Stochastic Inequalities
IMS Lecture Notes – Monograph Series
Volume 22 (1993)

MATRIX EXTREMES AND RELATED STOCHASTIC BOUNDS

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The spaces $F_{n\times k}$ and S_k^+ consisting of rectangular and positive definite matrices are developed as partially ordered sets having lower and upper bounds. Under Loewner (1934) ordering, spectral lower and upper bounds are constructed for pairs $\mathbf{A}, \mathbf{B} \in (S_k^+, \succeq_L)$ and are shown to be tight. Similar bounds are given for pairs in $(F_{n\times k}, \succeq)$ in terms of singular decompositions under an induced ordering. Applications pertaining to $(S_k^+, \succeq_L)$ include stochastic bounds for distributions of quadratic forms, minimal dispersion bounds in certain regular ensembles, and bounds on the peakedness of certain weighted vector sums. Applications to $(F_{n\times k}, \succeq)$ support the uniform improvement of any pair of first-order experimental designs.

1. Introduction

Extremal problems persist throughout applied probability and statistics. Their solutions often shed new light on structural aspects of the system at hand.

To fix ideas, we reexamine the concentration properties of measures $\mu(\cdot;\mathbf{p})$ induced by weighted sums $\sum_{i=1}^n p_i X_i$ of *iid* random scalars $\{X_1,\ldots,X_n\}$ having a symmetric log-concave density. Here $\mathbf{p} = [p_1,\ldots,p_n]$ satisfies $\{0 \le p_i \le 1, p_1 + \cdots + p_n = 1\}$, and we let $F(t;\mathbf{p}) = \mu([-t,t];\mathbf{p})$ with $t > 0$. Proschan (1965) has shown for each $t > 0$ that $F(t;\mathbf{p})$ is order-reversing under majorization, *i.e.*, if $\mathbf{p}$ majorizes $\mathbf{q}$, then $F(t;\mathbf{q}) \ge F(t;\mathbf{p})$ and thus $\mu(\cdot;\mathbf{q})$ is more peaked than $\mu(\cdot;\mathbf{p})$ in the sense of Birnbaum (1948).

Since linear functions arise in a variety of contexts not entailing ordered weights, we pose the further question: If neither $\mathbf{p}$ majorizes $\mathbf{q}$ nor $\mathbf{q}$ majorizes $\mathbf{p}$, what then may be said regarding the concentration properties of $\mu(\cdot;\mathbf{p})$ and $\mu(\cdot;\mathbf{q})$? One answer follows immediately on observing that the ordered simplex supporting majorization is a lattice with greatest lower

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AMS 1991 *subject classifications.* Primary 15A45, 60E15.

Key words and phrases. Matrix orderings, positive definite and rectangular matrices, monotone functions, stochastic bounds, applications.

bound *(glb)* and least upper bound *(lub)* given respectively by $\mathbf{p} \wedge \mathbf{q}$ and $\mathbf{p} \vee \mathbf{q}$. Proschan's (1965) result immediately gives the bounds

$$F(t; \mathbf{p} \vee \mathbf{q}) \leq \{F(t; \mathbf{p}), F(t; \mathbf{q})\} \leq F(t; \mathbf{p} \wedge \mathbf{q}) \tag{1.1}$$

for each $t > 0$. Accordingly, we may refer to $F(t; \mathbf{p} \vee \mathbf{q})$ as the *stochastic minorant*, and to $F(t; \mathbf{p} \wedge \mathbf{q})$ as the *stochastic majorant*, of measures $\{\mu(\cdot; \mathbf{p}), \mu(\cdot; \mathbf{q})\}$ when evaluated over symmetric sets $[-t, t]$. We return to this topic later.

In this paper we extend the foregoing concepts to include other spaces and other orderings. It is seen that lattice properties need not carry forward. Nonetheless, these spaces are developed as partially ordered sets having lower and upper bounds, and these bounds are shown to be tight. An outline of the paper follows.

Preliminary developments occupy Section 2. Our main results are set forth in Section 3, first with regard to an ordering for symmetric matrices due to Loewner (1934), then with regard to an induced ordering on the space of real rectangular matrices. Applications are developed in Section 4. For the positive semidefinite ordering these include stochastic bounds for distributions of quadratic forms, minimal dispersion bounds on vector estimators in certain regular ensembles, and bounds on the peakedness of certain weighted vector sums. Applications to rectangular matrices support the uniform improvement of any pair of first-order experimental designs.

2. Preliminaries

We establish conventions for notation and review basic properties of some ordered spaces and functions monotone on them.

2.1. *Notation*

Symbols include $\mathbb{R}^n$ as Euclidean n-space, $F_{n\times k}$ as the real $(n \times k)$ matrices with $n \geq k$, S_k as the real symmetric $(k \times k)$ matrices, and S_k^0, S_k^+ and D_k as their positive semidefinite, positive definite, and diagonal varieties, respectively. The simplex $R_n(c)$ in $\mathbb{R}^n$ is given by $R_n(c) = \{\mathbf{x} \in \mathbb{R}^n : x_1 \geq \cdots \geq x_n, x_1 + \cdots + x_n = c\}$, and the transpose of $\mathbf{x} \in \mathbb{R}^n$ is $\mathbf{x}' = [x_1, \ldots, x_n]$. Special arrays include the unit vector $\mathbf{1}_n = [1, \ldots, 1]' \in \mathbb{R}^n$, the unit matrix $\mathbf{I}_n$, and a typical diagonal matrix $\mathbf{D}_\alpha = \mathrm{Diag}(\alpha_1, \ldots, \alpha_k) \in D_k$. Groups of transformations on $\mathbb{R}^n$ include the general linear group $Gl(n)$ and the real orthogonal group $O(n)$.

The *spectral decomposition* $\mathbf{A} = \sum_{i=1}^k \alpha_i \mathbf{q}_i \mathbf{q}_i'$ of $\mathbf{A} \in S_k^+$ yields its *symmetric root* $\mathbf{A}^{1/2} = \sum_{i=1}^k \alpha_i^{1/2} \mathbf{q}_i \mathbf{q}_i'$. The *singular decomposition* of $\mathbf{X} \in F_{n\times k}$

is $\mathbf{X} = \sum_{i=1}^{k} \xi_i \mathbf{p}_i \mathbf{q}_i' = \mathbf{P}\mathbf{D}_\xi \mathbf{Q}'$ in which $\mathbf{P} = [\mathbf{p}_1, \ldots, \mathbf{p}_k]$ is semiorthogonal containing the *left singular vectors*, $\mathbf{Q} = [\mathbf{q}_1, \ldots, \mathbf{q}_k]$ is orthonormal containing the *right singular vectors*, and $\mathbf{D}_\xi = \mathrm{Diag}(\xi_1, \ldots, \xi_k)$ contains the ordered *singular values* of $\mathbf{X}$.

