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INEQUALITIES ON EXPECTATIONS BASED ON THE KNOWLEDGE OF MULTIVARIATE MOMENTS[1]

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The paper deals with discrete moment problems where the possible values of a random vector form a known finite set. First, some earlier results concerning the one dimensional discrete moment problem are summarized. Then, restricting the discussion to the two-dimensional case, for the sake of simplicity, two different discrete moment problems are formulated: (a) the known moments are those where the exponents of the random variables are chosen between 0 and some upper bounds; (b) the sum of the exponents is less than or equal to a given number. The bounds that can be obtained by our technique include bounds for probabilities and expectation.

1. Introduction

The one-dimensional discrete moment problem can be formulated in the following manner. Given a random variable ξ, the possible values of which are known to be $z_0 < z_1 < \cdots < z_n$ and a function $f(z)$, $z \in \{z_0, z_1, \ldots, z_n\}$. We want to give lower and upper bounds for $E[f(\xi)]$, based on the knowledge of the moments $\mu_k = E[\xi^k]$, $k = 1, 2, \ldots, m$, while the probability distribution of ξ is unknown.

Introducing the notations $p_i = P\{\xi = z_i\}$, $f_i = f(z_i)$, $i = 0, 1, \ldots, n$, $\mu_0 = 1$, we obtain the above mentioned bounds by solving the linear programming problems

$$\min(\max) \sum_{i=0}^{n} f_i p_i \tag{1.1}$$

subject to

$$\sum_{i=0}^{n} z_i^k p_i = \mu_k, \qquad k = 0, 1, \ldots, m$$

$$p_i \geq 0, \qquad i = 1, 2, \ldots, n,$$

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where we assume that $m < n$. Problems (1.1) are termed as discrete power moment problems. Replacing z_i^k by $\binom{z_i}{k}$ and μ_k by S_k, where $S_k = E[\binom{\xi}{k}]$, $k = 0, 1, \ldots, m$, we obtain the binomial moment problem which plays an important role in bounding probabilities of logical functions of events (Prékopa (1988, 1990a)). The discrete binomial moment problem can be transformed into the discrete power moment problem; hence we restrict ourselves to problems (1.1). (Note that if instead of the consecutive moments $\mu_1, \mu_2, \ldots, \mu_m$, the moments $\mu_{k_1}, \mu_{k_2}, \ldots, \mu_{k_m}$ would be known where $k_1 < k_2 < \cdots < k_m$ are non-consecutive integers then simple equivalence between the power and the binomial moment problems no longer exists.)

The duals of the problems (1.1) throw new light to this approach of bounding $E[f(\xi)]$. If (1.1) is a minimization problem, then its dual is

$$\max \sum_{k=0}^{m} \mu_k x_k \tag{1.2}$$

subject to

$$\sum_{k=0}^{m} z_i^k x_k \le f_i, \qquad i = 0, 1, \ldots, n,$$

and if (1.1) is a maximization problem, then its dual is

$$\min \sum_{k=0}^{m} \mu_k y_k \tag{1.3}$$

subject to

$$\sum_{k=0}^{m} z_i^k y_k \ge f_i, \qquad i = 0, 1, \ldots, n.$$

The optimum values of these problems are called the sharp lower and upper bounds for $E[f(\xi)]$. Since problems (1.1) have feasible solutions and finite optima, the duality theorem of linear programming ensures that so do problems (1.2), (1.3) and the optimum values of the primal-dual pairs coincide. Thus, we have the inequalities

$$\sum_{k=0}^{m} \mu_k x_k \le E\left[f(\xi)\right] \le \sum_{k=0}^{m} \mu_k y_k, \tag{1.4}$$

where $x_0, x_1, \ldots, x_m$ satisfy (1.2) and $y_0, y_1, \ldots, y_m$ satisfy (1.3). The bounds (1.4) are the best in the case of the optimal solutions x, y.

Among the choices of the function f, prominent are the following:

(1) f has positive divided differences of order $m + 1$ (for the definition of the divided differences see the next section).

(2) $f_r = 1$ and $f_i = 0$ for $i \neq r$.

(3) $f_0 = \cdots = f_{r-1} = 0,\ f_r = \cdots = f_n = 1.$

In cases (2) and (3) we are bounding $P\{\xi = z_r\}$ and $P\{\xi \geq z_r\}$, respectively.

If we did not know the possible values of ξ but we still would know the moments $\mu_1, \mu_2, \ldots, \mu_m$ then the general moment problem (see, e.g., Krein and Nudelman (1977)):

$$\min(\max) \int_a^b f(z)d\sigma \tag{1.5}$$

subject to

$$\int_a^b z^k d\sigma = \mu_k, \qquad k = 0, 1, \ldots, m$$

would provide us with bounds for $E[f(\xi)]$, where σ is the unknown probability distribution function on $[a, b]$. Assuming $a = z_0$, $b = z_n$, furthermore, designating by L_d, U_d and L_c, U_c the optimum values corresponding to problems (1.1) and (1.5), respectively, we have the relations

$$L_c \leq L_d \leq E[f(\xi)] \leq U_d \leq U_c. \tag{1.6}$$

This means that if the set of possible values of ξ is a known discrete set and we utilize it in the form of solving problems (1.1) then better bounds can be obtained than through solving problems (1.5). This is so even though the optimal solutions of problems (1.5) are discrete distributions. In fact, the supports of these distributions may not be subsets of $\{z_0, z_1, \ldots, z_n\}$.

Recent discovery by Samuels and Studden (1989) and by Prékopa (1988, 1990a, 1990b) of the fact that the sharp Bonferroni inequalities are essentially solutions of discrete moment problems, stresses the importance of the discrete case. A variety of applications of the discrete moment problem, ranging from communication or power system reliability calculations to approximations in queueing systems, can be mentioned.

2. Dual Feasible Bases and Lagrange Polynomials Associated with Problems (1.1)

In this section we further restrict ourselves to that special case of the objective function f where all $m+1$st divided differences are positive. The first order divided differences of f are

$$[z_{i_1}, z_{i_2}]f = \frac{f(z_{i_2}) - f(z_{i_1})}{z_{i_2} - z_{i_1}}, \qquad 0 \leq i_1 < i_2 \leq n. \tag{2.1}$$

