UNBIASEDNESS OF TESTS OF HOMOGENEITY WHEN ALTERNATIVES ARE ORDERED

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Let $X_1, X_2, \ldots, X_k$ be independent random variables whose densities are from an exponential family with parameters $\theta_1, \theta_2, \ldots, \theta_k$, respectively. That is, the densities are $f(x_i \mid \theta_i) = c(\theta_i)e^{x_i\theta_i}g(x_i)$. Assume that g is a Polya frequency function of order two (PF_2). Consider testing the null hypothesis $H : \theta_1 = \theta_2 = \ldots = \theta_k$ vs. the alternative $K : \theta_1 \geq \theta_2 \geq \ldots \geq \theta_k$. Write $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ and define a partial ordering $>>^*$ on $\Re^k$ by $\mathbf{x} >>^* \mathbf{y}$ if and only if $\Sigma_{i=1}^j x_i \geq \Sigma_{i=1}^j y_i$ for $j = 1, 2, \ldots, k-1$ and equality for $j = k$. A function $\varphi(\mathbf{x})$ is said to be ISO* if $\mathbf{x} >>^* \mathbf{y}$ implies $\varphi(\mathbf{x}) \geq \varphi(\mathbf{y})$. We prove that if $\varphi(\mathbf{x})$ is a similar test which is ISO* then φ is unbiased. In fact if $\varphi(\mathbf{x})$ is ISO* the power function of the test is conditionally monotone nondecreasing along rays orthogonal to the equiangular line. For cases where the distribution satisfies the semi-group property the power function is unconditionally monotone along these rays. Furthermore a way to generate unbiased tests with monotone power is given.

The result contrasts with and complements the result of Robertson and Wright (1982). They prove that when the density has the semi-group property (normal and Poisson, for example) the tests which are ISO* have ISO* power functions. Such a finding is different from ours. The class of distributions for which our result holds is larger than the class in Robertson and Wright.

Applications for particular distributions and particular tests are given. Also some admissibility results are given for particular distributions. For example, it is proven that Bartholomew's test is admissible for the normal case.

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1. Introduction and Summary. Let $X_1, X_2, \ldots, X_k$ be independent continuous or integer-valued random variables distributed according to a one-parameter exponential family with parameters θ_i, $i = 1, 2, \ldots, k$. That is, the joint density of the X_i is

$$f(\mathbf{x}, \theta) = \left(\Pi_{i=1}^{k} \beta(\theta_i)\right) e^{\Sigma_{i=1}^{k} x_i \theta_i} \left(\Pi_{i=1}^{k} g(x_i)\right), \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$. The dominating measure for each X_i is Lebesgue measure on $(-\infty, \infty)$ for the continuous case and counting measure on $\{0, \pm 1, \pm 2, \ldots\}$ for the case where the X_i are integer-valued. Assume that g is a Polya frequency function of order two (PF_2); that is, $g(\geq 0)$ is log concave on $(-\infty, \infty)$ or $\{0, \pm 1, \ldots\}$, respectively. The problem is to test

$$H : \theta_1 = \theta_2 = \ldots = \theta_k \text{ vs. } K : \theta_1 \geq \theta_2 \geq \ldots \geq \theta_k,$$

with at least one strict inequality under K. We study the unbiasedness and admissibility of tests.

Robertson and Wright (1982) study the problem of testing H vs. K when either (i) the distributions of the X_i come from a translation family, (ii) the distributions of the X_i satisfy the semi-group property (normal and Poisson, for example), or the distribution of $\mathbf{X} = (X_1, \ldots, X_k)$ is multinomial. They prove a result which yields a monotonicity property for the power functions of certain tests. This monotonicity property, called ISO*, implies unbiasedness of similar tests. More precisely, let $t_j(\mathbf{x}) = \Sigma_{i=1}^{j} x_i$, $j = 1, 2, \ldots, k$. We define the partial ordering $\mathbf{x} >>^* \mathbf{y}$ as follows: The vector $\mathbf{x} >>^* \mathbf{y}$ if and only if $t_j(\mathbf{x}) \geq t_j(\mathbf{y})$ for $j = 1, 2, \ldots, k-1$ and $t_k(\mathbf{x}) = t_k(\mathbf{y})$. A function h is said to be ISO* if $\mathbf{x} >>^* \mathbf{y}$ implies $h(\mathbf{x}) \geq h(\mathbf{y})$. For the distributions they study, Robertson and Wright (1982) prove that if $\varphi(\mathbf{x})$ is a test function which is ISO* then the power function of that test is ISO*.