Standard usage refers to independent, identically distributed *(iid)* variates and their cumulative distribution function *(cdf)*. $\mathcal{L}(\mathbf{X})$ denotes the distribution of $\mathbf{X}$, with $N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ as the Gaussian law on $\mathbb{R}^k$ having some mean $\boldsymbol{\mu}$ and dispersion matrix $\boldsymbol{\Sigma}$.

2.2. *Ordered Spaces*

A set $\mathcal{H}$ together with a binary relation $\succeq_0$ is said to be *linearly ordered* if the relation is reflexive, transitive, antisymmetric and complete. A *partial ordering* is reflexive, transitive and antisymmetric, and a *preordering* is reflexive and transitive. A partially ordered set $(\mathcal{H}, \succeq_0)$ is a *lower semi-lattice* if for any two elements x, y in $\mathcal{H}$, there is a greatest lower bound $x \wedge y$ in $\mathcal{H}$; an *upper semi-lattice* if there is a least upper bound $x \vee y$ in $\mathcal{H}$; and a *lattice* if it is both a lower and upper semi-lattice.

Ordered spaces of note include $(\mathbb{R}^k, \geq_k)$, with $\mathbf{x} \geq_k \mathbf{y}$ in $\mathbb{R}^k$ if and only if $\{x_i \geq y_i; 1 \leq i \leq k\}$, and the simplex $(R_n(c), \succeq_M)$ ordered by majorization (cf. Marshall and Olkin (1979)). The space $(S_k, \succeq_L)$ is ordered as in Loewner (1934) such that $\mathbf{A} \succeq_L \mathbf{B}$ if and only if $\mathbf{A} - \mathbf{B} \in S_k^0$, with $\mathbf{A} \succ_L \mathbf{B}$ whenever $\mathbf{A} - \mathbf{B} \in S_k^+$. The space $(F_{n \times k}, \succeq)$ has an induced ordering in which $\mathbf{X} \succeq \mathbf{Z}$ if and only if $\mathbf{X}'\mathbf{X} \succeq_L \mathbf{Z}'\mathbf{Z}$; see Jensen (1984). This ordering is invariant in the sense that $\mathbf{X} \succeq \mathbf{Z}$ if and only if $\mathbf{PXB} \succeq \mathbf{QZB}$ for any $\mathbf{P}, \mathbf{Q} \in O(n)$ and $\mathbf{B} \in Gl(k)$, and antisymmetry holds up to equivalence under $O(n)$ acting from the left.

Spaces with lower and upper bounds are germane to our studies. Clearly $(\mathbb{R}^k, \geq_k)$ is a lattice with $\mathbf{a} \wedge \mathbf{b} = [a_1 \wedge b_1, \ldots, a_k \wedge b_k]'$ and $\mathbf{a} \vee \mathbf{b} = [a_1 \vee b_1, \ldots, a_k \vee b_k]'$, where $\{a_i \wedge b_i = \min(a_i, b_i)$ and $a_i \vee b_i = \max(a_i, b_i); 1 \leq i \leq k\}$; see Vulikh (1967), for example. On working backwards from

$$\begin{aligned} v_1 &= x_1 \wedge y_1 \\ v_1 + v_2 &= (x_1 + x_2) \wedge (y_1 + y_2) \\ &\cdots \quad \cdots \\ v_1 + \cdots + v_k &= (x_1 + \cdots + x_k) \wedge (y_1 + \cdots + y_k) = c \end{aligned} \tag{2.1}$$

then from

$$\begin{aligned} u_1 &= x_1 \vee y_1 \\ u_1 + u_2 &= (x_1 + x_2) \vee (y_1 + y_2) \\ &\cdots \quad \cdots \\ u_1 + \cdots + u_k &= (x_1 + \cdots + x_k) \vee (y_1 + \cdots + y_k) = c, \end{aligned} \tag{2.2}$$

we conclude that $(R_k(c), \succeq_M)$ is a lattice with $\mathbf{x} \wedge \mathbf{y} = \mathbf{v}$ and $\mathbf{x} \vee \mathbf{y} = \mathbf{u}$. The space $(S_k, \succeq_L)$ is not a lattice (cf. Halmos (1958, p. 142)), nor can $(F_{n\times k}, \succeq)$ inherit lattice properties through its induced ordering. Nonetheless, both $(S_k^+, \succeq_L)$ and $(F_{n\times k}, \succeq)$ are shown subsequently to have lower and upper bounds that are tight.

2.3. *Monotone Functions*

A real-valued function $f(\cdot)$ on $(\mathcal{H}, \succeq_0)$ is said to be *order-preserving* if $x \succeq_0 y$ on $\mathcal{H}$ implies $f(x) \geq f(y)$ on $\mathbb{R}^1$, and to be *order-reversing* if $x \succeq_0 y$ on $\mathcal{H}$ implies $f(x) \leq f(y)$ on $\mathbb{R}^1$. Denote by $\Phi(\mathcal{H}, \succeq_0)$ the class of order-preserving functions, and by $\Phi^-(\mathcal{H}, \succeq_0)$ the order-reversing functions, on $(\mathcal{H}, \succeq_0)$. Specifically, $\Phi(\mathbb{R}^k, \geq_k)$ consists of functions $f(x_1, \ldots, x_k)$ nondecreasing in each argument, and functions in $\Phi(R_k(c), \succeq_M)$ comprise the Schur convex functions (cf. Marshall and Olkin (1979)). The class $\Phi(S_k, \succeq_L)$ is characterized in Marshall, Walkup and Wets (1967), and $\Phi(F_{n\times k}, \succeq)$ may be characterized through compositions as

$$\Phi(F_{n\times k}, \succeq) = \{\phi(\mathbf{X}) = \psi(\mathbf{X}'\mathbf{X}) : \psi \in \Phi(S_k, \succeq_L)\}; \tag{2.3}$$

for further details see Jensen (1984).

3. Matrix Extremes in S_k^+ and $F_{n\times k}$

Our principal findings are developed here. We first characterize the sets of lower and upper bounds for pairs of matrices in $(S_k^+, \succeq_L)$.

