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REGULAR, SAMPLE PATH AND STRONG STOCHASTIC CONVEXITY: A REVIEW

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Several notions of stochastic convexity and concavity and their properties are described in this survey. The notion of sample path stochastic convexity is a refinement of the well used notion of stochastic ordering, and it can be used to construct, on a common probability space, random variables which have desirable convexity (or concavity) properties with probability one.

Three open problems from the literature are described. These problems could not be resolved until the introduction of the stochastic convexity notions which are described in this survey. The solutions of these problems illustrate the strength and the usefulness of these notions. Each notion is accompanied by a description of some of its applications. References for more detailed study of these notions are given. Indications of further work in this area are included.

1. Introduction. Regular, Sample Path and Strong Stochastic Convexity notions are very valuable in many areas in probability and statistics such as queueing and reliability theory. Consider, for example, the following three open problems:

PROBLEM 1. Consider a single stage queueing system at which customers arrive according to a doubly stochastic Poisson (DSP) process. The stochastic intensity of the DSP is a Markov process on $\{\lambda_1, \lambda_2\}$ ($\lambda_i \geq 0, i = 1, 2$). The expected time this Markov process spends in state λ_i is $\theta r_i, i = 1, 2$ for some $r_i \geq 0, i = 1, 2$, and $\theta \geq 0$. The service times of the customers are independent and identically distributed random variables. Let $EW(\theta)$ be the average work load in this DSP/G/1 queueing system.

CONJECTURE 1. (Ross 1978). $EW(\theta)$ is a decreasing function of θ.

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PROBLEM 2. Let $N(\theta)$ be a generic random variable representing the stationary number of customers in a single stage M/M/c queueing system with c parallel servers, Poisson arrival process with rate θ and exponentially distributed service times with mean $\frac{1}{\mu}$. We assume that μ is fixed and θ takes on some value such that $0 \leq \theta < c\mu$. Grassmann (1983) showed that $EN(\theta)$ is increasing and convex. Thus we are led to the following conjecture.

CONJECTURE 2. The inventory carrying cost $Ef(N(\theta))$ is increasing and convex in θ for all increasing and convex inventory carrying cost functions f.

PROBLEM 3. Consider a closed queueing network of the Gordon-Newell type with m stations and k jobs. The number of parallel servers at station i is s_i, and the service times are independent and exponentially distributed with mean $\frac{1}{\mu_i}, i = 1, \ldots, m$. Jobs are served on a first come first served basis at each service station. The k jobs are routed from one station to another according to an irreducible stochastic matrix. Let $TH(k)$ be the throughput of this closed queueing network when all stations have ample waiting room so that no job will be blocked. Dowdy, Eager, Gordon and Saxton (1984) showed that when $s_i = 1, i = 1, 2, \cdots, m$, then $TH(k)$ is an increasing concave function of k. Thus we are led to the following conjecture.

CONJECTURE 3. The throughput $TH(k)$ is an increasing and concave function of the job population size k.

These conjectures have been resolved using the notions of stochastic convexity which are described in this survey. An outline to resolving these conjectures is given in Section 5.

Establishing convexity and concavity properties of the performance measures of stochastic systems is of great value in the optimal design and control of these systems. Usually one analyzes the second derivative or the second difference of the performance measures in order to establish these properties. This requires the knowledge of the performance measures, often in the form of an explicit function of the input parameters of the system (e.g., the mean number of customers in an M/M/c queue with the input parameter being the customer arrival rate – Grassmann 1983, Lee and Cohen 1983). Even then, the algebra involved in establishing the convexity or the concavity property is quite involved. Consequently convexity and concavity results of the performance measures of stochastic systems are very limited. These difficulties can be overcome by the use of the notions of stochastic convexity. The applications described in the survey demonstrate that the notions of stochastic convexity are very useful in establishing convexity properties of the performance measures of stochastic systems.

Bounds for the performance of stochastic system are very useful in the study of complex stochastic systems that do not lend themselves for explicit solutions. As we see in this survey, stochastic convexity can play a crucial role in obtaining such bounds.

