Stochastic Inequalities
IMS Lecture Notes – Monograph Series
Volume 22 (1993)

ON FKG–TYPE AND PERMANENTAL INEQUALITIES

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In this paper we survey results from Rinott and Saks (1990) on a "2m-function" inequality which generalizes the FKG and associated inequalities. We also present related conjectures and partial results on permanents and sums of permutation matrices. We hope that the motivation given in the first part of the paper, and the subsequent discussion will attract the attention of problem solvers to our conjectures.

1. Introduction

The FKG inequality (Fortuin, Kasteleyn and Ginibre (1971)) has been applied in many fields, including statistical mechanics, combinatorics, reliability theory and stochastic inequalities. In order to state it we need the following notation and definition: for $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_k)$ in $\mathbb{R}^k$, $\mathbf{x} \vee \mathbf{y}$ and $\mathbf{x} \wedge \mathbf{y}$ in $\mathbb{R}^k$ are defined to have coordinates $(\mathbf{x} \vee \mathbf{y})_j = \max(x_j, y_j)$ and $(\mathbf{x} \wedge \mathbf{y})_j = \min(x_j, y_j)$, $j = 1, \ldots, k$.

DEFINITION A σ-finite (nonnegative) measure μ on $\mathbb{R}^k$ is said to be an *FKG measure* if μ has a density function ϕ with respect to some product measure $d\sigma$ on $\mathbb{R}^k$, (that is, $d\sigma(\mathbf{x}) = \Pi_{j=1}^k d\sigma_j(x_j)$, and $d\mu(\mathbf{x}) = \phi(\mathbf{x})d\sigma(\mathbf{x})$), satisfying for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^k$,

$$\phi(\mathbf{x})\phi(\mathbf{y}) \le \phi(\mathbf{x} \vee \mathbf{y})\phi(\mathbf{x} \wedge \mathbf{y}). \tag{1}$$

Condition (1) is referred to as *multivariate total positivity of order 2* (MTP_2) in Karlin and Rinott (1980). It can be shown that if a positive density ϕ is TP_2 in every pair of variables, then (1) holds, i.e., ϕ is MTP_2. We now state the FKG inequality as follows:

[1]Work supported in part by National Science Foundation Grant DMS-9001274.

[2]Work supported in part by National Science Foundation Grant CCR89-11388.

AMS 1991 *subject classifications.* Primary 26D15; Secondary 15A15, 05A20, 60E15, 62H05.

Key words and phrases. Correlation inequalities, stochastic ordering, majorization, permutation matrices.

THEOREM 1.1 *Let* $\mathbf{X}$ *be a random vector in* $\mathbb{R}^k$ *whose distribution is an FKG probability measure. Then for any pair of nondecreasing real valued functions* α *and* β *defined on* $\mathbb{R}^k$, *we have*

$$E\{\alpha(\mathbf{X})\beta(\mathbf{X})\} \geq E\alpha(\mathbf{X}) \cdot E\beta(\mathbf{X}). \tag{2}$$

Note that when (2) holds, $\mathbf{X}$ is said to be *associated.*

Sarkar (1969) discovered that if $\mathbf{X}$ is a random vector having a density which is TP_2 in pairs then it is associated, a result which is very close to the FKG inequality.

Holley (1974) proved the following:

THEOREM 1.2 *Let* f_1, f_2, *be probability densities with respect to some product measure* $\mathbb{R}^k$, *satisfying* $f_1(\mathbf{x})f_2(\mathbf{y}) \leq f_1(\mathbf{x} \vee \mathbf{y})f_2(\mathbf{x} \wedge \mathbf{y})$. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be a random vectors in* $\mathbb{R}^k$ *having distributions with the densities* f_1 *and* f_2 *respectively. Then for any nondecreasing function* α *defined on* $\mathbb{R}^k$

$$E\alpha(\mathbf{X}) \geq E\alpha(\mathbf{Y}).$$

Next came the "4-function" Theorem of Ahlswede and Daykin (1978):