3.1. *Bounds in* $(S_k^+, \succeq_L)$

Given $(\mathbf{A}, \mathbf{B})$ in $(S_k^+, \succeq_L)$ we study first the lower bounds $H_L(\mathbf{A}, \mathbf{B}) = \{\mathbf{S} \in S_k^+ : \mathbf{S} \preceq_L \mathbf{A} \text{ and } \mathbf{S} \preceq_L \mathbf{B}\}$, and then the upper bounds $\mathbf{H}_U(\mathbf{A}, \mathbf{B}) = \{\mathbf{T} \in S_k^+ : \mathbf{T} \succeq_L \mathbf{A} \text{ and } \mathbf{T} \succeq_L \mathbf{B}\}$. The ordering $\mathbf{L} \preceq_L \{\mathbf{A}, \mathbf{B}\} \preceq_L \mathbf{U}$ always holds with $\mathbf{L} = \mathbf{0}$ and $\mathbf{U} = \mathbf{A} + \mathbf{B}$, and if $\mathbf{A} \preceq_L \mathbf{B}$, then $\mathbf{L} = \mathbf{A}$ and $\mathbf{U} = \mathbf{B}$. Since $\mathbf{A} \succeq_L \mathbf{S}$ if and only if $\mathbf{GAG}' \succeq_L \mathbf{GSG}'$ for any $\mathbf{G} \in Gl(k)$, it suffices to consider a canonical form in which $(\mathbf{A}, \mathbf{B}) \to (\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}, \mathbf{I}_k) \to (\mathbf{GAG}', \mathbf{GBG}') \to (\mathbf{D}_\gamma, \mathbf{I}_k)$, where $\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2} = \sum_{i=1}^k \gamma_i \mathbf{q}_i \mathbf{q}_i'$ is its spectral decomposition and $\mathbf{D}_\gamma = \mathrm{Diag}(\gamma_1, \ldots, \gamma_k)$ contains the ordered roots of $|\mathbf{A} - \gamma\mathbf{B}| = 0$. We thus seek $\mathbf{E} = \mathbf{GLG}'$ and $\mathbf{F} = \mathbf{GUG}'$ such that $\mathbf{E} \preceq_L \{\mathbf{D}_\gamma, \mathbf{I}_k\} \preceq_L \mathbf{F}$ or, equivalently, the classes $\mathbf{H}_L(\mathbf{D}_\gamma, \mathbf{I}_k)$ and $\mathbf{H}_U(\mathbf{D}_\gamma, \mathbf{I}_k)$.

First note that $\mathbf{A} \succeq_L \mathbf{B}$ on S_k^+ if and only if $\{\gamma_1 \geq \cdots \geq \gamma_k \geq 1\}$. If neither $\mathbf{A} \succeq_L \mathbf{B}$ nor $\mathbf{B} \succeq_L \mathbf{A}$, then at least one of two integers (r, s) can be found such that

$$\text{(3.1)} \qquad \{\gamma_1 \geq \cdots \geq \gamma_r > \gamma_{r+1} = 1 = \cdots = \gamma_{r+s} > \gamma_{r+s+1} \geq \cdots \geq \gamma_k > 0\}.$$

Now let $t = k - r - s$, and let $\{\epsilon_1 \geq \cdots \geq \epsilon_k > 0\}$ be the ordered eigenvalues of $\mathbf{E} \in S_k^+$. Essential properties of the lower bounds $\mathbf{H}_L(\mathbf{D}_\gamma, \mathbf{I}_k)$ are summarized in the following lemma.

LEMMA 1 *Let* $\mathbf{E} = [e_{ij}]$ *have eigenvalues* $\{\epsilon_1 \geq \cdots \geq \epsilon_k > 0\}$, *and consider the class* $\mathbf{H}_L(\mathbf{D}_\gamma, \mathbf{I}_k)$ *with* $\mathbf{D}_\gamma$ *fixed. In order that* $\mathbf{E} \in \mathbf{H}_L(\mathbf{D}_\gamma, \mathbf{I}_k)$ *it is necessary that*

(i) $\{\epsilon_i \leq 1; 1 \leq i \leq k\}$, *and that*

(ii) $\{e_{ii} \leq 1; 1 \leq i \leq r+s\}$ *and* $\{e_{ii} \leq \gamma_i; r+s+1 \leq i \leq k\}$.

Moreover, if $\{e_{ii}; 1 \leq i \leq k\}$ *are assigned their maximal values, a necessary and sufficient condition that* $\mathbf{E} \in \mathbf{H}_L(\mathbf{D}_\gamma, \mathbf{I}_k)$ *is that* $\mathbf{E}$ *should take the form*

(iii) $\mathbf{E}_M = \mathrm{Diag}(\mathbf{I}_r, \mathbf{I}_s, \gamma_{r+s+1}, \ldots, \gamma_k)$.

PROOF The equivalence of $\mathbf{I}_k \succeq_L \mathbf{E}$ and $\mathbf{I}_k \succeq_L \mathrm{Diag}(\epsilon_1, \ldots, \epsilon_k)$ gives conclusion (i). Conclusion (ii) follows on noting that the diagonal elements of the positive semidefinite matrices $\mathbf{D}_\gamma - \mathbf{E}$ and $\mathbf{I}_k - \mathbf{E}$ are necessarily nonnegative. To see necessity in conclusion (iii), assume first that $\mathbf{D}_\gamma - \mathbf{E} \in S_k^0$, so that $\mathbf{E}$ when assigned its maximal diagonal elements takes the form $\mathbf{E}_0 = \mathrm{Diag}(\mathbf{H}, \mathbf{I}_s, \gamma_{r+s+1}, \ldots, \gamma_k)$ such that $\mathbf{H} = [h_{ij}]$ with $\{h_{ii} = 1; 1 \leq i \leq r\}$, and $\mathrm{Diag}(\gamma_1, \ldots, \gamma_r) - \mathbf{H} \in S_k^0$. Other off-diagonal elements vanish since $\mathbf{D}_\gamma - \mathbf{E} \in S_k^0$ and the corresponding diagonal elements vanish. Now invoking the assumption $\mathbf{I}_k - \mathbf{E}_0 \in S_k^0$ stipulates further that $\mathbf{I}_r - \mathbf{H} \in S_r^0$, hence the off-diagonal elements of $\mathbf{H}$ must vanish also, giving $\mathbf{E}_M$ as in conclusion (iii). Sufficiency of the form (iii) follows since the diagonals of $\mathbf{E}_M$ take their maximal values, and both $\mathbf{D}_\gamma - \mathbf{E}_M$ and $\mathbf{I}_k - \mathbf{E}_M$ are positive semidefinite by construction. □

Turning to upper bounds for the pair $(\mathbf{D}_\gamma, \mathbf{I}_k)$, and thereby for $(\mathbf{A}, \mathbf{B})$, let $\{\eta_1 \geq \cdots \geq \eta_k > 0\}$ be the ordered eigenvalues of $\mathbf{F} \in S_k^+$. Without further proof, essential properties of $\mathbf{H}_U(\mathbf{D}_\gamma, \mathbf{I}_k)$ are as summarized in the following.

LEMMA 2 *Let* $\mathbf{F} = [f_{ij}]$ *have eigenvalues* $\{\eta_1 \geq \cdots \geq \eta_k > 0\}$, *and consider the class* $\mathbf{H}_U(\mathbf{D}_\gamma, \mathbf{I}_k)$ *with* $\mathbf{D}_\gamma$ *fixed. In order that* $\mathbf{F} \in \mathbf{H}_U(\mathbf{D}_\gamma, \mathbf{I}_k)$ *it is necessary that*

(i) $\{1 \leq \eta_i < \infty; 1 \leq i \leq k\}$, *and that*

(ii) $\{f_{ii} \geq \gamma_i; 1 \leq i \leq r\}$ *and* $\{f_{ii} \geq 1; r+1 \leq i \leq k\}$.

