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DEPENDENCE OF STABLE RANDOM VARIABLES

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The dependence structure of a multivariate normal distribution is characterized by its covariance matrix. However, in contrast to the normal case, discussion on dependence for α-stable random variables, $0 < \alpha < 2$, requires more care because variances do not exist. We review in this paper dependence concepts for α-stable random variables. A local measure of dependence is proposed. Also we illustrate how product–type stable laws arise naturally in applications.

1. Introduction

The study of dependence in random variables has yielded many useful results in statistical applications. For normal distributions, the dependence structure can be characterized by their covariance matrix. For example, Pitt (1982), Joag-Dev, Perlman and Pitt (1983) show that jointly normal random variables are associated if and only if their correlations are all nonnegative.

In contrast to normal vectors, a multivariate stable random vector cannot be specified in general by a finite number of numerical parameters. Moreover, when $0 < \alpha < 2$, no α-stable random variable has a finite second moment, and even the first moment does not exist when $\alpha \leq 1$. Therefore the investigation of dependence relationships among stable random variables is nontrivial. Using spectral measure as a tool, Lee, Rachev and Samorodnitsky (1990) derived necessary and sufficient conditions for association of stable random variables. In section 2, we will review some dependence results for stable random variables. Also we discuss the notion of geometric stable random variables.

In section 3 we focus on symmetric stable sub-Gaussian random variables. We show that except for the singular case, sub-Gaussian random vector

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cannot be associated. It is therefore of interest to derive a measure of local strength of dependence based on Bjerve and Doksum (1990)'s correlation curve. In section 4 we discuss the relationship between product-type stable random vectors and subordinated processes.

2. Stable Random Vectors and Dependence

2.1 *Association of Stable Vectors*

Stable laws are very useful in statistical applications, they have been used to model the distribution of stock price changes (see e.g. Akgiray and Booth (1988), Du Mouchel (1983), Fama (1965), Mandelbrot (1963), and Mittnik and Rachev (1991)), and the distribution of the frailty factor in the context of biostatistics (see e.g. Hougaard (1986)).

A random vector $\mathbf{X}=(X_1,X_2,\ldots,X_n)$ is called α-stable, $0<\alpha\leq 2$, if for any constants $A>0, B>0$, there is a $\mathbf{D}\in\mathbf{R}^n$ such that

$$A\mathbf{X}^{(1)}+B\mathbf{X}^{(2)}\stackrel{d}{=}(A^\alpha+B^\alpha)^{1/\alpha}\mathbf{X}+\mathbf{D},$$

where $\mathbf{X}^{(1)},\mathbf{X}^{(2)}$ are independent copies of $\mathbf{X}$.

Normal distributions are special cases of stable distributions with index of stability $\alpha=2$. An α-stable random vector is called strictly α-stable if $\mathbf{D}=\mathbf{0}$ for every A and B. An α-stable random vector $\mathbf{X}$ satisfying $\mathbf{X}\stackrel{d}{=}-\mathbf{X}$ is called symmetric α-stable ($S\alpha S$). Note that a $S2S$ vector is a zero-mean multivariate normal random vector.

Let $\phi_\alpha(\boldsymbol{\theta})=\phi_\alpha(\theta_1,\ldots,\theta_n)=E\exp\{i(\boldsymbol{\theta},\mathbf{X})\}=E\exp\{i\sum_{j=1}^n\theta_jX_j\}$ denote the characteristic function of an α-stable random vector in $\mathbf{R}^n$, where $(\boldsymbol{\theta},\mathbf{X})$ denotes the inner product. When $n=1$, the characteristic function of an α-stable random variable, $0<\alpha\leq 2$, has the form

$$(1)\qquad \phi_\alpha(\theta)=\begin{cases}\exp[-\sigma^\alpha|\theta|^\alpha(1-i\eta(\text{sign }\theta)\tan\frac{\pi\alpha}{2})+i\mu\theta] & \text{if } \alpha\neq 1\\ \exp[-\sigma|\theta|(1+i\eta\frac{2}{\pi}(\text{sign }\theta)\ln|\theta|)+i\mu\theta] & \text{if } \alpha=1.\end{cases}$$

where $\sigma\geq 0$ is typically referred to as scale parameter, $-1\leq\eta\leq 1$ is skewness parameter and $\mu\in R$ is location parameter (but do not rely too much on these names in the case $\alpha=1$). Conversely, a random variable X with characteristic function given by (1) is α-stable, and we say that X has a stable distribution $S_\alpha(\sigma,\eta,\mu)$.

When $\eta=1$, the random variable X is said to be *totally right skewed.* If also $0<\alpha<1$, and $\mu=0$, then X has the positive real line as its support, in which case it has the Laplace transform $E[\exp(-\theta X)]=\exp(-c\sigma^\alpha\theta^\alpha)$, where $c=(\cos\frac{\pi\alpha}{2})^{-1}$.