THEOREM 1.3 *Let* f_1, f_2 *and* g_1, g_2 *be nonnegative real valued functions defined on* $\mathbb{R}^k$ *that satisfy the following condition:* $f_1(\mathbf{x})f_2(\mathbf{y}) \leq g_1(\mathbf{x} \vee \mathbf{y})g_2(\mathbf{x} \wedge \mathbf{y})$. *Then, for any FKG measure* μ *on* $\mathbb{R}^k$:

$$\int_{\mathbb{R}^k} f_1(\mathbf{x})d\mu(\mathbf{x}) \int_{\mathbb{R}^k} f_2(\mathbf{x})d\mu(\mathbf{x}) \leq \int_{\mathbb{R}^k} g_1(\mathbf{x})d\mu(\mathbf{x}) \int_{\mathbb{R}^k} g_2(\mathbf{x})d\mu(\mathbf{x}). \tag{3}$$

It is an easy exercise to show that Ahlswede and Daykin's result implies Holley's theorem, which in turn implies the FKG inequality. For details, references and some examples and applications, see, e.g., Karlin and Rinott (1980), Graham (1982).

In the presence of theorems involving a single density (FKG), two densities (Holley), four functions (Ahlswede and Daykin), and further studies by Ahlswede and Daykin (1979) and Daykin (1980), it was natural to look for a more general result. This was done in Rinott and Saks (1990). In order to describe the next result we need the following notation. Given vectors $\mathbf{x}^i = (x_1^i, \ldots, x_k^i) \in \mathbb{R}^k$, $i = 1, \ldots, m$, define $\mathbf{x}^{[l]}$ to be the vector in $\mathbb{R}^k$ whose jth coordinate ($1 \leq j \leq k$) is the lth largest among $\mathbf{x}_j^i$, $i = 1, \ldots, m$. Formally we have $\mathbf{x}^{[l]} = \bigvee_{S:|S|=l} \bigwedge_{i \in S} \mathbf{x}^i$, $l = 1, \ldots, m$. In particular note that $\mathbf{x}^{[1]} = \vee_{i=1}^m \mathbf{x}^i$, $\mathbf{x}^{[m]} = \wedge_{i=1}^m \mathbf{x}^i$.

We can now quote the main result from Rinott and Saks (1990).

THEOREM 1.4 *Let $f_1, \dots, f_m$ and $g_1, \dots, g_m$ be nonnegative real valued functions defined on $\mathbb{R}^k$ satisfying the following condition: for every sequence $\mathbf{x}^1, \dots, \mathbf{x}^m$ of elements from $\mathbb{R}^k$*

$$f_1(\mathbf{x}^1)f_2(\mathbf{x}^2)\cdots f_m(\mathbf{x}^m) \le g_1(\mathbf{x}^{[1]})g_2(\mathbf{x}^{[2]})\cdots g_m(\mathbf{x}^{[m]}). \tag{4}$$

Then, for any FKG measure μ on $\mathbb{R}^k$:

$$\begin{aligned}\int_{\mathbb{R}^k} f_1(\mathbf{x})d\mu(\mathbf{x}) \quad \cdots \quad \int_{\mathbb{R}^k} f_m(\mathbf{x})d\mu(\mathbf{x}) \le \int_{\mathbb{R}^k} g_1(\mathbf{x})d\mu(\mathbf{x})\\ \cdots \quad \int_{\mathbb{R}^k} g_m(\mathbf{x})d\mu(\mathbf{x}).\end{aligned} \tag{5}$$

We are not concerned with issues of integrability in this paper and so we always assume that integrals are well-defined.

Note that in the case $m = 2$, condition (4) becomes $f_1(\mathbf{x})f_2(\mathbf{y}) \le g_1(\mathbf{x} \vee \mathbf{y})g_2(\mathbf{x} \wedge \mathbf{y})$. In this case Theorem 1.4 reduces to the "4-function" Theorem of Ahlswede and Daykin (1978).

The starting point of this paper is the one-dimensional version of Theorem 1.4 which we now state, with (hopefully) simpler notation.