2. Regular Stochastic Convexity. Let $\{P_\theta, \theta \in \Theta\}$ be a family of univariate distributions. Throughout this paper Θ is a convex set (i.e. an interval) of the real line $\mathcal{R}$ or of the set $\mathcal{N}_+ = \{0, 1, \ldots, \}$. Let $X(\theta)$ denote a random variable with distribution P_θ. It is convenient and intuitive to replace the notation $\{P_\theta, \theta \in \Theta\}$ by $\{X(\theta), \theta \in \Theta\}$ and this notation is used throughout this paper. Note that when we write $\{X(\theta), \theta \in \Theta\}$ we do not assume (and often we are not concerned with) any dependence (or independence) properties among the $X(\theta)$'s. We are only interested in the 'marginal distributions' $\{P_\theta, \theta \in \Theta\}$ of $\{X(\theta), \theta \in \Theta\}$ even when in some circumstances $\{X(\theta), \theta \in \Theta\}$ is a well-defined stochastic process. Note also that $X(\theta)$ does not mean that X is a function of θ; it only indicates that the distribution of $X(\theta)$ is P_θ. For example in an M/M/c queueing system $X(\theta)$ may be a generic random variable whose distribution is the stationary distribution of the number of jobs in the system when the arrival rate is θ or of the number of jobs in the M/M/c queueing system at time θ. The convexity and concavity properties that are of interest are then of $E\phi(X(\theta))$ with respect to θ for some suitably chosen (utility) function ϕ of the performance measure $X(\theta)$.

2.1. Definitions. In the following definitions SI, SCX, SCV, SICX, SD, SDCV etc. stand, respectively, for stochastically increasing, stochastically convex, stochastically concave, stochastically increasing and convex, stochastically decreasing, stochastically decreasing and concave, etc.

DEFINITION 2.1. Let $\{X(\theta), \theta \in \Theta\}$ be a set of random variables. Denote

(a) $\{X(\theta), \theta \in \Theta\} \in$ SI (or SD) if $E\phi(X(\theta))$ is increasing (or decreasing) for all increasing functions ϕ,

(b) $\{X(\theta), \theta \in \Theta\} \in$ SCX (or SCV) if $E\phi(X(\theta))$ is convex (or concave) for all convex (or concave) functions ϕ,

(c) $\{X(\theta), \theta \in \Theta\} \in$ SICX (or SICV) if $\{X(\theta), \theta \in \Theta\} \in$ SI and $E\phi(X(\theta))$ is increasing convex (or concave) in θ for all increasing convex (or concave) functions ϕ, and

(d) $\{X(\theta), \theta \in \Theta\} \in$ SDCX (or SDCV) if $\{X(\theta), \theta \in \Theta\} \in$ SD and $E\phi(X(\theta))$ is decreasing convex (or concave) in θ for all increasing convex (or concave) functions ϕ.

2.2. Closure Properties. The closure properties of these notions serve as the basis for studying the convexity and concavity properties of the performance measures of stochastic systems. We present some of the basic closure properties without proof.

THEOREM 2.2. *Suppose $\{X(\theta), \theta \in \Theta\}$ and $\{Y(\theta), \theta \in \Theta\}$ are two collections of random variables such that, for each θ, $X(\theta)$ and $Y(\theta)$ are independent. If $\{X(\theta), \theta \in \Theta\} \in$ SICX (or SICV) and $\{Y(\theta), \theta \in \Theta\} \in$ SICX (or SICV) then $\{X(\theta) + Y(\theta), \theta \in \Theta\} \in$ SICX (or SICV).*

(See Theorem 5.6 of Shaked and Shanthikumar 1990a).

THEOREM 2.3. *Let $\{X(\theta), \theta \in \Theta\}$ be a family of Λ-valued random variables where $\Lambda \subset \mathcal{R}$ is a convex set and let $\{Y(\lambda), \lambda \in \Lambda\}$ be another family of random variables.*

(a) If $\{X(\theta), \theta \in \Theta\} \in$ SICX (or SICV) and $\{Y(\lambda), \lambda \in \Lambda\} \in$ SICX (or SICV) then $\{Y(X(\theta)), \theta \in \Theta\} \in$ SICX (or SICV).

(b) If $\{X(\theta), \theta \in \Theta\} \in$ SDCX (or SDCV) and $\{Y(\lambda), \lambda \in \Lambda\} \in$ SDCX (or SDCV) then $\{Y(X(\theta)), \theta \in \Theta\} \in$ SDCX (or SDCV).

(See Theorem 3.2 of Shaked and Shanthikumar 1988a).

2.3. Applications. We next look at some of the applications of the above results to Markov chains. Let $\{X(n), n \in \mathcal{N}_+\}$ be a Markov chain with state space $S (S = \mathcal{R}_+ = [0, \infty)$ or $\mathcal{N}_+)$. Let $Y(x)$ and $Z(x)$ denote generic random variables representing $[X(n+1)|X(n) = x]$ and $[X(n+1) - x|X(n) = x]$ respectively. [Here, and elsewhere in this paper, for a random variable U and an event A, we denote by $[U|A]$ any random variable whose distribution is the conditional distribution of U given A.] Note that $Y(x) =^d x + Z(x), x \in S$, where $=^d$ denotes equality in law. Then one has (see Theorem 4.3 of Shaked and Shanthikumar (1988a)).