In the case $n \geq 2$, there is a similar representation for the characteristic function (ch.f) of α-stable vectors. Namely, a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is α-stable, $0 < \alpha \leq 2$, if ane only if there is a finite Borel measure m on the unit sphere S_n of $\mathbf{R}^n$ and a vector $\boldsymbol{\mu}^0 = (\mu_1^0, \ldots, \mu_n^0)$ in $\mathbf{R}^n$ such that:
(a) If $\alpha \neq 1$

$$(2) \quad \phi_\alpha(\boldsymbol{\theta}) = \exp\{-\int_{S_n} |(\boldsymbol{\theta}, \mathbf{s})|^\alpha (1 - i \operatorname{sign}((\boldsymbol{\theta}, \mathbf{s})) \tan \frac{\pi\alpha}{2}) m(ds) + i(\boldsymbol{\theta}, \boldsymbol{\mu}^0)\}$$

(b) If $\alpha = 1$
(2a)

$$\phi_\alpha(\boldsymbol{\theta}) = \exp\{-\int_{S_n} |(\boldsymbol{\theta}, \mathbf{s})|(1 + i\frac{2}{\pi} \operatorname{sign}((\boldsymbol{\theta}, \mathbf{s})) \ln |(\boldsymbol{\theta}, \mathbf{s})|) m(ds) + i(\boldsymbol{\theta}, \boldsymbol{\mu}^0)\},$$

where $\mathbf{s} = (s_1, \ldots, s_n) \in S_n$. The pair $(m, \boldsymbol{\mu})$ is unique when $0 < \alpha < 2$, the measure m is then called the *spectral measure* of the α-stable random vector $\mathbf{X}$.

Specifically, if $\mathbf{X}$ is symmetric α-stable, then it has characteristic function of the form

$$(3) \qquad \phi_\alpha(\boldsymbol{\theta}) = \exp\{-\int_{S_n} |\theta_1 s_1 + \ldots + \theta_n s_n|^\alpha m(ds)\}$$

where Γ is a finite symmetric measure on the Borel subsets of the unit sphere S_n.

Note that an α-stable random vector $\mathbf{X}$ has independent components if and only if its spectral measure m is discrete and concentrated on the intersection of the axes with the unit sphere S_n. See Samorodnitsky and Taqqu (1991) for a review on properties of multivariate stable random vectors.

Random variables $X_1, \ldots, X_n$ are called *associated* if for any functions $f, g\colon \mathbb{R}^n \to \mathbb{R}$, nondecreasing in each argument, we have $\operatorname{cov}(f(\mathbf{X}), g(\mathbf{X})) \geq 0$ whenever the covariance exists. The concept of *association* was introduced by Esary, Proschan, and Walkup (1967) to obtain bounds related to coherent functions(co-ordinatewise increasing) occurring in the theory of reliability. In a completely different context, Fortuin, Kasteleyn, and Ginibre (1971), considered the association concept for the Ising model of statistical physics. Association represents a strong form of positive dependence.

Pitt (1982), Joag-Dev, Perlman and Pitt (1983) show that nonnegatively correlated normal variables are associated. Inspired by their results, Lee, Rachev and Samorodnitsky (1990) derive the following theorem.

THEOREM 1 *Let $X_1, \ldots, X_n$ be jointly α-stable random variables, $0 < \alpha < 2$, with characteristic function given by (2). Then $X_1, \ldots, X_n$ are associated if and only if the spectral measure m satisfies the condition*

$$m(S_n^-) = 0$$

where $S_n^- = \{(s_1, \ldots, s_n) \in S_n$: *for some* $i, j \in \{1, \ldots, n\}, s_i > 0$ *and* $s_j < 0\}$.

Note that a result related to the sufficiency part of the above theorem was obtained by Resnick (1988) in terms of Poisson representation of an infinitely divisible random vector.

We list here some notions of dependence. A bivariate density function $f(x, y)$ of two arguments is said to be *totally positive of order 2* (abbreviated TP_2) if for all $x_1 < x_2$, and $y_1 < y_2$,

$$\begin{vmatrix} f(x_1, y_1) & f(x_1, y_2) \\ f(x_2, y_1) & f(x_2, y_2) \end{vmatrix} \geq 0.$$

A joint density function $f(x_1, \ldots, x_n)$ of n arguments is said to be TP_2 *in pairs* if $f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$ is TP_2 in (x_i, x_j) for all $i \neq j$ and all fixed values of the remaining arguments. If a random vector has a TP_2-in-pairs density then it is associated. See Karlin (1968), Barlow and Proschan (1981), and Tong (1990) for a review. Random variables $X_1, \ldots, X_n$ are called *positive upper orthant dependent* (PUOD) if

$$P(X_1 > x_1, \ldots, X_n > x_n) \geq P(X_1 > x_1) \ldots P(X_n > x_n)$$

for any $x_1, \ldots, x_n$, and they are called *positive lower orthant dependent* (PLOD) if

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) \geq P(X_1 \leq x_1) \ldots P(X_n \leq x_n)$$

for any $x_1, \ldots, x_n$. That is, if $X_1, \ldots, X_n$ are PUOD or PLOD, then they are more likely to take on larger values together or smaller values together. Lehmann (1966) shows that for the bivariate case, X_1, X_2 are PUOD if and only if they are PLOD; however, for higher dimensional cases, the equivalence no longer holds. It is also well known that association implies both PUOD and PLOD, but in general these implications cannot be reversed. For stable random variables, Lee, Samorodnitsky and Rachev (1990) show, as a result of theorem 1, that PLOD or PUOD implies association.

