Stochastic Orders and Decision under Risk
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ORDERINGS OF RISKS AND THEIR ACTUARIAL APPLICATIONS

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In actuarial theory a risk is a random variable describing a claim size (a single claim size, or the total claim amount of one contract in one period, or the aggregate claim of a portfolio of contracts in one period, e.g.). In the present contribution a number of (well-known as well as new) orderings of random variables are discussed. In particular, the relations between these orderings are investigated, and interpretations in terms of actuarial applications are given. Furthermore, the stability of the orderings with respect to convolutions and the forming of random sums is examined. Finally, it is shown that this approach can be used to generate formulas for risk premiums.

1. Introduction. The starting point of the present paper is given by the following two questions which are closely associated. First, is there a specific risk for which insurance companies are exposed to but for which companies in other economic branches are not (a "technical" or "actuarial" risk)? And if so, how can it be quantified? Second, how can the "dangerousness" of a risk (i.e., a claim variable) be described in the models of risk theory? While a treatment of the first problem involves economic aspects and goes beyond the scope of the present paper, the second question leads to the introduction of quantities and functions which induce order relations in the set of random variables describing risks.

An appropriate framework for such a study is given by the probabilistic models and methods of risk theory. Therefore, we start with a brief review of the so-called collective theory of risks. Subsequently, (first order) stochastic dominance and four order relations associated with stochastic dominance are introduced. For each of these orderings, interpretations in terms of actuarial applications are given. For specific distributions which are frequently applied in actuarial mathematics, necessary and sufficient conditions for domination in terms of the distribution parameters are derived.

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Furthermore, the relationships between the five order relations under consideration are investigated as well as the question whether domination of one distribution by another implies a corresponding domination of the moments of two distributions. Finally, we consider the convolution of distributions and the forming of random sums and examine the stability of the order relations with respect to these operations which are essential in actuarial calculations. In a short appendix we develop two inequalities for premium calculation.

These derivations can be used to give an answer to the above question for appropriate measures of dangerousness of risks. A decisive answer to this question cannot be given, however, since all order relations under consideration turn out to have both merits and shortcomings with regard to actuarial applications and interpretations.

A standard reference for orderings among risks is [2], Chapter 4. A forerunner of the present paper is [5].

2. The Collective Theory of Risks. In actuarial mathematics, in particular in risk theory, a risk is a random variable on some probability space $(\Omega, \mathcal{A}, I\!P)$ describing a claim size (a single claim size, or the total claim amount of one contract in one period, or the aggregate claim of a portfolio of contracts in one period, e.g., sometimes a claim amount per event, etc.). To model this, we consider an independent sequence $N, Y_1, Y_2, \cdots$ of random variables such that $Y_1, Y_2, \cdots$ are identically distributed. Here,

N denotes the number of claims,

Y_i denotes the size of the ith claim, $i = 1, 2, \cdots$.

Then

$X = \sum_{i=1}^{N} Y_i$ describes the total/aggregate claim.

(For notation and terminology see [6], e.g.)

Usual distributional assumptions are

$N \sim$	$\pi(\lambda)$	(Poisson),
	$B(n,p)$	(Binomial),
	$NB(r,p)$	(Negative Binomial)
	$L(p)$	(Logarithmic),
$Y \sim$	$\mathrm{Exp}(a)$	(Exponential),
	$\Gamma(a,b)$	(Gamma),
	$\mathrm{Par}(a,b)$	(Pareto),
	$LN(\mu,\sigma^2)$	(Lognormal).

In the following we shall also consider

$Y \sim (c)$ (one-point mass in $c > 0$),

$Y \sim (a, b; p)$ (two-point mass in a and b where $0 \leq a < b$, $0 < p < 1$, and $I\!P(Y = a) = 1 - I\!P(Y = b) = p$).

Note that for a risk X the quantities EX, $V(X)$ and $\sqrt{V(X)}$ can be interpreted as trivial scalar measures of dangerousness.

The coefficient of variation,

$$CV(X) = \frac{\sqrt{V(X)}}{EX},$$

is a familiar relative measure of variability, in particular in insurance economics, cf. [9].