Moreover, if $\{f_{ii}; 1 \leq i \leq k\}$ *are assigned their minimal values, a necessary and sufficient condition that* $\mathbf{F} \in \mathbf{H}_U(\mathbf{D}_\gamma, \mathbf{I}_k)$ *is that* $\mathbf{F}$ *should take the form*

(iii) $\mathbf{F}_m = \mathrm{Diag}(\gamma_1, \ldots, \gamma_r, \mathbf{I}_s, \mathbf{I}_t)$ *with* $t = k - r - s$.

All lower and upper bounds for $(\mathbf{A}, \mathbf{B})$ in $(S_k^+, \succeq_L)$ follow on mapping back to the original space. Since $\mathbf{A} = \mathbf{B}^{1/2}\mathbf{Q}\mathbf{D}_\gamma\mathbf{Q}'\mathbf{B}^{1/2}$ and $\mathbf{B} = \mathbf{B}^{1/2}\mathbf{Q}\mathbf{I}_k\mathbf{Q}'\mathbf{B}^{1/2}$, we conclude that $\mathbf{L} \in \mathbf{H}_L(\mathbf{A}, \mathbf{B})$ if and only if $\mathbf{L} = \mathbf{B}^{1/2}\mathbf{Q}\mathbf{E}\mathbf{Q}'\mathbf{B}^{1/2}$ for some $\mathbf{E} \in \mathbf{H}_L(\mathbf{D}_\gamma, \mathbf{I}_k)$. Similarly, $\mathbf{U} \in \mathbf{H}_U(\mathbf{A}, \mathbf{B})$ if and only if $\mathbf{U} = \mathbf{B}^{1/2}\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{B}^{1/2}$ for some $\mathbf{F} \in \mathbf{H}_U(\mathbf{D}_\gamma, \mathbf{I}_k)$.

3.2. *Spectral Bounds in* $(S_k^+, \succeq_L)$

If we require that lower and upper bounds for diagonal matrices should be diagonal, then $(D_k, \succeq_L)$ is seen to be a lattice on imbedding it in $(\mathbb{R}^k, \geq_k)$. Then the *glb* and *lub* of $(\mathbf{D}_\gamma, \mathbf{I}_k)$ are $\mathbf{D}_\gamma \wedge \mathbf{I}_k = \mathbf{E}_M$ and $\mathbf{D}_\gamma \vee \mathbf{I}_k = \mathbf{F}_m$, precisely as defined in Lemmas 1 and 2. This in turn prompts the following.

DEFINITION 1 The matrices given by $\mathbf{A} \wedge \mathbf{B} = \mathbf{B}^{1/2}\mathbf{Q}(\mathbf{D}_\gamma \wedge \mathbf{I}_k)\mathbf{Q}'\mathbf{B}^{1/2}$ and $\mathbf{A} \vee \mathbf{B} = \mathbf{B}^{1/2}\mathbf{Q}(\mathbf{D}_\gamma \vee \mathbf{I}_k)\mathbf{Q}'\mathbf{B}^{1/2}$ are called the *spectral glb* and the *spectral lub* for $(\mathbf{A}, \mathbf{B})$ in $(S_k^+, \succeq_L)$.

Properties of these spectral extremes are studied next. The main issues include the possible interchangeability of $\mathbf{A}$ and $\mathbf{B}$, and whether the spectral bounds are tight. Both are answered affirmatively in developments culminating in Theorem 1.

With regard to the reduction $(\mathbf{A}, \mathbf{B}) \to (\mathbf{D}_\gamma, \mathbf{I}_k)$, we may take instead $(\mathbf{A}, \mathbf{B}) \to (\mathbf{I}_k, \mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2}) \to (\mathbf{I}_k, \mathbf{D}_\theta)$. Here $\mathbf{D}_\theta$ is the diagonal matrix $\mathbf{D}_\theta = \mathrm{Diag}(\theta_1, \ldots, \theta_k)$ with $\{0 < \theta_1 \leq \cdots \leq \theta_k\}$ as the reverse-ordered roots of $|\mathbf{B} - \theta\mathbf{A}| = 0$, where $\mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2} = \sum_{i=1}^{k} \theta_i \mathbf{p}_i\mathbf{p}_i'$ is the spectral decomposition with $\mathbf{P} = [\mathbf{p}_1, \ldots, \mathbf{p}_k] \in O(k)$. Proceeding as before, define $\mathbf{B} \wedge \mathbf{A} = \mathbf{A}^{1/2}\mathbf{P}(\mathbf{I}_k \wedge \mathbf{D}_\theta)\mathbf{P}'\mathbf{A}^{1/2}$ and $\mathbf{B} \vee \mathbf{A} = \mathbf{A}^{1/2}\mathbf{P}(\mathbf{I}_k \vee \mathbf{D}_\theta)\mathbf{P}'\mathbf{A}^{1/2}$. To investigate whether the spectral bounds are invariant with regard to decomposition, *i.e.*, whether $\mathbf{A} \wedge \mathbf{B} = \mathbf{B} \wedge \mathbf{A}$ and $\mathbf{A} \vee \mathbf{B} = \mathbf{B} \vee \mathbf{A}$, we first note several duality relations. These are $\mathbf{D}_\theta = \mathbf{D}_\gamma^{-1}$, $\mathbf{I}_k \wedge \mathbf{D}_\theta = (\mathbf{D}_\gamma \vee \mathbf{I}_k)^{-1}$, $\mathbf{I}_k \vee \mathbf{D}_\theta = (\mathbf{D}_\gamma \wedge \mathbf{I}_k)^{-1}$, $(\mathbf{D}_\gamma \wedge \mathbf{I}_k)(\mathbf{D}_\gamma \vee \mathbf{I}_k) = \mathbf{D}_\gamma$, and $\mathbf{B}^{1/2}\mathbf{Q} = \mathbf{A}^{1/2}\mathbf{P}\mathbf{D}_\gamma^{-1/2}$ and $\mathbf{A}^{-1/2}\mathbf{P}\mathbf{D}_\gamma^{1/2} = \mathbf{B}^{-1/2}\mathbf{Q}$. The latter expressions follow on establishing relationships between the normalized eigenvectors of $\{\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}\mathbf{q}_i = \gamma_i\mathbf{q}_i; 1 \leq i \leq k\}$ and $\{\mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2}\mathbf{p}_i = \theta_i\mathbf{p}_i; 1 \leq i \leq k\}$. Our principal findings with regard to the lower and upper spectral bounds are as follow.