The higher order divided differences are defined recursively by

$$[z_{i_1}, z_{i_2}, \ldots, z_{i_k}, z_{i_{k+1}}]f = \frac{[z_{i_2}, \ldots, z_{i_{k+1}}]f - [z_{i_1}, \ldots, z_{i_k}]f}{z_{i_{k+1}} - z_{i_1}}, \tag{2.2}$$

$$0 \le i_1 < i_2 < \cdots < i_k < i_{k+1} \le n$$

for $k \ge 2$. It is known (see, e.g., Jordan (1965)) that if all divided differences of order k, corresponding to consecutive points, are positive then all divided differences of order k are positive, and we have the equality

$$[z_{i_1}, z_{i_2}, \ldots, z_{i_{k+1}}]f = \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_{i_1} & z_{i_2} & \cdots & z_{i_{k+1}} \\ \vdots & & \ddots & \\ z_{i_1}^{k-1} & z_{i_2}^{k-1} & \cdots & z_{i_{k+1}}^{k-1} \\ f(z_{i_1}) & f(z_{i_2}) & \cdots & f(z_{i_{k+1}}) \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_{i_1} & z_{i_2} & \cdots & z_{i_{k+1}} \\ \vdots & & \ddots & \\ z_{i_1}^{k-1} & z_{i_2}^{k-1} & \cdots & z_{i_{k+1}}^{k-1} \\ z_{i_1}^{k} & z_{i_2}^{k} & \cdots & z_{i_{k+1}}^{k} \end{vmatrix}}. \tag{2.3}$$

If f is defined for every z in $[z_0, z_n]$ and $f^{(m+1)}(z) > 0$ at every interior point z of this interval, then all $m+1$st divided differences of f are positive on $z_0, z_1, \cdots, z_n$ (see Jordan (1965)). The positivity of the first order divided differences means that f is increasing and the positivity of the second order divided differences means the convexity of the function f, i.e., the polygon connecting the points $(z_i, f(z_i)), i = 0, 1, \ldots, n$ in the plane, is convex. This implies that an equivalent formulation of the positivity of the second order divided differences is:

$$\frac{f(z_{i_3}) - f(z_{i_1})}{z_{i_3} - z_{i_1}} > \frac{f(z_{i_2}) - f(z_{i_1})}{z_{i_2} - z_{i_1}}, \qquad 0 \le i_1 < i_2 < i_3 \le n.$$

We assume that the $m+1$st divided differences of f are positive (while there is no condition on the lower order divided differences). Thus, we handle the type (1) of the objective function f, mentioned in the Introduction. The results corresponding to the others are presented in Prékopa (1990b).

Let $\mathbf{a}_i^T = (1, z_i, \cdots, z_i^m)$, $i = 0, \cdots, n$, and $A = (\mathbf{a}_0, \cdots, \mathbf{a}_n)$. Furthermore let B be an $(m+1) \times (m+1)$ part of the matrix A. Since B is a Vandermonde matrix, it is non-singular and thus, it represents a basis in the linear programming (minimization or maximization) problem (1.1). The columns of B will be called basic vectors.

Let $\mathbf{f}_\mathrm{B}$ designate the vector consisting of those f_i values as components which correspond to basic vectors $\mathbf{a}_i$. By definition, B is dual feasible if

(2.4) $f_j - \mathbf{f}_\mathrm{B}^\mathrm{T} B^{-1}\mathbf{a}_j \geq 0$ for all j, in the minimization problem,

(2.5) $f_j - \mathbf{f}_\mathrm{B}^\mathrm{T} B^{-1}\mathbf{a}_j \leq 0$ for all j, in the maximization problem.

If $\mathbf{a}_j$ is basic then we have equality in the above relations.

Let I be the set of subscripts of the basic vectors and $L_\mathrm{I}(z)$ the Lagrange polynomial corresponding to the points z_i, $i \in I$. Then

$$L_\mathrm{I}(z) = \sum_{i=0}^{m} L_{\mathrm{I}i}(z) f(z_i),$$

where

$$L_{\mathrm{I}i}(z) = \prod_{j \in I - \{i\}} \frac{(z - z_j)}{(z_i - z_j)}$$

is the ith fundamental polynomial. Furthermore let $\mathbf{b}^\mathrm{T}(z) = (1, z, \ldots, z^m)$. Clearly we have $\mathbf{b}(z_i) = \mathbf{a}_i$, $i = 0, 1, \ldots, n$ and

$$f(z) - \mathbf{f}_\mathrm{B}^\mathrm{T} B^{-1}\mathbf{b}(z) = \frac{1}{|B|} \begin{vmatrix} f(z) & \mathbf{f}_\mathrm{B}^\mathrm{T} \\ \mathbf{b}(z) & B \end{vmatrix}. \tag{2.6}$$

From (2.6) we first derive

$$\mathbf{f}_\mathrm{B}^\mathrm{T} B^{-1}\mathbf{b}(z) = L_\mathrm{I}(z). \tag{2.7}$$

In fact, $\mathbf{f}_\mathrm{B}^\mathrm{T} B^{-1}\mathbf{b}(z)$ is an mth degree polynomial that is equal to $f(z_i)$ if $z = z_i$. Another observation is that if $z \notin \{z_i, i \in I\}$ then the second determinant on the right hand side in (2.6) is different from 0. This follows from (2.3) and the assumption that all $m+1$st order divided differences of f are positive. This implies that if $z \notin \{z_i, i \in I\}$ then (2.6) is nonzero, in other words, no basis is dual-degenerate. Hence, in (2.4) and (2.5) we have equalities at basic points, otherwise we have strict inequalities. This means that the basis B is dual feasible in the minimization (maximization) problem (1.1) if and only if the Lagrange polynomial corresponding to the basic points $\{z_i, i \in I\}$ is strictly below (above) the function $f(z)$ at any nonbasic point z.

Using (2.6) and (2.3), we obtain the equation

$$f(z) - L_\mathrm{I}(z) = \prod_{j \in I}(z - z_j)\,[z, z_i, i \in I]\,f, \tag{2.8}$$

which is well-known in interpolation theory. We also mention Newton's form for the interpolating polynomial:

$$L_I(z) = f_0 + \sum_{k=0}^{m} \prod_{j \in I^{(k)}} (z - z_j) \left[z_j, j \in I^{(k)}\right] f,$$

where $I^{(k)}$ is the set of the first $k+1$ elements of I and f_0 is the function value corresponding to the first element in I.

In order to find necessary and sufficient condition that a basis is dual feasible in any of the problems (1.1), equation (2.8) can be used. Since $[z, z_i, i \in I]f > 0$ for every $z \notin \{z_i, i \in I\}$, we see that the basis is dual feasible in the minimization problem, if and only if

$$\prod_{j \in I}(z - z_j) > 0 \qquad \text{for all non-basic } z,$$

and is dual feasible in the maximization problem, if and only if

$$\prod_{j \in I}(z - z_j) < 0 \qquad \text{for all non-basic } z.$$

Thus, the basic vectors have to follow each other according to some patterns that can be best summarized in terms of their subscripts. If the basis B corresponds to the subscript set I, then sometimes we will write $B(I)$ instead of B.

THEOREM 2.1 *A basis $B(I)$ is dual feasible in the minimization (maximization) problem* (1.1) *if and only if the subscript set I, with elements arranged in increasing order, has the following structure:*

	$m+1$ *even*	$m+1$ *odd*
Minimization problem	$\{j, j+1, \ldots, k, k+1\}$	$\{0, j, j+1, \ldots, k, k+1\}$
Maximization problem	$\{0, j, j+1, \ldots, k, k+1, n\}$	$\{j, j+1, \ldots, k, k+1, n\}$.