REMARK 1. The use of the coefficient of variation as a measure of dangerousness of risks can be misleading. Consider the two random sums

$$X_1 = \sum_{i=1}^{N_1} Y_i, \quad X_2 = \sum_{i=1}^{N_2} Z_i$$

with the above independence assumptions and

$$N_1 \sim NB\left(10, \frac{9}{10}\right), \quad Y \sim \text{Exp}(a), \quad a > 0,$$
$$N_2 \sim NB\left(1, \frac{1}{10}\right), \quad Z \sim \text{Par}\left(a', \frac{8}{3}\right), \qquad a' > 0.$$

Then

$$CV(N_1) = 1 < \sqrt{\frac{10}{9}} = CV(N_2),$$
$$CV(Y) = 1 < 2 = CV(Z),$$

but

$$CV(X_1) = \sqrt{\frac{19}{10}} > \sqrt{\frac{14}{9}} = CV(X_2).$$

Hence the CV's of both the claim number and the claim amount distributon of X_1 are smaller than those of X_2, but the aggregate claims variable X_1 has a bigger CV than X_2.

REMARK 2. In [4] it is shown that there is a relationship between the degree of "dangerousness" of a distribution and the behavior of its failure rate; in short: distributions with decreasing failure rate (DFR) describe "dangerous" risks. Such distributions include, e.g., the Gamma distribution $\Gamma(a, b)$ and the Weibull distribution $W(a, b)$ with $b \leq 1$, respectively, and the Pareto distribution $\text{Par}(a, b)$.

In particular, DFR implies NWUE ("new worse than used in expectation"), and if X has the NWUE property it follows that $CV(X) \geq 1$, cf. also [10].

REMARK 3. Risks can be transformed (and the degree of their dangerousness can be changed) by providing for measures of risk sharing such as limits of liability, retentions, and reinsurance. For example, if the insured (or the first insurer) retains

$$\tilde{X} = \min(X, a)$$

and the first insurer (or the reinsurer) takes

$$\hat{X} = X - \tilde{X} = (X - a)^+,$$

we talk of a pure or straight deductible in the direct business and of an excess of loss reinsurance or stop loss reinsurance in the reinsurance business. For example, this leads to

$$E\hat{X} = \int (x - a)^+ F(dx) = \int_a^\infty (1 - F(x))dx$$

(where F is the distribution function of the nonnegative random variable X), the so-called net stop-loss premium with priority a, an expression which will reappear later.

REMARK 4. The main objective of risk theory is the study of the so-called risk process $(R_t, t \geq 0)$. Here we give a simple, but commonly used example. We denote by

R_0	the initial reserve,
N_t	the number of claims in the time interval $[0, t]$. We shall assume that $(N_t, t > 0)$ is a Poisson process, hence $N_t \sim \pi(\lambda t)$.
$X_t = \sum_{i=1}^{N_t} Y_i$	the aggregate claims in $[0, t]$, hence $X_t \sim CP(\lambda t, G)$, where CP stands for Compound Poisson, and G denotes the distribution function of Y.
$P = (1 + \delta)\lambda EY$	the premium rate per time unit (calculated according to the expected value principle with relative loading $\delta > 0$).

Then the reserve at time t is given by

$$R_t = R_0 + P \cdot t - X_t.$$

Now let T denote the time at which the reserve falls below the initial surplus for the first time, i.e.

$$T = \inf\{t > 0 : R_t < R_0\}.$$

Then for $0 < a < b$

$$\mathbb{P}(T < \infty, a < R_0 - R_T \leq b) = \frac{1}{(1 + \delta)EY} \int_a^b (1 - G(x))dx,$$

cf. [1]. In particular, the amount $\widetilde{Y}$ by which the reserve falls below the initial surplus, given that this occurs, has the density

$$\tilde{g} : x \to \frac{1}{EY}(1 - G(x))1_{(0,\infty)}(x).$$

(Note that $\tilde{g}$ is also the forward recurrence time density of a stationary renewal process with lifetime distribution function G.) This function will also be considered in the following. ∎

3. Different Orderings of Risks. In the following we shall introduce five different orderings of risks, beginning with the well-known (first order) stochastic dominance; the other four are based on and strongly associated with the concept of stochastic dominance. Throughout we shall consider random variables $X, Y, \cdots$ with distribution functions $F, G, \cdots$ such that $\mathbb{P}(X \geq 0) = 1$, $EX > 0$.