THEOREM 1 *Let* $\{\mathbf{A} \wedge \mathbf{B}, \mathbf{A} \vee \mathbf{B}\}$ *and* $\{\mathbf{B} \wedge \mathbf{A}, \mathbf{B} \vee \mathbf{A}\}$ *be spectral glb's and lub's as defined. Then for any* $(\mathbf{A}, \mathbf{B})$ *in* $(S_k^+, \succeq_L)$, *(i)* $\mathbf{A} \wedge \mathbf{B} \preceq_L \{\mathbf{A}, \mathbf{B}\} \preceq_L \mathbf{A} \vee \mathbf{B}$,

(ii) $\phi(\mathbf{A} \wedge \mathbf{B}) \leq \{\phi(\mathbf{A}), \phi(\mathbf{B})\} \leq \phi(\mathbf{A} \vee \mathbf{B})$ *for each* $\phi \in \Phi(S_k^+, \succeq_L)$, *and*

(iii) $\mathbf{A} \wedge \mathbf{B} = \mathbf{B} \wedge \mathbf{A}$ *and* $\mathbf{A} \vee \mathbf{B} = \mathbf{B} \vee \mathbf{A}$.

Moreover, the bounds are tight in the sense that

(iv) if $\{\mathbf{A}, \mathbf{B}\} \preceq_L \mathbf{T}$ *and* $\mathbf{T} \preceq_L \mathbf{A} \vee \mathbf{B}$, *then* $\mathbf{T} = \mathbf{A} \vee \mathbf{B}$, *and*

(v) if $\{\mathbf{A}, \mathbf{B}\} \succeq_L \mathbf{S}$ *and* $\mathbf{S} \succeq_L \mathbf{A} \wedge \mathbf{B}$, *then* $\mathbf{S} = \mathbf{A} \wedge \mathbf{B}$.

PROOF Conclusion (i) follows from Lemmas 1 and 2, and this implies (ii) in view of the monotonicity of $\Phi(S_k^+, \succeq_L)$. To see conclusion (iii), note that if $\mathbf{A} \preceq_L \mathbf{B}$, then $\mathbf{A} \wedge \mathbf{B} = \mathbf{A} = \mathbf{B} \wedge \mathbf{A}$ and $\mathbf{A} \vee \mathbf{B} = \mathbf{B} = \mathbf{B} \vee \mathbf{A}$. Otherwise let $\mathbf{B}_1' = \mathbf{B}^{1/2}\mathbf{Q}$ and $\mathbf{B}_2' = \mathbf{A}^{1/2}\mathbf{P}\mathbf{D}_\gamma^{-1/2}$; observe that $\mathbf{A} \wedge \mathbf{B} = \mathbf{B}_1'(\mathbf{D}_\gamma \wedge \mathbf{I}_k)\mathbf{B}_1$; and recall that $\mathbf{B} \wedge \mathbf{A} = \mathbf{A}^{1/2}\mathbf{P}(\mathbf{I}_k \wedge \mathbf{D}_\theta)\mathbf{P}'\mathbf{A}^{1/2}$. From the duality relations cited earlier it follows that $\mathbf{I}_k \wedge \mathbf{D}_\theta = (\mathbf{D}_\gamma \vee \mathbf{I}_k)^{-1} = (\mathbf{D}_\gamma \wedge \mathbf{I}_k)\mathbf{D}_\gamma^{-1}$, so that $\mathbf{B} \wedge \mathbf{A} = \mathbf{A}^{1/2}\mathbf{P}\mathbf{D}_\gamma^{-1/2}(\mathbf{D}_\gamma \wedge \mathbf{I}_k)\mathbf{D}_\gamma^{-1/2}\mathbf{P}'\mathbf{A}^{1/2} = \mathbf{B}_2'(\mathbf{D}_\gamma \wedge \mathbf{I}_k)\mathbf{B}_2$. But another duality asserts that $\mathbf{B}_1' = \mathbf{B}_2'$, so that $\mathbf{A} \wedge \mathbf{B} = \mathbf{B} \wedge \mathbf{A}$ as claimed. The assertion $\mathbf{A} \vee \mathbf{B} = \mathbf{B} \vee \mathbf{A}$ follows similarly. To establish (iv), first suppose that $\{\mathbf{A}, \mathbf{B}\} \preceq_L \mathbf{T}$ and $\mathbf{T} \preceq_L \mathbf{A} \vee \mathbf{B}$. Then since $\mathbf{D}_\gamma = \mathbf{Q}'\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}\mathbf{Q}$ and $\mathbf{D}_\gamma \vee \mathbf{I}_k = \mathbf{Q}'\mathbf{B}^{-1/2}(\mathbf{A} \vee \mathbf{B})\mathbf{B}^{-1/2}\mathbf{Q}$, the ordering $\mathbf{A} \preceq_L \mathbf{T} \preceq_L \mathbf{A} \vee \mathbf{B}$ implies

$$\mathbf{D}_\gamma \preceq_L \mathbf{Q}'\mathbf{B}^{-1/2}\mathbf{T}\mathbf{B}^{-1/2}\mathbf{Q} \preceq_L \mathbf{D}_\gamma \vee \mathbf{I}_k \tag{3.2}$$

whereas $\mathbf{B} \preceq_L \mathbf{T} \preceq_L \mathbf{A} \vee \mathbf{B}$ gives

$$\mathbf{I}_k \preceq_L \mathbf{Q}'\mathbf{B}^{-1/2}\mathbf{T}\mathbf{B}^{-1/2}\mathbf{Q} \preceq_L \mathbf{D}_\gamma \vee \mathbf{I}_k. \tag{3.3}$$

Letting $\mathbf{c}_i' = [0, \ldots, 0, 1, 0, \ldots, 0]$ have unity in the i^{th} coordinate and zeros elsewhere, we infer from (3.2) and (3.3) that

$$\gamma_i \leq \mathbf{c}_i'\mathbf{Q}'\mathbf{B}^{-1/2}\mathbf{T}\mathbf{B}^{-1/2}\mathbf{Q}\mathbf{c}_i \leq \gamma_i, \qquad 1 \leq i \leq r \tag{3.4}$$

$$1 \leq \mathbf{c}_i'\mathbf{Q}'\mathbf{B}^{-1/2}\mathbf{T}\mathbf{B}^{-1/2}\mathbf{Q}\mathbf{c}_i \leq 1, \qquad r+1 \leq i \leq k. \tag{3.5}$$

Combining these and letting $\mathbf{W} = [(\mathbf{D}_\gamma \vee \mathbf{I}_k) - \mathbf{Q}'\mathbf{B}^{-1/2}\mathbf{T}\mathbf{B}^{-1/2}\mathbf{Q}]$ we find that $\mathbf{c}_i'\mathbf{W}\mathbf{c}_i = 0$, so that the diagonal elements of $\mathbf{W}$ are zero. But since $\mathbf{W} \succeq_L \mathbf{0}$, this implies that $\mathbf{W} = \mathbf{0}$ and thus $\mathbf{T} = \mathbf{B}^{1/2}\mathbf{Q}(\mathbf{D}_\gamma \vee \mathbf{I}_k)\mathbf{Q}'\mathbf{B}^{1/2} = \mathbf{A} \vee \mathbf{B}$ as claimed in conclusion (iv). The proof for (v) proceeds similarly, to complete our proof. □

3.3. *Bounds on* $(F_{n \times k}, \succeq)$

We seek lower and upper bounds on $(F_{n \times k}, \succeq)$, now requiring that matrices in $F_{n \times k}$ should be of full rank $k \leq n$. We proceed constructively as follows, starting with $(\mathbf{X}, \mathbf{Z})$ in $(F_{n \times k}, \succeq)$.