PROPOSITION 1.1 *Let $f_1, f_2, \dots, f_m$ and $g_1, g_2, \dots, g_m$ be nonnegative real valued functions defined on $\mathbb{R}$ that satisfy the following condition: for every sequence $x_1, x_2, \dots, x_m$ of elements from $\mathbb{R}$,*

$$f_1(x_1)f_2(x_2)\cdots f_m(x_m) \le g_1(x_1^*)g_2(x_2^*)\cdots g_m(x_m^*), \tag{6}$$

where $(x_1^*, x_2^*, \dots, x_m^*)$ denotes the decreasing rearrangement of $(x_1, x_2, \dots, x_m)$. Then, assuming integrability,

$$\begin{aligned}&\int_{\mathbb{R}} f_1(x)d\mu(x)\int_{\mathbb{R}} f_2(x)d\mu(x)\cdots\int_{\mathbb{R}} f_m(x)d\mu(x)\\ &\le \int_{\mathbb{R}} g_1(x)d\mu(x)\int_{\mathbb{R}} g_2(x)d\mu(x)\cdots\int_{\mathbb{R}} g_m(x)d\mu(x),\end{aligned} \tag{7}$$

for any σ-finite measure μ on $\mathbb{R}$.

Theorem 1.4 can be deduced from Proposition 1.1 by induction arguments which may be of interest. However, they are quite standard and will not be reproduced here. They involve showing that the "marginal" functions defined by $p_i(\tilde{\mathbf{x}}) = \int_{\mathbb{R}} f_i(\tilde{\mathbf{x}}, x)d\mu_k(x)$ and $q_i(\tilde{\mathbf{x}}) = \int_{\mathbb{R}} g_i(\tilde{\mathbf{x}}, x)d\mu_k(x)$ satisfy the hypothesis of Theorem 1.4 as functions of $\tilde{\mathbf{x}} \in \mathbb{R}^{k-1}$. For details see Rinott and Saks (1990). The important part of the proof is the basis case $n = 1$, which is the content of Proposition 1.1.

In order to see the connection to permanents let us consider Proposition 1.1 in the simple case that $m = 2$, and $d\mu(x) = dx$. It says that if for all $x, y \in \mathbb{R}$

$$f_1(x)f_2(y) \le g_1(x \vee y)g_2(x \wedge y),$$

then, assuming integrability,

$$\int_{\mathbb{R}} f_1(x)dx \int_{\mathbb{R}} f_2(x)dx \le \int_{\mathbb{R}} g_1(x)dx \int_{\mathbb{R}} g_2(x)dx\,. \tag{8}$$

Starting on the l.h.s. of (8) we have

$$\begin{aligned}\int_{\mathbb{R}} f_1(x)dx \int_{\mathbb{R}} f_2(x)dx &= \int\int f_1(x)f_2(y)dxdy\\ &= \int\int_{x<y} \{f_1(x)f_2(y) + f_1(y)f_2(x)\}dxdy\\ &= \int\int \mathbf{Per}\begin{pmatrix} f_1(x) & f_2(x)\\ f_1(y) & f_2(y)\end{pmatrix} dxdy\,. \qquad (9)\end{aligned}$$

In the case that the integrals are taken with respect to a discrete measure $d\mu$, diagonal terms must also be considered, but as we shall see they do not pose any difficulty. Permanental inequalities enter the story because it is now clear that the inequality

$$\mathbf{Per}\begin{pmatrix} f_1(x) & f_2(x)\\ f_1(y) & f_2(y)\end{pmatrix} \le \mathbf{Per}\begin{pmatrix} g_1(x) & g_2(x)\\ g_1(y) & g_2(y)\end{pmatrix},$$

would imply (8). In the same way, we shall see that in order to prove (7) it would suffice to show that

$$\begin{aligned}&\mathbf{Per}\begin{pmatrix} f_1(x_1) & f_2(x_1) & \dots & f_m(x_1)\\ f_1(x_2) & f_2(x_2) & \dots & f_m(x_2)\\ \vdots & & & \vdots\\ f_1(x_m) & f_2(x_m) & \dots & f_m(x_m)\end{pmatrix} \le\\ &\qquad \mathbf{Per}\begin{pmatrix} g_1(x_1) & g_2(x_1) & \dots & g_m(x_1)\\ g_1(x_2) & g_2(x_2) & \dots & g_m(x_2)\\ \vdots & & & \vdots\\ g_1(x_m) & g_2(x_m) & \dots & g_m(x_m)\end{pmatrix}. \qquad (10)\end{aligned}$$

The latter inequality and related results and conjectures are the subject of the next section.