3. Bounds for $E[f(\xi)]$

Theorem 2.1 can be used to give bounds for $E[f(\xi)]$. These can be obtained in terms of formulas if m is small ($m \le 4$) or by algorithms if m is large.

Any dual feasible basis B provides us with a bound. If it is dual feasible in the minimization problem, then the corresponding objective function value is smaller than or equal to the optimum value. Hence,

$$\mathbf{f}_B^T B^{-1}\mu = E\left[L_I(\xi)\right] \le E\left[f(\xi)\right], \tag{3.1}$$

where $\mu^T = (1, \mu_1, \ldots, \mu_m)$. If, on the other hand, B is dual feasible in the maximization problem, then

$$\mathbf{f}_B^T B^{-1}\mu = E\left[L_I(\xi)\right] \ge E\left[f(\xi)\right]. \tag{3.2}$$

Note that the inequalities in (3.1) and (3.2) hold also for each possible value of ξ, if we remove the expectations.

A basis B is said to be primal feasible if $B^{-1}\mu \geq 0$. A basis that is both primal and dual feasible, is optimal. Combining these remarks with Theorem 2.1, we have

THEOREM 3.1 *Assume that the function f has all positive divided differences of order $m+1$. The following assertions hold true:*

(a) *If I has one of the two structures*

$$\{j, j+1, \ldots, k, k+1\}, \qquad \{0, j, j+1, \ldots, k, k+1\},$$

then

$$L_I(z) \leq f(z),$$

with strict inequality for all nonbasic z, and

$$E[L_I(\xi)] \leq E[f(\xi)].$$

This bound is sharp if $B(I)$ is a primal feasible basis in problem (1.1).

(b) *If I has one of the structures*

$$\{j, j+1, \ldots, k, k+1, n\}, \qquad \{0, j, j+1, \ldots, k, k+1, n\},$$

then

$$L_I(z) \geq f(z),$$

with strict inequality for all nonbasic z, and

$$E[L_I(\xi)] \geq E[f(\xi)].$$

This bound is sharp if $B(I)$ is a primal feasible basis in problem (1.1).

In order to obtain the sharp bound we need to check which one is that dual feasible basis $B(I)$ for which we also have primal feasibility, i.e., $[B(I)]^{-1}\mu \geq 0$. If m is small then Theorem 2.1 gives us a key to find this $B(I)$ by a formula (see Boros and Prékopa (1989), Prékopa (1990b)). However, if m is large then the sharp bound can be obtained only by an algorithm.

Instead of a general linear programming algorithm, the following very advantageous dual type algorithm can be used to solve any of the problems (1.1).

Step 0. Pick any dual feasible basis subscript set I, in accordance with Theorem 2.1.

Step 1. Check if $[B(I)]^{-1}\mu \geq 0$. If yes, then stop; optimal basis and optimal solution has been reached. Otherwise pick any j for which $([B(I)]^{-1})_j < 0$, and go to Step 2.

Step 2. Delete the jth vector from $B(I)$ (which is not necessarily the same as $\mathbf{a}_j$) and include that vector which restores the dual feasible basis structure. Go to Step 1.

Since no dual degeneracy occurs, the objective function values are strictly increasing and the algorithm terminates in a finite number of steps.

The above algorithm is of dual type but it is not exactly a special case of the dual algorithm of Lemke (1954). The difference is that here the incoming vector can be found very easily through a logical analysis of the subscript set I, rather than a costly procedure involving reduced prices. For more details of the algorithm see Prékopa (1990a).

NUMERICAL EXAMPLE We present sharp lower and upper bounds for the moment generating function $E\left(e^{t\xi}\right)$ at the point $t = 0.1$. We assume that the possible values of ξ are known to be $z_i = i$, $i = 0, 1, \cdots, 20$ and we know the first three moments of ξ: $\mu_1 = 9.73086229944$, $\mu_2 = 129.5641151$, $\mu_3 = 1903.250122$.

The function $f(z) = e^{0.1z}$ has positive derivative of any order at any z, hence the condition for f (that its fourth order divided differences, on the set of possible values of ξ, are positive) is fulfilled.

Using the algorithm described in this section, both the minimization and maximization problems (1.1) have been solved instantly on a 33MHz/486 PC. The code was written in APL language which is very suitable to these problems. Below we present the subscript sets of the bases encountered in the subsequent iterations, together with the optimal solutions.

Minimization problem

Initial basis	6	7	8	9
	7	8	9	10
	8	9	10	11
	7	8	10	11
	7	8	11	12
	6	7	11	12
	5	6	11	12
	5	6	12	13
	4	5	12	13
	4	5	13	14
	3	4	13	14
	3	4	14	15
Optimal basis	3	4	15	16

The optimal solution is:

$$p_3 = 0.3170498444, \qquad p_4 = 0.1397544435, \qquad p_{15} = 0.4704357076,$$

$$p_{16} = 0.0727600045, \qquad p_i = 0, \quad \text{for any other } i.$$

The optimum value is:

$$e^{(0.1)3}p_3 + e^{(0.1)4}p_4 + e^{(0.1)15}p_{15} + e^{(0.1)16}p_{16} = 3.105190886.$$

Maximization problem

Initial basis	0	1	2	20
	0	2	3	20
	0	3	4	20
	0	4	5	20
	0	5	6	20
	0	6	7	20
	0	7	8	20
	0	8	9	20
	0	9	10	20
Optimal basis	0	10	11	20

The optimal solution is:

$$p_0 = 0.20523691, \qquad p_{10} = 0.2755241346, \qquad p_{11} = 0.3787952662,$$

$$p_{20} = 0.140443686, \qquad p_i = 0, \quad \text{for any other } i.$$

The optimum value is:

$$e^{(0.1)}p_0 + e^{(0.1)10}p_{10} + e^{(0.1)11}p_{11} + e^{(0.1)20}p_{20} = 3.129899305.$$

Thus, we have the sharp bounds:

$$3.105190886 \le E\left(e^{0.1\xi}\right) \le 3.129899305.$$

4. Multivariate Discrete Moment Problems

For the sake of simplicity we restrict ourselves to the discussion of the bivariate case. The results generalize to the multivariate case in a straightforward manner.

Let ξ_1 and ξ_2 be two discrete random variables with known finite supports which are z_{ij}, $j = 0, 1, \dots, n_i$, $i = 1, 2$, and assume that some of the bivariate moments

$$\mu_{\alpha\beta} = E\left[\xi_1^\alpha \xi_2^\beta\right] \tag{4.1}$$

are known, where α and β are nonnegative integers, while the probabilities

$$p_{ij} = P\{ \xi_1 = z_{1i}, \xi_2 = z_{2j} \}$$

are unknown. Note that the support of the random vector (ξ_1, ξ_2) is part of the set $\{z_{10}, \cdots, z_{1n_1}\} \times \{z_{20}, \cdots, z_{2n_2}\}$ but we do not assume any further knowledge about it. Let furthermore $f(z_1, z_2)$ be a function on the set $\{z_{10}, \cdots, z_{1n_1}\} \times \{z_{20}, \cdots, z_{2n_2}\}$. We intend to give lower and upper bounds for $E[f(\xi_1, \xi_2)]$ under some conditions regarding the moments (4.1) and the function f.