One of the main results of this paper (Theorem 2.3) is that for the distributions in (1) with g PF_2, if $\varphi(\mathbf{x})$ is a test function which is similar and ISO* then φ is unbiased. Furthermore φ is such that its conditional power function (conditioned on $T_k = \Sigma_{i=1}^{k} X_i$) is monotone nondecreasing along rays orthogonal to the equiangular line. Still further if the distribution in (1) has the semi-group property then φ has a power function that is unconditionally monotone along rays orthogonal to the equiangular line.

In Theorem 2.3 we actually prove that the conditional power function of a similar ISO* test function satisfies a monotonicity property with respect to a certain partial ordering. The partial ordering is different from the one considered by Robertson and Wright (1982). Neither partial ordering implies the other. Points on rays orthogonal to the equiangular line will be ordered by both partial orderings mentioned above. Thus a power function which is conditionally monotone with respect to either partial ordering will be conditionally monotone along rays orthogonal to the equiangular line. What is important however is that the class of distributions for which Theorem 2.3 holds is larger than the class studied by

Robertson and Wright. Our class of distributions (1) includes the normal family with common known variance and means θ_i, the Poisson family with means related to θ_i, the binomial family with common sample size n and probabilities related to θ_i, the gamma family with common shape parameter (≥ 1) and scale parameters related to θ_i (which includes the chi-square distribution with two or more degrees of freedom), and many others. The binomial family is *not* covered by the results of Robertson and Wright (1982). In fact, at the end of Section 3 we present a counterexample to show that in the binomial case an ISO* test function need *not* have an ISO* power function, at least conditionally.

Hereafter, when there is no confusion it is convenient to let monotone power function mean a power function which is conditionally monotone along rays orthogonal to the equiangular line.

In Cohen and Sackrowitz (1987b), the class of distributions in (1) were considered for the problem of testing H vs. K': not H. In that study, test functions which were similar and Schur convex were shown to be unbiased. The method of proof used here will be different (and simpler) than the method used there.

In addition to providing a sufficient condition for a test to be unbiased and have a monotone power function for H vs. K, we give a method of generating such tests in Theorem 2.6.

Whereas the model of the paper is stated in terms of a single observation for each population it will be seen that all results remain true when we have n observations from each population, provided that each X_i is replaced by the sample mean $\bar{X}_i$ from the ith population. This follows as a special case of Theorem 2.7, a result on unbiasedness of tests for the important case of unequal numbers of observations from each population.

Applications to particular tests and to particular distributions will be made in Section 3. For gamma distributions with common shape parameter (≥ 1), unbiasedness is established for a certain natural class of tests for equality of the scale parameters.

For the statistical model of this paper one can easily derive an essentially complete class of test procedures from the result of Eaton (1970). For the normal case we prove that the likelihood ratio test derived by Bartholomew (1959) is admissible. We indicate other admissible tests for the normal case and indicate admissible tests for the binomial and Poisson cases as well.

Unbiasedness and monotone power results are given in Section 2. Section 3 contains applications to specific distributions and tests, while admissibility of tests is discussed in Section 4.

2. Unbiasedness of Tests. For the statistical model described near (1) we note first that any unbiased test for H vs. K must be similar and therefore must have Neyman structure with respect to $T \equiv \Sigma_{i=1}^{k} X_i$ (see Lehmann (1986), Theorem 2, p. 144). Hence any unbiased test of size α must have conditional size α (given $T = t$). It is clear that tests which are conditionally unbiased of size α (for every t) are unbiased of size α. Our plan will therefore be to show that the similar tests under study will be conditionally unbiased.

