ACCUMULATION POINTS OF A PARTICULAR NORMALIZED RANDOM WALK

Andre Adler
Illinois Institute of Technology

Abstract

It is shown that a certain hypothesis is unnecessary when obtaining a generalized one-sided Law of the Iterated Logarithm. Also, as in the independent and identically distributed case the [almost certain] set of accumulation points is an interval. It turns out that our set of limit points is the entire interval $[c, \infty]$, for some finite nonzero constant c.

Results. Herein we establish the almost sure set of accumulation points of normalized weighted sums of independent copies of an unbounded asymmetrical random variable. The random variables under examination are of a particularly peculiar type. They permit the existence of a [weighted] Strong Law of Large Numbers, where the limit is not the common expectation (which may not even exist). We will only consider random variables with either mean zero or without finite first absolute moment. For a more extensive survey of previous results see Adler (1989) and Adler (1990). From Adler and Rosalsky (1989) it is clear that the weights $\{a_n, n \geq 1\}$ should satisfy $n|a_n| \uparrow$ and $\Sigma_{k=1}^n |a_k| = O(n|a_n|)$. So, as in Adler (1990) we consider weights of the form $a_n = n^\alpha$, $\alpha > -1$.

Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables and $S_n = \Sigma_{k=1}^n k^\alpha X_k$ denote our partial sums. Define

$$\tilde{\mu}(x) = \int_x^\infty P\{|X| > t\}\, dt \quad \text{if} \quad EX = 0$$

and

$$\mu(x) = \int_0^x P\{|X| > t\}\, dt \quad \text{when} \quad E|X| = \infty.$$

Our hypotheses are $E(X^- / \tilde{\mu}(X^-)) < \infty$ and $\tilde{\mu}(x) \sim \tilde{\mu}(x \log_2 x)$ when the mean is zero. Similarly we assume that $E(X^- / \mu(X^-)) < \infty$ and $\mu(x) \sim \mu(x \log_2 x)$ when $E|X| = \infty$. (It is customary to define log log x as $\log_2 x$.)

Next, we establish the sequence $\{c_n, n \geq 1\}$ by way of

$$c_n = \begin{cases} n\tilde{\mu}(c_n) & \text{if } \quad EX = 0 \\ \\ n\mu(c_n) & \text{if } \ E|X| = \infty . \end{cases}$$

Then our norming sequence is obtained via $b_n = n^{\alpha}c_n$. Lastly, purely for simplicity, we choose to call our limit inferior L_{α}. Hence

$$L_{\alpha} = \begin{cases} -(\alpha+1)^{-1} & \text{if } \quad EX = 0 \\ \\ (\alpha+1)^{-1} & \text{if } \ E|X| = \infty . \end{cases}$$

In generalizing the results of Adler (1990) it is shown that the hypothesis $\tilde{\mu}(b_n) = O(\tilde{\mu}(c_n))$, when $\alpha \varepsilon(-1, -1/2]$, is unnecessary.

THEOREM. If $\alpha > -1$, then with probability one

$$\liminf_{n\to\infty} \frac{S_n}{b_n} = L_{\alpha} \quad \text{and} \quad \limsup_{n\to\infty} \frac{S_n}{b_n} = \infty.$$

PROOF. In view of Adler (1990) it suffices to show that

$$\frac{\Sigma_{k=1}^{n} k^{\alpha}[X_k - EXI(|X| \le c_n)]}{b_n} \overset{P}{\to} 0,$$

when EX = 0 and $-1 < \alpha \le -1/2$. This will be proved via Adler, Rosalsky and Taylor (1991). Hence we must verify

$$c_n \uparrow, \ \frac{c_n}{n} \downarrow, \sum_{k=1}^{n} k^{\alpha P} = o(b_n^P), \text{and} \sum_{k=1}^{n} \frac{b_k^P}{k^{2+\alpha P}} \cdot \sum_{j=1}^{n} j^{\alpha P} = O(b_n^P),$$

for some $p \in (1, -1/\alpha)$.

Clearly $c_n \uparrow$ and $c_n/n \downarrow$. Let $p \in (1, -1/\alpha)$. Since $\tilde{\mu}(c_n)$ is slowly varying at infinity

$$\sum_{k=1}^{n} k^{\alpha p} \le \frac{(n+1)^{\alpha p+1}}{\alpha p+1} = o([n^{\alpha+1}\tilde{\mu}(c_n)]^P) = o(b_n^P)$$

and

$$\sum_{k=1}^{n} \frac{b_k^p}{k^{2+\alpha p}} \sum_{j=1}^{n} j^{\alpha p} = \sum_{k=1}^{n} \frac{[k^{\alpha+1}\tilde{\mu}(c_k)]^P}{k^{2+\alpha p}} \cdot \sum_{j=1}^{n} j^{\alpha p}$$

$$= \sum_{k=1}^{n} k^{p-2}[\tilde{\mu}(c_k)]^p \cdot \sum_{j=1}^{n} j^{\alpha p}$$
$$= O(n^{p-1}[\tilde{\mu}(c_n)]^p) \cdot O(n^{\alpha p+1})$$
$$= O([n^{\alpha+1}\tilde{\mu}(c_n)]^p) = O(b_n^p).$$

This completes the proof. □

Utilizing techniques similar to those that can be found in Fernholz (1980) it follows from the above theorem that every point in the interval $[L_\alpha, \infty]$ is an almost certain accumulation point of S_n/b_n. This can be accomplished by the verification of a couple of claims. The first is that the hypothesis $\tilde{\mu}(x) \sim \tilde{\mu}(x\log_2 x)$ (or $\mu(x) \sim \mu(x\log_2 x)$) implies that $c_n \sim c_{n+1}$ and hence $b_n \sim b_{n+1}$. Next one shows that $X_n^- = o(c_n)$ a.s. (This follows immediately from Fernholz (1980).) These two together with

$$P\{\frac{S_{n+1}}{b_{n+1}} \le x \le \frac{S_n}{b_n}, \text{ i.o. } (n)\} = 1,$$

for all x ε (L_α, ∞), leads us to the desired conclusion.

References

[1] Adler, A. (1989). Exact sequences for sums of independent random variables. In *Almost Everywhere Convergence* (G. Edgar and L. Sucheston, Ed.), Academic Press, Boston, 11-29.

[2] Adler, A. (1990). Generalized one-sided laws of the iterated logarithm for random variables barely with or without finite mean. *J. Theor. Probab.*, 3, 587-597.

[3] Adler, A. and Rosalsky, A. (1989). On the Chow-Robbins "fair" games problem. *Bull. Inst. Math. Acad. Sinica, 17*, 211-227.

[4] Adler, A., Rosalsky, A. and Taylor, R. L. (1991). A weak law for normed weighted sums of random elements in Rademacher type p Banach spaces. *J. Mult. Analy.* 37.

[5] Fernholz, L. T. (1980). Limit points associated with generalized iterated logarithm laws. *Ann. Probab., 8*, 390-392.