As regards the moments $\mu_{\alpha\beta}$, we consider two cases:

(a) there exist positive integers m_1, and m_2 such that $\mu_{\alpha\beta}$ are known for all α and β satisfying $0 \le \alpha \le m_1$, $0 \le \beta \le m_2$;

(b) there exists a positive integer m such that $\mu_{\alpha\beta}$ are known for all $\alpha \ge 0$, $\beta \ge 0$, $\alpha + \beta \le m$.

The corresponding linear programming problems providing us with the sharp lower and upper bounds for $E[f(\xi_1, \xi_2)]$, are the following (let $f_{ij} = f(z_{1i}, z_{2j})$):

$$\min(\max) \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} f_{ij} p_{ij} \tag{4.2}$$

subject to

$$\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} z_{1i}^{\alpha} z_{2j}^{\beta} p_{ij} = \mu_{\alpha\beta}$$

$$0 \le \alpha \le m_1, \qquad 0 \le \beta \le m_2$$

$$p_{ij} \ge 0, \qquad 0 \le i \le n_1, \qquad 0 \le j \le n_2$$

and

$$\min(\max) \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} f_{ij} p_{ij} \tag{4.3}$$

subject to

$$\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} z_{1i}^{\alpha} z_{2j}^{\beta} p_{ij} = \mu_{\alpha\beta}$$

$$\alpha \ge 0, \qquad \beta \ge 0, \qquad \alpha + \beta \le m$$

$$p_{ij} \ge 0, \qquad 0 \le i \le n_1, \qquad 0 \le j \le n_2.$$

Regarding the function f, the technique developed in Prékopa (1990b) for the univariate discrete moment problem and partly outlined in the previous sections, allows for handling problem (4.2) in three different cases which are analogous with the cases (1), (2) and (3) mentioned in Section 1. In this paper, however, we restrict ourselves to the two-dimensional version of case (1).

Our condition on f is formulated for the case of problem (4.2). Later, we will use the results concerning problem (4.2), to obtain results for problem (4.3). First we introduce some notations. Let $I_1 \subset \{0, 1, \ldots, n_1\}$, $I_2 \subset \{0, 1, \ldots, n_2\}$, $|I_1| = m_1 + 1$, $|I_2| = m_2 + 1$ be some subscript sets and

$$L_{I_1}^{(1)}(z_1, z_2) = \sum_{i \in I_1} f(z_{1i}, z_2) L_{I_1 i}^{(1)}(z_1)$$

$$L_{I_2}^{(2)}(z_1, z_2) = \sum_{j \in I_2} f(z_1, z_{2j}) L_{I_2 j}^{(2)}(z_2)$$

$$L_{I_1 I_2}(z_1, z_2) = \sum_{i \in I_1} \sum_{j \in I_2} f(z_{1i}, z_{2j}) L_{I_1 i}^{(1)}(z_1) L_{I_2 j}^{(2)}(z_2)$$

where

$$L_{I_1 i}^{(1)}(z_1) = \prod_{j \in I_1 - \{i\}} \frac{z_1 - z_{1j}}{z_{1i} - z_{1j}},$$

$$L_{I_2 i}^{(2)}(z_2) = \prod_{j \in I_2 - \{i\}} \frac{z_2 - z_{2j}}{z_{2i} - z_{2j}}.$$

We use the order (m_1+1, m_2+1) divided differences of f over $\{z_{10}, \ldots, z_{1n_1}\} \times \{z_{20}, \ldots, z_{2n_2}\}$ which are defined in a natural way through the subsequent applications of the divided difference operations. This property of f is ensured if it is defined on $[z_{10}, z_{1n_1}] \times [z_{20}, z_{2n_2}]$ and

$$\frac{\partial^{m_1+m_2+2} f(z_1, z_2)}{\partial z_1^{m_1+1} \partial z_2^{m_2+1}} > 0$$

for every interior point of the rectangle (see Popoviciu (1945)).

Conditions on f in problems (4.2)

Let I_1, I_2 be a pair of dual feasible subscript sets, both in the minimization (maximization) problem (1.1), using m_1, and m_2, respectively, instead of m. We assume that at least one of the conditions (i), (ii), (iii), (iv), presented below, is satisfied.

$(i\,a)$ For any fixed $z_2 \in \{z_{20}, \ldots, z_{2n_2}\}$, the function of the variable z_1 : $f(z_1, z_2)$, has all positive divided differences of order $m_1 + 1$.

$(i\,b)$ For any fixed $z_1 \in \{z_{10}, \ldots, z_{1n_1}\}$, the function of the variable z_2 : $L_{I_1}^{(1)}(z_1, z_2)$, has all positive divided differences of order $m_2 + 1$.

(*ii a*) For any fixed $z_1 \in \{z_{10}, \dots, z_{1n_1}\}$, the function of the variable z_2 : $f(z_1, z_2)$, has all positive divided differences of order $m_2 + 1$.

(*ii b*) For any fixed $z_2 \in \{z_{20}, \dots, z_{2n_2}\}$, the function of the variable z_1 : $L^{(2)}_{I_2}(z_1, z_2)$, has all positive divided differences of order $m_1 + 1$.

(*iii*) The function $f(z_1, z_2)$ has all positive divided differences of order $(m_1 + 1, m_2 + 1)$ and both (*i b*) and (*ii b*) hold.

(*iv*) Let $I_1^{(i)}$ and $I_2^{(j)}$ designate the sets of the first $i+1$ and $j+1$ elements in I_1 and I_2, respectively and $[z_{1h},\, h \in I_1^{(i)},\, z_{2k},\, k \in I_2^{(j)}]f$ the divided difference of order (i, j) of the function f, corresponding to the points z_{1h}, $h \in I_1^{(i)}$; z_{2k}, $k \in I_2^{(j)}$. We assume that the inequality (let f_0 be the function value corresponding to the first elements in I_1 and I_2):

$$\begin{aligned} f(z_1, z_2) \quad > \quad & f_0 + \sum_{\substack{i=0 \\ i+j\geq 1}}^{m_1} \sum_{j=0}^{m_2} \left[z_{1h},\, h \in I_1^{(i)}; z_{2k},\, k \in I_2^{(j)}\right] f \\ \times \quad & \prod_{h\in I_1^{(i)}} (z_1 - z_{1h}) \prod_{k\in I_2^{(j)}} (z_2 - z_{2k}) \end{aligned}$$

holds for every $(z_1, z_2) \notin \{z_{1i}, i \in I_1\} \times \{z_{2j}, j \in I_2\}$, if I_1, I_2 correspond to minimization problems. If I_1, I_2 correspond to maximization problems then the opposite inequality is assumed to hold. Note that the sum has the same value for any ordering of the points in the sets $\{z_{1i}, i \in I\}$ and $\{z_{2j}, j \in I_2\}$, and is obtained so that we write up the Newton's form of the polynomial $L_{I_1 I_2}(z_1, z_2)$ subsequently for z_1 and z_2.

