \end{pmatrix},$$

where $U_1, U_2, \ldots, U_n$ are mutually independent with distributions depending on θ; $V_1, V_2, \ldots, V_n$ are independent and identically distributed, independent of the U_j's, and with a common distribution depending on θ; $U_1', U_2', \ldots, U_n', V_1'$ are all independent; and for each θ,

$$U_j' =_d U_j \text{ and } V_1' =_d V_1, \quad j = 1, 2, \ldots, n.$$

It is then seen that the Y_j's are independent whereas the X_j's are positively associated [e.g., in the sense of Esary, Proschan and Walkup (1967)].

In the following theorem it is assumed that all the univariate random variables mentioned above have densities of the form (1) with respect to a σ-finite measure μ and for some fixed functions c, α, and ϕ. More specifically we suppose that the density of U_j and U_j' is $f_{t_1,\theta}$ and that the density of V_j is $f_{t_2,\theta}$, $j = 1, 2, \ldots, n$ (thus, in particular, the density of V_1 is $f_{t_2,\theta}$). As before, μ will be either the Lebesgue measure or the counting measure. The proof of the theorem can be found in Shaked and Tong (1990).

THEOREM 5 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors as described in the preceding paragraph. Then*

$$\mathbf{X} \leq_{(i)} \mathbf{Y}. \tag{2}$$

Theorem 5 complements Corollary 4. In the latter two vectors of independent random variables are compared. In the former only one of the vectors consists of independent random variables, but the two random vectors have equal marginals.

Theorem 5 shows how some multivariate Poisson random vectors [as described, e.g., in Johnson and Kotz (1969, Chapter 11, Section 4) and references therein] can be compared in the comparison-of-experiments ordering. Similarly, multivariate gamma distributions [as, e.g., in Johnson and Kotz (1972, Chapter 40, Section 2)] and multivariate negative-binomial distributions can be compared. Theorem 5 also shows (after some calculations) that a normal random vector with independent and identically distributed components, all with mean θ, is more informative than a similar normal random vector with the same marginals but with positively correlated permutation symmetric components, when the common variance is known. In the next section we discuss even a more general result for normal random vectors.

4. Comparison of Normal Vectors with a Common Marginal Distribution

Shaked and Tong (1990) proved the following monotonicity result. Note that in the following result not only it is seen that independence is more informative than positive dependence, but more than that, it is seen there that negative dependence is even more informative than independence.

PROPOSITION 6 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two vectors of* n *normal random variables with means* θ*, a common known variance* $\sigma^2 > 0$*, and common correlation coefficients* ρ_2 *and* ρ_1*, respectively. Then*

$$\mathbf{X} \leq_{(i)} \mathbf{Y} \text{ for all } -\frac{1}{n-1} \leq \rho_1 \leq \rho_2 \leq 1.$$

Let $\mathbf{X}_\rho$ denote a multivariate normal random vector with means θ, a common variance σ^2, and a common correlation coefficient $\rho \geq -1/(n-1)$. Here the unknown parameter is θ and the parameter ρ indexes the distribution of $\mathbf{X}_\rho$. From the proof of Proposition 6 in Shaked and Tong (1990) it is seen that, for a fixed ρ, the permutation symmetric multivariate normal distribution is monotone in the sense $\leq_{(i)}$ as a function of the sample size n (the larger n is, the more informative is the vector $\mathbf{X}_\rho$ provided $\rho < 1$).

The Fisher's information corresponding to the distribution of $\mathbf{X}_\rho$ is

$$E\left(\frac{\partial}{\partial\theta}\log f_\rho(\mathbf{X})\right)^2 = \frac{1}{\mathrm{Var}(\bar{X}_\rho)}, \quad \rho \in (-\frac{1}{n-1}, 1),$$

where $\bar{X}_\rho$ is the average of the components of $\mathbf{X}_\rho$ and

$$f_\rho(\mathbf{x}) = \frac{1}{(2\pi\sigma)^{n/2}|\mathbf{R}(\rho)|^{1/2}} \exp\{-\frac{1}{2}(\mathbf{x}-\theta\mathbf{1})'\mathbf{R}^{-1}(\rho)(\mathbf{x}-\theta\mathbf{1})\}.$$

Here $\mathbf{R}(\rho)$ is the covariance matrix of $\mathbf{X}_\rho$. Thus, in this case, the two experiments based on $\mathbf{X}_{\rho_1}$ and $\mathbf{X}_{\rho_2}$ can be ordered in the ordering $\leq_{(i)}$ if, and only if, they can be ordered according to the Fisher's information. In general the equivalence of these two orderings is not always true (see, e.g., Hansen and Torgersen (1974)).