We now discuss the notion of multivariate totally positive distribution as done in Karlin and Rinott (1980). This notion is closely related to the FKG inequality. Let $f(x)$ be a nonnegative function defined on $\mathcal{X}^{(k)} \equiv \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_k$, where each $\mathcal{X}_i$ is a totally ordered subset of $\Re^1$, satisfying

$$f(\mathbf{x} \vee \mathbf{y}) f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x}) f(\mathbf{y}), \tag{2}$$

where $\vee$ and $\wedge$ are the corresponding lattice operations on $\mathcal{X}^{(k)}$, i.e.,

$$\begin{aligned} \mathbf{x} \vee \mathbf{y} &= (\max(x_1, y_1), \max(x_2, y_2), \ldots, \max(x_k, y_k)), \\ \mathbf{x} \wedge \mathbf{y} &= (\min(x_1, y_1), \min(x_2, y_2), \ldots, \min(x_k, y_k)). \end{aligned}$$

A function f with the property (2) is called multivariate totally positive of order 2 (MTP_2) on $\mathcal{X}^{(k)}$. In this paper, either $\mathcal{X}_i = (-\infty, \infty)$ for $i = 1, \ldots, k$ or $\mathcal{X}_i = \{0, \pm 1, \ldots\}$ for $i = 1, \ldots, k$.

From Karlin and Rinott (1980) we note that if $f(\mathbf{x})$ and $g(\mathbf{x})$ are MTP_2 on $\mathcal{X}^{(k)}$ then $f(\mathbf{x})g(\mathbf{x})$ is MTP_2 on $\mathcal{X}^{(k)}$. Also, if $f(\mathbf{x}) = g(x_i, x_j)$ where g is TP_2 on $\mathcal{X}_i \times \mathcal{X}_j$, then f is MTP_2 on $\mathcal{X}^{(k)}$, hence products of such functions are MTP_2 on $\mathcal{X}^{(k)}$.

Now define $T_j = \Sigma_{i=1}^{j} X_i$, $j = 1, 2, \ldots, k$, $\mathbf{T} = (T_1, \ldots, T_k)$, and $\mathbf{T}^{(k-1)} = (T_1, \ldots, T_{k-1})$. The range of $\mathbf{T}$ is again $\mathcal{X}^{(k)}$ while the range of $\mathbf{T}^{(k-1)}$ is $\mathcal{X}^{(k-1)}$. Let $f_\theta(\mathbf{t}^{(k-1)} \mid t_k)$ denote the conditional density of $\mathbf{T}^{(k-1)}$ given $T_k = t_k$.

LEMMA 2.1. *Assume (1) with g PF_2. Then for any θ, $f_\theta(\mathbf{t}^{(k-1)} \mid t_k)$ is MTP_2 on $\mathcal{X}^{(k-1)}$.*

PROOF. The density of $f_\theta(\mathbf{t}^{(k-1)} \mid t_k)$ satisfies

$$\begin{aligned} f_\theta(\mathbf{t}^{(k-1)} \mid t_k) &= C_{t_k}(\theta) e^{(\theta_1 - \theta_2)t_1 + \cdots + (\theta_{k-1} - \theta_k)t_{k-1}} \cdot \\ &\qquad g(t_1) g(t_2 - t_1) \cdots g(t_k - t_{k-1}), \end{aligned} \tag{3}$$

for $\mathbf{t}^{(k-1)} \in \mathcal{X}^{(k-1)}$. Since g is log concave on $\mathcal{X}_i \Leftrightarrow g(x_{i+1} - x_i)$ is TP_2 on $\mathcal{X}_i \times \mathcal{X}_{i+1}$, it follows from (3) that for fixed t_k, $f_\theta(\mathbf{t}^{(k-1)} \mid t_k)$ is MTP_2 on $\mathcal{X}^{(k-1)}$. ||

Now let $r_{\theta^{(2)}, \theta^{(1)}}(\mathbf{t}^{(k-1)} \mid t_k) = f_{\theta^{(2)}}(\mathbf{t}^{(k-1)} \mid t_k) / f_{\theta^{(1)}}(\mathbf{t}^{(k-1)} \mid t_k)$ where $\theta^{(2)}, \theta^{(1)}$ lie in the parameter space and let H_k denote the family of componentwise nondecreasing functions on $\mathcal{X}^{(k)}$ i.e. if $h(\mathbf{x}) \varepsilon H_k$, then $h(x_1, x_2, \cdots x_i, \cdots x_k)$ is nondecreasing in x_i while $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k$ are fixed, for $i = 1, 2, \ldots k$. Also for $\theta^{(2)}$ lying in the alternative space and $\theta^{(1)}$ lying in the null space or the alternative space let $\xi = \theta^{(2)} - \theta^{(1)}$.