3.1. *First Order Stochastic Dominance.* This well-known order relation is defined by

$$X \prec Y :\Leftrightarrow 1 - F(x) \leq 1 - G(x) \;\text{ for all }\; x \geq 0.$$

Actuarial interpretation. For any premium, retention limit, priority, $\cdots$ x the probability that the actual claim is above x is higher in case of Y than in case of X.

Of course,

$$X \prec Y \Rightarrow \int_x^\infty (1 - F(t))dt \leq \int_x^\infty (1 - G(t))dt \;\text{ for all }\; x \geq 0.$$

Actuarial interpretation. For any priority x, the net stop loss premium is greater for Y than for X.

In particular, we obtain the well-known implication

$$X \prec Y \Rightarrow EX \leq EY.$$

Actuarial application. Generation of an implicit premium loading, e.g. in life assurance: Instead of the original risk X, consider a risk Y such that $X \prec Y$, and apply the net risk principle to Y.

$$P(Y) = EY = \int_0^\infty (1-G(x))dx = \underbrace{\Bigg(1 + \underbrace{\frac{\int_0^\infty (F(x)-G(x))dx}{\int_0^\infty (1-F(x))dx}}_{\text{relative loading}}\Bigg)EX}_{\text{expected value principle applied to } X}$$

This means, for instance, that the use of valuation tables in life assurance is nothing else but the transition from the net premium principle to the expected value principle.

3.2. *Stop Loss Dominance.* From the foregoing discussion we know that for any risk X and for $x \geq 0$

$$m_X(x) := m(x) := \int_x^\infty (1 - F(t))dt = \int_x^\infty (t - x)F(dt)$$

is the net stop loss premium with priority x. Now we introduce

$$X <_m Y :\Leftrightarrow m_X(x) \leq m_Y(x) \text{ for all } x \geq 0.$$

(See the interpretation in the preceding paragraph.)

Of course,

$$X <_m Y \Rightarrow EX \leq EY$$

and since

$$\int_0^\infty m(t)dt = \int_0^\infty t(1 - F(t))dt = \frac{1}{2}\int_0^\infty t^2 F(dt) = \frac{1}{2}EX^2,$$

it follows that

$$X <_m Y \Rightarrow EX^2 \leq EY^2,$$

$$X <_m Y, \ EX = EY \Rightarrow V(X) \leq V(Y).$$

(These implications are well-known, cf. [7].)

Hence the order relation $<_m$ can be used to compare risks in terms of their first two moments. In particular, if $P(X) = EX + \delta V(X)$, $\delta > 0$, is the variance principle of premium calculation, we obtain the implication

$$X <_m Y, \ EX = EY \Rightarrow P(X) \leq P(Y).$$

Furthermore, the (μ, σ)-rule in decision theory is a necessary condition for $<_m$.

3.3. In Remark 4 of Section 2 we considered the density function

$$\tilde{f} : x \rightarrow \frac{1}{EX}(1 - F(x))1_{(0,\infty)}(x).$$

Let $\widetilde{X}$ be a random variable having this density and let $\widetilde{F}$ be its corresponding distribution function. Obviously,

$$\ell_X(x) := \ell(x) := 1 - \widetilde{F}(x) = \frac{1}{EX}\int_x^\infty (1 - F(t))dt = \frac{m(x)}{EX}.$$

Now we introduce the order relation

$$X <_\ell Y :\Leftrightarrow \widetilde{X} \prec \widetilde{Y} \Leftrightarrow \ell_X(x) \leq \ell_Y(x) \text{ for all } x \geq 0.$$

Since

$$E\widetilde{X} = \int_0^\infty \ell_X(t)dt = \frac{EX^2}{2EX} = \frac{V(X) + (EX)^2}{2EX},$$

it follows that

$$X <_\ell Y,\; EX = EY \Rightarrow EX^2 \leq EY^2,\; V(X) \leq V(Y).$$

Actuarial application. Premium calculation.