First transform $(\mathbf{X},\mathbf{Z}) \to [\mathbf{X}(\mathbf{Z}'\mathbf{Z})^{-1/2}, \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1/2}]$, observing that $\mathbf{H} = \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1/2}$ is semiorthogonal such that $\mathbf{H}'\mathbf{H} = \mathbf{I}_k$ and $\mathbf{H}\mathbf{H}'$ is idempotent of rank k. We next undertake the singular decompositions $\mathbf{X}(\mathbf{Z}'\mathbf{Z})^{-1/2} = \sum_{i=1}^k \lambda_i \mathbf{p}_i \mathbf{q}_i' = \mathbf{P}\mathbf{D}_\lambda \mathbf{Q}'$ and $\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1/2} = \sum_{i=1}^k \mathbf{u}_i \mathbf{q}_i' = \mathbf{U}\mathbf{I}_k\mathbf{Q}'$ such that $\mathbf{D}_\lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_k)$, and $\mathbf{P}$ and $\mathbf{U}$ are semiorthogonal, whereas $\mathbf{Q} \in O(k)$. It is essential to note that $\mathbf{X}$ and $\mathbf{Z}$ may be recovered as $\mathbf{X} = \mathbf{P}\mathbf{D}_\lambda\mathbf{Q}'(\mathbf{Z}'\mathbf{Z})^{1/2}$ and $\mathbf{Z} = \mathbf{U}\mathbf{I}_k\mathbf{Q}'(\mathbf{Z}'\mathbf{Z})^{1/2}$. Corresponding to earlier usage we now define provisional lower and upper bounds, to be verified subsequently, as

$$\mathbf{X} \wedge \mathbf{Z} = \mathbf{P}(\mathbf{D}_\lambda \wedge \mathbf{I}_k)\mathbf{Q}'(\mathbf{Z}'\mathbf{Z})^{1/2} \tag{3.6}$$

$$\mathbf{X} \vee \mathbf{Z} = \mathbf{P}(\mathbf{D}_\lambda \vee \mathbf{I}_k)\mathbf{Q}'(\mathbf{Z}'\mathbf{Z})^{1/2}. \tag{3.7}$$

Subject to later justification we set forth the following.

DEFINITION 2 The matrix $\mathbf{X} \wedge \mathbf{Z}$ in (3.6) is called a *singular lower bound* for $(\mathbf{X},\mathbf{Z})$ in $(F_{n\times k}, \succeq)$, and $\mathbf{X} \vee \mathbf{Z}$ in (3.7) is called a *singular upper bound.*

Basic properties of the singular bounds are given in the following.

THEOREM 2 *Consider matrices* $\mathbf{X}\wedge\mathbf{Z}$ *and* $\mathbf{X}\vee\mathbf{Z}$ *as constructed from* $(\mathbf{X},\mathbf{Z})$ *in* $(F_{n\times k}, \succeq)$. *Then*

(i) $\mathbf{X} \wedge \mathbf{Z} \preceq \{\mathbf{X},\mathbf{Z}\} \preceq \mathbf{X} \vee \mathbf{Z}$;

(ii) $\phi(\mathbf{X} \wedge \mathbf{Z}) \leq \{\phi(\mathbf{X}), \phi(\mathbf{Z})\} \leq \phi(\mathbf{X} \vee \mathbf{Z})$ *for each* $\phi \in \Phi(F_{n\times k}, \succeq)$*; and*

(iii) $\mathbf{X} \wedge \mathbf{Z}$ *and* $\mathbf{X} \vee \mathbf{Z}$ *are determined up to equivalence under* $O(n)$ *acting from the left. Moreover, the bounds are tight in the sense that*

(iv) if $\{\mathbf{X},\mathbf{Z}\} \preceq \mathbf{T}$ *and* $\mathbf{T} \preceq \mathbf{X} \vee \mathbf{Z}$, *then* $\mathbf{T}$ *is equivalent to* $\mathbf{X} \vee \mathbf{Z}$, *and*

(v) if $\{\mathbf{X},\mathbf{Z}\} \succeq \mathbf{S}$ *and* $\mathbf{S} \succeq \mathbf{X} \wedge \mathbf{Z}$, *then* $\mathbf{S}$ *is equivalent to* $\mathbf{X} \wedge \mathbf{Z}$.

PROOF The conclusions follow from Theorem 1 since the orderings "$\mathbf{X} \succeq \mathbf{Y}$ on $(F_{n\times k}, \succeq)$" and "$\mathbf{X}'\mathbf{X} \succeq_L \mathbf{Y}'\mathbf{Y}$ on $(S_k^+, \succeq_L)$" are equivalent. In particular, with $\mathbf{X}'\mathbf{X} = \mathbf{A}$, $\mathbf{Z}'\mathbf{Z} = \mathbf{B}$ and $\mathbf{D}_\gamma = \mathbf{D}_\lambda^2$, conclusion (i) follows from its counterpart in Theorem 1. Conclusion (ii) follows from (i) and the monotonicity of $\Phi(F_{n\times k}, \succeq)$. Conclusion (iii) is apparent since $\mathbf{X} \succeq \mathbf{Y}$ and $\mathbf{P}\mathbf{X} \succeq \mathbf{U}\mathbf{Y}$ are equivalent for any $\mathbf{P}, \mathbf{U}$ in $O(n)$. Conclusions (iv) and (v) follow from their counterparts in Theorem 1 together with the foregoing conclusion (iii). □

In view of Theorem 2 we now see that there are equivalence classes of singular lower and upper bounds for $(\mathbf{X},\mathbf{Z})$ in $(F_{n\times k}, \succeq)$. Thus $\mathbf{X} \wedge \mathbf{Z}$ and $\mathbf{X} \vee \mathbf{Z}$ in (3.6) and (3.7) may be replaced by their equivalents

$$\{\mathbf{X} \wedge \mathbf{Z}\} = \{\mathbf{R}(\mathbf{D}_\lambda \wedge \mathbf{I}_k)\mathbf{Q}'(\mathbf{Z}'\mathbf{Z})^{1/2};\ \mathbf{R} \in O(n)\} \tag{3.8}$$

$$\{\mathbf{X} \vee \mathbf{Z}\} = \{\mathbf{R}(\mathbf{D}_\lambda \vee \mathbf{I}_k)\mathbf{Q}'(\mathbf{Z}'\mathbf{Z})^{1/2};\ \mathbf{R} \in O(n)\}. \tag{3.9}$$

4. Some Applications

We illustrate stochastic bounds arising from Theorems 1 and 2.