THEOREM 2.4. *Suppose $X(0) = 0$ a.s. If $\{Z(x), x \in S\} \in$ SD and $Z(x) \geq 0$ a.s. for each $x \in S$, then $\{X(n), n \in \mathcal{N}_+\} \in$ SICV.*

We next see how Theorem 2.4 can be applied to record values. Let $X(n)$ be the nth record value of a sequence of independent and identically distributed random variables $\{D_n, n \in \mathcal{N}_+\}$. That is $X(n) = \max\{X(n-1), D_n\}$. Then one has

THEOREM 2.5. *If $X(0) = 0$ a.s. then $\{X(n), n \in \mathcal{N}_+\} \in$ SICV.*

PROOF. We apply Theorem 2.4. Here $Y(x) =^d \max\{D_n, x\}$ and $Z(x) =^d \max\{D_n - x, 0\}$. Clearly $\{Y(x), x \geq 0\}$ and $\{Z(x), x \geq 0\}$ satisfy the conditions of Theorem 2.4. ∎

3. Sample Path Convexity. Establishing closure properties for regular stochastic convexity is sometimes very difficult. For this reason we define a notion of sample path stochastic convexity that is stronger than regular stochastic convexity, but is easy to work with. In addition, several families of random variables such as exponential, normal and gamma random variables possess sample path convexity properties with respect to the appropriate parameters.

3.1. Definitions. Consider a family $\{X(\theta), \theta \in \Theta\}$ of random variables. Let $\theta_i \in \Theta$, $i = 1,2,3,4$, be any four values such that $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$.

DEFINITION 3.1. If there exist four random variables $\hat{X}_i$, $i = 1,2,3,4$, defined on a common probability space, such that $\hat{X}_i =^d X(\theta_i)$, $i = 1,2,3,4$, and

(a) (i) $\max[\widehat{X}_2, \widehat{X}_3] \leq \widehat{X}_4$ a.s. and (ii) $\widehat{X}_2 + \widehat{X}_3 \leq \widehat{X}_1 + \widehat{X}_4$ a.s., then $\{X(\theta), \theta \in \Theta\}$ is said to be *stochastic increasing and convex in the sample path sense* (denoted $\{X(\theta), \theta \in \Theta\} \in$ SICX(sp));

(b) (i) $\widehat{X}_1 \leq \min[\widehat{X}_2, \widehat{X}_3]$ a.s. and (ii) $\widehat{X}_1 + \widehat{X}_4 \leq \widehat{X}_2 + \widehat{X}_3$ a.s., then $\{X(\theta), \theta \in \Theta\}$ is said to be *stochastically increasing and concave in the sample path sense* (denoted $\{X(\theta), \theta \in \Theta\} \in$ SICV (sp));

(c) (i) $\widehat{X}_1 \geq \max[\widehat{X}_2, \widehat{X}_3]$ a.s. and (ii) $\widehat{X}_1 + \widehat{X}_4 \geq \widehat{X}_2 + \widehat{X}_3$ a.s., then $\{X(\theta), \theta \in \Theta\}$ is said to be *stochastically decreasing and convex in the sample path sense* (denoted $\{X(\theta), \theta \in \Theta\} \in$ SDCX (sp));

(d) (i) $\widehat{X}_4 \leq \min[\widehat{X}_2, \widehat{X}_3]$ a.s. and (ii) $\widehat{X}_1 + \widehat{X}_4 \leq \widehat{X}_2 + \widehat{X}_3$ a.s., then $\{X(\theta), \theta \in \Theta\}$ is said to be *stochastically decreasing and concave in the sample path sense* (denoted $\{X(\theta), \theta \in \Theta\} \in$ SDCV(sp)).

Though Condition (i) in the above definitions requires stochastic monotonicity in $X_i, i = 1,2,3,4$, we do not require the construction of $\widehat{X}_i, i = 2,3$, to satisfy any a.s. monotonicity property (i.e., we do not require either that $\widehat{X}_2 \geq \widehat{X}_3$ a.s. or $\widehat{X}_2 \leq \widehat{X}_3$ a.s. be satisfied).

The following proposition is then immediate (see Proposition 3.2 of Shaked and Shanthikumar 1988b):

PROPOSITION 3.2.

(a) *If $\{X(\theta), \theta \in \Theta\} \in$ SICX (sp) [or SICV (sp)] and if ϕ is an increasing convex [or concave] function, then $\{\phi(X(\theta)), \theta \in \Theta\} \in$ SICX (sp) [or SICV (sp)].*

(b) *If $\{X(\theta), \theta \in \Theta\} \in$ SDCX (sp) [or SDCV (sp)] and if ϕ is increasing convex [or concave] function, then $\{\phi(X(\theta)), \theta \in \Theta\} \in$ SDCX (sp) [or SDCV (sp)].*

The above proposition shows that the sample path notions imply the regular notions of stochastic concavity. Counterexamples can be constructed to show that the reverse need not be true (e.g. see Counterexample 3.4 of Shaked and Shanthikumar 1990a). Hence we have

COROLLARY 3.3.