LEMMA 2.2. *Suppose $\xi' = (\xi_1, \xi_2, \cdots \xi_k)$ is such that $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_k$. Then for fixed t_k the ratio $r_{\theta^{(2)}, \theta^{(1)}}(\mathbf{t}^{(k-1)} \mid t_k)$ lies in H_{k-1}.*

PROOF. From (3) we find

$$r_{\theta^{(2)},\theta^{(1)}}(\mathbf{t}^{(k-1)} \mid t_k) = C_{t_k}(\theta^{(2)}, \theta^{(1)})e^{(\xi_1-\xi_2)t_1+\ldots+(\xi_{k-1}-\xi_k)t_{k-1}},$$

where $C_{t_k}(\theta^{(2)}, \theta^{(1)}) > 0$ whenever the marginal density of T_K at t_k is positive. The result is immediate. ||

The well-known FKG inequality for $w_1, w_2 \in H_k$ and an MTP$_2$ density function f on $\mathcal{X}^{(k)}$ states that

$$Ew_1(\mathbf{T})w_2(\mathbf{T}) \geq Ew_1(\mathbf{T})Ew_2(\mathbf{T}). \tag{4}$$

See, for example, Karlin and Rinott (1980).

THEOREM 2.3. *Let $\theta^{(2)}$ be a parameter point in the alternative space and let $\theta^{(1)}$ be a parameter point in the null space or alternative space. Let $\xi = \theta^{(2)} - \theta^{(1)}$. Let $\varphi(\mathbf{x})$ be a similar size α test which is ISO* in $\mathbf{x}$. If $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_k$, then $E_{\theta^{(2)}}(\varphi(\mathbf{x}) \mid T) \geq E_{\theta^{(1)}}(\varphi(\mathbf{x}) \mid T)$. Also for any $\theta^{(1)}$ lying in the null space $E_{\theta^{(2)}}\varphi(\mathbf{x}) \geq \alpha = E_{\theta^{(1)}}\varphi(\mathbf{x})$, which implies $\varphi(\mathbf{x})$ is unbiased.*

PROOF. It suffices to show that φ satisfies the above inequalities conditionally given $T_k = t_k$. Define $\psi(\mathbf{t}) = \varphi(t_1, t_2-t_1, \cdots, t_k-t_{k-1})$. Since φ is ISO*, $\varphi \in H_{k-1}$ as a function of $\mathbf{t}^{(k-1)}$ for fixed t_k. Then by Lemma 2.2 and (4),

$$\begin{aligned}
E_{\theta^{(2)}}(\varphi(\mathbf{x}) \mid T_k = t_k) &= E_{\theta^{(2)}}(\psi(\mathbf{t}) \mid T_k = t_k) \\
&= \int \psi(\mathbf{t}) r_{\theta^{(2)},\theta^{(1)}}(\mathbf{t}^{(k-1)} \mid t_k) f_{\theta^{(1)}}(\mathbf{t}^{(k-1)} \mid t_k) d\mu(\mathbf{t}^{(k-1)}) \\
&\geq \int \psi(\mathbf{t}) f_{\theta^{(1)}}(\mathbf{t}^{(k-1)} \mid t_k) d\mu(\mathbf{t}^{(k-1)}) \\
&\times \int f_{\theta^{(1)}}(\mathbf{t}^{(k-1)} \mid t_k) d\mu(\mathbf{t}^{(k-1)}) \\
&= E_{\theta^{(1)}}(\psi(\mathbf{t}) \mid T_k = t_k) \\
&= E_{\theta^{(1)}}(\varphi(\mathbf{x}) \mid T_k = t_k). \quad ||
\end{aligned}$$

Now let $\theta^{(2)}$ be any point in the alternative space and let $\bar{\theta}^{(2)} = (\bar{\theta}^{(2)}, \bar{\theta}^{(2)}, \ldots, \bar{\theta}^{(2)})$ be the projection of $\theta^{(2)}$ onto the equiangular line. Let $\theta^{(1)} = \lambda\theta^{(2)} + (1-\lambda)\bar{\theta}^{(2)}$, for $0 < \lambda < 1$; i.e., $\theta^{(1)}$ lies in the alternative space on the ray orthogonal to the equiangular line connecting $\theta^{(2)}$ and $\bar{\theta}^{(2)}$.