Introducing the notations (let $\mu_{00} = 1$):

$$A_i = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_{i0} & z_{i1} & \cdots & z_{in_i} \\ \vdots & & \ddots & \\ z_{i0}^{m_i} & z_{i1}^{m_i} & \cdots & z_{in_i}^{m_i} \end{pmatrix}, \qquad i = 1, 2,$$

$$A = A_1 \otimes A_2 = \begin{pmatrix} A_1 & A_1 & \cdots & A_1 \\ z_{20}A_1 & z_{21}A_1 & \cdots & z_{2n_2}A_1 \\ \vdots & & \ddots & \\ z_{20}^{m_2}A_1 & z_{21}^{m_2}A_1 & \cdots & z_{2n_2}^{m_2}A_1 \end{pmatrix},$$

$$\begin{aligned} \mathbf{b}^{\mathrm{T}} &= E\left[(1, \xi_1, \dots, \xi_1^{m_1}) \otimes (1, \xi_2, \dots, \xi_2^{m_2})\right] \\ &= (\mu_{00}, \mu_{10}, \dots, \mu_{m_1 0}, \mu_{01}, \mu_{11}, \dots), \\ \mathbf{p}^{\mathrm{T}} &= (p_{ij},\, 0 \le i \le n_1,\, 0 \le j \le n_2), \\ \mathbf{f}^{\mathrm{T}} &= (f_{ij},\, 0 \le i \le n_1,\, 0 \le j \le n_2), \end{aligned}$$

we can rewrite problems (4.2) in the concise form:

$$\min(\max)\,\mathbf{f}^{\mathrm{T}}\mathbf{p} \tag{4.4}$$

subject to

$$\begin{aligned} A\mathbf{p} &= \mathbf{b}, \\ \mathbf{p} &\geq \mathbf{0}. \end{aligned}$$

5. Bounds for $E[f(\xi_1, \xi_2)]$

Let I_1, I_2 be basis subscript sets from $\{0, 1, \ldots, n_1\}$ and $\{0, 1, \ldots, n_2\}$, respectively. The set $I = I_1 \times I_2$ represents a basis subscript set for problem (4.4) and $B(I) = B_1(I_1) \otimes B_2(I_2)$ is a basis from A. Bases of this type will be called rectangular. The vector $\mathbf{f}_{\mathrm{B}} = \mathbf{f}_{\mathrm{B}_1\mathrm{B}_2}$ designates that part of f which corresponds to the basis vectors in B. We prove

THEOREM 5.1 *If f satisfies one of the conditions (i), (ii), (iii), (iv) and I_1, I_2 correspond to minimization (maximization) problem, then we have the relations*

$$f(z_1, z_2) = L_{\mathrm{I}_1\mathrm{I}_2}(z_1, z_2) \tag{5.1}$$

for $(z_1, z_2) \in \{z_{1i}, i \in I_1\} \times \{z_{2j}, j \in I_2\}$, *and*

$$f(z_1, z_2) > (<)\, L_{\mathrm{I}_1\mathrm{I}_2}(z_1, z_2), \tag{5.2}$$

otherwise. Furthermore, $I_1 \times I_2$ is a dual feasible basis subscript set in the minimization (maximization) problem.

PROOF Let $B_1(I_1)$ and $B_2(I_2)$ designate that $(m_1 + 1) \times (m_1 + 1)$ and $(m_2 + 1) \times (m_2 + 1)$ parts of A_1 and A_2 which correspond to the columns with subscript sets I_1 and I_2, respectively. Then, as it is easy to show, we have the relations:

$$\frac{1}{|B_1(I_1) \otimes B_2(I_2)|} \begin{vmatrix} f(z_1, z_2) & \mathbf{f}^{\mathrm{T}}_{\mathrm{B}_1\mathrm{B}_2} \\ \begin{pmatrix} 1 \\ z_1 \\ \vdots \\ z_1^m \end{pmatrix} \otimes \begin{pmatrix} 1 \\ z_2 \\ \vdots \\ z_2^m \end{pmatrix} & B_1(I_1) \otimes B_2(I_2) \end{vmatrix}$$

$$\begin{aligned} &= f(z_1, z_2) - L_{\mathrm{I}_1\mathrm{I}_2}(z_1, z_2) \\ &= f(z_1, z_2) - \left(f_0 + \sum_{\substack{i=0 \\ i+j \geq 1}}^{m_1} \sum_{j=0}^{m_2} \left[z_{1h}, h \in I_1^{(i)}, z_{2k}, k \in I_{2(j)} \right] f \right. \\ &\qquad \left. \prod_{h \in I_1^{(i)}} (z_1 - z_{1h}) \prod_{k \in I_2^{(j)}} (z_2 - z_{2k}) \right). \end{aligned} \tag{5.3}$$

Now the expression in the first line of (5.3) is the reduced price (traditionally designated by $c - z$ in linear programming theory) corresponding to the basis with subscript set $I_1 \times I_2$ in problem (4.2) and the point (z_1, z_2). The dual feasibility in the minimization (maximization) problem means that these values are nonnegative (nonpositive) for every nonbasic (z_1, z_2). We will prove positivity (negativity), i.e., also the dual non-degeneracy of the basis. Note that (5.1) holds trivially.

To prove (5.2) under (i) and (ii) is simple. We only have to repeat the reasoning, applied to the one-dimensional case.

The assertion under (iv) is a consequence of the equality between the first and third lines in (5.3).

The assertion under (iii) is a consequence of the equality (see Popoviciu (1945)):

$$\begin{aligned} & f(z_1, z_2) - \left\{ L^{(1)}_{I_1}(z_1, z_2) + L^{(2)}_{I_2}(z_1, z_2) - L_{I_1 I_2}(z_1, z_2) \right\} \\ & \quad = f(z_1,z_2) - L_{I_1\, I_2}(z_1,z_2) - \Big\{ L^{(1)}_{I_1}(z_1,z_2) \\ & \qquad - L_{I_1\, I_2}(z_1,z_2) + L^{(2)}_{I_2}(z_1,z_2) - L_{I_1\, I_2}(z_1,z_2) \Big\} \\ & \quad = \prod_{i\in I_1} (z_1 - z_{1i}) \prod_{j\in I_2} (z_2 - z_{2j}) [z_{1i},\, i \in I_1;\, z_{2j},\, j \in I_2]\, f \end{aligned} \tag{5.4}$$

and the equality between the first and the second lines in (5.3). We only have to apply (2.8) and the rest of the proof is simple. □

THEOREM 5.2 *Suppose that f satisfies one of the conditions (i), (ii), (iii), and (iv). Then the following assertions hold true:*

(a) *If I_1 and I_2 both have one of the structures:*

$$\{ j, j+1, \ldots, k, k+1 \}, \qquad \{ 0, j, j+1, \ldots, k, k+1 \},$$

then

$$L_{I_1 I_2}(z_1, z_2) \le f(z_1, z_2),$$

with strict inequality for all nonbasic (z_1, z_2), and

$$E\left[L_{I_1 I_2}(\xi_1, \xi_2)\right] \le E\left[f(\xi_1, \xi_2)\right]. \tag{5.5}$$

This bound is sharp if $B(I_1 \times I_2)$ is a primal feasible basis in problem (4.2).