(i) *SICX (sp)* $\Longrightarrow$ *SICX*

(ii) SICV (sp) $\Longrightarrow$ *SICV*

(iii) SDCX (sp) $\Longrightarrow$ *SDCX*

(iv) SDCV (sp) $\Longrightarrow$ *SDCV.*

The sample path convexity notion has been used in Chang, Chao and Pinedo (1988) and Shanthikumar (1990) to resolve Conjecture 1; used in Shaked and Shanthikumar (1988b) to resolve Conjecture 2 and used in Shanthikumar and Yao (1988) to resolve Conjecture 3. For an outline to resolving these conjectures see Section 5.

3.2. Closure Properties. In this section we summarize two of the closure properties of the sample path convexity notions. The proofs can be found in Shaked and Shanthikumar (1988a Theorems 2.1 and 3.1 respectively).

THEOREM 3.4. *Let* $\{X(\theta), \theta \in \Theta\}$ *and* $\{Y(\theta), \theta \in \Theta\}$ *be two families of random variables such that for each* $\theta \in \Theta$, $X(\theta)$ *and* $Y(\theta)$ *are independent. Then*

(i) $\{X(\theta), \theta \in \Theta\} \in$ *SICX (sp) and*
$\{Y(\theta), \theta \in \Theta\} \in$ *SICX (sp)* $\Longrightarrow$
$\{X(\theta) + Y(\theta), \theta \in \Theta\} \in$ *SICX (sp),*

(ii) $\{X(\theta), \theta \in \Theta\} \in$ *SICV (sp) and*
$\{Y(\theta), \theta \in \Theta\} \in$ *SICV (sp)* $\Longrightarrow$
$\{X(\theta) + Y(\theta), \theta \in \Theta\} \in$ *SICV (sp),*

(iii) $\{X(\theta), \theta \in \Theta\} \in$ *SDCX (sp) and*
$\{X(\theta), \theta \in \Theta\} \in$ *SDCX (sp)* $\Longrightarrow$
$\{X(\theta) + Y(\theta), \theta \in \Theta\} \in$ *SDCX (sp), and*

(iv) $\{X(\theta), \theta \in \Theta\} \in$ *SDCV (sp) and*
$\{Y(\theta), \theta \in \Theta\} \in$ *SDCV (sp)* $\Longrightarrow$
$\{X(\theta) + Y(\theta), \theta \in \Theta\} \in$ *SDCV (sp).*

THEOREM 3.5. *Let* $\{X(\theta), \theta \in \Theta\}$ *be a family of* Λ*-valued random variables, where* $\Lambda \subset \mathcal{R}$ *is a convex set. Also let* $\{Y(\lambda), \lambda \in \Lambda\}$ *be another family of random variables.*

(a) If

(i) $\{X(\theta), \theta \in \Theta\} \in$ *SICX (sp) [or SICV (sp)] and*

(ii) $\{Y(\lambda), \lambda \in \Lambda\} \in$ *SICX (sp) [or SICV (sp)],*

then $\{Y(X(\theta)), \theta \in \Theta\} \in$ *SICX (sp) [or SICV (sp)].*

(b) If

(i) $\{X(\theta), \theta \in \Theta\} \in$ *SDCX (sp) [or SDCV (sp)] and*

(ii) $\{Y(\lambda), \lambda \in \Lambda\} \in$ *SICX (sp) [or SICV (sp)],*

then $\{Y(X(\theta)), \theta \in \Theta\} \in$ *SDCX (sp) [or SDCV (sp)].*

3.3. Applications. Two applications of the sample path convexity notions in Markov processes are given next. Let $\{X(n), n \in \mathcal{N}_+\}$ be a Markov chain with state space S ($S = \mathcal{R}_+$ or $\mathcal{N}_+$). Let $Y(x) =^d [X(n+1)|X(n) = x]$ and $Z(x) = Y(x) - x, x \in S$. Then one has (see Theorem 4.2 of Shaked and Shanthikumar 1988a).

THEOREM 3.6. *Suppose* $X(0) = x_0$ a.s. *If* $Z(x) \geq 0$ *for each* $x \in S$ *and* $\{Z(x), x \in S\} \in$ *SI, then* $\{X(n), n \in \mathcal{N}_+\} \in$ *SICX (sp).*

Next consider a Galton-Watson branching process $\{X(n), n \in \mathcal{N}N_+\}$ in discrete time. Let $D_i, i = 1, 2, \ldots$, be independent and identically distributed random variables such that D_i has the same distribution as the number of offsprings of an ancestor. Then $Y(x) =^d \sum_{i=1}^{x} D_i, x \in \mathcal{N}_+$.