In Shaked and Tong (1985) some notions of partial ordering of exchangeable random variables by positive dependence are introduced, and for exchangeable normal variables the orderings reduce to the ordering of the correlation coefficients. Consequently, from Proposition 6 it follows that if exchangeable normal variables are more positively dependent then the experiment is less informative. A question of interest is then what can be said for normal variables which are not exchangeable.

Shaked and Tong (1992) provide an answer to this question by showing how a more general partial ordering of positive dependence yields a monotonicity result for nonexchangeable normal variables with a common marginal distribution.

In order to introduce the partial ordering of positive dependence for multivariate normal vectors we first need some preliminaries. Consider an n-dimensional vector of nonnegative integers given by

$$\mathbf{k} = (k_1, \ldots, k_r, 0, \ldots, 0),\ k_1 \geq k_2 \geq \cdots \geq k_r \geq 1,\ \sum_{s=1}^{r} k_s = n, \tag{3}$$

for some $r \leq n$. (The assumption of monotonicity of k_s in s is not an essential restriction. If it does not hold then the random variables can always be relabeled, if necessary, yielding the assumed monotonicity.) For arbitrary but fixed $0 \leq \rho_1 \leq \rho_2 \leq 1$ let us define a correlation matrix $\mathbf{R}(\mathbf{k})$ given by

$$\rho_{ij}(\mathbf{k}) = \begin{cases} 1 & \text{for } i = j, \\ \rho_2 & \text{for } i \neq j \text{ and } \sum_{s=0}^{m} k_s + 1 \leq i, j \leq \sum_{s=0}^{m+1} k_s, \\ & \qquad m \in \{1, 2, \ldots, r-1\}, \\ \rho_1 & \text{otherwise,} \end{cases} \tag{4}$$

where $k_0 \equiv 0$.

If $\mathbf{X}$ has a correlation matrix $\mathbf{R}(\mathbf{k})$ then its components belong to r groups, with group sizes $k_1, k_2, \ldots, k_r$, respectively, such that the correlations within groups are ρ_2 and the correlations between groups are ρ_1. This type of correlation matrices arise in many applied problems in linear models and multivariate analysis. For example, if in a family of four children the first two [the last two] are brothers/sisters, but any pair between the two groups are half brothers/sisters, then under the additive genetic model the vector of measurements (X_1, X_2, X_3, X_4), of a certain biological variable, will have means θ, variances σ^2, and a correlation matrix $\mathbf{R}(\mathbf{k})$ with

$k = (2, 2, 0, 0)$. For references on the applications of such a correlation matrix in an agricultural genetic selection problem see, e.g., Tong (1990, pp. 129-130).

Let $\mathbf{k}^*$ be another vector of nonnegative integers such that

$$(5) \quad \mathbf{k}^* = (k_1^*, \ldots, k_{r^*}^*, 0, \ldots, 0),\ k_1^* \geq k_2^* \geq \cdots \geq k_{r^*}^* \geq 1,\ \sum_{s=1}^{r} k_s^* = n,$$

for some $r^* \leq n$, and let $\mathbf{R}(\mathbf{k}^*)$ be defined similarly. Let $\mathbf{X}$ and $\mathbf{Y}$ have, respectively, the multinormal distributions

$$(6) \qquad \mathcal{N}_n(\theta\mathbf{1}, \sigma^2\mathbf{R}(\mathbf{k})) \text{ and } \mathcal{N}_n(\theta\mathbf{1}, \sigma^2\mathbf{R}(\mathbf{k}^*))$$