2. Permanents: Results and Conjectures

First, observe that the permanent on the l.h.s. of (10) is equal to $\sum_{\pi\in S_m}\prod_{i=1}^m f_i(x_{\pi(i)})$. At the end of the Introduction we said that (10) would imply (7); we now restate this implication formally as

LEMMA 2.1 *Let f_i and g_i be real valued functions defined on $\mathbb{R}$, satisfying*

$$(11) \qquad \sum_{\pi \in S_m} \prod_{i=1}^{m} f_i(x_{\pi(i)}) \leq \sum_{\pi \in S_m} \prod_{i=1}^{m} g_i(x_{\pi(i)})$$

for any x_i in $\mathbb{R}$, $i = 1, \ldots, m$. Then for any σ-finite measure μ on $\mathbb{R}$ we have (assuming integrability)

$$\prod_{i=1}^{m} \int_{\mathbb{R}} f_i(x) d\mu(x) \leq \prod_{i=1}^{m} \int_{\mathbb{R}} g_i(x) d\mu(x).$$

PROOF Simply take an m-fold integral with respect to the measure $\prod_{i=1}^{m} d\mu(x_i)$ on both sides of (11), to obtain $m! \prod_{i=1}^{m} \int_{\mathbb{R}} f_i(x) d\mu(x) \leq m! \prod_{i=1}^{m} \int_{\mathbb{R}} g_i(x) d\mu(x)$.

Lemma 2.1 and the above discussion show that in order to prove Proposition 1.1, it would suffice to prove

CONJECTURE 2.1 *Let $f_1, f_2, \ldots, f_m$ and $g_1, g_2, \ldots, g_m$ be nonnegative real valued functions defined on $\mathbb{R}$ that satisfy the following condition: for every sequence $x_1, x_2, \ldots, x_m$ of elements from $\mathbb{R}$,*

$$(12) \qquad f_1(x_1) f_2(x_2) \cdots f_m(x_m) \leq g_1(x_1^*) g_2(x_2^*) \cdots g_m(x_m^*),$$

where $(x_1^*, x_2^*, \ldots, x_m^*)$ denotes the decreasing rearrangement of $(x_1, x_2, \ldots, x_m)$. Then for any sequence $x_1, x_2, \ldots, x_m$,

$$(13) \qquad \begin{aligned} \mathbf{Per}||f_j(x_i)|| &= \sum_{\pi \in S_m} \prod_{i=1}^{m} f_i(x_{\pi(i)}) \\ &\leq \sum_{\pi \in S_m} \prod_{i=1}^{m} g_i(x_{\pi(i)}) = \mathbf{Per}||g_j(x_i)|| \end{aligned}$$

Conjecture 2.1 can be given a more appealing matrix formulation which we now describe. Defining $A_{i,j} = f_j(x_i)$ and $B_{i,j} = g_j(x_i)$, $i, j = 1, 2, \ldots, m$, we have

$$\mathbf{Per}(A) = \sum_{\pi \in S_m} \prod_{i=1}^{m} f_i(x_{\pi(i)})$$

and

$$\mathbf{Per}(B) = \sum_{\pi \in S_m} \prod_{i=1}^{m} g_i(x_{\pi(i)}).$$

Thus (13) is equivalent to $\mathbf{Per}(A) \leq \mathbf{Per}(B)$, where $x_1, x_2, \ldots, x_m$ is an arbitrary sequence of real numbers. By the invariance of permanents under row permutations we may, without loss of generality, assume that $x_1 \geq x_2 \geq$

$\ldots \geq x_m$. Condition (12) applied to the sequence $x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(m)}$ for any permutation $\pi \in S_m$, becomes

$$\begin{aligned} & A_{\pi(1),1}A_{\pi(2),2}\cdots A_{\pi(m),m} = f_1(x_{\pi(1)})f_2(x_{\pi(2)})\cdots f_m(x_{\pi(m)}) \\ & \leq g_1(x_1)g_2(x_2)\cdots g_m(x_m) = B_{1,1}B_{2,2}\cdots B_{m,m}. \end{aligned} \tag{14}$$