THEOREM 3.7. *Suppose* $D_i \geq 1$ a.s. and $P\{D_i > 1\} > 0$. *If* $X(0) \geq 1$ a.s., *then* $\{X(n), n \in \mathcal{N}_+\} \in$ *SICX (sp).*

PROOF. First condition on $X(0) = x_0$. Since $Z(x) = Y(x) - x =^d \sum_{i=1}^{x}(D_i - 1)$ and $D_i \geq 1$ a.s. one sees that $Z(x) \geq 0$ a.s. Also it is easily seen that $\{Z(x), x \in \mathcal{N}_+\} \in$ SI. Then conditioned on $X(0) = x_0$, the result of Theorem 3.7 follows immediately from Theorem 3.6.

From the definition of sample path convexity, it is clear that by unconditioning with respect to $X(0)$, the sample path convexity of $\{X(n), n \in \mathcal{N}_+\}$ is preserved (this is Theorem 3.9 of Shaked and Shanthikumar 1988b). ∎

4. Strong Stochastic Convexity. As we have pointed out earlier, sample path stochastic convexity has proven to be a valuable notion in establishing convexity properties of stochastic systems. Since most of the analysis uses sample path constructions, establishing closure properties for the sample path notions become very easy. However, in order to apply these results one has to first establish that the input family of random variables satisfies the sample path stochastic convexity notion. At present we have to verify this property for a family of random variables through a sample path construction (see Definition 3.1) which is not simple in some applications. It could be useful to have a definition, equivalent to Definition 3.1, which is functional in the sense that it defines the sample path convexity notion by monotonicity and convexity properties of $E\phi(X(\theta))$ as a function of θ for some function ϕ

(such as Definition 2.1). So far our attempt to develop a functional definition equivalent to Definition 3.1 has not been fruitful. For this reason a functional definition of stochastic convexity that is stronger than the sample path notions is introduced in Shaked and Shanthikumar (1990b). In this section we give the definition of this strong stochastic convexity and present some of its closure properties and applications.

4.1. Definitions. Let $\{X(\theta), \theta \in \Theta\}$ be a family of random variables with survival functions $\overline{F}_\theta(x) = P\{X(\theta) > x\}, \theta \in \Theta$.

DEFINITION 4.1. $\{X(\theta), \theta \in \Theta\}$ is said to be *(strongly) stochastically increasing [decreasing] and convex [concave] in the sense of usual stochastic ordering* if $E\phi(X(\theta))$ is increasing [decreasing] and convex [concave] for all increasing functions ϕ. We denote this by $\{X(\theta), \theta \in \Theta\} \in$ SICX (st) [SICV (st), SDCX (st), SDCV (st)].

It is easily seen that the following definition is equivalent.

PROPOSITION 4.2. *$\{X(\theta), \theta \in \Theta\} \in$ SICX (st) [SICV (st), SDCX (st), SDCV (st)] if and only if $\overline{F}_\theta(x)$ is increasing and convex [increasing and concave, decreasing and convex, decreasing and concave] in θ for each fixed x.*

The following equivalent definition of these notions is established in Shaked and Shanthikumar (1990b, Theorem 3.10):

THEOREM 4.3. *$\{X(\theta), \theta \in \Theta\} \in$ SICX (st) [SICV (st), SDCX (st), SDCV (st)] $\Longleftrightarrow$ for any $\theta_i \in \Theta, i = 1,2,3,4$, such that $\theta_1 \le \theta_2 \le \theta_3 \le \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$, there exist four random variables $\widehat{X}_i, i = 1,2,3,4$, defined on a common probability space such that $\widehat{X}_i =^d X(\theta_i)$, $i = 1,2,3,4$, and*

$$\widehat{X}_1 \le [\le, \ge, \ge] \widehat{X}_4 \text{ a.s.},$$
$$\min\{\widehat{X}_1, \widehat{X}_4\} \ge [\le, \ge, \le] \min\{\widehat{X}_2, \widehat{X}_3\} \text{ a.s.},$$
$$\max\{\widehat{X}_1, \widehat{X}_4\} \ge [\le, \ge, \le] \max\{\widehat{X}_2, \widehat{X}_3\} \text{ a.s., and hence}$$
$$\widehat{X}_1 + \widehat{X}_4 \ge [\le, \ge, \le] \widehat{X}_2 + \widehat{X}_3 \text{ a.s.}$$

Observing that $\{\psi(\theta), \theta \in \Theta\} \in$ SICX (sp) for any increasing convex function ψ, and that it is not SICX (st), it is immediate from Theorem 4.3 and Corollary 3.3 that one has

COROLLARY 4.4.