for some $\mathbf{k}, \mathbf{k}^*$ satisfying (3) and (5), respectively, where $\theta \in \mathbb{R}$ is the common mean, $\sigma^2 > 0$ is the common known variance, and $\mathbf{1} = (1, \ldots, 1)$. Clearly the X_j's and the Y_j's defined in (6) have a common univariate $\mathcal{N}(\theta, \sigma^2)$ distribution. In the special case $\mathbf{k} = (n, 0, \ldots, 0)$ and $\mathbf{k}^* = (1, 1, \ldots, 1)$ both $\mathbf{X}$ and $\mathbf{Y}$ are exchangeable normal vectors with correlation coefficients ρ_2, ρ_1, respectively. However they are not exchangeable otherwise. A result of Tong (1989) states that if $\mathbf{k} \succ \mathbf{k}^*$, where '$\succ$' denotes the majorization ordering, then the X_j's tend to "hang together" more than the Y_j's, hence are more positively dependent in the sense that

$$(7) \qquad E\prod_{i=1}^{n} \phi(X_i) \geq E\prod_{i=1}^{n} \phi(Y_i) \text{ for all } \phi : \mathbb{R} \to [0, \infty),$$

provided the expectations exist. [Note that (7) implies that $\mathrm{Corr}(\phi(X_i), \phi(X_j)) \geq \mathrm{Corr}(\phi(Y_i), \phi(Y_j))$ for all ϕ.] The question of interest is whether this partial ordering of positive dependence also provides a partial ordering for information on θ in the sense of Definition 1. Shaked and Tong (1992) answered this question in the following theorem:

THEOREM 7 *Assume that* $\mathbf{X}$ *and* $\mathbf{Y}$ *satisfy* (6) *where* $\theta \in \mathbb{R}$ *is the unknown parameter,* $\sigma^2 > 0$ *is the common known variance, and* $0 \leq \rho_1 \leq \rho_2 \leq 1$ *are arbitrary but fixed. If* $\mathbf{k} \succ \mathbf{k}^*$*, then* $\mathbf{X} \leq_{(i)} \mathbf{Y}$.

The proof of Theorem 7 given by Shaked and Tong (1992) depends on an application of Torgersen (1984). Eaton (1991, Remark 2.3) noted that it is also possible to give an alternative proof of the theorem based on a direct verification.

A related result which follows from the remarks after Proposition 2.2 of Eaton (1991) is the following. Let $\mathbf{R}(\mathbf{k})$ be as defined in (4), but now, in order to point out the dependence of $\mathbf{R}(\mathbf{k})$ on ρ_1 and on ρ_2 we write it as $\mathbf{R}_{\rho_1,\rho_2}(\mathbf{k})$ where $0 \leq \rho_1 \leq \rho_2 \leq 1$.

THEOREM 8 *Assume that* $\mathbf{X}$ *has the multinormal distribution* $\mathcal{N}_n(\theta\mathbf{1}, \sigma^2\mathbf{R}_{\rho_1,\rho_2}(\mathbf{k}))$ *and that* $\mathbf{Y}$ *has the multinormal distribution* $\mathcal{N}_n(\theta\mathbf{1}, \sigma^2\mathbf{R}_{\rho_1',\rho_2'}(\mathbf{k}))$ *where* $\theta \in \mathbb{R}$ *is the unknown parameter,* $\sigma^2 > 0$ *is the common known variance and* $\mathbf{k}$ *is fixed partition vector as described in* (3).
(a) If $\rho_2 = \rho_2' \geq \rho_1 \geq \rho_1'$ *then* $\mathbf{X} \leq_{(i)} \mathbf{Y}$.
(b) If $\rho_2 \geq \rho_2' \geq \rho_1 = \rho_1'$ *then* $\mathbf{X} \leq_{(i)} \mathbf{Y}$.