Note that in Conjecture 2.1, (12) is assumed also when $(x_1, x_2, \ldots, x_m)$ is replaced by $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ for any nondecreasing sequence $1 \leq i_1 \leq i_2 \leq \ldots \leq i_m \leq m$ of integers. For this choice, which allows equalities among the x_i's, condition (12) becomes

$$\begin{aligned} & A_{i_{\pi(1)},1}A_{i_{\pi(2)},2}\cdots A_{i_{\pi(m)},m} = f_1(x_{i_{\pi(1)}})f_2(x_{i_{\pi(2)}})\cdots f_m(x_{i_{\pi(m)}}) \\ & \leq g_1(x_{i_1})g_2(x_{i_2})\cdots g_m(x_{i_m}) = B_{i_1,1}B_{i_2,2}\cdots B_{i_m,m}. \end{aligned} \tag{15}$$

This leads to

Definition Let A and B be $m \times m$ nonnegative matrices. We say that the relation

$$A \ll B \tag{16}$$

holds if for any nondecreasing sequence $1 \leq i_1 \leq i_2 \leq \ldots \leq i_m \leq m$ of integers and any permuation π of $\{1, 2, \ldots, m\}$:

$$A_{i_{\pi(1)},1}A_{i_{\pi(2)},2}\cdots A_{i_{\pi(m)},m} \leq B_{i_1,1}B_{i_2,2}\cdots B_{i_m,m}\,. \tag{17}$$

Note that in (17) we allow equalities between the i_j's. It is easily seen that the relation $\ll$ is transitive and reflexive, and thus defines a quasi-order on the set of $m \times m$ matrices. Conjecture 2.1 reduces to

Conjecture 2.2 *Let A and B be $m \times m$ nonnegative matrices. If $A \ll B$, then*

$$\mathbf{Per}(A) \leq \mathbf{Per}(B).$$

If the support of $d\mu$ in Proposition 1.1 consists of only two points, say $\{0, 1\}$, then the matrix $A_{i,j} = f_j(x_i)$ defined above has only two distinct rows: $(f_1(1), \ldots, f_m(1))$, and $(f_1(0), \ldots, f_m(0))$. In this case we can prove Conjecture 2.2. This is the content of the next lemma:

Lemma 2.2 *Conjecture 2.2 holds for the case that A is an $m \times m$ nonnegative matrix such that for some r between 1 and m, A consists of r identical rows followed by $m - r$ identical rows, and B has the same structure.*

We briefly discuss Lemma 2.2 before proving it. It turns out that Lemma 2.2 leads to a *complete* proof of Theorem 1.4 and Proposition 1.1. In fact, as we saw, Lemma 2.2 suffices to prove Proposition 1.1 in the case that the measure $d\mu$ has support on $\{0, 1\}$. This is the basis for an induction argument (to which we briefly referred in the Introduction) which proves Theorem 1.4 when the support of $d\mu$ is $\{0,1\}^n$, for any n. In order to prove Proposition 1.1 for the case that $d\mu$ has a finite support of cardinality s, say, we embed this support in $\{0,1\}^{s-1}$ and apply the previous result. Thus, even Proposition 1.1, which is one-dimensional, requires the multivariate result and the induction argument. For $d\mu$ of infinite support a further approximation argument is required. The details of these arguments are given in Rinott and Saks (1990).

Recall that Proposition 1.1 would follow directly from either Conjecture 2.1, or Conjecture 2.2, without any need for induction, embedding, and approximation. The rest of this paper concerns Conjecture 2.2. We hope that the above discussion and the partial results and variations we present next, will generate interest in our conjectures.

For the proof of Lemma 2.2 we need a simple majorization-type lemma.

LEMMA 2.3 *Let α_l and β_l be nonnegative numbers, $l = 1,\ldots,n$. Assume that for any $V \subseteq \{1,\ldots,n\}$ there exists a set $W \subseteq \{1,\ldots,n\}$ with $|V| = |W|$ and $\prod_{l\in V}\alpha_l \le \prod_{l\in W}\beta_l$. Then $\sum_{l=1}^n \alpha_l \le \sum_{l=1}^n \beta_l$.*

PROOF This follows readily from Theorem 3.C.1.b. in Marshall and Olkin (1979) and the fact that the conditions of the Lemma are equivalent to $(\log(\alpha_1),\ldots,\log(\alpha_n))$ being weakly majorized by $(\log(\beta_1),\ldots,\log(\beta_n))$.