(i) *SICX (st) $\Longrightarrow$ SICX (sp) $\Longrightarrow$ SICX*

(ii) SICV (st) $\Longrightarrow$ SICV (sp) $\Longrightarrow$ SICV

(iii) SDCX (st) $\Longrightarrow$ SDCX (sp) $\Longrightarrow$ SDCX

(iv) SDCV (st) $\Longrightarrow$ SDCV (sp) $\Longrightarrow$ SDCV.

All the above implications are strict.

4.2. Closure Properties. Unlike the two previous notions, strong stochastic convexity does not have many closure properties. For example there are no counterparts to Theorems 2.2 and 2.3 or Theorems 3.4 and 3.5 for the strong stochastic convexity notion. Instead we present some specialized closure properties under random summation.

THEOREM 4.5. *Let $\{N(\theta), \theta \in \Theta\}$ be a family of discrete random variables on $\mathcal{N}_+$ and let $\{X(n), n = 1, 2, \cdots\}$ be a sequence of independent and identically distributed non-negative random variables and let $X(0) = 0$. Suppose $\{N(\theta), \theta \in \Theta\}$ and $\{X(n), n \in \mathcal{N}_+\}$ are mutually independent. Set $Y(\theta) = \sum_{n=0}^{N(\theta)} X(n), \theta \in \Theta$. If $\{N(\theta), \theta \in \Theta\} \in$ SICX (st) [SICV (st), SDCX (st), SDCV(st)] then $\{Y(\theta), \theta \in \Theta\} \in$ SICX (st) [SICV (st), SDCX (st), SDCV (st)].*

PROOF. Consider the case $\{N(\theta), \theta \in \Theta\} \in$ SICX (st). The other three cases can be similarly proven. From Theorem 4.3 one knows that for any $\theta_i \in \Theta$, $i = 1, 2, 3, 4$, such that $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$, and $\theta_1 + \theta_4 = \theta_2 + \theta_3$, there exist four random variables $\widehat{N}_i$, $i = 1, 2, 3, 4$, defined on a common probability space, such that $\widehat{N}_i =^d N(\theta_i)$, $i = 1, 2, 3, 4$ and

$$\widehat{N}_4 \geq \widehat{N}_1 \quad \text{a.s.} \tag{4.1}$$

$$\min\{\widehat{N}_1, \widehat{N}_4\} \geq \min\{\widehat{N}_2, \widehat{N}_3\} \quad \text{a.s.} \tag{4.2}$$

$$\max\{\widehat{N}_1, \widehat{N}_4\} \geq \max\{\widehat{N}_2, \widehat{N}_3\} \quad \text{a.s., and hence} \tag{4.3}$$

$$\widehat{N}_1 + \widehat{N}_4 \geq \widehat{N}_2 + \widehat{N}_3 \quad \text{a.s.} \tag{4.4}$$

Define $\widehat{Y}_i = \sum_{n=0}^{\widehat{N}_i} X(n)$, $i = 1, 2, 3, 4$. Then, clearly, $\widehat{Y}_i =^d Y(\theta_i)$, $i = 1, 2, 3, 4$. Furthermore, from (4.1) to (4.4), one sees that

$$\widehat{Y}_4 \geq \widehat{Y}_1 \quad \text{a.s.} \tag{4.5}$$

$$\min\{\widehat{Y}_1, \widehat{Y}_4\} \geq \min\{\widehat{Y}_2, \widehat{Y}_3\} \quad \text{a.s.} \tag{4.6}$$

$$\max\{\widehat{Y}_1, \widehat{Y}_4\} \geq \max\{\widehat{Y}_2, \widehat{Y}_3\} \quad \text{a.s., and hence} \tag{4.7}$$

$$\widehat{Y}_1 + \widehat{Y}_4 \geq \widehat{Y}_2 + \widehat{Y}_3 \quad \text{a.s.} \tag{4.8}$$

Theorem 4.5 then follows from Theorem 4.3. ∎

THEOREM 4.6. *Consider $\{X(\theta), \theta \in \Theta\}$ and $\{Y(\theta), \theta \in \Theta\}$ and suppose that for each $\theta, X(\theta)$ and $Y(\theta)$ are independent. Define*

$$V(\theta) = \max\{X(\theta), Y(\theta)\}$$

and

$$W(\theta) = \min\{X(\theta), Y(\theta)\}.$$

(i) If $\{X(\theta), \theta \in \Theta\}$ and $\{Y(\theta), \theta \in \Theta\} \in$ SICX (st) [SDCX (st)] then $\{W(\theta), \theta \in \Theta\} \in$ SICX (st) [SDCX (st)].

(ii) If $\{X(\theta), \theta \in \Theta\}$ and $\{Y(\theta), \theta \in \Theta\} \in$ SICV (st) [SDCV (st)] then $\{V(\theta), \theta \in \Theta\} \in$ SICV (st) [SDCV (st)].