(b) *If I_1 and I_2 both have one of the structures:*

$$\{ j, j+1, \ldots, k, k+1, n \}, \qquad \{ 0, j, j+1, \ldots, k, k+1, n \},$$

then

$$L_{I_1 I_2}(z_1, z_2) \ge f(z_1, z_2),$$

with strict inequality for all nonbasic (z_1, z_2), and

$$E\left[L_{I_1 I_2}(\xi_1, \xi_2)\right] \geq E\left[f(\xi_1, \xi_2)\right]. \tag{5.6}$$

This bound is sharp if $B(I_1 \times I_2)$ is a primal feasible basis in problem (4.2).

PROOF The theorem is a consequence of Theorem 4.1 and the fact that a both primal and dual feasible basis is optimal. □

REMARK It is not sure that among the bases of the form $B_1(I_1) \otimes B_2(I_2)$ there is one which is primal feasible. This is ensured if ξ_1 and ξ_2 are independent because in this case $\mu_{\alpha\beta} = E[\xi_1^\alpha]E[\xi_2^\beta]$ and the constraints in problem (4.2) split into two separate sets of constraints, where there are primal feasible bases $B_1(I_1)$ and $B_2(I_2)$. If ξ_1 and ξ_2 are dependent random variables then the sharp inequalities may not be among those in (5.5) and (5.6). Still, we can obtain the sharp inequalities if we use $B_1(I_1) \otimes B_2(I_2)$ as an initial dual feasible basis and apply the dual method for the solution of the problem.

For the case of problem (4.3) the bounds obtained for problem (4.2) can be used in the following manner. First we observe that if both I_1 and I_2 are dual feasible basis subscript sets in the minimization problem, $|I_1| + |I_2| = m + 2$, and f satisfies one of the conditions (*i*), (*ii*), (*iii*), and (*iv*), with $m_1 = |I_1|$, $m_2 = |I_2|$, then

$$L_{I_1 I_2}(z_1, z_2) \leq f(z_1, z_2), \tag{5.7}$$

for any (z_1, z_2). Similarly, if I_1, I_2 are dual feasible basis subscript sets in the maximization problem, $|J_1| + |J_2| = m + 2$, and f satisfies one of the conditions (*i*), (*ii*), (*iii*), and (*iv*), with $m_1 = |J_1|$, $m_2 = |J_2|$,then

$$L_{J_1 J_2}(z_1, z_2) \geq f(z_1, z_2) \tag{5.8}$$

for any (z_1, z_2). Then we replace (z_1, z_2) by (ξ_1, ξ_2), take expectations in (5.7) and (5.8), and let I_1, I_2, J_1, J_2 vary so that the best bounds for $E[f(\xi_1, \xi_2)]$ are obtained. This result is summarized in

THEOREM 5.3 *We have the inequalities*

$$\begin{aligned}
&\max\{E\left[L_{I_1 I_2}(\xi_1, \xi_2)\right] \mid I_1, I_2 \textit{ dual feasible for the min problem,}\\
&\qquad |I_1| + |I_2| = m + 2\} \leq E\left[f(\xi_1, \xi_2)\right]\\
&\quad \leq \max\{E\left[L_{J_1 J_2}(\xi_1, \xi_2)\right] \mid J_1, J_2 \textit{ dual feasible for the max problem,}\\
&\qquad |J_1| + |J_2| = m + 2\},
\end{aligned}$$

where we assume that the function f satisfies the condition mentioned in Section 4, *for all I_1, I_2 and J_1, J_2, respectively, that are allowed in the above inequalities.*

REMARK When constructing the bounds presented in Theorem 5.3 we may restrict ourselves to some of the rectangular bases with $|I_1| + |I_2| = m + 2$ ($|J_1| + |J_2| = m + 2$). In this case the bounds become weaker but we impose less condition on f.

We illustrate the above bounds in the case of $m = 2$. Problem (4.2) is now the following

$$\text{Minimize } \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} f_{ij}p_{ij} \tag{5.9}$$

subject to

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} p_{ij} = \mu_{00} \tag{5.9a}$$

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} z_{1i}p_{ij} = \mu_{10} \tag{5.9b}$$

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} z_{2j}p_{ij} = \mu_{01} \tag{5.9c}$$

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} z_{1i}^2p_{ij} = \mu_{20} \tag{5.9d}$$

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} z_{2j}^2p_{ij} = \mu_{02} \tag{5.9e}$$

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} z_{1i}z_{2j}p_{ij} = \mu_{11} \tag{5.9f}$$

$$p_{ij} \ge 0, \qquad i = 0,1,\dots,n_1, \quad j = 0,1,\dots,n_2.$$

For a given pair I_1, I_2, with $|I_1| + |I_2| = m + 2 = 4$ we pick a subset of the set of constraints of problem (4.3) so that the matrix of the new constraints is a tensor product of two matrices with sizes $|I_1| \times (n_1 + 1)$ and $|I_2| \times (n_2 + 1)$, respectively. There are three possibilities to do this concerning problem (5.9).

The first one is to pick the constraints (5.9a), (5.9b), (5.9d). In this case $|I_1| = 3$, $|I_2| = 1$.

The second one is to pick the constraints (5.9a), (5.9c), (5.9e). In this case $|I_1| = 1$, $|I_2| = 3$.

The third one is to pick the constraints (5.9a), (5.9b), (5.9c), (5.9f). In this case $|I_1| = 2$, $|I_2| = 2$.