COROLLARY 2.4. *Let $\varphi(\mathbf{x})$ be a similar size α test which is ISO* in $\mathbf{x}$. Then the power function of $\varphi(\mathbf{x})$ is conditionally monotone nondecreasing along rays orthogonal to the equiangular line. If the density in (1) satisfies the semi-group property, then the power function is unconditionally monotone.*

PROOF. Let $\theta^{(2)}$ be any point in the alternative space and let $\theta^{(1)}$ be defined as above, i.e. $\theta^{(1)} = \lambda\theta^{(2)} + (1-\lambda)\bar{\theta}^{(2)}$. Then $\theta^{(2)} - \theta^{(1)} = (1-\lambda)(\theta^{(2)} - \bar{\theta}^{(2)}) = \xi$.

Since $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_k$, the conditional monotonicity property follows from Theorem 2.3. If the density in (1) satisfies the semigroup property then the marginal distribution of T_k under $\theta^{(2)}$ is the same as the marginal distribution under $\theta^{(1)} = \lambda\theta^{(2)} + (1-\lambda)\bar{\theta}^{(2)}$. Thus Theorem 2.3 implies that unconditionally the power function is monotone.

REMARK 2.5. Theorem 2.3 can be proved by the arguments used in Cohen and Sackrowitz (1987b). The argument there is more difficult because the FKG inequality approach fails in that problem.

The next theorem identifies a general class of unbiased tests for H vs. K. Let $D = \{\mathbf{x} \mid x_1 \geq x_2 \geq \ldots \geq x_k\}$ and let $P(\mathbf{x} \mid D)$ be the unique point $\mathbf{z} \equiv (x_1, \ldots, z_k)$ in D which minimizes $\Sigma_1^k (x_i - z_i)^2$, i.e., $P(\mathbf{x} \mid D) \equiv P(\mathbf{x})$ is the projection of $\mathbf{x}$ onto D. See Brunk (1965).

THEOREM 2.6. *Let $\varphi(\mathbf{x})$ be a test function such that φ is Schur convex in $\mathbf{x}$. Let $\varphi^*(\mathbf{x}) = \varphi(P(\mathbf{x}))$. Then if $\varphi^*(\mathbf{x})$ is similar, $\varphi^*(\mathbf{x})$ is unbiased and has monotone power for H vs. K.*

PROOF. Suppose that $\mathbf{x} >>^* \mathbf{y}$. Then by Corollary 2.3 of Robertson and Wright (1982), $P(\mathbf{x}) >>^* P(\mathbf{y})$. Since $P(\mathbf{x})$ and $P(\mathbf{y})$ both lie in D it follows that $P(\mathbf{x})$ majorizes $P(\mathbf{y})$. Also since φ is Schur convex we have that

$$\varphi^*(\mathbf{x}) = \varphi(P(\mathbf{x})) \geq \varphi(P(\mathbf{y})) = \varphi^*(\mathbf{y}).$$

Thus φ^* is ISO* and the result follows from Theorem 2.3. ||

Theorem 2.6 provides a method of constructing unbiased tests for H vs. K. Suppose that $h(\mathbf{x})$ is a Schur convex test statistic used to test H vs. K': not H. Let φ be the size α test function based on h and suppose that φ is similar for H, hence has Neymann structure. Therefore

$$\begin{aligned} \varphi(\mathbf{x}) &= 1 \quad \text{if } h(\mathbf{x}) > C_\alpha(t_k) \\ &= \gamma(\mathbf{x}) \quad \text{if } h(\mathbf{x}) = C_\alpha(t_k) \\ &= 0 \quad \text{if } h(\mathbf{x}) < C_\alpha(t_k), \end{aligned}$$

where $C_\alpha(t_k)$ and $\gamma(\mathbf{x})$ are such that φ has conditional size α given $T_k = t_k$. Next define $h^*(\mathbf{x}) = h(P(\mathbf{x}))$ and now let

$$\begin{aligned} \varphi^*(\mathbf{x}) &= 1 \quad \text{if } h^*(\mathbf{x}) > C_\alpha^*(t_k) \\ &= \gamma^*(\mathbf{x}) \quad \text{if } h^*(\mathbf{x}) = C_\alpha^*(t_k) \\ &= 0 \quad \text{if } h^*(\mathbf{x}) < C_\alpha^*(t_k), \end{aligned}$$

where $C_\alpha^*(t_k)$ and $\gamma^*(\mathbf{x})$ are such that φ^* has conditional size α. Then φ^* is ISO* and by Theorem 2.6, φ^* is unbiased and has monotone power for H vs. K. We note that the critical values change from the φ test to the φ^* test.