PROOF. The desired results follow immediately from the observation that (i) the survival function of $W(\theta)$ at x is equal to $P\{X(\theta) > x\}P\{Y(\theta) > x\}$, (ii) the survival function of $V(\theta)$ at x is equal to $1 - (1 - P\{X(\theta) > x\})(1 - P\{Y(\theta) > x\})$, and Proposition 4.2. ∎

4.3. Applications. In this section we consider two applications of strong stochastic convexity in queueing and reliability theory. First consider a single server queueing system M/G/1 where jobs arrive according to Poisson process with rate λ and the service times $X_n, n = 1, 2, \cdots$ are independent and identically distributed random variables with mean one. Let $W(\lambda)$, for $0 < \lambda < 1$, represent the stationary waiting time of an arbitrary job. It is then well known that

$$W(\lambda) =^d \sum_{n=0}^{N(\lambda)} Y_n, \tag{4.9}$$

where $Y_0 = 0, \{Y_n, n = 1, 2, \cdots\}$ is a sequence of independent and identically distributed non-negative random variables with distribution equal to the distribution of the stationary excess life of $\{X_n, n = 1, 2, \ldots\}$ (i.e. the density function of Y_n, $f_Y(x) = P\{X_n > x\}/E[X_n]$), and $N(\lambda)$ is a geometric random variable such that $P\{N(\lambda) > k\} = (1 - \lambda)\lambda^k, 0 < \lambda < 1$.

THEOREM 4.7. *$\{W(\lambda), 0 < \lambda < 1\} \in$ SICX (st).*

PROOF. It is clear that $\{N(\lambda), 0 < \lambda < 1\} \in$ SICX(st). The result then follows from (4.9) and Theorem 4.5. ∎

REMARK 4.8. This result, for the special case of the M/G/1 queue, is stronger than that given in Shaked and Shanthikumar (1988b, Theorem 5.1) for GI/G/1 queues.

Next consider the imperfect repair model of Cleroux et al. (1979). A new item with an absolutely continuous survival function $\overline{F}$ undergoes an imperfect repair upon each time it fails before it is scrapped. With probability p the repair is unsuccessful and the item is scrapped. With probability $1 - p$ the repair is successful and minimal, that is, after a successful repair at time t the item is as good as a working item at age t.

If $X(p)$ denote the time to scrap then the survival function of $X(p)$ is $\overline{F}^p$ (Berg and Cleroux 1982). Then the following result is immediate:

THEOREM 4.9. *Let $\overline{F}$ be an absolutely continuous survival function such that $\overline{F}(0) = 1$. Then $\{X(p), p \in (0,1)\} \in SDCX\ (st)$.*

5. Resolving the Conjectures To resolve Conjecture 1 we need additional notions and results than what we have described here. The following stochastic convexity properties of the workload process $\{V(t), t \geq 0\}$ of an M/G/1 queueing system plays a crucial role in resolving Conjecture 1: let $V(t : x, \lambda)$ be the workload in an M/G/1 queueing system at time t when the customer arrival rate is λ and the initial workload $V(0 : x, \lambda)$ at time 0 is x. Then $\{V(t : x, \lambda), \lambda \geq 0\} \in \text{SICX}(sp)$, $\{V(t : x, \lambda), x \geq 0\} \in \text{SICX}(sp)$, and before the first arrival time T, $\{V(t : x, \lambda), T \geq t \geq 0\} \in \text{SDCX}(sp)$. For the details of the application of these results to resolve Conjecture 1, see Chao, Chang and Pinedo (1988) and Shanthikumar (1990).

To resolve Conjecture 2 let $N(t)$ be the number of customers at time t in an M/M/c queueing system with arrival rate λ and service rate μ. Then $\{N(t), t \geq 0\}$ is a continuous time Markov process. Let $\{N_n, n = 0, 1, \ldots\}$ be the Markov chain obtained from $\{N(t), t \geq 0\}$ by uniformizing it with rate $\eta > \lambda + c\mu$. For any four values of $\lambda^{(i)} \in \mathcal{R}_+$, $i = 1, 2, 3, 4$, such that $\lambda^{(1)} \leq [\lambda^{(2)}, \lambda^{(3)}] \leq \lambda^{(4)}$ and $\lambda^{(1)} + \lambda^{(4)} = \lambda^{(2)} + \lambda^{(3)}$, let $\{N_n^{(i)}, n = 0, 1, 2, \ldots\}$, $i = 1, 2, 3, 4$, be the corresponding uniformized Markov chains. Then sample path construction is used in Shaked and Shanthikumar (1988b) to show that there exist four processes $\{\widehat{N}_n^{(i)}, n = 0, 1, 2, \ldots\}$, $i = 1, 2, 3, 4$, defined on a common probability space such that $\{\widehat{N}_n^{(i)}, n = 0, 1, 2, \ldots\} =^d \{N_n^{(i)}, n = 0, 1, 2, \ldots\}$, $i = 1, 2, 3, 4$, and (a) $\widehat{N}_n^{(1)} \leq [\widehat{N}_n^{(2)}, \widehat{N}_n^{(3)}] \leq \widehat{N}_n^{(4)}$ and (b) $\widehat{N}_n^{(1)} + \widehat{N}_n^{(4)} \geq \widehat{N}_n^{(2)} + \widehat{N}_n^{(3)}$, $n = 0, 1, 2, \ldots$ The latter two inequalities are proved through induction starting with $\widehat{N}_0^{(i)} = N(0)$, $i = 1, 2, 3, 4$, which trivially satisfy these inequalities. The conjecture then follows from Corollary 3.3.