In order to simplify the formulation of the next theorem we introduce the notation:

$$Z_1 = \{z_{10}, \cdots, z_{1n_1}\}, \quad Z_2 = \{z_{20}, \cdots, z_{2n_2}\}, \quad Z = Z_1 \times Z_2, \quad z = (z_1, z_2).$$

THEOREM 5.4 *Suppose that f has positive divided differences of orders $(1,0)$, $(0,1)$, $(2,0)$, $(0,2)$, $(3,0)$, $(0,3)$, $(2,1)$, $(1,2)$. Then we have the following bounds on $f(z_1, z_2)$:*

$$(5.10) \qquad \sum_{j \in I_2} f(z_{10}, z_{2j}) L^{(2)}_{I_2 j}(z) \leq f(z_1, z_2), \qquad z \in Z$$

for any $I_2 = \{0, l, l+1\}$, $1 \leq l \leq n_2 - 1$, with strict inequality for $z \notin \{z_{10}\} \times \{z_{2j}, j \in I_2\}$;

$$(5.11) \qquad \sum_{i \in I_1} f(z_{1i}, z_{20}) L^{(1)}_{I_1 i}(z_1) \leq f(z_1, z_2), \qquad z \in Z,$$

for any $I_1 = \{0, k, k+1\}$, $1 \leq k \leq n_1 - 1$, with strict inequality for $z \notin \{z_{1i}, i \in I_1\} \times \{z_{20}\}$;

$$(5.12) \qquad \sum_{i \in K_1, j \in K_2} f(z_{1i}, z_{2j}) L^{(1)}_{k_1 i}(z_1) L^{(2)}_{K_2 j}(z_2) \leq f(z_1, z_2), \qquad z \in K,$$

for any $K_1 = \{r, r+1\}$, $0 \leq r \leq n_1 - 1$, $K_2 = \{s, s+1\}$, $0 \leq s \leq n_2 - 1$, with strict inequality for $z \notin \{z_{1i}, i \in I_1\} \times \{z_{2j}, j \in I_2\}$;

$$(5.13) \qquad \sum_{j \in I_2} f(z_{1n_1}, z_{2j}) L^{(2)}_{I_2 j}(z_2) \geq f(z_1, z_2), \qquad z \in Z,$$

for any $I_2 = \{l, l+1, n_2\}$, $0 \leq l \leq n_2 - 2$, with strict inequality for $z \notin \{z_{1n_1}\} \times \{z_{2j}, j \in I_2\}$;

$$(5.14) \qquad \sum_{i \in I_1} f(z_{1i}, z_{2n_2}) L^{(1)}_{I_1 i}(z_1) \geq f(z_1, z_2), \qquad z \in Z,$$

for any I_1 of the form $I_1 = \{k, k+1, n_1\}$, $0 \leq k \leq n_1 - 1$, with strict inequality for $z \notin \{z_{1i}, i \in I_1\} \times \{z_{2n_2}\}$;

$$(5.15) \qquad \sum_{i \in K_1, j \in K_2} f(z_{1i}, z_{2j}) L^{(1)}_{K_1 i}(z_1) L^{(2)}_{K_2 j}(z_2) \geq f(z_1, z_2), \qquad z \in Z,$$

for $K_1 = \{0, n_1\}$, $K_2 = \{0, n_2\}$, with strict inequality for $z \notin K_1 \times K_2$.

PROOF Inequalities (5.10), (5.11), (5.13) and (5.14) are consequences of Theorem 3.1 and the assumption that for any z_{1i} the function $f(z_{1i}, z_2)$ is strictly increasing in z_2 and for any z_{2j} the function $f(z_1, z_{2j})$ is strictly increasing in z_1. We have utilized the assumption that all divided differences of orders $(0,1)$, $(1,0)$, $(0,3)$ and $(3,0)$ are positive.

Inequality (5.15) is a simple consequence of the positivity of the $(2,0)$, $(0,2)$ order divided differences.

To prove (5.12) assume first that $z_1 \notin \{z_{1r}, z_{1r+1}\}$, $z_2 \notin \{z_{2s}, z_{2s+1}\}$. Since all (1, 2) and (2, 1) order divided differences are positive, we derive

$$
\begin{aligned}
&\frac{f(z_1,z_2)+f(z_{1r},z_{2s})-f(z_{1r},z_2)-f(z_1,z_{2s})}{(z_1-z_{1r})(z_2-z_{2s})} \\
&\qquad > \frac{f(z_{1r+1},z_2)+f(z_{1r},z_{2s})-f(z_{1r},z_2)-f(z_{1r+1},z_{2s})}{(z_{1r+1}-z_{1r})(z_2-z_{2s})} \\
&\qquad > \frac{f(z_{1r+1},z_{2s+1})+f(z_{1r},z_{2s})-f(z_{1r},z_{2s+1})-f(z_{1r+1},z_{2s})}{(z_{1r+1}-z_{1r})(z_{2s+1}-z_{2s})}.
\end{aligned}
\tag{5.16}
$$

On the other hand, the positivity of the (0, 2), (2, 0) order divided differences imply that

$$
f(z_{1r},z_2) > f(z_{1r},z_{2s}) + \frac{f(z_{1r},z_{2s+1})-f(z_{1r},z_{2s})}{z_{2s+1}-z_{2s}}(z_2-z_{2s}) \tag{5.17}
$$

$$
f(z_1,z_{2s}) > f(z_{1r},z_{2s}) + \frac{f(z_{1r+1},z_{2s})-f(z_{1r},z_{2s})}{z_{1r+1}-z_{1r}}(z_1-z_{1r}). \tag{5.18}
$$

Picking the inequality that exists between the first and third lines in (5.16) and utilizing (5.17) and (5.18), we obtain:

$$
\begin{aligned}
f(z_1,z_2) > f(z_{1r},z_{2s}) &+ \frac{f(z_{1r},z_{2s+1})-f(z_{1r},z_{2s})}{z_{2s+1}-z_{2s}}(z_2-z_{2s}) \\
&+\frac{f(z_{1r+1},z_{2s})-f(z_{1r},z_{2s})}{z_{1r+1}-z_{1r}}(z_1-z_{1r}) \\
&+\frac{f(z_{1r+1},z_{2s+1})+f(z_{1r},z_{2s})-f(z_{1r},z_{2s+1})}{(z_{1r+1}-z_{1r})(z_{2s+1}-z_{2s})} \\
&-\frac{f(z_{1r+1},z_{2s})}{(z_{1r+1}-z_{1r})(z_{2s+1}-z_{2s})}(z_1-z_{1r})(z_2-z_{2s}).
\end{aligned}
\tag{5.19}
$$

Inequality (5.19) is the same as (5.12).

Considering the case where either $z_1 \in \{z_{1r}, z_{1r+1}\}$ or $z_2 \in \{z_{2s}, z_{2s+1}\}$ holds, we can easily check the validity of (5.19) in all possible cases. Since (5.19) is the same as (5.12), the proof of the theorem is complete. □

Theorem 5.4 provides us with a tool to establish lower and upper bounds for $E[f(\xi_1,\xi_2)]$. We only have to plug ξ_1 and ξ_2 in the place of z_1 and z_2, respectively, in all inequalities and pick the best lower and upper bounds.

Each expectation inequality that we obtain from (5.10), (5.11), (5.13), (5.14) can be optimized by the use of the algorithm presented in Section 3. The expectation inequality that we obtain from (5.15) is already optimal. There is no easy algorithm, however, to find the best pair K_1, K_2 to optimize the expectation inequality that we obtain from (5.12). We may try out each pair K_1, K_2 and pick that one which provides us with the largest lower bound. Instead of doing this, the following may be suggested.