COROLLARY 1 *Let* $X_1, \ldots, X_n$ *be jointly* α*-stable. Then the notion of association is equivalent to PLOD or PUOD.*

Following Alam and Saxena (1981), we call random variables $X_1, \ldots, X_n$ *negatively associated* if for any $1 \leq k < n$, and $f: \mathbb{R}^k \to \mathbb{R}, g: \mathbb{R}^{n-k} \to \mathbb{R}$, nondecreasing in each argument, $\text{cov}(f(\mathbf{Y}), g(\mathbf{Z})) \leq 0$ whenever the covariance exists, where $\mathbf{Y}$ and $\mathbf{Z}$ are any k and $(n-k)$-dimensional random vectors correspondingly, representing a partition of the set $(X_1, \ldots, X_n)$ into two subset of sizes k and $n-k$ accordingly. In the normal case, negative association has been characterized by Joag-dev and Proschan (1983). Lee, Samorodnitsky and Rachev (1990) derive the following theorem.

THEOREM 2 *Let $X_1, \ldots, X_n$ be jointly α-stable random variables $0 < \alpha < 2$, with characteristic function given by (2). Then $X_1, \ldots, X_n$ are negatively associated if and only if the spectral measure m satisfies the condition*

$$m(S_n^+) = 0$$

where $S_n^+ = \{(s_1, \ldots, s_n) \in S_n\colon$ for some $i \neq j,\ s_i \cdot s_j > 0\}$.

Recently, some other properties related to dependence concepts of stable random vectors have also been discussed. Multiple regressions on stable random vectors have been considered by Wu and Cambanis (1991) and Samorodnitsky and Taqqu (1991). A version of Slepian-type inequalities, due to Fernique (1975), was extended to stable random vectors in terms of Levy measures by Samorodnitsky and Taqqu (1990).

2.2. *Geometric Stable Vectors*

A random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ is called *geometric stable* if there exist (1) a sequence of i.i.d. random vectors $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots$, (2) independent of the $\mathbf{X}^{(i)}$'s a geometric r.v. $T(p)$ with mean $1/p$ $(0 < p < 1)$, and (3) constants $\mathcal{A}(p) > 0$ and $\mathcal{C}(p) \in \mathbb{R}^n$ such that

$$\mathcal{A}(p) \sum_{i=1}^{T(p)} (\mathbf{X}^{(i)} + \mathcal{C}(p)) \xrightarrow{d} \mathbf{Y}, \text{ as } p \to 0 \tag{4}$$

Geometric stable random vectors (GSRV's) are used in reliability queuing theory, financial modelling and its study goes back to the works of A. Renyi, H. Robbins and B. V. Gnedenko, see the surveys in Rachev (1991), and Rachev and Sengupta (1991).

We now characterize the class of GSRV's. The characterization is in terms of "dual" representation (see (b) below) of the ch.f's of GSRV, and α-stable random vectors, which, as it follows (see (a)-(c)), share one and the same domain of attraction.

LEMMA 1 *For a random vector $\mathbf{Y}$ the following are equivalent:*
(a) $\mathbf{Y}$ is GSRV.
(b) The characteristic function $f_{\mathbf{Y}}$ of $\mathbf{Y}$ admits the representation

$$f_{\mathbf{Y}}(\boldsymbol{\theta}) = \frac{1}{1 - \ln \phi_\alpha(\boldsymbol{\theta})},\ \boldsymbol{\theta} \in \mathbb{R}^n, \tag{5}$$

where $\phi_\alpha(\boldsymbol{\theta})$ is the ch.f of α-stable vector (see (2), (2a)*).*
(c) $\mathbf{X}^{(1)}$ *in* (4) *belongs to the domain of attraction of α-stable random vector with ch.f. ϕ_α defined by* (5).
(d) $\mathbf{X}^{(1)}$ *in* (4) *has polar coordinates ρ and Θ such that, as $R \to \infty$,*

$$P(\rho > R, \Theta \in A)/P(\rho > kR, \Theta \in B)$$

$$\to k^\alpha \tilde{m}(A)/\tilde{m}(B) \tag{6}$$

for any $k > 0$ and any Borel sets A and B of S_n with $\tilde{m}(B) \neq 0$, $\tilde{m}$ stands for the spectral measure of ϕ_α in (5) *rewritten in polar coordinates.*

PROOF The proof is essentially given in Mittnik and Rachev (1991). The limit relation (6) follows from the characterization of the domain of attraction of multivariate α-stable law (see Rvačeva (1962), Resnick and Greenwood (1979)).