The last result in this section concerns the case where n_i observations are taken from population i. That is, let r, $n_1, \ldots, n_r$ be positive integers such that $n_1 + n_2 + \ldots + n_r = k$ and suppose in (1) that

$$\theta_{n_1+\ldots+n_{i-1}+1} = \ldots = \theta_{n_1+\ldots+n_i} \equiv \theta_{(i)}, \quad 1 \le i \le r,$$

where $n_0 = 0$. Consider the problem of testing

$$H : \theta_{(1)} = \ldots = \theta_{(r)} \text{ vs. } \tilde{K} : \theta_{(1)} \ge \ldots \ge \theta_{(r)}$$

with at least one strict inequality under $\tilde{K}$. (Note that H is the same null hypothesis as before.) Clearly $\bar{\mathbf{X}} \equiv (\bar{X}_1, \ldots, \bar{X}_r)$ is a sufficient statistic for $(\theta_{(1)}, \ldots, \theta_{(r)})$, where

$$\bar{X}_i = \frac{1}{n_i} \Sigma_{q=1}^{n_i} X_{n_1+\ldots+n_{i-1}+q}, \quad 1 \le i \le r.$$

In Theorem 2.7 we present a condition under which a similar test $\varphi(\bar{\mathbf{x}})$ is unbiased with monotone power for H vs. $\tilde{K}$.

As in Robertson and Wright (1982), Section 5, define the weighted ordering $>>_W^*$ on $\Re^r$ as follows: $\mathbf{y} >>_W^* \mathbf{z}$ if and only if $\Sigma_{i=1}^j n_i y_i \ge \Sigma_{i=1}^j n_i z_i$, $j = 1, 2, \ldots, r-1$, with equality for $j = r$. A function f on $\Re^r$ is said to be ISO_W^* if $\mathbf{y} >>_W^* \mathbf{z}$ implies $f(\mathbf{y}) \ge f(\mathbf{z})$. Clearly, if $n_1 = \ldots = n_r$, then $>>_W^*$ and ISO_W^* reduce to the unweighted versions $>>^*$ and ISO*, respectively.

THEOREM 2.7. *Let $\varphi(\bar{\mathbf{x}})$ be a similar size α test function for H vs. $\tilde{K}$ which is ISO_W^* in $\bar{\mathbf{x}}$. Then φ is unbiased and has monotone power.*

PROOF. If $\varphi(\bar{\mathbf{x}})$ is ISO_W^* in $\bar{\mathbf{x}} \in \Re^r$, then it is easily seen that as a function of $\mathbf{x} \equiv (x_1, \ldots, x_k)$, φ is ISO* on $\Re^k$. The result then follows immediately from Theorem 2.3. ||

3. Applications. Robertson and Wright (1982), Section 4, describe a class of contrast statistics which are ISO*. Furthermore, they discuss a class of statistics based on an ℓ_2 distance between estimates satisfying the inequalities defining the alternative hypothesis K (or $\tilde{K}$). For example, in the case $n_1 = \ldots = n_r$ of equal sample sizes for each population (cf. the end of Section 2), let

$$S = \Sigma_{i=1}^r (\bar{\theta}_{(i)} - m(\bar{\mathbf{x}}))^2,$$

where $\bar{\theta} \equiv (\bar{\theta}_{(1)}, \ldots, \bar{\theta}_{(r)}) = P(\bar{\mathbf{x}} \mid D)$, and $m(\bar{\mathbf{x}}) \equiv \Sigma n_i \bar{x}_i / \Sigma n_i$ is the overall mean. Then S is ISO* by Theorem 2.6. In fact, as remarked after Theorem 2.6, we can generate an ISO* test function for H vs. K (or $\tilde{K}$) from any Schur convex test function for H vs. K'. In Cohen and Sackrowitz (1987b), test functions which were Schur convex were studied. These same test functions are ISO* when $\mathbf{x}$ is replaced by $P(\mathbf{x} \mid D)$.