EXAMPLE 9 In order to illustrate the result in Theorem 7 let us consider the special case $n = 4$. Let $\mathbf{R}_4$ $[\mathbf{R}_1]$ be the 4×4 correlation matrix with all the correlation coefficients being ρ_2 $[\rho_1]$, and let

$$\mathbf{R}_3 = \begin{pmatrix} 1 & \rho_2 & \rho_2 & \rho_1 \\ \rho_2 & 1 & \rho_2 & \rho_1 \\ \rho_2 & \rho_2 & 1 & \rho_1 \\ \rho_1 & \rho_1 & \rho_1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{R}_2 = \begin{pmatrix} 1 & \rho_2 & \rho_1 & \rho_1 \\ \rho_2 & 1 & \rho_1 & \rho_1 \\ \rho_1 & \rho_1 & 1 & \rho_2 \\ \rho_1 & \rho_1 & \rho_2 & 1 \end{pmatrix}.$$

If $0 \leq \rho_1 \leq \rho_2 \leq 1$ and $\mathbf{X}$ and $\mathbf{Y}$ have the distributions $\mathcal{N}_4(\theta\mathbf{1}, \sigma^2\mathbf{R}_{j+1})$ and $\mathcal{N}_4(\theta\mathbf{1}, \sigma^2\mathbf{R}_j)$ respectively, where $\sigma^2 > 0$ is known, then $\mathbf{X} \leq_{(i)} \mathbf{Y}$ holds for $j = 1, 2, 3$. This follows from Theorem 7 and the fact that $(4,0,0,0) \succ (3,1,0,0) \succ (2,2,0,0) \succ (1,1,1,1)$.

When we combine Theorem 7 with existing results, other useful results can be obtained. For example, if $\mathbf{X}$ and $\mathbf{Z}$ have the $\mathcal{N}_n(\theta\mathbf{1}, \sigma^2\mathbf{R}(\mathbf{k}))$ and $\mathcal{N}_n(\theta\mathbf{1}, \mathbf{\Sigma})$ distributions, respectively, and if there exists a correlation matrix $\mathbf{R}(\mathbf{k}^*)$ such that $\mathbf{k} \succ \mathbf{k}^*$ and $\sigma^2\mathbf{R}(\mathbf{k}^*) - \mathbf{\Sigma}$ is either positive definite or positive semidefinite, then $\mathbf{X} \leq_{(i)} \mathbf{Z}$ holds.

It is worthwhile to note that if $\mathbf{X}$ and $\mathbf{Y}$ satisfy (6) then, by a simple calculation, the amounts of Fisher's information on θ in the density functions of $\mathbf{X}$ and $\mathbf{Y}$ are, respectively, $\sigma^{-2}(\mathbf{1}(\mathbf{R}(\mathbf{k}))^{-1}\mathbf{1}')$ and $\sigma^{-2}(\mathbf{1}(\mathbf{R}(\mathbf{k}^*))^{-1}\mathbf{1}')$. Furthermore, the variances of the least-squares estimators of θ based on $\mathbf{X}$ and $\mathbf{Y}$ are, respectively, $\sigma^2(\mathbf{1}(\mathbf{R}(\mathbf{k}))^{-1}\mathbf{1}')^{-1}$ and $\sigma^2(\mathbf{1}(\mathbf{R}(\mathbf{k}^*))^{-1}\mathbf{1}')^{-1}$. Thus Theorem 7 yields a partial ordering for the Fisher's information and for the variances of the least-squares estimators via a majorization ordering of $\mathbf{k}$ and $\mathbf{k}^*$ in the correlation matrices.

Finally we point out that smaller correlations are not necessarily indicators of larger amounts of information. More specifically, if $\mathbf{X}$ and $\mathbf{Y}$ are multivariate normal random vectors with the same marginal distributions and with correlation matrices $\{\rho_{i,j}\}_{i,j=1}^n$ and $\{\eta_{i,j}\}_{i,j=1}^n$, respectively, such that $\rho_{i,j} \geq \eta_{i,j}$ for all i and j, then, in general, when σ^2 is known, it is not necessarily true that $\mathbf{X} \leq_{(i)} \mathbf{Y}$; some conditions must be imposed on the structures of the correlation matrices in order to assure that $\mathbf{X} \leq_{(i)} \mathbf{Y}$.

For example, such conditions can be found in Proposition 6 and in Theorem 8. Eaton (1991, Example 2.1) has shown that there exist multivariate normal random vectors **X** and **Y** such that **X** and **Y** have the same univariate marginals, the coordinates of **Y** are independent and identically distributed, the coordinates of **X** are positively correlated, but $\mathbf{Y} \leq_{(i)} \mathbf{X}$.

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