The failure rate function of $\widetilde{X}$ is given by

$$\frac{1 - F(x)}{EX} \Big/ \frac{\int_x^\infty (1 - F(t))dt}{EX} = \frac{1 - F(x)}{\int_x^\infty (1 - F(t))dt}.$$

The expression obviously equals the reciprocal value of the (conditional) mean residual life time function at x. Hence we have the implication

$$X \text{ has } \left\{ \begin{matrix} \text{decreasing} \\ \text{increasing} \end{matrix} \right\} \text{ mean residual life} \Rightarrow \widetilde{X} \text{ has } \left\{ \begin{matrix} \text{increasing} \\ \text{decreasing} \end{matrix} \right\} \text{ failure rate.}$$

Now consider the premium calculation principle

$$P(X) = \frac{EX^2}{EX} = 2E\widetilde{X}.$$

In view of the above derivation, this premium principle can now be characterized as follows. In order to calculate the premium for a risk X which has increasing mean residual life, we transform X into a risk $\widetilde{X}$ with decreasing failure rate (which is a cautious step), and then apply the expected value principle with relative loading $\delta = 1$ to $\widetilde{X}$.

3.4. For a risk X let F^* be the distribution function of $X^* := \frac{X}{EX}$. Of course, $F^*(x) = F(xEX)$ and

$$\tilde{\tilde{f}} : x \to (1 - F^*(x))1_{(0,\infty)}(x)$$

is a density of some random variable $\tilde{\tilde{X}}$ with distribution function $\tilde{\tilde{F}}$, say.

Obviously,

$$k_X(x) := k(x) := 1 - \tilde{\tilde{F}}(x) = 1 - \int_0^x (1 - F(tEX))dt$$
$$= \frac{1}{EX} \int_{xEX}^{\infty} (1 - F(t))dt = \ell(xEX) = \frac{1}{EX} m(xEX).$$

Now we introduce the order relation

$$X <_k Y :\Leftrightarrow \tilde{\tilde{X}} \prec \tilde{\tilde{Y}} \Leftrightarrow k_X(x) \leq k_Y(x) \text{ for all } x \geq 0.$$

Note that in general $\tilde{\tilde{X}} \neq \widetilde{(\tilde{X})}$ and that

$$\mathbb{P}(\tilde{X} \leq x) = \tilde{F}(x) = 1 - \ell(x) = 1 - k\left(\frac{x}{EX}\right)$$
$$= \tilde{\tilde{F}}\left(\frac{x}{EX}\right) = \mathbb{P}((EX) \cdot \tilde{\tilde{X}} \leq x).$$

Hence the distributions of $\tilde{X}$ and $(EX)\tilde{\tilde{X}}$ are the same.

Furthermore

$$\int_0^{\infty} k_X(t)dt = \frac{1}{EX} \int_0^{\infty} \ell_X(t)dt = \frac{EX^2}{2(EX)^2}$$
$$= \frac{1}{2} \frac{V(X) + (EX)^2}{(EX)^2} = \frac{1}{2}((CV(X))^2 + 1),$$

hence

$$X <_k Y \Rightarrow CV(X) \leq CV(Y),$$
$$X <_k Y,\ EX = EY \Rightarrow EX^2 \leq EY^2,\ V(X) \leq V(Y).$$

Remark 5. We do not know of any direct actuarial interpretation of the order relation $<_k$. By definition, $<_k$ is related to $\prec$. Moreover, it can easily be shown that

$$X <_k Y \Leftrightarrow \frac{X}{EX} <_m \frac{Y}{EY},$$

cf. [5]. (See also the theorem in Section 4.)

Obviously, the order relation $<_k$ is scale independent.