PROOF OF LEMMA 2.2 For $\pi \in S_m$ define $\alpha_\pi = \Pi_{i=1}^m A_{\pi(i),i}$, $\beta_\pi = \Pi_{i=1}^m B_{\pi(i),i}$. Then, $\mathbf{Per}(A) \le \mathbf{Per}(B)$ is equivalent to $\sum_\pi \alpha_\pi \le \sum_\pi \beta_\pi$. By Lemma 2.3 it suffices to show that for any $V \subseteq S_m$ there exists a set $W \subseteq S_m$ with $|V| = |W|$ and

$$\prod_{\pi\in V}\alpha_\pi \le \prod_{\pi\in W}\beta_\pi . \tag{18}$$

Recall that A has (at most) two distinct rows, $(s_1, s_2, \ldots, s_m)$ and $(t_1, t_2, \ldots, t_m)$, say. Then $\alpha_\pi = \prod_{j=1}^m s_j^{a_j} t_j^{b_j}$, where a_j and b_j are either 0 or 1 and $a_j + b_j = 1$. Therefore

$$\prod_{\pi\in V}\alpha_\pi = \prod_{j=1}^m s_j^{k_j} t_j^{l_j} \tag{19}$$

for some nonnegative integers satisfying $k_j + l_j = |V|, j = 1,\ldots,m$. Denote the two distinct rows of B by $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_m)$, with $\mathbf{u}$ preceding $\mathbf{v}$. Let $\eta \in S_m$ be such that $k_{\eta(1)} \ge k_{\eta(2)} \ge \cdots \ge k_{\eta(m)}$. We can rewrite the r.h.s. of (19) as $s_{\eta(1)}^{k_{\eta(1)}} \cdots s_{\eta(m)}^{k_{\eta(m)}} t_{\eta(1)}^{l_{\eta(1)}} \cdots t_{\eta(m)}^{l_{\eta(m)}}$. Now observe that the latter expression can be written as a product of terms of the form $s_{\eta(1)} \cdots s_{\eta(r)} t_{\eta(r+1)} \cdots t_{\eta(m)}$. Condition (17) applied to the present case becomes $s_{\eta(1)} \cdots s_{\eta(r)} t_{\eta(r+1)} \cdots t_{\eta(m)} \le u_1 \cdots u_r v_{r+1} \cdots v_m$. We thus obtain

$$\prod_{j=1}^{m} s_j^{k_j} t_j^{l_j} \le \prod_{j=1}^{m} u_j^{k_{\eta(j)}} v_j^{l_{\eta(j)}} . \tag{20}$$

Finally, note that the quantity on the r.h.s. of (20) equals $\prod_{\pi \in W} \beta_\pi$ for the coset $W = V\eta = \{\pi\eta : \pi \in V\}$. Thus (18) is established and the proof is complete.

In view of Lemma 2.3 and the subsequent discussion (see (18)), the following conjecture would imply Conjecture 2.2.

CONJECTURE 2.3 *Let A and B be $m \times m$ nonnegative matrices satisfying $A \ll B$. Then for every subset V of the permutation group S_m there exists a subset W of S_m with $|V| = |W|$ and*

$$\prod_{\pi \in V} \alpha_\pi \le \prod_{\pi \in W} \beta_\pi , \tag{21}$$

where $\alpha_\pi = \Pi_{i=1}^{m} A_{\pi(i),i}$ and $\beta_\pi = \Pi_{i=1}^{m} B_{\pi(i),i}$.