Taking into account the characterization (5) of GSRV $\mathbf{Y}$ we say that $\mathbf{Y}$ and ϕ_α share one and the same spectral measure Γ and index of stability α given in (2), (2a).

Necessary conditions for association of GSRV's follow from lemma 1 and the condition for association of stable vectors in the previous section. Sufficient conditions, however, are not obvious, and this problem is still open.

3. Product-type Stable Random Vectors

3.1. *Stable Laws Derived from Products*

Let $\mathbf{Z} = (Z_1, \dots, Z_n)$ be an arbitrary symmetric α'-stable random vector. Let T be a positive α/α'-stable random variable, $0 < \alpha < \alpha'$, independent of $\mathbf{Z}$ and having Laplace transform $E\exp(-\theta T) = \exp(-\theta^{\alpha/\alpha'})$, $\theta \geq 0$. Then the random vector defined by

$$\mathbf{X} = T^{1/\alpha'}\mathbf{Z} \tag{7}$$

is symmetric α-stable. They are sometimes referred to as product-type stable random vectors. In section 4, we will see that these product-type stable random vectors can be obtained naturally from stable processes directed by an operational time stable process.

It is clear that components of the product-type stable vector $\mathbf{X}$ are conditionally independent. If one further assumes that $Z_1, \dots, Z_n$ are i.i.d., then components of $\mathbf{X}$ are positively dependent by mixture as considered by Shaked (1977), Tong (1977, 1980), and Shaked and Tong (1985). We note that components of the derived vectors $\mathbf{X}$ can be strongly dependent. This fact can be demonstrated by the following example.

EXAMPLE 1 For any fixed positive integers $k_1 > 1$ and $k_2 > 1$, assume that $\{Z_i, \ i = 1, \ldots, n\}$ are n i.i.d. totally right skewed strictly $1/k_2$-stable random variables. Let T be a totally right skewed strictly $1/k_1$-stable variable, independent of $\{Z_i, i = 1, \ldots, n\}$. Then the derived vector $(X_1, \ldots, X_n) = T^{1/\alpha'}\mathbf{Z}$, with $\alpha' = 1/k_1$, is $1/k_1k_2$ stable and its components are TP_2 in pairs. See Theorem 6 in section 4 for general results. We show in the following section that there are many cases where components of a product-type stable random vector are neither associated nor positive orthant dependent.

On the other hand, note that if $\mathbf{X} = E^{1/\alpha'}\mathbf{Z}$ where $\mathbf{Z}$ is a symmetric α'-stable random vector and if (1) E is independent of $\mathbf{Z}$, (2) E is an exponential random variable then $\mathbf{X}$ is GSRU, see equation (4). If E has arbitrary distribution on $\mathbf{R}_+$ then $\mathbf{X} = E^{1/\alpha'}\mathbf{Z}$ is called Robbins mixture: for applications of Robbins mixtures to reliability theory, queueing and finance modelling we refer to Szasz (1972), Szynal (1976), Karolev (1988), Melamed (1988), Rachev and Ruschendorf (1991), Rachev and Samorodnitsky (1991).

3.2. *Sub-Gaussian Random Vectors*

When $\mathbf{Z} \equiv \mathbf{G} = (G_1, \ldots, G_n)$ is a zero mean Gaussian vector (i.e. S_2S) in $\mathbf{R}^n$, independent of the positive α/α'-stable random vector T in (7), then the derived vector $\mathbf{X} = T^{1/2}\mathbf{G}$ is called a *sub-Gaussian $S\alpha S$* random vector, with governing Gaussian vector $\mathbf{G}$. It is shown in section 4 that sub-Gaussian vectors arise naturally in stable processes which are subordinated to Gaussian processes. Sub-Gaussian vectors form a special class of stable vectors which, unlike general stable vectors, can be characterized by finitely many parameters. Specifically, the sub-Gaussian vector $\mathbf{X}$ derived above has a characteristic function of the form

$$\phi_\alpha(\boldsymbol{\theta}) = \exp\{-|\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\theta_i\theta_j R_{ij}|^{\alpha/2}\}, \tag{8}$$

where $R_{ij} = E(G_iG_j), \ i,j = 1, \ldots, n$ are the covariances of the underlying Gaussian vector $(G_1, \ldots, G_n)$.