Remark 6. Let $X \sim \text{Exp}(a)$, $Y \sim \text{Par}(a', b)$. Then it can be shown that

$$X <_k Y \text{ for all } a, a' > 0,\ b > 1.$$

Hence the order relation $<_k$ is in accordance with the opinion that Pareto or exponential distributions can be used to describe more or less dangerous claim size distributions, respectively. ∎

3.5. Consider the function

$$
\begin{aligned}
r_X(x) :&= r(x) := E[(X-x)^+ \mid X > x] \\
&= \frac{E(X-x)^+}{1-F(x)} = \frac{1}{1-F(x)} \int_x^\infty (1-F(t))dt \\
&\quad \text{for } x \geq 0 \text{ such that } F(x) < 1, \\
r_X(x) :&= r(x) := 0 \text{ in case } F(x) = 1.
\end{aligned}
$$

In the language of reliability theory, r is the (conditional) mean residual lifetime function; in terms of actuarial mathematics, it equals the conditional expected claim above the retention limit (deductible, priority, $\cdots$) x. A thorough discussion of the function r and its properties is given in [3].

Now we consider the following order relation:

$$X <_r Y :\Leftrightarrow r_X(x) \leq r_Y(x) \text{ for all } x \geq 0.$$

REMARK 7. Clearly, the functions k, l and m are nonincreasing. This is not necessarily the case with r. For example, for $X \sim \mathrm{Par}(a,b)$ with $b > 1$ we obtain

$$r_X(x) = \frac{a+x}{b-1}, \quad x \geq 0. \qquad \blacksquare$$

3.6. Assume that the distributions of X and Y belong to the same class, say exponential. Then it would be helpful to know that a dominance relation between X and Y was reflected by a certain relation between the parameters of the distributions of X and Y. In Table 1 we have compiled conditions on the parameters of some classes of distributions that are necessary and sufficient for dominance. Since the proofs are elementary (though involved in some cases) we omit them.

4. Relationships Between the Orderings. For rating and comparing the above 5 order relations it is of course important to know whether these orderings imply one another. The relationships between the first four orderings have already been studied in [5] except for the fact that there it was assumed that the risks under consideration were strictly positive almost surely – an assumption which in some cases is crucial.

The results – including counterexamples – are given in Table 2. Since the proofs are straightforward or even trivial, they are omitted.

The following result which is a direct consequence of the above derivations is not contained in Table 2.

TABLE 1

Necessary and sufficient conditions for dominance for specific distributions (cf. [5]).

Relation	$X \prec Y$	$X <_m Y$	$X <_\ell Y$	$X <_k Y$	$X <_r Y$
Distribution	necessary and sufficient conditions for dominance				
$X \sim (c)$ $Y \sim (c')$	$c \le c'$	$c \le c'$	$c \le c'$	always fulfilled	$c \le c'$
$X \sim (a,b;p)$ $c := EX$ $Y \sim (a',b';p')$ $c' := EY$	$a \le a'$, $b \le b'$, $p \ge p'$ or $a' \ge b$	$a \le a'$, $b \le b'$ and $(1-p)(b-a')$ $\le (1-p')(b'-a')$ or $a' \le a$, $b \le b'$, $c \le c'$	$a \le a'$, $b \le b'$ and $\frac{(1-p)(b-a')}{c} \le \frac{(1-p')(b'-a')}{c'}$ or $a' \le a$, $b \le b'$, $c \le c'$	$\frac{a'}{c'} \le \frac{a}{c}$, $\frac{b}{c} \le \frac{b'}{c'}$	$a \le a'$, $b \le b'$, $b \le c'$ or or $a' \le a$, $b \le b'$, $c \le c'$
$X \sim \pi(\lambda)$ $Y \sim \pi(\lambda')$	$\lambda \le \lambda'$	$\lambda \le \lambda'$	$\lambda \le \lambda'$	$[\Rightarrow \lambda \ge \lambda'(!)]$	$\lambda \le \lambda'$
$X \sim L(p)$ $Y \sim L(p')$	$p \le p'$	$p \le p'$	$p \le p'$	$[\Rightarrow p \le p']$	$p \le p'$
$X \sim U(a,b)$ $Y \sim U(a',b')$	$a \le a'$, $b \le b'$	$a+b \le a'+b'$, $b \le b'$	$a+b \le a'+b'$,	$\frac{b}{a+b} \le \frac{b'}{a'+b'}$ $b \le b'$	$a+b \le a'+b'$, $b \le b'$
$X \sim \text{Exp}(a)$ $Y \sim \text{Exp}(a')$	$a \ge a'$	$a \ge a'$	$a \ge a'$	always fulfilled	$a \ge a'$
$X \sim \text{Par}(a,b)$ $Y \sim \text{Par}(a',b')$	$\frac{b}{b'} \ge \max\left(\frac{a}{a'},1\right)$	$\frac{b-1}{b'-1} \ge \max\left(\frac{a}{a'},1\right)$ $(\min(b,b') > 1)$	$\frac{b-1}{b'-1} \ge \max\left(\frac{a}{a'},1\right)$ $(\min(b,b') > 1)$	$b \ge b'$ $(\min(b,b') > 1)$	$\frac{b-1}{b'-1} \ge \max\left(\frac{a}{a'},1\right)$ $(\min(b,b') > 1)$