To resolve Conjecture 3 consider a two station (say stations 1 and 2) cyclic queueing network with state dependent service rates $\eta_i, i = 1, 2$, and let $D_i(t)$ be the number of customers departed from station i during $(0, t]; i = 1, 2$. Then given the number of customers at stations 1 and 2 at time 0, $\{(D_1(t), D_2(t)), t \geq 0\}$ is a continuous time Markov chain. Let $\{(D_{1:n}, D_{2:n}), n = 0, 1, \ldots\}$ be the Markov chain obtained from $\{(D_1(t), D_2(t)), t \geq 0\}$ by uniformizing it with rate $\lambda >> \eta_1 + \eta_2$. For any four values of $k_i \in \mathcal{N}_+$, $i = 1, 2, 3, 4$, such that $k_1 \leq [k_2, k_3] \leq k_4$ and $k_1 + k_4 = k_2 + k_3$ let $\{(D_{1:n}^{(j)}, D_{2:n}^{(j)}), n = 0, 1, \ldots\}$, $j = 1, 2, 3, 4$, be the corresponding uniformized Markov chains. Then sample path construction is used in Shanthikumar and Yao (1988) to show that if η_i is increasing and concave for both i =1 and 2 then there exist four processes $\{(\widehat{D}_{1:n}^{(j)}, \widehat{D}_{2:n}^{(j)}), n = 0, 1, \ldots\}$, $j = 1, 2, 3, 4$, defined on a common probability space such that $\{(\widehat{D}^{(j)}_{1:n}, \widehat{D}^{(j)}_{2:n}), n = 0, 1, \ldots\} =^d \{(D^{(j)}_{1:n}, D^{(j)}_{2:n}), n = 0, 1, \ldots\}$, $j = 1, 2, 3, 4$, and (a) $\widehat{D}^{(1)}_{i:n} \leq [\widehat{D}^{(2)}_{i:n}, \widehat{D}^{(3)}_{i:n}] \leq \widehat{D}^{(4)}_{i:n}$ and (b) $\widehat{D}^{(1)}_{i:n} + \widehat{D}^{(4)}_{i:n} \leq \widehat{D}^{(2)}_{i:n} + \widehat{D}^{(3)}_{i:n}, i = 1, 2; n = 0, 1, 2, \ldots$. The latter two inequalities are proved through induction starting with $\widehat{D}^{(j)}_{i:0} = D_i(0)$, $i = 1, 2; j = 1, 2, 3, 4$ which trivially satisfy these inequalities. The conjecture for a two station closed queueing network then follows from the observation that for $i = 1, 2$, the throughput $TH(k) = \lim_{t\to\infty} E[D_i(t)]/t$ and that $\eta_i(l) = \min\{s_i, l\}\mu_i$ is increasing and concave in l. Then Norton's reduction can be used to extend this result to any number of stations.

6. Summary and Conclusions. In this paper we have reviewed three (regular, sample path and strong) notions of stochastic convexity. Some of their closure properties are presented and a sample of applications are given. While the notion of regular stochastic convexity seems to be the most natural one we found it technically convenient to work with sample path stochastic convexity. To verify sample path convexity of a given family of random variables, the notions of strong stochastic convexity is technically the most useful. The following implications play a crucial role in the usage of these notions:

Strong stochastic convexity $\Longrightarrow$

Sample path stochastic convexity $\Longrightarrow$

Regular stochastic convexity.

All the examples and applications discussed in this paper are restricted to a single parameter and univariate random variables. In attempting to extend these notions to multiple parameters we found the notion of directional convexity (see Chang, Chao, Pinedo and Shanthikumar 1990, Meester 1990, Meester and Shanthikumar 1990b, Shaked and Shanthikumar 1990a and Shanthikumar 1990) technically very useful. Also the notions of strong (sample path) stochastic convexity introduced in Meester and Shanthikumar (1990a), and Shanthikumar and Yao (1989, 1991) are technically well suited to handle multiple parameters as well as multivariate random variables.

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