Let m_0 be a uniform (i.e. rotationally invariant) finite Borel measure on S_n. Then for some $c \geq 0$

$$\int_{S_n} |\sum_{j=1}^{n}\theta_j s_j|^\alpha m_0(ds) = c\ [\frac{1}{2}\sum_{j=1}^{n}\theta_j^2]^{\alpha/2}.$$

Therefore, a $S\alpha S$ random vector , $\alpha < 2$, is sub-Gaussian with a governing Gaussian vector having i.i.d. components if and only if its spectral measure is uniform on the unit sphere S_n. A uniform spectral measure does not satisfy the required condition for association as was stated in Theorem 1. As a consequence, we have

COROLLARY 2 *If a $S\alpha S$ random vector $\mathbf{X}$ is sub-Gaussian with a governing Gaussian vector having i.i.d. $N(0,\sigma^2)$ components, then components of $\mathbf{X}$ are positively dependent by mixture but they are neither associated nor positively orthant dependent.*

In general, the spectral measure of a sub-Gaussian vector is a transform of the uniform measure m_0 on S_n. Hence we have

THEOREM 3 *A non-degenerate (i.e. having non-zero components) sub-Gaussian vector $(T^{1/2}G_1,\ldots,T^{1/2}G_n)$ with governing Gaussian vector $\mathbf{G}$, as defined in equation (7), is associated if and only if $G_1 = c_2G_2 = \ldots = c_nG_n$ a.e. for some $c_2 > 0,\ldots,c_n > 0$.*

PROOF Suppose $P(G_1 = c_2G_2) = 0$ for any $c_2 \in R$. Assume that the stable random vector $(T^{1/2}G_1,\ldots,T^{1/2}G_n)$ is associated. Then the sub-vector $(T^{1/2}G_1,T^{1/2}G_2)$ is also associated. Hence $(T^{1/2}G_1,T^{1/2}G_2)$ has a spectral measure which is concentrated on the parts of the unit circle specified in Theorem 1, namely, in the first and third quadrant of the unit circle. On the other hand, any Gaussian vector can be written as a linear combination of i.i.d. $N(0,1)$'s. The condition $P(G_1 = c_2G_2) = 0$ for any $c_2 \in R$ implies that this linear transformation for the vector (G_1,G_2) is of full rank. Therefore the spectral measure of $(T^{1/2}G_1,T^{1/2}G_2)$ is a rigid transformation of the uniform measure on the unit sphere, and it maps the unit sphere onto the entire unit sphere. This leads to a contradiction. Similarly, the case $G_1 = c_2G_2$ for some $c_2 < 0$ can be ruled out.

3.3. *Conditional Moments for Sub-Gaussians Random Vectors*

Samorodnitsky and Taqqu (1991) derived regression equations for general stable random vectors. In particular, they show that when $(X_1,X_2) = (T^{1/2}G_1,T^{1/2}G_2)$ in equation (7) is a non-degenerate sub-Gaussian $S\alpha S$, (even when $\alpha \leq 1$), random vector with governing Gaussian vector $\mathbf{G}$,

$$E(X_2|X_1 = x) = \frac{\mathrm{Cov}(G_1,G_2)}{\mathrm{Var}G_1}x = R_{12}{R_{11}}^{-1}x \qquad a.e. \tag{9}$$

and $E(X_2^2|X_1 = x) < \infty$ a.e. if $1 < \alpha < 2$ (note that the unconditional second moment is infinite.).

We can calculate the conditional variance as follows.

$$\begin{aligned} E(G_2|G_1) &= R_{12}R_{11}{}^{-1}G_1, \\ \mathrm{Var}(G_2|G_1) &= R_{22} - R_{12}^2 R_{11}^{-1}, \\ E(X_2^2|X_1 = x) &= E(E(TG_2^2|T, G_1)|T^{1/2}G_1 = x) \\ &= E(T|X_1 = x)\mathrm{Var}\ (G_2|G_1) + R_{12}^2 R_{11}^2{}^{-1} x^2. \end{aligned}$$

Hence

$$\mathrm{Var}\ (X_2|X_1 = x) = E(T|X_1 = x)(R_{22} - R_{12}^2 R_{11}^{-1}). \tag{10}$$

Note that from equation (9) we see that the conditional mean of X_2 given X_1 is completely determined by their governing Gaussian vector $\mathbf{G}$. Equation (10), however, demonstrates how the random variable T influences the variance of the conditional law of X_2 given X_1.

Wu and Cambanis (1991) show that

$$\mathrm{Var}(X_2|X_1 = x) = (1/2)(R_{22} - R_{12}^2 R_{11}^{-1}) f(x;\alpha)^{-1} \int_{|x|}^{\infty} u f(u;\alpha) du, \tag{10a}$$

where $f(x;\alpha)$ is the density function of a $S_\alpha(1,0,0)$ random variable (i.e. having the characteristic function $\exp[-|t|^\alpha]$).

Moreover, if T in $\mathbf{X} = \sqrt{T}\mathbf{G}$ is standard exponentially distributed rather than positive stable distributed, then $\mathbf{X}$ has a multivariate Laplace distribution, that is, its ch.f has the form

$$\phi_\alpha(\boldsymbol{\theta}) = \frac{1}{1 + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \theta_i \theta_j R_{ij}}.$$

See Feldman and Rachev (1991) and Rachev and Sengupta (1991) for application of Laplace distributions in random fields, U-statistics and modelling commodity prices.