Starting from any rectangular basis $B_1(K_1) \otimes B_2(K_2)$ as initial dual feasible basis of the $4 \times [(n_1+1)(n_2+1)]$ size linear programming (where we minimize the objective function of problem (5.9) subject to (5.9a), (5.9b), (5.9c), (5.9f) and the nonnegativity restrictions), we carry out the dual method and obtain a (not necessarily rectangular) optimal basis. The corresponding optimum value is the best lower bound on $E[f(\xi_1, \xi_2)]$, using μ_{10}, μ_{01}, μ_{11}.

NUMERICAL EXAMPLE Assume that the possible values of any of the random variables ξ_1, ξ_2 are known to be $0, \cdots, 9$. Assume furthermore that

$$\mu_{10} = 4.8, \qquad \mu_{20} = 31.5$$

$$\mu_{01} = 4.1, \qquad \mu_{02} = 27.5, \qquad \mu_{11} = 19.95$$

and let

$$f(z_1, z_2) = e^{0.005(z_1^2 + z_2^2 + z_1 z_2)}.$$

Considering one subproblem of problem (5.9), let (i, j) represent the column vector consisting of the coefficients of $z_{1i}^{\alpha} z_{2j}^{\beta}$, as components, for the allowed α, β values. Thus, any basis of any subproblem is a collection of subscript pairs (i, j).

The results are summarized in the following tables:

	Optimal basis of the reduced problem	Nonzero elements of the optimal probability distribution	Optimum value: Bound
Lower bound based on μ_{10}, μ_{20}	(0,0), (6,0), (7,0)	$p_{00} = 0.2642857143$ $p_{60} = 0.35$ $p_{70} = 0.3857142857$	1.176108584
Upper bound based on μ_{10}, μ_{20}	(2,9), (3,9), (9,9)	$p_{29} = 0.1285714286$ $p_{39} = 0.55$ $p_{99} = 0.3214285714$	2.285735942
Lower bound based on μ_{01}, μ_{02}	(0,0), (0,6), (0,7)	$p_{00} = 0.3857142857$ $p_{06} = 0.2$ $p_{07} = 0.4142857143$	1.154458017
Upper bound based on μ_{01}, μ_{02}	(1,9), (2,9), (9,9)	$p_{19} = 0.05$ $p_{29} = 0.6428571429$ $p_{99} = 0.3071428571$	2.189880833

Table 1. Optimal lower and upper bounds on $E\,[f(\xi_1,\xi_2)]$ using only univariate first and second order moments.

		Nonzero elements of the corresponding probability distribution	Bound
Lower bound based on μ_{10}, μ_{01}, μ_{11}	Best rectangular basis (4,4), (4,5) (5,4), (5,5)	$p_{00} = 0.45$ $p_{45} = -0.25$ $p_{54} = 0.45$ $p_{55} = 0.35$	1.352634113
Optimal lower bound using μ_{10}, μ_{01}, μ_{11}	Optimal basis (0,0), (4,4) (5,4), (5,5)	$p_{00} = 0.0125$ $p_{44} = 0.1375$ $p_{54} = 0.7$ $p_{55} = 0.15$	1.355182972
Optimal upper bound using μ_{10}, μ_{01}, μ_{11}	Optimal basis (0,0), (0,9) (9,0), (9,9)	$p_{00} = 0.2574074074$ $p_{09} = 0.2092592593$ $p_{90} = 0.287037037$ $p_{99} = 0.2462962963$	1.831596631

Table 2. Lower and upper bounds on $E\,[f(\xi_1,\xi_2)]$ using first order moments and the expectation of the product of the random variables. Observe that the best rectangular basis is not primal feasible because $p_{45} < 0$.

Using the largest lower bound and smallest upper bound, we obtain the inequalities:

$$1.355182972 \le E\left[f(\xi_1, \xi_2)\right] \le 1.831596631.$$

Figures 1, 2, 3 and 4 serve to illustrate the structures of the dual feasible bases that appear in Tables 1 and 2.

9	o	o	□	□	o	o	o	o	o	□
8	o	o	o	o	o	o	o	o	o	o
7	o	o	o	o	o	o	o	o	o	o
6	o	o	o	o	o	o	o	o	o	o
5	o	o	o	o	o	o	o	o	o	o
4	o	o	o	o	o	o	o	o	o	o
3	o	o	o	o	o	o	o	o	o	o
2	o	o	o	o	o	o	o	o	o	o
1	o	o	o	o	o	o	o	o	o	o
0	•	o	o	o	o	o	•	•	o	o
	0	1	2	3	4	5	6	7	8	9

Figure 1. • (□) means: optimal basis producing lower (upper) bound using μ_{10}, μ_{20}

9	o	o	o	o	o	o	o	o	o	□
8	o	o	o	o	o	o	o	o	o	o
7	•	o	o	o	o	o	o	o	o	o
6	•	o	o	o	o	o	o	o	o	o
5	o	o	o	o	o	o	o	o	o	o
4	o	o	o	o	o	o	o	o	o	o
3	o	o	o	o	o	o	o	o	o	o
2	o	o	o	o	o	o	o	o	o	□
1	o	o	o	o	o	o	o	o	o	□
0	•	o	o	o	o	o	o	o	o	o
	0	1	2	3	4	5	6	7	8	9

Figure 2. • (□) means: optimal basis producing lower (upper) bound using μ_{01}, μ_{02}

9	□	o	o	o	o	o	o	o	o	□
8	o	o	o	o	o	o	o	o	o	o
7	o	o	o	o	o	o	o	o	o	o
6	o	o	o	o	o	o	o	o	o	o
5	o	o	o	o	●	●	o	o	o	o
4	o	o	o	o	●	●	o	o	o	o
3	o	o	o	o	o	o	o	o	o	o
2	o	o	o	o	o	o	o	o	o	o
1	o	o	o	o	o	o	o	o	o	o
0	□	o	o	o	o	o	o	o	o	□
	0	1	2	3	4	5	6	7	8	9

Figure 3. ● means: best rectangular basis producing lower bound using μ_{10}, μ_{01}, μ_{11}
□ means: optimal basis producing upper bound using μ_{10}, μ_{01}, μ_{11}

9	o	o	o	o	o	o	o	o	o	o
8	o	o	o	o	o	o	o	o	o	o
7	o	o	o	o	o	o	o	o	o	o
6	o	o	o	o	o	o	o	o	o	o
5	o	o	o	o	o	△	o	o	o	o
4	o	o	o	o	△	△	o	o	o	o
3	o	o	o	o	o	o	o	o	o	o
2	o	o	o	o	o	o	o	o	o	o
1	o	o	o	o	o	o	o	o	o	o
0	△	o	o	o	o	o	o	o	o	o
	0	1	2	3	4	5	6	7	8	9

Figure 4. △ means: optimal basis producing lower bound using μ_{10}, μ_{01}, μ_{11}

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