TABLE 2
Relationships between order relations.

	$X \prec Y$	$X <_m Y$	$X <_\ell Y$	$X <_k Y$	$X <_r Y$
$X \prec Y$	$\Leftrightarrow$	$\Rightarrow$	$\not\Rightarrow$ $^*X \sim (1,4;\frac{1}{6})$ $Y \sim (2,4;\frac{1}{6})$	$\not\Rightarrow$ $^*X \sim U(0,2)$ $Y \sim U(1,2)$	$\not\Rightarrow$ $^*X \sim (1,4;\frac{1}{6})$ $Y \sim (2,4;\frac{1}{6})$
$X <_m Y$	$\not\Rightarrow$ $^*X \sim \mathrm{Par}(7,3)$ $Y \sim \mathrm{Par}(4,2)$	$\Leftrightarrow$	$\not\Rightarrow$ $^*X \sim (1,4;\frac{1}{6})$ $Y \sim (2,4;\frac{1}{6})$	$\not\Rightarrow$ $^*X \sim (1,3;\frac{1}{2})$ $Y \sim (1,3;\frac{1}{3})$	$\not\Rightarrow$ $^*X \sim (1,4;\frac{1}{6})$ $Y \sim (2,4;\frac{1}{6})$
$X <_\ell Y$	$\not\Rightarrow$ $^*X \sim \mathrm{Par}(7,3)$ $Y \sim \mathrm{Par}(4,2)$	$\Rightarrow^{**}$ $X \sim \mathrm{Exp}(2)$ $Y \sim \mathrm{Exp} * (1;\frac{1}{3})$	$\Leftrightarrow$	$\not\Rightarrow$ $^*X \sim (1,3;\frac{1}{2})$ $Y \sim (1,3;\frac{1}{3})$	$\not\Rightarrow$ $^*X \sim (1,3;\frac{1}{2})$ $Y \sim (1,3;\frac{1}{3})$
$X <_k Y$	$\not\Rightarrow$ $^*X \sim \mathrm{Exp}(1)$ $Y \sim \mathrm{Exp}(2)$	$\not\Rightarrow$ $^*X \sim \mathrm{Exp}(1)$ $Y \sim \mathrm{Exp}(2)$	$\not\Rightarrow$ $^*X \sim \mathrm{Exp}(1)$ $Y \sim \mathrm{Exp}(2)$	$\Leftrightarrow$	$\not\Rightarrow$ $^*X \sim \mathrm{Exp}(1)$ $Y \sim \mathrm{Exp}(2)$
$X <_r Y$	$\not\Rightarrow$ $^*X \sim U(1,2)$ $Y \sim U(0,4)$	$\Rightarrow^{**}$ $X \sim \mathrm{Exp}(2)$ $Y \sim \mathrm{Exp} * (1;\frac{1}{3})$	$\Rightarrow$	$\not\Rightarrow$ $^*X \sim U(0,2)$ $Y \sim U(1,2)$	$\Leftrightarrow$

*counterexample

**statement is valid for $\mathbb{P}(Y > 0) = 1$, counterexample for the case $\mathbb{P}(Y = 0) > 0$.

$\mathrm{Exp} * (a;p)$ denotes the mixture of $\mathrm{Exp}(a)$ and the one-point mass in 0 with weights p and $1-p$, respectively.