3.4. *Local Correlation Functions*

In an effort to characterize the dependence for sub-Gaussian vectors, we derive in this section a notion of local correlation for sub-Gaussian random vectors.

Bjerve and Doksum (1990) introduced a measure of local strength of dependence by combining ideas from nonparametric regression and of Galton (1888). They define the function

$$\rho(x) = \frac{\sigma\beta(x)}{[\{\sigma\beta(x)\}^2 + \sigma_{X_2|X_1=x}^2]^{1/2}} \tag{11}$$

where $\sigma^2 = \mathrm{Var}(X_1)$, $\sigma^2_{X_2|X_1=x} = \mathrm{Var}(X_2|X_1 = x)$, and $\beta(x) = \frac{d}{dx}E(X_2|X_1 = x)$ is the slope of the nonparametric regression. More generally, they note that the conditional mean $E(X_2|X_1 = x)$ can be replaced by a location function. The variances $\sigma^2 = \mathrm{Var}(X_1)$ and $\sigma^2_{X_2|X_1=x} = \mathrm{Var}(X_2|X_1 = x)$ can be replaced by squares of corresponding scale functions.

For an $S\alpha S$ random vector (X_1, X_2), Samorodnitsky and Taqqu (1991) show that in many cases the first, and when $1 < \alpha \leq 2$, second conditional moments exist if a certain integrability condition holds. When $1 < \alpha < 2$, the *covariation* is designed to replace the covariance. It is a useful quantity and it appears naturally in the context of regression for stable random variables.

DEFINITION Let X_1 and X_2 be jointly $S\alpha S$ with $\alpha > 1$ and let m be the spectral measure of the random vector (X_1, X_2). The covariation of X_1 and X_2 is defined to be the real number

$$[X_1, X_2]_\alpha = \int_{S_2} s_1 |s_2|^{\alpha-1} \mathrm{sign}(s_2) m(ds).$$

As a result, we have

$$[X_1, X_1]_\alpha = \int_{S_2} |s_1|^\alpha m(ds) = \sigma_1^\alpha,$$

where σ_1 is the scale parameter of the $S\alpha S$ random variable X_1. Let $\mathcal{F}_\alpha$ be a linear space of jointly $S\alpha S$ random variables. Then when $\alpha > 1$, the covariation induces a norm on $\mathcal{F}_\alpha$ such that

$$\|X\|_\alpha = ([X, X]_\alpha)^{1/\alpha}.$$

Convergence in $\|.\|_\alpha$ is equivalent to convergence in probability and convergence in L^p for any $p < \alpha$. Moreover, for sub-Gaussian vector $(X_1, \ldots, X_n)$ with ch.f defined in (8), we have

$$\begin{aligned} [X_i, X_j]_\alpha &= 2^{-\alpha/2} R_{ij} R_{jj}^{(\alpha-2)/2}, \quad i, j = 1, \ldots, n, \text{ and} \\ \|X_i\|_\alpha &= 2^{-1/2} R_{ii}^{1/2}. \end{aligned}$$

Note that when $\alpha = 2$, we have $[X_1, X_2]_2 = \frac{1}{2}\,\mathrm{Cov}(X_1, X_2)$, and $[X_1, X_1]_2 = \frac{1}{2}\,\mathrm{Var}(X_1) = \sigma_1^2$. For an $S\alpha S$ random variable X, $1 < \alpha < 2$, we use $k_\alpha \|X\|_\alpha$, with

$$k_\alpha = [\frac{\alpha\Gamma(1 - 1/\alpha)}{\Gamma(1/\alpha)}]^{1/2},$$

to replace the notion of $\{\mathrm{Var}(X)\}^{1/2}$ in the calculation of a notion of local correlation function. Note that $k_2 = \sqrt{2}$ is consistent with the above computation.

Hence we can define, for an $S\alpha S$ random vector (X_1, X_2), the local correlation curve as

$$\rho(x) = \frac{\beta(x)k_\alpha \|X_1\|_\alpha}{[\{\beta(x)k_\alpha\|X_1\|_\alpha\}^2 + \sigma^2_{X_2|X_1=x}]^{\frac{1}{2}}} \tag{12}$$

whenever the conditional variance is finite *a.e.* Note that, for the special case when $\alpha = 2$, $\rho(x) \equiv$ the correlation coefficient of (X_1, X_2).