The case $m = 2$ of Conjecture 2.3 is already covered in the proof of Lemma 2.2. However let us verify it directly. As usual, for distinct indices $1 \le i_1, i_2, \ldots, i_m \le m$, let $(i_1, i_2, \ldots, i_m)$ denote the permutation π with $\pi(j) = i_j, j = 1, \ldots, m$. In the case $m = 2$ of Conjecture 2.3 we have to consider three possible sets V: $\{(1,2)\}, \{(2,1)\}$ and $\{(1,2),(2,1)\}$. It is very easy to see that for the first two, Conjecture 2.3 holds with $W = \{(1,2)\}$. For the last take $W = V$ and use the fact that for any i, $\prod_{j=1}^{m} A_{i,j} \le \prod_{j=1}^{m} B_{i,j}$, so $\prod_{\pi \in S_2} \alpha_\pi = A_{1,1}A_{2,2}A_{2,1}A_{1,2} = (A_{1,1}A_{1,2})(A_{2,1}A_{2,2}) \le (B_{1,1}B_{1,2})(B_{2,1}B_{2,2}) = \prod_{\pi \in S_2} \beta_\pi$, where the inequality is obtained by applying (17) twice. We note that for $m > 2$, we have examples showing that the set W in Conjecture 2.3 need not be unique.

In order to discuss Conjecture 2.3 for $m > 2$, we need some notation. A vector $\mathbf{s} = (s_1, \ldots, s_m) \in \mathbb{R}^m$ is an *m-sequence* if all of its entries are integers in the range 1 to m. We use lower case Greek letters to denote m sequences whose entries are distinct, i.e., that represent permutations. Each m-sequence $\mathbf{s}$ is associated to a matrix $\Psi_{\mathbf{s}}$ which has a 1 in position (s_i, i) for each $i = 1, \ldots, m$, and all other entries are 0. In particular, for a permutation π, Ψ_π is its associated permutation matrix. If T is a set of m-sequences, we set $\Psi_T = \sum_{\mathbf{s} \in T} \Psi_{\mathbf{s}}$.

Let $C_{m,k}$ denote the class of $m \times m$ nonnegative integer matrices with all row and column sums equal to k. It can be shown that this class consists of all matrices that can be written as a sum of k $m \times m$ permutation matrices (which need not be *distinct*). This is proved in the same way as the well known theorem of Birkhoff and Von Neumann about doubly stochastic matrices, see, e.g., Marshall and Olkin (1979, p. 36 2.F.1).

We denote by $\mathcal{U}_{m,k}$ the subset of $\mathcal{C}_{m,k}$ consisting of matrices which can be expressed as the sum of k *distinct* permutation matrices. Note that $\mathcal{U}_{m,k} = \{\Psi_V : V \subseteq S_m, |V| = k\}$.

For $P \in \mathcal{C}_{m,k}$, set $A^P = \prod_{i,j} A_{i,j}^{P_{i,j}}$. Note that for a subset V of S_m, we have

$$A^{\Psi_V} = \prod_{\pi \in V} \alpha_\pi . \tag{22}$$

We can now rewrite Conjecture 2.3 as follows:

CONJECTURE 2.4 *Let A and B be $m \times m$ nonnegative matrices satisfying $A \ll B$. Then for every subset V of S_m there exists a subset W of S_m, with $|V| = |W|$, such that*

$$A^{\Psi_V} \leq B^{\Psi_W}, \tag{23}$$

Given an m-sequence $\mathbf{s}$, let $\mathbf{s}_*$ denote its *increasing* (more precisely, non-decreasing) rearrangement. The set of defining conditions for $A \ll B$ is easily seen to be equivalent to the set of conditions: $A^{\Psi_{\mathbf{s}}} \leq B^{\Psi_{\mathbf{s}_*}}$ for all m-sequences $\mathbf{s}$. Furthermore, it is clear that if $\{(P_i, Q_i)\}$ is any indexed family of matrix pairs for which $A^{P_i} \leq B^{Q_i}$ and α_i are nonnegative constants, then $A^{\sum_i \alpha_i P_i} \leq B^{\sum_i \alpha_i Q_i}$. This suggests the following

DEFINITION Let $\mathcal{P}$ denote the convex cone (in $\mathbb{R}^{m^2} \times \mathbb{R}^{m^2}$) consisting of all pairs of matrices of the form $(\sum \alpha_{\mathbf{s}} \Psi_{\mathbf{s}}, \sum \alpha_{\mathbf{s}} \Psi_{\mathbf{s}_*})$ where the sum extends over all m-sequences $\mathbf{s}$, and all $\alpha_{\mathbf{s}}$ are nonnegative.