THEOREM. *For nonnegative random variables X, Y with $EX = EY > 0$ the following holds:*

$$X <_m Y \Leftrightarrow X <_\ell Y \Leftrightarrow X <_k Y. \qquad \blacksquare$$

5. Moment Inequalities Implied by Dominance. The most important characteristics of distributions are their moments. In particular, moments (up to order three at most) are used for the purpose of setting rates in insurance. Generally, in the actual practice of determining premiums, only the expected value and the variance of the risk under consideration are required (and available).

Compatibility of any order relation $<$ in the set of risks, and a premium calculation principle P would mean

$$X < Y \Leftrightarrow P(X) \leq P(Y).$$

If we only require the implication

$$X < Y \Rightarrow P(X) \leq P(Y) \qquad (*)$$

we immediately obtain results such as the following.

For $P(X) = (1 + \delta)EX$, $\delta \geq 0$, $(*)$ is fulfilled for $< \in \{\prec, <_m\}$, and for $< \in \{<_\ell, <_r\}$ in case that Y is strictly positive a.s.

For $P(X) = EX + \delta V(X)$ and for $P(X) = EX + \delta\sqrt{V(X)}$, $\delta \geq 0$, $(*)$ is fulfilled for $< \in \{\prec, <_k, <_\ell, <_m, <_r\}$ under the additional assumption that $EX = EY$.

For the exponential principle $P(X) = \frac{1}{a} \ln Ee^{aX}$, $a > 0$, $(*)$ is fulfilled for $< \in \{\prec, <_m\}$, and for $< \in \{<_\ell, <_r\}$ in case that Y is strictly positive a.s.

Note that in case of the percentile principle

$$P_\epsilon(X) = F^{-1}(1 - \epsilon) = \inf\{x : F(x) \geq 1 - \epsilon\}, \quad \epsilon \in [0, 1),$$

the following characterization is valid:

$$X \prec Y \Leftrightarrow P_\epsilon(X) \leq P_\epsilon(Y) \text{ for all } \epsilon \in [0, 1).$$

A weaker requirement would be that $X < Y$ implies only the corresponding relations between the moments of X and Y which are involved in the calculation of P. In Table 3 it is shown whether such implications are valid for the orderings under consideration and certain types of moments. In the negative cases, counterexamples are given. Most of the affirmative statements can be verified easily.

6. Stability of Orderings with Respect to Summation. In many applications of probability theory, the formation of sums of random variables is of utmost importance. In particular, in assurance risks are aggregated (and their claims added) in order to accomplish the so-called balance (or stability) by number. Of course, a suitable ordering of risks should be preserved under summation of random variables (or convolution of distributions). Since in the collective theory of risks random sums are particularly important, an ordering of risks should also be stable with respect to the formation of random sums.

A number of results in this direction are compiled in Table 4 where N, N' and $X, X_1, X_2, \cdots, Y, Y_1, Y_2, \cdots, Z$ are random variables describing the number of claims and the (aggregate) claim sizes, respectively. All random variables are assumed to be stochastically independent except for X and Y, X_1 and Y_1, X_2 and $Y_2, \cdots$, respectively. F and G denote the distribution functions of X and Y, respectively.

The following relationships between the properties considered in Table 4 are valid.

$$(P1) \Rightarrow (P2a) \Leftrightarrow (P2b) \Rightarrow (P3)$$

and

$$(P4) \wedge (P5) \Leftrightarrow (P6).$$

The proofs are straightforward though sometimes involved. To prove some of the statements one can follow the lines of [2], or make use of the theory of Tchebycheff systems (cf. [8], e.g.).

7. Appendix: Premium Calculation. In this final section we indicate how some of the functions introduced above can be used to obtain lower bounds for risk premiums. We employ the Compound Poisson model (cf. Remark 4 in Section 2) with premium rate $P > EX_1 = \lambda\ EY$. We know that the (unconditional) amount $\widetilde{Y}$ of the first surplus below the initial level is

$$0 \text{ with probability } 1 - \frac{\lambda\ EY}{P},$$

in $(a, b] \subset (0, \infty)$ with probability $\int_a^b \frac{\lambda(1-G(y))}{P} dy$.