4.1. *Distributions of Quadratic Forms*

Quadratic forms in Gaussian variates arise in a variety of settings, often under one of several alternative models. Specifically, with $\mathbf{A} \in S_k^0$, let $G_A(t;\mathbf{\Sigma})$ be the *cdf* of $V = \mathbf{X}'\mathbf{A}\mathbf{X}$ under $\mathcal{L}(\mathbf{X}) = N_k(\mathbf{0},\mathbf{\Sigma})$. Standard results include the following. (i) $\mathcal{L}(\mathbf{X}'\mathbf{A}\mathbf{X}) = \mathcal{L}\left(\sum_{i=1}^{k}\gamma_i Z_i^2\right)$, where $\{Z_1,\ldots,Z_k\}$ are *iid* $N_1(0,1)$ and $\{\gamma_1,\ldots,\gamma_k\}$ are the roots of $|\mathbf{A}-\gamma\mathbf{\Sigma}^{-1}| = 0$; see Johnson and Kotz (1970), including series expansions for $G_A(t;\mathbf{\Sigma})$. (ii) For fixed $\mathbf{A} \in S_k^0$ and $t > 0$, it is clear that $G_A(t,\mathbf{\Sigma}) \in \Phi^-(S_k^+,\succeq_L)$ when considered as a function on S_k^+, as may be seen on applying a result of Anderson (1955). Given two models, $N_k(\mathbf{0},\mathbf{\Sigma})$ and $N_k(\mathbf{0},\mathbf{\Omega})$, our earlier developments support the bounds

$$(4.1)\qquad G_A(t;\mathbf{\Sigma}\vee\mathbf{\Omega}) \leq \{G_A(t;\mathbf{\Sigma}), G_A(t;\mathbf{\Omega})\} \leq G_A(t;\mathbf{\Sigma}\wedge\mathbf{\Omega})$$

for each $t > 0$. As t varies, this provides an envelope bounding the two *cdf's* $G_A(t,\mathbf{\Sigma})$ and $G_A(t;\mathbf{\Omega})$. Moreover, these bounds may be evaluated numerically in particular cases using known series expansions; see Johnson and Kotz (1970).

To continue, consider an ensemble $\{N_k(\mathbf{0},\Xi);\Xi \in K_0\}$, for which $K_0 \subset S_k^+$ contains a minimal element Ξ_m and a maximal element Ξ_M under the ordering $\succeq_L$. Corresponding to (4.1) we have stochastic bounds given by

$$(4.2)\qquad G_A(t;\Xi_M) \leq \{G_A(t;\Xi);\Xi \in K_0\} \leq G_A(t;\Xi_m)$$

for each $t > 0$. For fixed $\mathbf{A}$, this gives an envelope of curves for all such *cdf's* as t varies over $[0,\infty)$.

4.2. *Minimal Dispersion Bounds*

We consider the regular estimation of vector parameters. Specifically, consider a family of dominated probability measures having density functions $\{f(\mathbf{X};\theta);\theta \in \Theta\}$ with $\Theta \subset \mathbb{R}^r$; let $\mathbf{G}(\theta) = [g_1(\theta),\ldots,g_k(\theta)]'$ be estimable functions on Θ having the partial derivatives $\Delta(\theta) = [\delta_{ij}(\theta)] = [\partial g_i(\theta)/\partial\theta_j]$ for all $\theta \in \Theta$; and let $\widehat{\mathbf{G}}(\mathbf{X}) = [\widehat{g}_1(\mathbf{X}),\ldots,\widehat{g}_k(\mathbf{X})]'$ be any unbiased estimator for $\mathbf{G}(\theta)$ having some dispersion matrix $V(\widehat{\mathbf{G}}(\mathbf{X}))$. Under regularity conditions $\mathcal{R}$, given as (i)–(vi) on page 194 of Zacks (1971), for example, a standard result is a minimal dispersion bound given by $V(\widehat{\mathbf{G}}(\mathbf{X})) \succeq_L$

$\Delta(\theta)[\mathcal{I}(\theta)]^{-1}\Delta'(\theta)$, where $\mathcal{I}(\theta) = [\mathcal{I}_{ij}(\theta)]$ is the Fisher information matrix. Moreover, this bound holds for every unbiased estimator $\widehat{\mathbf{G}}(\mathbf{X})$.

If two such regular families have information matrices $\mathcal{I}_1(\theta)$ and $\mathcal{I}_2(\theta)$, then our earlier construction applies directly. The resulting bound, namely,

$$V(\widehat{\mathbf{G}}(\mathbf{X})) \succeq_L \Delta(\theta)[\mathcal{I}_1(\theta) \vee \mathcal{I}_2(\theta)]^{-1}\Delta'(\theta) \tag{4.3}$$

applies to any unbiased estimator for $\mathbf{G}(\theta)$ taken from either model.

These developments extend to any regular ensemble $\{f_\tau(\mathbf{X};\theta); \theta \in \Theta, \tau \in T\}$ for which there is a maximal Fisher information matrix $\mathcal{I}_M(\theta)$. Corresponding to (4.3) we now have the bound

$$V(\widehat{\mathbf{G}}(\mathbf{X})) \succeq_L \Delta(\theta)[\mathcal{I}_M(\theta)]^{-1}\Delta'(\theta) \tag{4.4}$$

uniformly for every unbiased estimator $\widehat{\mathbf{G}}(\mathbf{X})$ for $\mathbf{G}(\theta)$ taken from any family in the ensemble.

4.3. *Peakedness of Vector Sums*

Consider the measures $\mu(\cdot;\mathbf{p})$ on $\mathbb{R}^k$ induced by weighted sums $\sum_{i=1}^n p_i\mathbf{X}_i$ of *iid* random vectors $\{\mathbf{X}_1,\ldots,\mathbf{X}_n\}$ having a symmetric log-concave density. Let $\mathbf{C}_0^k$ be the class of all compact convex subsets of $\mathbb{R}^k$ symmetric under reflection through $\mathbf{0} \in \mathbb{R}^k$, *i.e.* $\mathbf{x} \in A$ implies $-\mathbf{x} \in A$ for each $A \in \mathbf{C}_0^k$. It is known for each $A \in \mathbf{C}_0^k$ that $\mu(A;\mathbf{p})$ is order-reversing when considered as a function on $(R_n(c), \succeq_M)$, *i.e.*, $\mu(A;\mathbf{p}) \in \Phi^-(R_n(c), \succeq_M)$; see Olkin and Tong (1988) and Chan, Park and Proschan (1989). Thus if $\mathbf{p} \succeq_M \mathbf{q}$, then $\mu(\cdot;\mathbf{q})$ is more peaked about $\mathbf{0} \in \mathbb{R}^k$ than $\mu(\cdot;\mathbf{p})$ in the sense of Sherman (1955). Corresponding to (1.1) we therefore have the bounds

$$\mu(A;\mathbf{p} \vee \mathbf{q}) \leq \{\mu(A;\mathbf{p}), \mu(A;\mathbf{q})\} \leq \mu(A;\mathbf{p} \wedge \mathbf{q}) \tag{4.5}$$

for each $A \in \mathbf{C}_0^k$. Accordingly, we may refer to $\mu(\cdot;\mathbf{p} \vee \mathbf{q})$ as the stochastic minorant, and to $\mu(\cdot;\mathbf{p} \wedge \mathbf{q})$ as the stochastic majorant, of measures $\{\mu(\cdot;\mathbf{p}), \mu(\cdot;\mathbf{q})\}$ when evaluated over sets in $\mathbf{C}_0^k$.