Our next goal is to show that Conjectures 2.3-2.4 can be recast in terms of the cone $\mathcal{P}$ without involving the relation $\ll$. For this purpose we need the following straightforward Proposition which provides an alternative characterization of the relation $\ll$.

PROPOSITION 2.1 *$A \ll B$ if and only if $A^P \leq B^Q$ for all (P, Q) is in $\mathcal{P}$.*

REMARK If (P, Q) is a pair of matrices such that $A^P \leq B^Q$ holds whenever A and B are matrices satisfying $A \ll B$, then (P, Q) must belong to $\mathcal{P}$. To see this, note that $A \ll B$ if and only if the $2m^2$-dimensional vector $(-\log A, \log B)$ has nonnegative dot-product with every vector in $\mathcal{P}$. Now apply a separating hyperplane argument (e.g., Farkas' Lemma, see Papadimitriou and Steiglitz (1982, p. 74)): If C is a cone and D is the set of vectors whose dot-product with each vector in C is non-negative, then any vector $\mathbf{v}$ whose dot-product with every vector in D is non-negative belongs to C.

In view of Proposition 2.1 and the remark following it, verifying Conjecture 2.4 is equivalent to proving that for any $P = \Psi_V$ in $\mathcal{U}_{m,k}$ there exists a matrix $Q = \Psi_W$ in $\mathcal{U}_{m,k}$ such that $(P, Q) \in \mathcal{P}$. We thus avoid the condition $\ll$ and reduce Conjectures 2.3-2.4 to a conjecture on sums of permutation matrices:

CONJECTURE 2.5 *For every matrix P in $\mathcal{U}_{m,k}$ there is a matrix Q in $\mathcal{U}_{m,k}$ such that $(P,Q) \in \mathcal{P}$.*

Clearly, this last conjecture would imply all the previous ones.

In order to prove Conjecture 2.5 it would be sufficient to show:

(*) any matrix P in $\mathcal{U}_{m,k}$ has a representation $P = \sum_{l=1}^{k} \Psi_{\mathbf{s}^{(l)}}$, where $\mathbf{s}^{(l)}$ are m-sequences, such that the matrix $Q = \sum_{l=1}^{v} \Psi_{\mathbf{s}_*^{(l)}}$ is also in $\mathcal{U}_{m,k}$.

It is easy to see that any P in $\mathcal{C}_{m,k}$ can be represented as $P = \sum_{l=1}^{k} \Psi_{\mathbf{s}^{(l)}}$. Generally the representation is not unique. However we do not have a way of constructing a representation that guarantees that if P is in $\mathcal{U}_{m,k}$, then so is $Q = \sum_{l=1}^{v} \Psi_{\mathbf{s}_*^{(l)}}$. Thus, we cannot prove Conjecture 2.5 in general. For $m = 2$ and $m = 3$ we can verify (*) by exhausting all cases of $P \in \mathcal{U}_{m,k}$, and constructing for each P a suitable matrix Q. Since $\cup_k \mathcal{U}_{m,k} = \{\Psi_V : V \subseteq S_m\}$, an exhaustive search of $\mathcal{U}_{m,k}$ for all k requires checking $2^{m!}$ subsets V of S_m. Such a search is not hard for $m = 3$, but becomes very time consuming for larger values of m.

REMARK It is not hard to show that any matrix P in $\mathcal{C}_{m,k}$ has a (unique) representation of the form $P = \sum_{l=1}^{k} \Psi_{\mathbf{s}^{(l)}}$ where $\mathbf{s}^{(l)}$ are m-sequences satisfying $\mathbf{s}^{(1)} \leq \mathbf{s}^{(2)} \leq \ldots \leq \mathbf{s}^{(k)}$. This representation appeared to us to be a natural candidate for satisfying (*). However, we have examples showing that the resulting Q matrix need not be in $\mathcal{U}_{m,k}$,

REMARK Results similar to ours were obtained independently by R. Aharoni and U. Keich about a year after we obtained our results. We benefited from discussions of the subject with them.

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