To achieve unbiasedness and monotone power we recall that the critical values of the test statistics must be chosen conditionally for each value of $T_k = \Sigma_{i=1}^k X_i$. This is so since we prove unbiasedness by proving conditional unbiasedness and requiring that all conditional sizes are α. One exception is the normal case where the statistic may be chosen to be independent of T_k under H. Such is the case for Bartholomew's test ($\equiv$ the likelihood ratio test) which is based on the statistic S. It is known that the distribution of S under H is chi-bar squared and is independent of T_k. (A simple proof using Basu's theorem (see Lehmann, (1983), p. 46) establishes the independence.) Other statistics in the normal case could be chosen which are also ISO* and independent of T_k under H.

The binomial case is an important one because it arises frequently in practice. Ordered alternatives appear to be natural in experiments involving increased doses of a drug or increased learning of a subject. The current most popular approach is to use either the likelihood ratio test statistic for H vs. K or a χ^2-type statistic for H vs. K, comparing it with a critical value determined by the asymptotic distribution of the statistic under H. See Barlow, Bartholomew, Bremner, and Brunk (1972), pp. 192–193 for a discussion of these two tests. Use of the asymptotic distribution requires that the sample size in each population tends to ∞. Such a test not only requires large sample sizes but it is not unbiased. It is not unbiased because one critical value is used for all T_k and thus the test will not have Neyman structure. To find an unbiased test one can instead proceed as follows: Consider the conditional distribution of $\mathbf{X}$ (or $\bar{\mathbf{X}}$) given T_k. Test H vs. K' by a likelihood ratio test or a chi-square test. (The chi-square test is the one used for a $2 \times k$ contingency table.) Finally replace $\mathbf{x}$ (or $\bar{\mathbf{x}}$) by $P(\mathbf{x} \mid D)$ (or $P(\bar{\mathbf{x}} \mid D)$) to test H vs. K. It would be necessary to consider the conditional distribution of the statistic under H to determine critical values for each value of T_k.

Most remarks above concerning the binomial case also pertain to the Poisson case. Conditionally, in this latter case we deal with testing whether multinomial probabilities are equal to $(1/k, \ldots, 1/k)$ (or to $(1/r, \ldots, 1/r)$).

For the case of gamma random variables with a common shape parameter and unknown scale parameters $\theta_1, \ldots, \theta_k$, consider the following two-parameter family of tests for H vs. K' studied in Cohen and Strawderman (1971) and in Marshall and Olkin (1979), p. 387:

$$(5) \qquad \varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } R(\lambda, \eta) \equiv \left(\Sigma_{i=1}^k x_i^\lambda\right)^{1/\lambda} / \left(\Sigma_{i=1}^k x_i^\eta\right)^{1/\eta} > C_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where C_α is a constant determined by the size α. (This test is similar because it is scale invariant.) In (5), replace $\mathbf{x}$ by $P(\mathbf{x} \mid D) \equiv P(\mathbf{x})$ to obtain the statistic $R^*(\lambda, \eta)$ and the test function $\varphi^*(\mathbf{x}) \equiv \varphi(P(\mathbf{x}))$ for H vs. K. ||

THEOREM 3.1. *For $\lambda \geq 0 \geq \eta$, the test φ^* based on $R^*(\lambda, \eta)$ is unbiased and has monotone power for H vs. K.*

PROOF. Let $Y_i = \log X_i$, so that the distribution of $\mathbf{Y} \equiv (Y_1, \ldots, Y_k)$ form a translation-parameter family with parameter $(\log\theta_1, \log\theta_2, \ldots, \log\theta_k)$. Theorem 3.1 of Robertson and Wright (1982) implies that if $R^*(\lambda, \eta)$ is ISO* in $\mathbf{y}$ then the test based on it is unbiased. But $R(\lambda, \eta)$ is Schur convex in $\mathbf{y}$ (cf. Marshall and Olkin (1979), p. 387) so the argument used in the proof of our Theorem 2.6 implies that $\varphi(P(\mathbf{x}))$ is ISO* in $\mathbf{y}$ and hence unbiased with monotone power. (Robertson and Wright's theorem establishes the property of an ISO* power function.) ||