Consequently,

$$I\!P(\widetilde{Y} \le c) = 1 - \frac{\lambda\ EY}{P} + \int_0^c \frac{\lambda(1 - G(y))}{P} dy, \quad c > 0,$$

TABLE 3
Properties implied by dominance.

	$EX^n \leq EY^n$ $\forall\, n \in \mathbb{N}$	$EX^2 \leq EY^2$	$EX \leq EY$	$E(X \mid X > 0) \leq E(Y \mid Y > 0)$	$V(X) \leq V(Y)$	$CV(X) \leq CV(Y)$	$[EX = EY \Rightarrow V(X) \leq V(Y)]$
$X \prec Y$	Yes	Yes	Yes	No	No	No	Yes
$X <_m Y$	Yes	Yes	Yes	No	No	No	Yes
$X <_\ell Y$	No*	No*	No*	Yes	No	No	Yes
$X <_k Y$	No	No	No	No	No	Yes	Yes
$X <_r Y$	No*	No*	No*	Yes	No	No	Yes

*implication is valid if $\mathbb{P}(Y > 0) = 1$.

TABLE 4

Stability of dominance relations with respect to convolution.

	implication	relation	$\prec$	$<_m$	$<_\ell$	$<_k$	$<_r$
(P0)	$X < Y$	$\Rightarrow \alpha X < \alpha Y \;\forall\, \alpha > 0$	Yes	Yes	Yes	Yes	Yes
(P1)	$X_i < Y_i, i = 1, 2$	$\Rightarrow \alpha X_1 + \beta X_2 < \alpha Y_1 + \beta Y_2 \;\forall\, \alpha, \beta > 0$	Yes	Yes	No	No	No
(P2a) (P2b)	$X < Y$ $X_i < Y_i, i = 1, \cdots, n$	$\Rightarrow X + Z < Y + Z$ $\Rightarrow X_1 + \cdots + X_n < Y_1 + \cdots + Y_n$	Yes	Yes	No	No	No
(P3)	$F < G$	$\Rightarrow F^{*n} < G^{*n}$	Yes	Yes	No	Yes	No
(P4)	$X_i < Y_i, i = 1, 2, \cdots$	$\Rightarrow \sum_{i=1}^{N} X_i < \sum_{i=1}^{N} Y_i$	Yes	Yes	No Yes**	No Yes*	No
(P5)	$N < N'$	$\Rightarrow \sum_{i=1}^{N} X_i < \sum_{i=1}^{N'} X_i$	Yes	No Yes*	No Yes*	No	No
(P6)	$N < N'$ $X_i < Y_i, i = 1, 2, \cdots$	$\Rightarrow \sum_{i=1}^{N} X_i < \sum_{i=1}^{N'} Y_i$	Yes	No Yes*	No Yes**	No	No

*if $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ are identically distributed, respectively.

**if $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ are identically distributed, respectively, and if (P2b) is fulfilled.

and

$$E\widetilde{Y} = \int_0^\infty y \frac{\lambda(1 - G(y))}{P} dy = \frac{\lambda\, EY^2}{2P}.$$

We now impose for some $c \geq 0$, $\gamma \in (0, 1)$, the condition that

$$\mathbb{P}(\widetilde{Y} > c) \leq \gamma$$

which is equivalent to

$$\frac{\lambda\, EY}{P} - \frac{\lambda}{P} \int_0^c (1 - G(y)) dy \leq \gamma.$$

Solving with respect to P, we obtain the condition

$$P \geq \frac{\lambda}{\gamma} \int_c^\infty (1 - G(y)) dy = (1 + \delta)\lambda \int_c^\infty (1 - G(y)) dy \text{ with } \delta := \frac{1}{\gamma} - 1.$$

Hence if $Z_1 <_m Z_2$, the lower bound obtained in the preceding calculation is lower for Z_1 than for Z_2 for all $c \geq 0$.

If instead we impose the condition $E\widetilde{Y} \leq c$ for some $c > 0$, we obtain the lower bound

$$P \geq \frac{\lambda\, EY^2}{2c} = \frac{V(X_1)}{2c} \quad (\text{if } c < \frac{EY^2}{2EY}).$$

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