For $S\alpha S$ sub-Gaussian vector $(X_1, X_2) = (T^{1/2}G_1,\ T^{1/2}G_2)$ with governing Gaussian vector (G_1, G_2) we have by (9),(10) and (10a)

$$\rho(x) = \frac{k_\alpha R_{12}}{[k_\alpha^2 R_{12}^2 + (R_{11}R_{22} - R_{12}^2)f(x;\alpha)^{-1}\int_{|x|}^{\infty} uf(u;\alpha)du]^{1/2}}. \tag{13}$$

It follows that $\rho(x)$ is an even function, $\rho(0) = R_{12}/(R_{11}R_{22})^{1/2}$ = the correlation coefficient of G_1 and G_2. Observe that

$$\rho(x) \sim \frac{k_\alpha}{\alpha(\alpha-1)^{1/2}} R_{12}(R_{11}R_{22} - R_{12}^2)^{-1/2}|x|^{-1} \text{ as } |x| \to \infty. \tag{14}$$

Furthermore, $\rho(x) \equiv 0$ when G_1 and G_2 are independent; $\rho(x) \equiv 1$, when $G_1 = G_2$.

4. Subordination

In this section we will show that product-type stable random vectors considered in section 3 can appear naturally in applications. Let $\{\mathbf{X}(t)\}$ be a Markov process with continuous transition probabilities and $\{\mathbf{T}(t)\}$ a process with nonnegative independent increments, then $\{\mathbf{X}(\mathbf{T}(t))\}$ is again a Markovian process. The process $\{\mathbf{X}(\mathbf{T}(t))\}$ is said to be *subordinate to the parent process* $\{\mathbf{X}(t)\}$ *using the operational time* $\mathbf{T}(t)$. The process $\{\mathbf{T}(t)\}$ is called the *directing process.* The role of the directing process is to inject some additional randomness into the parent process through its time parameter t. In equipment usage, for example, $\{\mathbf{X}(t)\}$ may represent cumulative wear on a machine component after t hours of operation and $\{\mathbf{T}(t)\}$ may represent the number of hours that the machine has operated after t hours of calendar time have passed. The process $\{\mathbf{T}(t)\}$ thereby captures the random delays and accelerations of operational use of the machine over calendar time. The term *subordination* was first introduced by Bochner (1955). Subordinated processes have also been referred to as *derived processes* by Cohen (1962). Various properties of derived processes were investigated by Stam (1965). See Mandelbrot and Taylor (1967), Clark (1973), Rachev and Ruschendorf (1991), Rachev and Samorodnitsky (1991), for modelling stock returns and option pricing via subordinated processes. Whitmore and Lee (1991) considered statistical inferences for subordinated processes and applications.

DEFINITION A stochastic process $\{\mathbf{X}(t)\}$ is called an *α-stable Lévy motion* with skewness parameter η and scale parameter σ if
(1) $\mathbf{X}(0) = 0$.
(2) $\{\mathbf{X}(t)\}$ has stationary independent increments.
(3) $\mathbf{X}(t) - \mathbf{X}(s)$ has the distribution $S_\alpha(\sigma(t-s)^{1/\alpha}, \eta, 0)$ for any $\sigma > 0$, $0 \le s < t < \infty$, and for some $0 < \alpha \le 2$, and $-1 \le \eta \le 1$.
When $\sigma = 1$, $\{\mathbf{X}(t)\}$ is said to be a *standard α-stable Lévy motion.* An α-stable Lévy motion is $1/\alpha$-self similar unless both $\alpha = 1$ and $\eta \ne 0$. The role that stable Lévy motion plays among stable processes is similar to the role that Brownian motion plays among Gaussian processes. See Samorodnitsky and Taqqu (1992) for properties of Lévy motions.

The following results was given in Lee and Whitmore (1991). Related results can also be found in Stam (1966, p. 137-138), and Samorodnitsky and Taqqu (1992).

THEOREM 4 *Assume that $\{\mathbf{X}(t)\}$ is a standard α-stable Lévy motion with $\alpha \ne 1$ and skewness parameter η, and that $\{\mathbf{T}(t)\}$ is a standard β-stable Lévy motion with $0 < \beta < 1$ and the skewness parameter* 1. *Assume also that processes $\{\mathbf{X}(t)\}$ and $\{\mathbf{T}(t)\}$ are independent. Then*
(a) *If $\alpha\beta \ne 1$, then the process $\{\mathbf{X}(\mathbf{T}(t))\}$ is an $\alpha\beta$-stable Lévy motion such that*

$$\mathbf{X}(\mathbf{T}(t)) - \mathbf{X}(\mathbf{T}(s)) \sim S_{\alpha\beta}((\kappa(t-s))^{1/\alpha\beta}, \xi, 0),$$

with $\xi = \tan(\zeta\beta)/(\tan\frac{\alpha\beta\pi}{2})$, $\kappa = (\cos\zeta\beta)(1+\eta^2\tan^2\frac{\pi\alpha}{2})^{\beta/2}(\cos\frac{\pi\alpha}{2})^{-1}$, *and* $\zeta = \arctan(\eta\tan\frac{\pi\alpha}{2})$.
(b) *If $\alpha\beta = 1$ then the process $\{\mathbf{X}(\mathbf{T}(t))\}$ is of the form $\kappa(L(t) + t\tan\beta\zeta)$, where L is a standard symmetric* 1*-stable (Cauchy) motion, and κ and ζ are as in* (a).
(c) *The process $\{\mathbf{X}(\mathbf{T}(t))\}$ has the same one-dimensional distributions as that of the process $\{(\mathbf{T}(t))^{1/\alpha}\mathbf{X}(1)\}$.*