To continue, we consider a possibly contaminated Gaussian sample as follows. Let $\{\mathbf{Y}_1,\ldots,\mathbf{Y}_n\}$ be *iid* $N_k(\mathbf{0},\mathbf{\Sigma})$ and let $\{\mathbf{Z}_1,\ldots,\mathbf{Z}_n\}$ be *iid* $N_k(\mathbf{0},\mathbf{\Omega})$. Now consider a possibly contaminated sample $\{\mathbf{X}_1,\ldots,\mathbf{X}_n\}$ in which $\{\mathbf{X}_i = \delta_i\mathbf{Y}_i + (1-\delta_i)\mathbf{Z}_i; 1 \leq i \leq n\}$ with $\{\delta_i \in \{0,1\}; 1 \leq i \leq n\}$. Further let $\mu(\cdot;\mathbf{p},\mathbf{\Sigma})$ be the measure induced by $\sum_{i=1}^n p_i\mathbf{Y}_i$. A basic result of Anderson (1955) shows that for each fixed $A \in \mathbf{C}_0^k$ and $\mathbf{p} \in R_n(c)$, the measure $\mu(A;\mathbf{p},\mathbf{\Sigma})$ is order-reversing when considered as a function on $(S_k^+, \succeq_L)$.

On combining the foregoing developments with Theorem 1, we now have the following bounds on the measure $\mu_X(\cdot;\mathbf{p})$ induced by $\sum_{i=1}^n p_i\mathbf{X}_i$ in a possibly contaminated sample, as

$$\mu(A;\mathbf{p} \vee \mathbf{q};\mathbf{\Sigma} \vee \mathbf{\Omega}) \leq \{\mu_X(A;\mathbf{p}), \mu_X(A;\mathbf{q})\} \leq \mu(A;\mathbf{p} \wedge \mathbf{q},\mathbf{\Sigma} \wedge \mathbf{\Omega}) \tag{4.6}$$

for each $A \in \mathbf{C}_0^k$, where the brackets in the center encompass all 2^n possible choices for $\{\delta_1, \ldots, \delta_n\}$ in the contaminated model. Equality is achieved on the left when $\mathbf{p} \succeq_M \mathbf{q}$ and $\boldsymbol{\Sigma} \succeq_L \boldsymbol{\Omega}$ at $\delta_1 = \cdots = \delta_n = 1$, and on the right at $\delta_1 = \cdots = \delta_n = 0$.

With regard to the peakedness of vector sums, the results of Olkin and Tong (1988) and of Chan, Park and Proschan (1989) have been extended by Eaton (1988) to include concentration properties in Gauss-Markov estimation. Our developments here, applied in Eaton's setting, would appear to provide lower and upper bounds for concentration probabilities in two or more linear models having dispersion parameters not ordered in $(S_k^+, \succeq_L)$. Earlier work by the author (Jensen (1979)) complements Eaton's work in demonstrating that Gauss-Markov estimators are most peaked among median-unbiased linear estimators, even without first or second moments as assumed by Eaton (1988).

4.4. *First-Order Experimental Designs*

Consider models $\mathbf{Y} = \alpha_1 \mathbf{1}_n + \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$ and $\mathbf{Y} = \alpha_2 \mathbf{1}_n + \mathbf{Z}\boldsymbol{\beta} + \mathbf{e}$ having design matrices $\mathbf{X}, \mathbf{Z} \in F_{n \times k}$ in centered form as deviations from their column means. We are concerned with inferences regarding the elements of $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_k]'$ in the two models, based on the Gauss-Markov estimators $\widehat{\boldsymbol{\beta}}(\mathbf{X}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$, and similarly $\widehat{\boldsymbol{\beta}}(\mathbf{Z}) = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}$, when $\mathcal{L}(\mathbf{e}) = N_n(\mathbf{0}, \sigma^2 \mathbf{I}_n)$. The directed Fisher efficiency of design $\mathbf{Z}$ relative to $\mathbf{X}$ in estimating $\mathbf{a}'\boldsymbol{\beta}$ is given by

$$E_F(\mathbf{Z}, \mathbf{X}; \mathbf{a}) = \mathbf{a}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{a}/\mathbf{a}'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{a}. \tag{4.7}$$

The Pitman efficiency of $\mathbf{Z}$ relative to $\mathbf{X}$, in normal-theory tests for H : $\mathbf{A}\boldsymbol{\beta} = \delta_0$ against $K : \mathbf{A}\boldsymbol{\beta} \neq \boldsymbol{\delta}_0$ is given by

$$E_P(\mathbf{Z}, \mathbf{X} \mid \mathbf{A}) = \frac{(\mathbf{A}\boldsymbol{\beta} - \boldsymbol{\delta}_0)'[\mathbf{A}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{A}']^{-1}(\mathbf{A}\boldsymbol{\beta} - \boldsymbol{\delta}_0)}{(\mathbf{A}\boldsymbol{\beta} - \boldsymbol{\delta}_0)'[\mathbf{A}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{A}']^{-1}(\mathbf{A}\boldsymbol{\beta} - \boldsymbol{\delta}_0)}. \tag{4.8}$$

It is clear that design $\mathbf{Z}$ is more efficient than $\mathbf{X}$, uniformly for all $\mathbf{a} \in \mathbb{R}^k$ under Fisher efficiency, and for all $\{\mathbf{A} \in F_{r \times k}; 1 \leq r \leq k\}$ under Pitman efficiency, if and only if $\mathbf{Z} \succeq \mathbf{X}$ on $(F_{n \times k}, \succeq)$.

If neither $\mathbf{Z} \succeq \mathbf{X}$ nor $\mathbf{X} \succeq \mathbf{Z}$, then Theorem 2 supports the construction of a new design dominating both $\mathbf{X}$ and $\mathbf{Z}$ in efficiency. This is precisely the design $\mathbf{X} \vee \mathbf{Z}$ given in expression (3.7), or any equivalent design from expression (3.9). In summary, given any specified pair of first-order experimental designs in $F_{n \times k}$, we may construct numerically a new design dominating both designs in its efficiency for inferences regarding $\boldsymbol{\beta}$.

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