Note that when $\lambda = 1$ and $\eta = 0$, the test φ in (5) is equivalent to the likelihood ratio test for H vs. K'. When $\mathbf{x}$ is replaced by $P(\mathbf{x} \mid D)$, the resulting test φ^* is the likelihood ratio test for H vs. K. When $\lambda = \infty$ and $\eta = -\infty$, φ is Hartley's test for H vs. K' and φ^* is its analog for the ordered alternative K. When $\lambda = 2$ and $\eta = 1$, the test φ is the locally most powerful unbiased test for H vs. K' (see Cohen and Sackrowitz (1987b)) while when $\lambda = \infty$ and $\eta = 1$, φ corresponds to Cochran's test. Although Theorem 3.1 does not apply to these latter two cases, Theorems 2.3 and 2.6 imply that they are in fact unbiased as long as the gamma density is PF_2, i.e., whenever the shape parameter is at least one.

We conclude this section by mentioning two counterexamples. The first already appears in Cohen, Sackrowitz, and Strawderman (1985), Example 5.4. It is an example where a test function φ for H vs. K is similar and ISO* and where the underlying density is of exponential type, but the test is not conditionally unbiased. Theorem 2.3 does not apply here because the function g in (1) is *not* PF_2 in this example.

The second counterexample presents a test function for H vs. K which is ISO* but which does not have (conditionally) an ISO* power function when the underlying distribution is binomial. In fact, take $k = 3$ and let $X_i \sim B(2, p_i)$, $i = 1, 2, 3$. Conditional on $T_3 \equiv X_1 + X_2 + X_3 = 2$, the six possible sample points are (2,0,0), (1,1,0), (0,2,0), (1,0,1), (0,1,1) and (0,0,2). The test function which rejects H for (2,0,0), (1,1,0) and (0,2,0) is conditionally ISO* with conditional size 0.4. The conditional power at $\mathbf{p}^{(1)} = (.88, .01, .01)$ is .8841, while the conditional power at $\mathbf{p}^{(2)} = (.45, .44, .01)$ is .9708, although $\mathbf{p}^{(1)} >>^* \mathbf{p}^{(2)}$. This example shows that the method of Robertson and Wright (1982) cannot work conditionally for the binomial case, while our method does establish unbiasedness. An open question is whether the power function is *unconditionally* ISO* for the binomial case when the test function is ISO*.

4. Admissibility of Tests. It follows from a theorem of Eaton (1970) that tests which conditionally are ISO* with convex acceptance sections for fixed T_k form an essentially complete class of tests. In the binomial and Poisson cases one can use the technique in Matthes and Truax (1967), Section 4(b) to establish that these tests are actually admissible. (The Poisson case requires only a bit more argument but an example of such an argument appears in Cohen and Sackrowitz (1987a).)

For the continuous cases, however, conditional admissibility does not imply unconditional admissibility except for the case $k = 2$. Nevertheless, we can prove the following theorem for the normal case.

THEOREM 4.1. *Let* $X_1, \ldots, X_k$ *be independent with* $X_i \sim N(\theta_i, 1)$, $i = 1, \ldots, k$. *The likelihood ratio test for* H *vs.* K *(Bartholomew's test) is admissible.*

PROOF. As discussed in Section 3 the likelihood ratio statistic S is independent of $T_k = \Sigma_{i=1}^k X_i$. Furthermore each acceptance section is convex by the argument in Birnbaum (1955), Section 9. Hence the overall acceptance region is convex. Since the test is ISO* as well as convex this permits application of the theorem of Stein (1956) to establish admissibility.

REMARK 4.2. It is clear that for the model (1) there are many ISO* tests for H vs. K that are unbiased but are inadmissible. Any similar test function $\varphi(\mathbf{x})$ which is ISO* but yet has <u>non</u> convex acceptance sections for fixed T_k is unbiased (by Theorem 2.3) and inadmissible (by the complete class theorem of Mattes and Truax (1967) or of Eaton (1970)). One simple example is to take $k = 3$, $T_3 = t_3$ and the acceptance region to be the union of $\{X_1 < C_1\}$ and $\{X_3 > C_3\}$.

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