As a special case of Theorem 4, consider k independent Brownian motions $\{\mathbf{X}_j(t)\}, j = 1, 2, \ldots, k$, such that $\mathbf{X}_j(0) = 0$ and $\mathbf{X}_j(t) \sim N(0, t/c)$ for $j = 1, 2, \ldots, k$, where $c = (\cos\frac{\pi\alpha}{4})^{-1}$ and $\alpha < 2$. Assume that $\{\mathbf{T}(t)\}$ is a standard totally right skewed $\alpha/2$ stable Lévy motion, independent of processes $\{\mathbf{X}_j(t)\}, j = 1, 2, \ldots, k$. Then for any $t > 0$ fixed, the random vector $\langle \mathbf{X}_1(\mathbf{T}(t)), \ldots, \mathbf{X}_k(\mathbf{T}(t))\rangle$ equals in distribution the symmetric α-stable random vector $\langle \mathbf{T}(t)^{1/2}\mathbf{X}_1(1), \ldots, \mathbf{T}(t)^{1/2}\mathbf{X}_k(1)\rangle$. The latter vector is a sub-Gaussian vector with a governing Gaussian vector having i.i.d. $N(0, t/c)$ components as was discussed in section 3.

Markov processes with TP_2 transition densities are useful in shock models. For example, inverse Gaussian processes, gamma processes and some right skewed stable processes have TP_2 transition densities. Lee and Whitmore (1991) show that if the transition densities of both the parent process and the operational time process are TP_2, then transition density of the derived subordinated process is also TP_2. They have the following results.

THEOREM 5 *Assume that the process* $\{\mathbf{X}_j(t)\}$ *has a transition density function* $f_{j,t}(x)$ *which is* TP_2 *in* t *and* x, *for* $j = 1, \ldots, k$, *and that* $\{\mathbf{T}(t)\}$ *has a transition density function* $u_t(s)$ *which is* TP_2 *in* t *and* s. *If the process* $\{\mathbf{T}(t)\}$ *is independent of processes* $\{\mathbf{X}_j(t)\}, j = 1, 2, \ldots, k$, *then*
(a) *The transition density function* $h_{j,t}(x)$ *of the subordinated process* $\{\mathbf{X}_j(\mathbf{T}(t))\}$ *is* TP_2 *in* t *and* x, *for* $j = 1, \ldots, k$.
(b) *For any* $t > 0$ *fixed, the random vector* $\langle \mathbf{X}_1(\mathbf{T}(t)), \ldots, \mathbf{X}_k(\mathbf{T}(t)) \rangle$ *is* TP_2 *in pairs.*

THEOREM 6 *For any two positive integers* $k_1 > 1, k_2 > 1$ *fixed, assume that* $\{\mathbf{X}_i(t)\}$ *are* n *i.i.d. totally right skewed* $1/k_2$*-stable Lévy motions,* $i = 1, \ldots, n$. *Let* $\{\mathbf{T}(t)\}$ *be a totally right skewed* $1/k_1$*-stable Lévy motion independent of processes* $\{\mathbf{X}_i(t)\}$, $i = 1, \ldots, n$. *Then, for any* $t > 0$ *fixed, the vector* $\langle \mathbf{X}_1(\mathbf{T}(t)), \ldots, \mathbf{X}_n(\mathbf{T}(t)) \rangle$ *is* $1/(k_1k_2)$*-stable and is* TP_2 *in pairs.*

Finally, as a special case of subordination, consider an exponential time-change of α-stable Lévy motion $\{\mathbf{X}(t)\}$. Let $\mathbf{Y}(t) = \mathbf{X}(Et)$, $t \geq 0$ where E is standard exponential random variable independent of $\mathbf{X}$. Then by the self-similarity of Lévy motion, $\mathbf{Y} \stackrel{d}{=} E^{1/\alpha}\mathbf{X}$, unless $\alpha = 1$ and $\eta \neq 0$. The finite dimensional distributions of the process $\{\mathbf{Y}(t)\}$ are geometric stable. In fact, readily from Lemma 1 (c) we get

$$E \exp(i\{(Y(t_1), \ldots, Y(t_k)), \boldsymbol{\theta}\}) = \frac{1}{1 + \int_{S_n} |(\boldsymbol{\theta}, \mathbf{s})|^\alpha \Gamma_{t_1, \ldots, t_k}(d\mathbf{s})},$$

where $m_{t_1, \ldots, t_k}$ is the spectral measure of $(X(t_1), \ldots, X(t_k))$. See Rachev and Resnick (1991) for similar results on max-stable random processes and exponential time change leading to geometric max-stable processes.

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